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Robust Dynamic Output Feedback Second-Order Sliding Mode Controller for Uncertain Systems

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Abstract: This paper addresses the problem of designing a dynamic output feedback sliding mode control algorithm to stabilize a linear MIMO uncertain system having relative degree two. Introducing a suitable dynamic compensator into the sliding variable, the additional degree of freedom can be used to robustly guarantee the closed-loop system stability once the system is in the sliding mode. A modified asymptotically stable second-order sliding mode control is analyzed and the proposed controller can obtain the real second-order sliding mode. Finally, the feasibility of the proposed method is illustrated by a numerical example.

Keywords: Dynamic output feedback, relative degree two, second-order, sliding mode.

1. INTRODUCTION

Previous researches [1-7] have concentrated on designs for output feedback controllers via sliding mode technique to stabilize multivariable plants with matched uncertainties. Early on, Zak and Hui [1] developed an algorithm that uses the eigenstructure method to design an output-dependent sliding variable for uncertain systems. Kwan [3] presented an adapted dynamic output feedback controller to remove two major limits from the scheme of Zak and Hui's method [1]. Further, Edwards and Spurgeon [4] have synthesized output feedback controllers for uncertain systems with reference to the ideas of sliding mode. Of the basis of analyzing static output feedback sliding mode control design, two conditions presented here are used for checking for the existence of a stable controller. The first is that the system must be minimum phase. The second is a rank condition in which the relative degree of the transfer function matrix is one. For a mechanical system using the position information only, the static output feedback sliding mode control algorithm cannot be directly implemented, because of the lack of the rank condition. Hence, these two important conditions limit the practical applications of the abovementioned approaches.

The concept of high order sliding mode as the generation of conventional sliding mode has been recently developed. For example, the case of second-order sliding mode corresponds to the control acting on the second derivative of the sliding variable. Several such second-order sliding mode algorithms have been presented in these papers [8-13]. Levant [8,9] presented the twisting algorithm to stabilize second-order nonlinear systems but used knowledge of the output-derivative. Bartolini [11] developed an optimized version of the twisting algorithm. The super twisting algorithm [8,9] does not require the output derivative to be measured but it has been originally developed and analyzed for system with relative degree one. A robust exact finite time convergence differentiator is proposed in [12], which is based on this controller. Fridman et al. [13] applied the similar technique to construct the velocity observer for mechanical systems.

An alternative output feedback second-order sliding mode controller for relative degree two MIMO systems is proposed in this paper. The developed control algorithm does not include any explicit differentiator. We first propose a modified second-order sliding mode control in which it can guarantee the global asymptotically stability and does not require the derivative of the sliding variable. A suitable dynamic compensator is introduced into the sliding variable in which the effect of the derivative action can be obtained by the additional compensator. Once the system is in the sliding mode, the additional degree of freedom can be used to robustly stabilize the closed-loop system and obtain the desired system performance. The proposed control law theoretically provides the real second-order sliding mode.

2. PROBLEM FORMULATION

Consider an uncertain system that satisfies the matched condition of the form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\big(\mathbf{u}(t) + \mathbf{d}(\mathbf{x}, t)\big), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \end{aligned} \tag{1}$$

where $\mathbf{x} \in \Re^n$, $\mathbf{u} \in \Re^m$, $\mathbf{d} \in \Re^m$, and $\mathbf{y} \in \Re^p$ are the state position vector, the control forces, the unknown matched disturbance vector and the output vector, respectively. Without loss generality, we assume that $rank(\mathbf{C}) = p$ and $rank(\mathbf{B}) = m$ where $p \geq m$. Suppose

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that the pairs (**A**, **B**) and (**A**, **C**) are stabilizable and detectable, respectively. If the two conditions (1) the triple (**C**, **A**, **B**) is minimum phase and (2) *rank*(**CB**) = *m* hold, Spurgeon and Edwards [4] have shown that a static output-dependent sliding variable can be designed to stabilize the reduced-order system. When the mechanical system uses position measurement only, the transfer matrix function has relative degree two, so conventional static output feedback sliding mode control methods [1-4] cannot be directly implemented in mechanical systems without using velocity measurements. In this paper, we consider a dynamic output feedback sliding mode control algorithm in which the proposed procedure is capable of being used in the system with relative degree two. A modified robust globally asymptotically stable second-order sliding mode is presented and discussed. Introducing an additional dynamic compensator into the sliding variable and using the concept of second-order sliding mode, both robust stability of the closed-loop system and external disturbance attenuation can be guaranteed once the system is in the sliding mode. Before introducing the proposed method, the following assumptions are made throughout this paper.

Assumption 1: The matched disturbance $\mathbf{d}(\mathbf{x}, t)$ has the upper bound.

Assumption 2: The matrix ***CAB*** is of full rank.

Assumption 3: System (1) is minimum phase.

3. GLOBAL STABILITY OF PERTURBED SECOND-ORDER SYSTEMS

In this section, we present a preliminary result that will be useful in designing the controller. Consider the following system:

$$\ddot{\sigma}(t)+l_1\dot{\sigma}(t)+l_2\sigma(t)=-ksign(\sigma(t))+f(t), \tag{2}$$

where $\sigma \in \Re$, l_1, l_2 and k are positive constants designed by the user. Moreover, $f(t)$ is an external perturbation with the bound

$$|f(t)| \le \eta, \tag{3}$$

where $\eta > 0$ is a known constant.

