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Average Consensus Seeking of High-order Continuous-time Multi-agent Systems with Multiple Time-varying Communication Delays

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Abstract: The average consensus problem of high-order multi-agent systems with multiple time-varying communication delays is investigated in this paper. By using the idea of state decomposition, the condition for guaranteeing average consensus is converted into verifying the stability of zero equilibrium of disagreement system. Considering multiple time-varying communication delays, Lyapunov-Krasovskii approach in time-domain is employed to analyze the stability of zero equilibrium. With the help of Free-weighting Matrices (FWM) approach, the tolerant upper bounds on communication delays can be obtained through solving feasible linear matrix inequalities (LMIs). Delay-dependent stability criteria for both strongly-connected fixed and switching topologies are provided in the main results. Further, the conclusion is extended to the case of jointly-connected switching topologies. Numerical examples and simulation results are given to demonstrate the effectiveness and the benefit on reducing conservativeness of the proposed method.

Keywords: Average consensus, free-weighting matrices, linear matrix inequality, Lyapunov-Krasovskii functional.

1. INTRODUCTION

Distributed consensus problem becomes an interesting and important topic in the field of multi-agent coordination in recent years. The tasks of consensus are to make each agent achieve agreement based on different input, and make sure the same upper and lower bounds on coordination number. Many researchers paid much attention to the theory of distributed consensus for the merits of decentral control, local information exchange and simple behavior coordination. So far, it has been widely used in the areas of formation control [1], flocking [2], rendezvous [3] in multi-agent systems, fusion estimation [4], collaborative decision-making [5] and coupled oscillator synchronization [6].

Convergence property is one of the focused topics in distributed consensus theory. In many applications, communication delays should be considered in consensus problem because of agents moving, communication congestion, or finite transmission distance. Many works in the literature focused on the stability conditions for guaranteeing that the agents achieve consensus with time-delays, see e.g., [7-14]. One is the analysis methodologies in frequency domain, where the stability criterion can be derived from the distribution of the eigenvalues in complex plane, e.g., [7,8]. They provided sufficient and necessary conditions for the upper bounds of both uniform and non-uniform time-varying delays. However, it is difficult to find the common or multiple Lyapunov functionals in the case of switching topologies [9]. Therefore, they are only valid for fixed topology in most cases.

Another familiar methodology is the time domain analysis through building Lyapunov-Krasovskii or Lyapunov-Razumikhin functional. The convergence property or the stability can be judged from the negative definite of the Lyapunov functional derivative, e.g., [10-13]. However, it is still possible to improve the criteria in time domain because they are all sufficient conditions to achieve average consensus. Recently, [14] discussed the consensus for high-order dynamics systems with uniform communication delays and strongly-connected switching topologies.

Although a large body of work has been produced to the convergence property of the average consensus problem, there are still some topics which deserve further research by the analysis methodologies in time domain. These aspects include: (i) **Agent's Dynamics**. The literature aforementioned only focused on the average consensus for first-order or second-order dynamic, e.g., [11-13]; (ii) **Communication Conditions**. There are rigorous restrictions to the communication delay such as uniform delay [14] or strongly connected topologies [11,12,14]; (iii) **Conservativeness**. The stability criteria have higher conservativeness, e.g., the "basic inequality" was adopted in [11-13] to justify the negative definite of

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the Lyapunov functional derivative.

The aim of this paper is to provide the stability criteria for average consensus with multiple time-varying delays. This criteria can be employed to justify the convergence property with the consideration of the multiple time-varying communication delays and fixed/switching topologies. They can be regarded as the generalization of the work [11-14]. Different from the existing results, the proposed stability criteria mainly focuses on the following aspects: (i) discuss the multi-agent systems with high order chain integrator dynamics; (ii) relax the restrictions to the communication conditions, including multiple time-varying communication delays and jointly-connected switching topologies; (iii) obtain the lower conservativeness criterion with the help of FWM method [15-17]; and find the ultimate criteria with brief expression, which can be solved conveniently by mathematical tool.