Lemma 1: Consider the unperturbed system as

$$\ddot{\sigma}(t)+l_1\dot{\sigma}(t)+l_2\sigma(t)=-ksign(\sigma(t)). \tag{4}$$

If the roots of the characteristic equation $s^2+l_1 s+l_2=0$ are stable, then the two variables σ(t) and $\dot{\sigma}(t)$ asymptotically converge to zero for k > 0.

Proof: First, we choose the parameters l_1 and l_2 such that the roots of the characteristic equation, $s^2+l_1 s+l_2=0$, are located in the left-half plane. Assume now for simplicity that the initial conditions are $\sigma(t_0)=0$ and $\dot{\sigma}(t_0)>0$. Thus the trajectory enters the half-plane $\sigma(t)>0$ (quadrant I). When $\sigma(t)>0$, we have $\ddot{\sigma}(t)+l_1\dot{\sigma}(t)+l_2\sigma(t)=-k$ and obtain its equivalent point as $(\sigma,\dot{\sigma})=(-k/l_2,0)$. Since the roots of the characteristic equation are all stable, the curve will hit the axis $\sigma(t)=0$ in finite time. Let the trajectory of

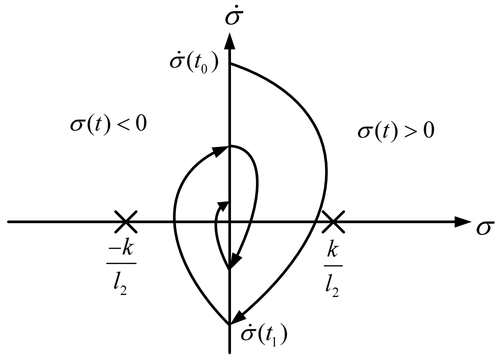

Fig. 1. Phase paths of the second-order system.

system (4) intersect next time with the axis $\sigma(t)=0$ at the point $\dot{\sigma}(t_1)$. Then the trajectory enters the half-plane $\sigma(t)<0$ (quadrant III). When $\sigma(t)<0$, we have $\ddot{\sigma}(t)+l_1\dot{\sigma}(t)+l_2\sigma(t)=k$ and obtain its equivalent point as $(\sigma,\dot{\sigma})=(k/l_2,0)$. Since the system is stable, it follows that the system will hit the axis $\sigma(t)=0$ in finite time. Therefore, its solutions cross the axis $\sigma(t)=0$ from quadrant II to quadrant I, and from quadrant IV to quadrant III. After gluing these paths along the line $\sigma(t)=0$, we obtain the phase portrait of the system, as shown in Fig. 1. Then we choose a Lyapunov function as

$$V(t)=\frac{\dot{\sigma}^2(t)}{2}+\frac{l_2\sigma^2(t)}{2}+k|\sigma(t)|,$$

and then obtain its time derivative as

$$\begin{aligned}\dot{V}(t)&=\dot{\sigma}(t)\left(-l_1\dot{\sigma}(t)-l_2\sigma(t)-ksign(\sigma(t))\right)\\&\quad+l_2\dot{\sigma}(t)\sigma(t)+ksign(\sigma(t))\dot{\sigma}(t)\\&=-l_1\dot{\sigma}^2(t).\end{aligned}$$

From the above equation, we can obtain that the variables $\sigma(t)$ and $\dot{\sigma}(t)$ asymptotically converge to zero. The proof of the lemma is finished.

Theorem 1: Consider system (2) with satisfying (3). If the parameters l_1 and l_2, and the gain k are chosen to satisfy the following conditions:

$$l_2<\frac{l_1^2}{4} \text{ and } k>\eta, \tag{5}$$

then the two variables $\sigma(t)$ and $\dot{\sigma}(t)$ asymptotically converge to zero.

Proof: Since the parameters l_1 and l_2 are chosen to satisfy the condition (5), the roots of the characteristic equation, $s^2+l_1 s+l_2=0$, are stable. When $\sigma(t)>0$, equation (2) becomes

$$\ddot{\sigma}(t)+l_1\dot{\sigma}(t)+l_2\sigma(t)=-k+f(t).$$

Let $v_1=\sigma+\dfrac{k}{l_2}$ and $v_2=\dot{v}_1=\dot{\sigma}$. It follows that

$$\dot{\mathbf{v}}(t)=\begin{bmatrix}0 & 1\\ -l_2 & -l_1\end{bmatrix}\begin{bmatrix}v_1(t)\\ v_2(t)\end{bmatrix}+\begin{bmatrix}0\\ f(t)\end{bmatrix}=\Phi\mathbf{v}(t)+\mathbf{b}f(t),$$

where $\mathbf{v} = \begin{bmatrix} v_1^T & v_2^T \end{bmatrix}^T$, $\Phi = \begin{bmatrix} 0 & 1 \\ -l_2 & -l_1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Write the above dynamic equation as

$$\mathbf{v}(t) = \mathbf{e}^{\Phi t}\mathbf{v}(0) + \int_0^t \mathbf{e}^{\Phi \tau}\mathbf{b} f(t-\tau)d\tau.$$

Since two parameters l_1 and l_2 are chosen to satisfy (5), we know that it has two distinct real roots $\lambda_{1,2} = -\alpha, -\beta$ where $\beta > \alpha > 0$, $l_1 = \alpha + \beta$ and $l_2 = \alpha\beta$. Under this condition, we have

$$\mathbf{e}^{\Phi t} = \begin{bmatrix} \frac{1}{\beta-\alpha}(\beta e^{-\alpha t} - \alpha e^{-\beta t}) & \frac{1}{\beta-\alpha}(e^{-\alpha t} - e^{-\beta t}) \\ \frac{\alpha\beta}{\alpha-\beta}(e^{-\alpha t} - e^{-\beta t}) & \frac{1}{\alpha-\beta}(\alpha e^{-\alpha t} - \beta e^{-\beta t}) \end{bmatrix}.$$

The upper bound of $v_1(t)$ can be constructed as

$$\begin{aligned} |v_1(t)| &\le C_1 e^{-\alpha t} + \frac{\eta}{\beta-\alpha}\int_0^t \left| e^{-\alpha\tau} - e^{-\beta\tau} \right| d\tau \\ &= C_1 e^{-\alpha t} + \frac{\eta}{\beta-\alpha}\left(\frac{\beta-\alpha}{\alpha\beta}\right) = C_1 e^{-\alpha t} + \frac{\eta}{l_2}, \end{aligned}$$

where $C_1 > 0$ is a constant. It follows that

$$|v_1(t)| = |\sigma(t) + k/l_2| \le C_1 e^{-\alpha t} + \eta / l_2. \qquad (6)$$

Equation (6) shows that the ball of radius $r = \eta / l_2$ with center located at $(-k/l_2, 0)$ is an attractor B_{s1}. Similar to the work, we have, when $\sigma(t) < 0$, the ball of radius r, with center located at $(k/l_2, 0)$ is another attractor B_{s2}. Choose the gain k to satisfy the inequality $k > \eta$ and then we have

$$\left(\frac{k}{l_2} - r\right) = \frac{k}{l_2} - \frac{\eta}{l_2} > 0 \text{ and } \left(-\frac{k}{l_2} + r\right) = \frac{-k}{l_2} + \frac{\eta}{l_2} < 0.$$

It follows from the above two inequalities that the two attractors B_{s1} and B_{s2} do not intersect each other, and the behavior of the perturbed system (2) will be qualitatively similar to the behavior of the nominal system (4). Therefore, the perturbed system converges to the origin in the same way of the nominal system and the condition that $\sigma(t)$ and $\dot{\sigma}(t)$ asymptotically go to zero can be guaranteed. We complete the proof of this theorem.

Given a sliding variable σ, however ideal sliding mode is achieved by means of a control signal switching at infinite frequency, which cannot be attained in real plants. It was proven [8] that the best possible sliding accuracy attainable with discrete measurements is

$$|\sigma(t)| = O(T^2) \text{ and } |\dot{\sigma}(t)| = O(T), \qquad (7)$$

where the sampling interval is $T > 0$ and the magnitude of a variable v is said to be of order $O(T^n)$ if

$$\lim_{T\to 0}\frac{v}{T^n} \ne 0 \text{ and } \lim_{T\to 0}\frac{v}{T^{n-1}} = 0, \qquad (8)$$

where n is an integer. We define that $O(T^0) = O(1)$. Moreover, the condition (7) is also called the 'real second-order sliding mode'.

Lemma 2 [14]**:** If g is continuously differentiable with respect to its all arguments and satisfies

$$G_{diff} \equiv \sup\left(\frac{dg(t)}{dt}\right) = O(T^n),$$

then for any number $\zeta_1 > 0$ and $\zeta_2 > 0$, $|\zeta_1 - \zeta_2| = O(T)$, we have

$$g(\zeta_2) - g(\zeta_1) = O(T^{n+1}).$$

Theorem 2: Consider the perturbed system as (2). If the conditions of Theorem 1 hold and the variable σ is sampled with a constant sampling interval T, then, after a finite time, the system ensures the establishment of a real second-order sliding mode, i.e.,

$$|\sigma(t)| = O(T^2) \text{ and } |\dot{\sigma}(t)| = O(T),$$

where the sampling interval T is sufficiently small such that the two operations $O(T^n) + O(T^{n+1}) \approx O(T^n)$ and $O(T^n)O(1) \approx O(T^n)$ are hold.

Proof: Based on the result of Theorem 1, we know that the behavior of the perturbed system (2) will be qualitatively similar to the behavior of the nominal system (3). Let $|\ddot{\sigma}(t)| = O(T^k)$ where k, decided in the latter, is a positive constant. It follows from Lemma 2 that $|\dot{\sigma}(t)| = O(T^{k+1})$ and $|\sigma(t)| = O(T^{k+2})$. Using two approximation operations, we can obtain

$$\begin{aligned} |\ddot{\sigma} + l_1\dot{\sigma} + l_2\sigma| &\le |\ddot{\sigma}| + l_1|\dot{\sigma}| + l_2|\sigma| \\ &= O(T^k) + O(1)O(T^{k+1}) + O(1)O(T^{k+2}) \\ &\approx O(T^k). \end{aligned}$$

Since $|ksign(\sigma(t))| = O(1)$, we from the above equation have $k = 0$ and then obtain

$$|\sigma(t)| = O(T^2) \text{ and } |\dot{\sigma}(t)| = O(T).$$

The proof of this theorem is finished.