This paper is organized as follows. In Section 2, we give the background and necessary mathematical preliminaries. Section 3 describes the problem and consensus protocol. In Section 4, the main theoretical results including delay-dependent stability criteria for fixed and switching topologies are presented. Numerical examples and discussion are given in Section 5, and concluding remarks are stated in Section 6

Throughout this paper, the notation $\star$ represents the symmetric part in a symmetric matrix; $\mathcal{D} > (\geq, <, \leq) 0$ denotes that the matrix $\mathcal{D}$ is positive definite(positive semidefinite, negative, negative semidefinite); **1** represents $[1,1,...,1]^T$ with appropriate dimensions and 0 or **0** denotes zero value or zero matrix with appropriate dimensions. $\otimes$ denotes the Kronecker product.

2. BACKGROUND AND PRELIMINARIES

2.1. Graph theory

Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ denote the relationship between multiple agents with the set of nodes $\mathcal{V} = \{\upsilon_1, \dots, \upsilon_n\}$, the set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ and adjacent matrix $\mathcal{A} = [a_{ij}]$. The node indices belongs to a finite index set $\mathcal{I} = \{1, 2, \dots, n\}$. The edge can be depicted by (υ_i, υ_j), and the value of a_{ij} corresponds to the edge of the graph, i.e. $(\upsilon_i, \upsilon_j) \in \mathcal{E} \Leftrightarrow a_{ij} > 0$. The neighbors of node υ_i is defined by $\mathcal{N}_i = \{\upsilon_j \in \mathcal{V} : (\upsilon_i, \upsilon_j) \in \mathcal{E}\}$. Let $\mathbf{L} = [l_{ij}]$ denote the Laplacian matrix corresponding to the topology of the graph, where $l_{ii} = \sum_{j=1}^{n} a_{ij}$ and $l_{ij} = -a_{ij}, i \neq j$. Let $x_i \in \mathbb{R}^{N_d}$ represent the state value of node υ_i with N_d dimensions(such as position or temperature etc.), and the multi-agent systems with the state vector $x = (x_1, \dots, x_n)^T$ and the topology G can be described by G_x=(G, x) (usually denote G for compactness). We say nodes υ_i and υ_j agree in multi-agent systems if and only if $x_i = x_j$. The nodes achieve consensus if and only if $x_i = x_j$ for all $i, j \in \mathcal{I}, i \neq j$. Whenever the nodes are all in agreement, the common value of all nodes is called the *group decision value* [7].

2.2. Several definitions and lemmas

Before giving the main results, we introduce some definitions and lemmas which play an important role in the proof of our main theoretical results.

Definition 1 (Strongly Connected) [7]**:** If there is a directed path from every node to every other node, the graph is said to be strongly connected (connected for undirected graph).

Definition 2 (Balanced Graphs) [7]**:** We say the node υ_i of a digraph $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is balanced if and only if its in-degree and out-degree is equal, i.e., $deg_{out}(\upsilon_i) = deg_{in}(\upsilon_i)$. A graph $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is called balanced if and only if all of nodes are balanced, or $\sum_j a_{ij} = \sum_j a_{ji}$, $\forall i \in \mathcal{I}$.

Definition 3 (Balanced Matrix) [12]**:** A square matrix $\mathbf{F} \in \mathbb{R}^{n \times n}$ is said to be a balanced matrix if and only if $\mathbf{1}_n^T \mathbf{F} = 0$ and $\mathbf{F}\mathbf{1}_n = 0$.

Definition 4 (Jointly-connected) [18]**:** By the union of a collection of simple graphs $\{G_1, G_2, ..., G_{N_{ss}}\}$, each with nodes set $\mathcal{V}$, is meant the simple graph G with nodes set $\mathcal{V}$ and edges equaling the union of the edge sets of all of the graphs in the collection. We say that such a collection is jointly-connected if the union of its members is a strongly connected graph (connected for undirected graph).

Lemma 1 [7]**:** If the graph G of multi-agent systems is strongly connected, then its Laplacian **L** satisfies:

1) rank(**L**)= n-1;

2) zero is one eigenvalue of **L**, and $\mathbf{1}_n$ is the corresponding eigenvector, i.e., $\mathbf{L}\mathbf{1}_n = 0$;

3) the rest n-1 eigenvalues all have positive real-parts. Particularly, if the graph G is undirected, they are all positive and real.