4. OUTPUT FEEDBACK CONTROLLER DESIGN

Chang [15] has proposed a second-order sliding mode control for MIMO systems, where an additional dynamics is imposed on the sliding variable, but the resulting controller requires the derivative of the sliding variable. In this section, we develop a modified second-order sliding mode controller using output feedback only in which the proposed algorithm does not require the derivative of the sliding variable. An output-dependent PID sliding variable with two independent gain parameters is introduced to obtain the effect of the derivative action throughout the additional compensator. The proposed control law can obtain the desired system performance once the system is in the sliding mode.

Since Assumption 2 holds, the matrix $\mathbf{F} \in \Re^{m\times p}$ which is of full rank should be found such that the matrix

FCAB is invertible. Let $\mathbf{L}_1 \in \Re^{m\times m}$ be the positive definite diagonal matrix given by $\mathbf{L}_1 = diag(l_{11},\cdots,l_{1m})$ and design the sliding variable as

$$\dot{\mathbf{w}}(t) = -\mathbf{L}_1\mathbf{w}(t) - \mathbf{L}_1\mathbf{F}\mathbf{y}(t) + \mathbf{K}_D\mathbf{y}(t) + \mathbf{K}_P\int_0^t \mathbf{y}(\tau)d\tau, \quad \mathbf{s}(t) = \mathbf{F}\mathbf{y}(t) + \mathbf{w}(t), \tag{9}$$

where $\mathbf{w} = [w_1 \cdots w_m]^T$, $\mathbf{s} = [s_1 \cdots s_m]^T$, and the gain matrices $\mathbf{K}_D \in \Re^{m\times q}$ and $\mathbf{K}_P \in \Re^{m\times q}$ are designed in the latter. From (9), we have

$$\dot{\mathbf{s}}(t) + \mathbf{L}_1\mathbf{s}(t) = \mathbf{F}\dot{\mathbf{y}}(t) + \mathbf{K}_D\mathbf{y}(t) + \mathbf{K}_P\int_0^t \mathbf{y}(\tau)d\tau. \tag{10}$$

Taking the time derivative of (10) and substituting the system dynamics into it can obtain

$$\ddot{\mathbf{s}}(t) + \mathbf{L}_1\dot{\mathbf{s}}(t) = \left(\mathbf{FCA}^2 + \mathbf{K}_D\mathbf{CA}\right)\mathbf{x}(t) + \mathbf{K}_P\mathbf{y}(t) + \mathbf{FCAB}\left(\mathbf{u}(t) + \mathbf{d}(\mathbf{x},t)\right). \tag{11}$$

The controller is designed as

$$\mathbf{u}(t) = -\left(\mathbf{FCAB}\right)^{-1}\left(\left(\mathbf{FCA}^2 + \mathbf{K}_D\mathbf{CA}\right)\hat{\mathbf{x}}(t) + \mathbf{K}_P\mathbf{y}(t) + \mathbf{L}_2\mathbf{s}(t) + \mathbf{K}sign\left(\mathbf{s}(t)\right)\right), \tag{12}$$

where $\mathbf{L}_2 = diag(l_{21},\cdots,l_{2m})$, $\mathbf{K} = diag(k_1,\cdots,k_m)$, and $sign(\mathbf{s}(t)) = [sign(s_1(t))\cdots sign(s_m(t))]^T$. The estimation states are generated by the following Luenberger observer

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}\left(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t)\right). \tag{13}$$

Since the pair $(\mathbf{A}, \mathbf{C})$ is detectable, a gain matrix $\mathbf{L} \in \Re^{n\times p}$ should be found to stabilize the matrix $\mathbf{A} - \mathbf{LC}$. Substituting the control input (12) into (11), we obtain

$$\ddot{\mathbf{s}}(t) + \mathbf{L}_1\dot{\mathbf{s}}(t) + \mathbf{L}_2\mathbf{s}(t) = -\mathbf{K}sign\left(\mathbf{s}(t)\right) + \mathbf{f}(t), \tag{14}$$

where $\mathbf{f}(t) = (\mathbf{FCA}^2 + \mathbf{K}_D\mathbf{CA})\tilde{\mathbf{x}}(t) + \mathbf{FCABd}(\mathbf{x},t)$ and $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$ is the estimation error. The dynamic equation of $\tilde{\mathbf{x}}$ is given by

$$\dot{\tilde{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{LC})\tilde{\mathbf{x}}(t) + \mathbf{Bd}(\mathbf{x},t). \tag{15}$$