Lemma 2 [12]**:** Consider the Laplacian matrix of complete graph

$$\begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix}.$$

If $\mathbf{E}_c$ is the matrix of eigenvectors of the Laplacian matrix of complete graph, it is an orthogonal matrix. Given any positive semi-definite balanced matrix $\mathbf{F} \in \mathbb{R}^{n \times n}$, the following holds,

$$\mathbf{E}_c^T \mathbf{F} \mathbf{E}_c = \begin{bmatrix} \widetilde{\mathbf{F}}_{(n-1)\times(n-1)} & 0 \\ 0 & 0 \end{bmatrix},$$

where $\widetilde{\mathbf{F}} = \mathbf{E}_{c1}^T \mathbf{F} \mathbf{E}_{c1}$ is a positive semi-definite matrix, $\mathbf{E}_{c1}$ is the matrix of the eigenvectors corresponding to the non-zero eigenvalues of the Laplacian of complete graph.

Lemma 3 [19]**:** For given symmetric matrix $\mathbf{Z}$ with the form $\mathbf{Z}=[Z_{ij}]$, $Z_{11}\in\mathbb{R}^{r\times r}$, $Z_{12}\in\mathbb{R}^{r\times(n-r)}$, $Z_{22}\in\mathbb{R}^{(n-r)\times(n-r)}$, then $\mathbf{Z}<0$ if and only if $Z_{11}<0$, $Z_{22}-Z_{21}Z_{11}^{-1}Z_{12}<0$ or $Z_{22}<0$, $Z_{11}-Z_{12}Z_{22}^{-1}Z_{21}<0$.

3. PROBLEM DESCRIPTION

In this paper, we will discuss the dynamics of agent i described by high order chain integrator in continuous-time domain

$$\begin{aligned}\dot{x}_i^{(0)}(t)&=x_i^{(1)}(t),\\ &\vdots\\ \dot{x}_i^{(l-2)}(t)&=x_i^{(l-1)}(t),\\ \dot{x}_i^{(l-1)}(t)&=u_i(t),\end{aligned}\tag{1}$$

where $x_i^{(k)}(t)$ is the high order state of agent i and $u_i(t)$ is the consensus protocol.

For the purpose of reaching average consensus, the consensus protocol with multiple communication delays is adopted as follows:

$$\begin{aligned}u_i(t)=&-\sum_{p=1}^{l-1}\beta_p x_i^{(p)}-\sum_{q=1}^{m}\sum_{j\in N_i}a_{ij}\\ &\times\beta_0[x_j^{(0)}(t-\tau_q(t))-x_i^{(0)}(t-\tau_q(t))],\end{aligned}\tag{2}$$

where β_p, $p=0,1,...,l-1$ are the positive constant coefficients and $\tau_q(t), q=1,2,...,m$ are the communication delays. Suppose that for m given communication delays such that

$$0\le\tau_i(t)\le\bar{\tau}_i, i=1,2,...,m \tag{3}$$

and their derivatives

$$0\le\mu_i\le\bar{\mu}_i, i=1,2,...,m. \tag{4}$$

Therefore, the collective dynamics of the multi-agent system with m communication delays can be depicted in matrix form under fixed communication topology

$$\dot{\mathbf{x}}(t)=(\mathbf{I}_n\otimes\mathbf{H})\mathbf{x}(t)+\sum_{q=1}^{m}(\mathbf{L}_q\otimes\Gamma)\mathbf{x}(t-\tau_q),\tag{5}$$

where

$$\begin{aligned}&\mathbf{x}(t)=[\mathbf{x}_1^T(t),\cdots,\mathbf{x}_n^T(t)]^T,\\ &\mathbf{x}_i(t)=[x_i^{(0)}(t),\cdots,x_i^{(l-1)}(t)]^T,\quad i\in\mathcal{I},\\ &\mathbf{H}=\begin{bmatrix}0 & \mathbf{I}_{l-1}\\ 0 & -\beta\end{bmatrix}_{l\times l},\quad \Gamma=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ -\beta_0 & 0\end{bmatrix}_{l\times l},\\ &\beta=[\beta_1,\cdots\beta_{l-1}].\end{aligned}$$

$\mathbf{L}_q$ is the corresponding Laplacian matrix of the communication topology for the q-th delay, and it satisfies that $\sum_{q=1}^{m}\mathbf{L}_q=\mathbf{L}$.