Since the matrix $\mathbf{A} - \mathbf{LC}$ is Hurwitz and the matched disturbance is bounded, the estimation error $\tilde{\mathbf{x}}(t)$ has the upper bound and hence, the vector $\mathbf{f}(t)$ is also bounded. Let the vector $\mathbf{f}(t) = [f_1 \cdots f_m]^T$ have the upper bound and then obtain $\left|f_i(t)\right| \le \eta_i$ for $i = 1,\cdots,m$ where $\eta_i > 0$ is a known constant. We first express system (14) as a set of second-order systems with the form

$$\ddot{s}_i(t) + l_{1i}\dot{s}(t) + l_{2i}s(t) = -k_i sign\left(s_i(t)\right) + f_i(t). \tag{16}$$

Since the signal $f_i(t)$ has the upper bound, we choose the parameters l_{1i}, l_{2i} and k_i are chosen to satisfy the following conditions:

$$l_{2i} < \frac{l_{1i}^2}{4} \text{ and } k_i > \eta_i, \text{ for } i = 1,\cdots,m. \tag{17}$$

Based on the result of Theorem 2 and (17), we know that the system (26) satisfies the condition of real second-order sliding mode and hence, can obtain from the concept of equivalent control

$$\left[\mathbf{K}sign\left(\mathbf{s}(t)\right)\right]_{eq} = \left(\mathbf{FCA}^2 + \mathbf{K}_D\mathbf{CA}\right)\tilde{\mathbf{x}}(t) + \mathbf{FCABd}(\mathbf{x},t). \tag{18}$$

Then the control input in the sliding mode becomes

$$\mathbf{u}_{eq}(t) = -\left(\mathbf{FCAB}\right)^{-1}\left(\mathbf{K}_P\mathbf{y}(t) + \left(\mathbf{FCA}^2 + \mathbf{K}_D\mathbf{CA}\right)\mathbf{x}(t)\right) - \mathbf{d}\left(\mathbf{x},t\right). \tag{19}$$

Substitute this term (19) into system (1) to obtain the closed-loop system in the sliding mode as

$$\dot{\mathbf{x}} = \left(\mathbf{A} - \mathbf{B}\left(\mathbf{FCAB}\right)^{-1}\left(\mathbf{FCA}^2 + \mathbf{K}_D\mathbf{CA} + \mathbf{K}_P\mathbf{C}\right)\right)\mathbf{x} = \left(\bar{\mathbf{A}} - \mathbf{BG}\bar{\mathbf{C}}\right)\mathbf{x}, \tag{20}$$

where $\bar{\mathbf{C}} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix}$, $\bar{\mathbf{A}} = \mathbf{A} - \mathbf{B}(\mathbf{FCAB})^{-1}\mathbf{FCA}^2$, and $\mathbf{G} = (\mathbf{FCAB})^{-1}[\mathbf{K}_P \quad \mathbf{K}_D]$. From (20), we know that the estimation error dynamics and the matched disturbance do not affect the system behavior. Although system (1) has relative degree two, the proposed algorithm can obtain the effect of the desired differentiator and the additional degree of freedom is capable of stabilizing system (20).

Lemma 3: Consider the following system

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}(\mathbf{FCAB})^{-1}\mathbf{FCA}^2)\mathbf{x}(t) + \mathbf{Bu}(t), \quad \mathbf{y}(t) = \bar{\mathbf{C}}\mathbf{x}(t). \tag{21}$$

If Assumptions 2 to 3 hold, then system (21) is minimum phase.

Proof: In this proof, we shall show that the invariant zeros of systems (1) and (21) are the same. First,

$$\begin{bmatrix} \lambda\mathbf{I}_n - \mathbf{A} + \mathbf{B}(\mathbf{FCAB})^{-1}\mathbf{FCA}^2 & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \lambda\mathbf{I}_n - \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ (\mathbf{FCAB})^{-1}\mathbf{FCA}^2 & \mathbf{I} \end{bmatrix}.$$

If $\lambda \in \Re$ is an invariant zero of system (1), then it follows that

$$rank\left(\begin{bmatrix} \lambda\mathbf{I}_n - \mathbf{A} + \mathbf{B}(\mathbf{FCAB})^{-1}\mathbf{FCA}^2 & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}\right) < n + m.$$

Hence, there exist vectors $\mathbf{x}^0 \neq \mathbf{0}$ and $\mathbf{g}^0$ such that the following equation is satisfied:

$$\begin{bmatrix} \lambda\mathbf{I}_n - \mathbf{A} + \mathbf{B}(\mathbf{FCAB})^{-1}\mathbf{FCA}^2 & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{x}^0 \\ \mathbf{g}^0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

From the above equation, we have $(\lambda\mathbf{I}-\mathbf{A}+\mathbf{B}(\mathbf{FCAB})^{-1}\mathbf{FCA}^2)\mathbf{x}^0+\mathbf{Bg}^0=\mathbf{0}$ and $\mathbf{Cx}^0=\mathbf{0}$. Multiplying the equation $(\lambda\mathbf{I}-\mathbf{A}+\mathbf{B}(\mathbf{FCAB})^{-1}\mathbf{FCA}^2)\mathbf{x}^0+\mathbf{Bg}^0=\mathbf{0}$ from the left by $\boldsymbol{C}$ and using the conditions $\mathbf{Cx}^0=\mathbf{0}$ and $\mathbf{CB}=\mathbf{0}$ can be shown that $\mathbf{CAx}^0=\mathbf{0}$. Hence, we can obtain that the triple $\lambda\in\Re$, $\mathbf{x}^0$, $\mathbf{g}^0$ satisfy the following equation:

$$\begin{bmatrix}\lambda\mathbf{I}-\mathbf{A}+\mathbf{B}(\mathbf{FCAB})^{-1}\mathbf{FCA}^2 & \mathbf{B}\\ \bar{\mathbf{C}} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x}^0\\ \mathbf{g}^0\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix},$$

and conclude that $\lambda\in\Re$ is also an invariant zero of system (21). If $\lambda_2\in\Re$ is an invariant zero of system (21), then there exist vectors $\mathbf{x}^2\neq\mathbf{0}$ and $\mathbf{g}^2$ satisfying

$$\begin{bmatrix}\lambda_2\mathbf{I}-\mathbf{A}+\mathbf{B}(\mathbf{FCAB})^{-1}\mathbf{FCA}^2 & \mathbf{B}\\ \bar{\mathbf{C}} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x}^2\\ \mathbf{g}^2\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}.$$

From the above equation, we have $(\lambda_2\mathbf{I}-\mathbf{A}+\mathbf{B}(\mathbf{FCAB})^{-1}\mathbf{FCA}^2)\mathbf{x}^2+\mathbf{Bg}^2=\mathbf{0}$ and $\mathbf{Cx}^2=\mathbf{0}$. Let $\mathbf{g}^3=(\mathbf{FCAB})^{-1}\mathbf{FCA}^2\mathbf{x}^2+\mathbf{g}^2$ and then obtain

$$\begin{bmatrix}\lambda_2\mathbf{I}_n-\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x}^2\\ \mathbf{g}^3\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}.$$

It follows that $\lambda_2\in\Re$ is also an invariant zero of system (1). From the above analysis, we can conclude that the invariant zeros of system (1) and system (21) are equivalent. As a result, system (21) is minimum phase. We complete the proof of this lemma.

Lemma 4: If Assumptions 2 to 3 hold, then the pairs $(\bar{\mathbf{A}},\mathbf{B})$ and $(\bar{\mathbf{A}},\bar{\mathbf{C}})$ are stabilizable and detectable, respectively.

Proof: Since state feedback cannot change the controllability, from $\bar{\mathbf{A}}=\mathbf{A}-\mathbf{B}(\mathbf{FCAB})^{-1}\mathbf{FCA}^2$, we can conclude that the pair $(\bar{\mathbf{A}},\mathbf{B})$ is stabilizable. It follows from Lemma 3 that

$$rank\left(\begin{bmatrix}s\mathbf{I}_n-\bar{\mathbf{A}} & \mathbf{B}\\ \bar{\mathbf{C}} & \mathbf{0}\end{bmatrix}\right)=rank\left(\begin{bmatrix}s\mathbf{I}_n-\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{0}\end{bmatrix}\right)=n+m \quad \forall s\in C^+.$$

Since $rank(\bar{\mathbf{C}}\mathbf{B})=rank(\mathbf{FCAB})=m$, the realization $(\bar{\mathbf{C}},\bar{\mathbf{A}},\mathbf{B})$ can be written as [4]

$$\bar{\mathbf{A}}=\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}\mathbf{B}_1\\ \mathbf{0}\end{bmatrix},\ \bar{\mathbf{C}}=\begin{bmatrix}\mathbf{C}_1 & \mathbf{C}_2\end{bmatrix},$$

where the matrix $\mathbf{B}_1\in\Re^{m\times m}$ is invertible and the matrix $\mathbf{C}_1\in\Re^{2p\times m}$ has full rank. Hence,

$$rank\left(\begin{bmatrix}s\mathbf{I}_n-\bar{\mathbf{A}} & \mathbf{B}\\ \bar{\mathbf{C}} & \mathbf{0}\end{bmatrix}\right)=rank\left(\begin{bmatrix}-\mathbf{A}_{21} & s\mathbf{I}_{n-m}-\mathbf{A}_{22}\\ \mathbf{C}_1 & \mathbf{C}_2\end{bmatrix}\right)+m.$$

It follows that

$$rank\left(\begin{bmatrix}-\mathbf{A}_{21} & s\mathbf{I}_{n-m}-\mathbf{A}_{22}\\ \mathbf{C}_1 & \mathbf{C}_2\end{bmatrix}\right)=n \quad \forall \operatorname{Re}(s)>0.$$

From linear algebraic theory and the above rank condition, we can obtain

$$rank\left(\begin{bmatrix}s\mathbf{I}_n-\bar{\mathbf{A}}\\ \bar{\mathbf{C}}\end{bmatrix}\right)=rank\left(\begin{bmatrix}s\mathbf{I}_m-\mathbf{A}_{11} & -\mathbf{A}_{12}\\ -\mathbf{A}_{21} & s\mathbf{I}_{n-m}-\mathbf{A}_{22}\\ \mathbf{C}_1 & \mathbf{C}_2\end{bmatrix}\right)=n \quad \forall \operatorname{Re}(s)>0.$$

As a result, the pair $(\bar{\mathbf{A}},\bar{\mathbf{C}})$ is detectable. We complete the proof of the lemma.