Replace $\mathbf{L}_q$ by $\mathbf{L}_{\sigma q}$, then the dynamics for switching communication topologies can be described as follows

$$\dot{\mathbf{x}}(t)=(\mathbf{I}_n\otimes\mathbf{H})\mathbf{x}(t)+\sum_{q=1}^{m}(\mathbf{L}_{\sigma q}\otimes\Gamma)\mathbf{x}(t-\tau_q),\tag{6}$$

where $\mathbf{L}_{\sigma q}$ is the topology determined by switching signal with successive times $\sigma(t):[0,\infty]\to\{1,2,...,N_{ss}\}$ (σ in short). If the protocol (2) asymptotically solves the average consensus of multi-agent systems with high order dynamics if and only if $\lim_{t\to\infty}\left\|x_i^{(k)}(t)-x_j^{(k)}(t)\right\|=0$, $\forall i,j\in\mathcal{I}$ and $\lim_{t\to\infty}x_i^{(k)}(t)=\frac{1}{n}\sum_{i=1}^{n}x_i^{(k)}(0),\forall i\in\mathcal{I}$ for $k=0,1,...,l-1$.

4. MAIN RESULTS

In this section,we will give the delay-dependent stability criteriafor multi-agent system under strongly-connected fixed/switching topologies. Then the conclusion is extended to the case of jointly-connected switching topologies which has a week assumption on the connectivity of the interconnection among agents.

4.1. Strongly-connected fixed topology

Before giving the main results of stability, we have the following assumption:

Assumption 1: the topology of multi-agent systems is strongly connected;

Assumption 2: its corresponding graph is balanced;

Assumption 3: the parameter $\beta_0,\beta_1,\cdots,\beta_{l-1}$ make the polyno-mial $\beta_{l-1}s^{l-1}+\beta_{l-2}s^{l-2}+\cdots+\beta_1 s+\beta_0$ Hurwitz stable.

Theorem 1: Consider the multi-agent systems with m communication delays under Assumptions 1, 2 and 3 hold. Given the upper bound on communication delays $\bar{\tau}_i$ and the upper bound on their derivative $\bar{\mu}_i$, $i=1,2,...,m$, the protocol (2) globally asymptotically solves the average consensus of multi-agent systems (5) if there exist matrices with proper dimension

$$\begin{aligned}&\mathbf{P}=\mathbf{P}^T>0,\ \mathbf{Q}_i^{(1)}=\left[\mathbf{Q}_i^{(1)}\right]^T\ge 0,\ \mathbf{Q}_i^{(2)}=\left[\mathbf{Q}_i^{(2)}\right]^T\ge 0,\\ &\mathbf{R}_i^{(1)}=\left[\mathbf{R}_i^{(1)}\right]^T\ge 0,\ \mathbf{R}_i^{(2)}=\left[\mathbf{R}_i^{(2)}\right]^T\ge 0,\quad i=1,2,...,m\end{aligned}$$

and free-weighting matrices

$$\begin{aligned}&\mathbf{N}_i=\left[\mathbf{N}_i^{(1)},\mathbf{N}_i^{(2)},\mathbf{N}_i^{(3)}\right]^T,\quad \mathbf{M}_i=\left[\mathbf{M}_i^{(1)},\mathbf{M}_i^{(2)},\mathbf{M}_i^{(3)}\right]^T,\\ &\mathbf{S}_i=\left[\mathbf{S}_i^{(1)},\mathbf{S}_i^{(2)},\mathbf{S}_i^{(3)}\right]^T,\quad i=1,2,...,m\end{aligned}$$

such that the following LMI hold:

$$\sum_{i=1}^{m}\Theta_i<0,\tag{7}$$

where

$$\Theta_i = \begin{bmatrix} \Theta_i^{(1)} + \Theta_i^{(2)} & \Theta_i^{(3)} & \overline{\tau}_i \mathbf{N}_i & \overline{\tau}_i \mathbf{S}_i & \overline{\tau}_i \mathbf{M}_i \\ \star & \Theta_i^{(4)} & 0 & 0 & 0 \\ \star & \star & -\overline{\tau}_i \mathbf{R}_i^{(1)} & 0 & 0 \\ \star & \star & \star & -\overline{\tau}_i \mathbf{R}_i^{(1)} & 0 \\ \star & \star & \star & \star & -\overline{\tau}_i \mathbf{R}_i^{(2)} \end{bmatrix},$$

$$\Theta_i^{(1)} = \begin{bmatrix} \mathbf{P}\widetilde{\mathbf{H}} + \widetilde{\mathbf{H}}^T \mathbf{P} + \mathbf{Q}_i^{(1)} + \mathbf{Q}_i^{(2)} & \mathbf{P}\tilde{\Gamma}_i & 0 \\ \star & -(1-\overline{\mu}_i)\mathbf{Q}_i^{(1)} & 0 \\ \star & \star & -\mathbf{Q}_i^{(2)} \end{bmatrix},$$

$$\Theta_i^{(2)} = \left[\mathbf{N}_i + \mathbf{M}_i, -\mathbf{N}_i + \mathbf{S}_i, -\mathbf{S}_i - \mathbf{M}_i\right] + \left[\mathbf{N}_i + \mathbf{M}_i, -\mathbf{N}_i + \mathbf{S}_i, -\mathbf{S}_i - \mathbf{M}_i\right]^T,$$

$$\Theta_i^{(3)} = \overline{\tau}_i [\widetilde{\mathbf{H}}, \tilde{\Gamma}_i, 0]^T (\mathbf{R}_i^{(1)} + \mathbf{R}_i^{(2)}),$$

$$\Theta_i^{(4)} = -\overline{\tau}_i (\mathbf{R}_i^{(1)} + \mathbf{R}_i^{(2)}),$$

$$\widetilde{\mathbf{H}} = \mathbf{I}_{n-1} \otimes \mathbf{H}, \quad \tilde{\Gamma}_i = (\mathbf{E}_{c1}^T \mathbf{L}_i \mathbf{E}_{c1}) \otimes \Gamma,$$

and $\mathbf{E}_{c1}$ is the matrix of the eigenvectors corresponding to the non-zero eigenvalues of the Laplacian of complete graph with dimension $n \times n$.

Proof: According to conditions of multi-agent systems achieving average consensus, $\frac{1}{n}\sum_{i=1}^{n} x_i^{(k)}$ for $k = 0,1,...,l-1$ are invariance vectors. Hence, using the decomposition idea in [7,20], $\mathbf{x}(t)$ can be expressed as a combination of the following two decoupled subspaces

$$\mathbf{x}(t) = Ave(\mathbf{x}(0))\mathbf{1}_n \otimes [\beta_0, \underbrace{0...0}_{l-1}]^T + \delta(t), \tag{8}$$

where $Ave(\mathbf{x}(0)) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i(0)$ and $\delta(t)$ denotes the disagreement systems.

That means the stability problem of collective dynamics systems(5) can be transformed into verifying whether systems (9) has a stable zero equilibrium

$$\dot{\delta}(t) = (\mathbf{I}_n \otimes \mathbf{H})\delta(t) + \sum_{q=1}^{m} (\mathbf{L}_q \otimes \Gamma)\delta(t - \tau_q). \tag{9}$$

Denote $\mathcal{M}_1 \triangleq span\{\mathbf{1}_n \otimes [\beta_0, \cdots, \beta_{l-1}]^T\}^{\perp}$ and $\mathcal{M}_2 \triangleq span\{[\beta_0, \cdots, \beta_{l-1}, \underbrace{0, \cdots, 0}_{l(n-1)}]^T\}^{\perp}$. By Lemma 1 and Lemma 2, the following relationship holds [14]

$$\mathcal{M}_1 = (\mathbf{E}_c \otimes \mathbf{I}_l)\mathcal{M}_2.$$

Then, for any $\delta \in \mathcal{M}_1$, there exists $\tilde{\delta} = [\tilde{\delta}_1^T, \tilde{\delta}_2^T]^T \in \mathcal{M}_2$ such that $\delta = (\mathbf{E}_c \otimes I_l)\tilde{\delta}$ with the definitions as follows

$$\tilde{\delta}_1 = [\tilde{\delta}_1^{(0)}, \cdots, \tilde{\delta}_1^{(l-1)}]^T,$$

$$\tilde{\delta}_2 = [\tilde{\delta}_2^{(0)}, \cdots, \tilde{\delta}_2^{(l-1)}, \cdots, \tilde{\delta}_n^{(0)}, \cdots, \tilde{\delta}_n^{(l-1)}]^T.$$

According to the dynamics (9), we have

$$\dot{\tilde{\delta}}_1(t) = \mathbf{H}\tilde{\delta}_1(t), \tag{10}$$

$$\dot{\tilde{\delta}}_2(t) = (\mathbf{I}_{n-1} \otimes \mathbf{H})\tilde{\delta}_2(t) + \sum_{q=1}^{m} (\tilde{\mathbf{L}}_\mathbf{q} \otimes \Gamma)\tilde{\delta}_2(t - \tau_q), \tag{11}$$

where $\tilde{\mathbf{L}}_\mathbf{q} = \mathbf{E}_{c1}^T \mathbf{L}_\mathbf{q} \mathbf{E}_{c1}$, $q = 1, \cdots, m$, and $\mathbf{E}_{c1}$ is the matrix of eigenvectors of the Laplacian matrix of complete graph with dimension $n \times n$.