From $\bar{\mathbf{C}}\mathbf{B}=\begin{bmatrix}\mathbf{CB}\\ \mathbf{CAB}\end{bmatrix}$ and Lemma 3, we can know that system (21) is minimum phase and has relative degree one. When the system is minimum phase and has relative degree one, Schumacher [16] have shown that it can be stabilized by direct output feedback alone. As a result, the static output feedback design techniques [17,18] can be used to search the gain matrix $\boldsymbol{G}$ such that the matrix $\bar{\mathbf{A}}-\mathbf{B}\mathbf{G}\bar{\mathbf{C}}$ is Hurwitz. Having designed the matrix $\boldsymbol{G}$, we can from $\mathbf{G}=(\mathbf{FCAB})^{-1}[\mathbf{K}_D \quad \mathbf{K}_P]$ obtain that the parameters $\mathbf{K}_D$ and $\mathbf{K}_P$. Since the matrix $\bar{\mathbf{A}}-\mathbf{B}\mathbf{G}\bar{\mathbf{C}}$ is Hurwitz, we can from (20) obtain that the system performance satisfies the following property:

$$\mathbf{y}(t)\to\mathbf{0} \quad \text{as} \quad t\to\infty.$$

5. NUMERICAL EXAMPLE

To demonstrate the design techniques, an inverted pendulum mechanical system is considered in this section, where the values of the physical parameters are the same as those in Edwards's book [4]. Let r, $\dot{r}$, θ, and $\dot{\theta}$ be the system states and assume that only r and θ are available for measurement. Moreover, the unknown matched disturbance is set as $d(t)=1.5\sin(\pi t)+0.5\cos(5t)$. For this case, the conventional static output feedback sliding mode controllers [1-4] cannot be successfully implemented. Since $\begin{bmatrix}\mathbf{C}\\ \mathbf{CA}\end{bmatrix}=\mathbf{I}_4$, we first assign the desired eigenvalues of the closed-loop system $\bar{\mathbf{A}}-\mathbf{B}[\mathbf{K}_P \quad \mathbf{K}_D]$ as $\{-2,-3\pm i,-1.5\}$ to obtain $\mathbf{K}_D=[-5.3681 \quad -11.1012]$ and $\mathbf{K}_P=[-3.0305 \quad -37.6977]$. Choosing $\mathbf{L}_1=15$ and $\mathbf{L}_2=1$, we design the sliding variable as

$$\begin{aligned}s(t)&=[0.2857 \quad -0.8999]y(t)+w(t),\\ \dot{w}(t)&=-15w(t)+[-9.6536 \quad 2.3970]y(t)\\ &\quad-[3.0305 \quad 37.6977]\int_0^t y(\tau)d\tau.\end{aligned}$$

The controller using the sign function is given by

$$\begin{aligned}u(t)&=[0 \quad 33.8275 \quad 11.5681 \quad 10.9382]\hat{\mathbf{x}}(t)\\ &\quad+[3.0305 \quad 37.6977]\mathbf{y}(t)+s(t)+6sign(s(t)),\end{aligned}$$

where the observer is designed as

$$\dot{\hat{\mathbf{x}}}(t)=\begin{bmatrix}-6.6303 & -1.2027 & 1 & 0\\ -0.4463 & -12.2042 & 0 & 1\\ -4.3855 & -3.0011 & -1.9872 & 0.0091\\ -12.2680 & -42.1505 & 6.2589 & -0.1783\end{bmatrix}\hat{\mathbf{x}}(t)$$

$$+\mathbf{B}u(t)+\begin{bmatrix}6.6303 & 1.2027\\ 0.4463 & 12.2042\\ 4.3855 & 1.0678\\ 12.2680 & 79.1276\end{bmatrix}\mathbf{y}(t).$$

The simulation is carried out at a fixed step size of 0.1 milliseconds and the initial states are set as $\mathbf{x}(0)=[0.1\quad 0\quad 0\quad 0]^T$. Figs. 2-3 show the responses of the system output and the sliding variable, respectively. The phase plot of the sliding variable is shown in Fig. 4. The steady state error in the sliding variable, as shown in Fig. 5, is of the order of 10^{-8} and that of $\dot{\mathbf{s}}(t)$ is of the order of 10^{-4}. As can be seen from these figures, the proposed method produces the 'real second-order sliding mode' Figs. 6-8 gives the simulation results using the saturation function $sat(s,\varepsilon)$ instead of the sign function where $\varepsilon=0.005$. The responses of the system output and the control input are illustrated in Figs. 6 and 7, respectively. Fig. 8 shows that the system is finally constrained in the sliding layer. Although the dynamic output feedback control law raises control complexity and requires additional software, the proposed control scheme globally guarantees the robust stability of the closed-loop system and the property of disturbance attenuation in the proposed algorithm is evident. Without using the velocity feedback, the proposed control law can stabilize the inverted pendulum mechanical system very well.

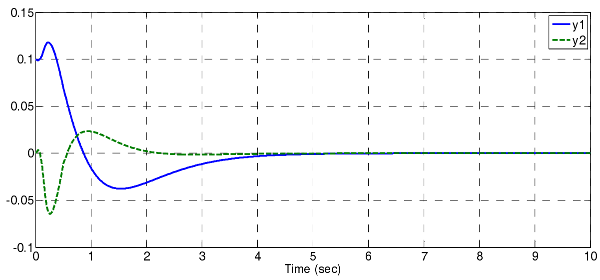

Fig. 2. System outputs using the sign function.

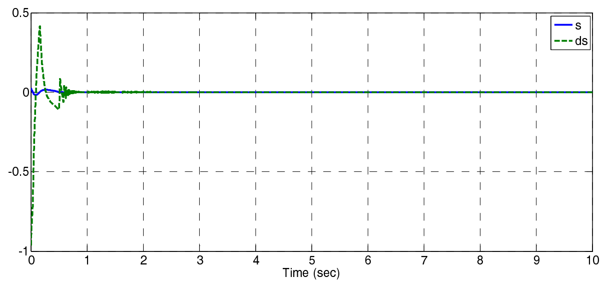

Fig. 3. Responses of $\mathbf{s}(t)$ and $\dot{\mathbf{s}}(t)$ using the sign function.

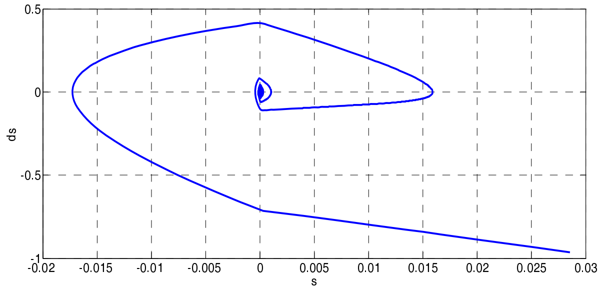

Fig. 4. Phase plot using the sign function.

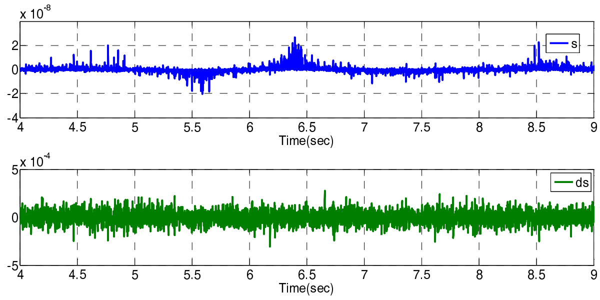

Fig. 5. Responses of $\mathbf{s}(t)$ and $\dot{\mathbf{s}}(t)$ using the sign function for $4\le t\le 9$.

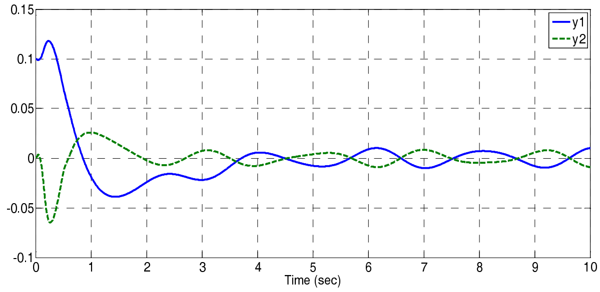

Fig. 6. System outputs using the saturation function.

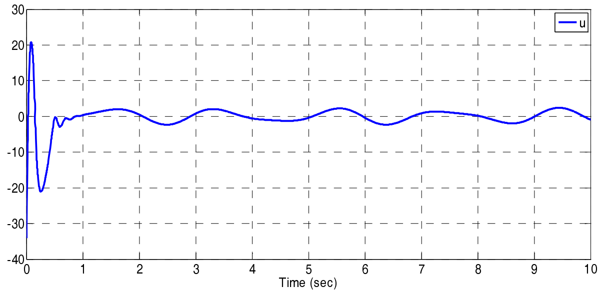

Fig. 7. Control input using the saturation function.

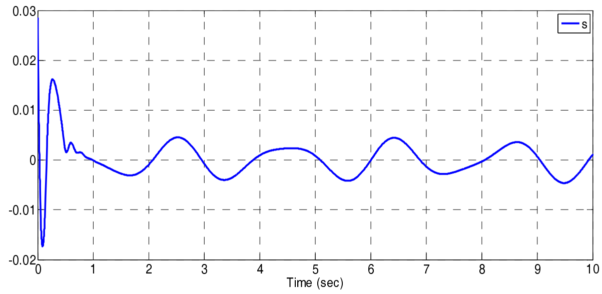

Fig. 8. Sliding variable using the saturation function.

6. CONCLUSION

In this paper we have proposed a modified second-order sliding mode control algorithm for a MIMO uncertain system with relative degree two. Introducing a suitable dynamic compensator into the sliding variable, the addition degree of freedom can be used to stabilize the closed-loop system once the system is in the sliding mode. Using the developed sliding mode controller, it is shown that the real second-order sliding mode can be guaranteed. Finally, an inverted pendulum using position measurement only is used to demonstrate the algorithm. The simulation results demonstrate that the proposed control scheme exhibits reasonably good system performance.

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