For any $\tilde{\delta} \in \mathcal{M}_2$, we have $\beta_0 \tilde{\delta}_1^{(0)} +, ..., + \beta_{l-1}\tilde{\delta}_1^{(l-1)} = 0$. From the assumption that the parameter $\beta_0, \beta_1, \cdots, \beta_{l-1}$ make the polynomial $\beta_{l-1}s^{l-1} + \beta_{l-2}s^{l-2} + \cdots + \beta_1 s + \beta_0$ Hurwitz stable, the conclusion can be achieved that subsystem (10) is globally asymptotically stable. Next, the remaining work is to prove the stability of subsystem (11).

Define the following candidate quadratic Lyapunov-Krasovskii functional

$$\begin{aligned} V(t, \tilde{\delta}_2) &= \tilde{\delta}_2^T(t)\mathbf{P}\tilde{\delta}_2(t) + \sum_{i=1}^{m} \int_{t-\tau_i(t)}^{t} \tilde{\delta}_2^T(s)\mathbf{Q}_i^{(1)}\tilde{\delta}_2(s)\,ds \\ &+ \sum_{i=1}^{m} \int_{t-\overline{\tau}_i}^{t} \tilde{\delta}_2^T(s)\mathbf{Q}_i^{(2)}\tilde{\delta}_2(s)\,ds \\ &+ \sum_{i=1}^{m} \int_{-\overline{\tau}_i}^{0} \int_{t+\theta}^{t} \dot{\tilde{\delta}}_2^T(s)(\mathbf{R}_i^{(1)} + \mathbf{R}_i^{(2)})\dot{\tilde{\delta}}_2(s)\,ds\,d\theta, \end{aligned} \tag{12}$$

where $\mathbf{P} = \mathbf{P}^T > 0$, $\mathbf{Q}_i^{(1)} = [\mathbf{Q}_i^{(1)}]^T \geq 0$, $\mathbf{Q}_i^{(2)} = [\mathbf{Q}_i^{(2)}]^T \geq 0$, $\mathbf{R}_i^{(1)} = [\mathbf{R}_i^{(1)}]^T \geq 0$, $\mathbf{R}_i^{(2)} = [\mathbf{R}_i^{(2)}]^T \geq 0$, $i = 1, 2, ..., m$ are all balanced matrices with appropriate dimension to be determined.

By Newton-Leibniz formula

$$f(t) - f(t - r) = \int_{t-r}^{t} \dot{f}(s)\,ds. \tag{13}$$

Hence, given any free-weighting matrices with proper dimension

$$\mathbf{N}_i = \begin{bmatrix} \mathbf{N}_i^{(1)} \\ \mathbf{N}_i^{(2)} \\ \mathbf{N}_i^{(3)} \end{bmatrix}, \quad \mathbf{M}_i = \begin{bmatrix} \mathbf{M}_i^{(1)} \\ \mathbf{M}_i^{(2)} \\ \mathbf{M}_i^{(3)} \end{bmatrix}, \quad \mathbf{S}_i = \begin{bmatrix} \mathbf{S}_i^{(1)} \\ \mathbf{S}_i^{(2)} \\ \mathbf{S}_i^{(3)} \end{bmatrix}, \quad i = 1, 2, ..., m,$$

we have

$$\sum_{i=1}^{m} 2\xi_i^T(t)\mathbf{N}_i[\tilde{\delta}_2(t) - \tilde{\delta}_2(t - \tau_i(t)) - \int_{t-\tau_i(t)}^{t} \dot{\tilde{\delta}}_2(s)ds] = 0,$$

$$\sum_{i=1}^{m} 2\xi_i^T(t)\mathbf{M}_i[\tilde{\delta}_2(t - \tau_i(t)) - \tilde{\delta}_2(t - \overline{\tau}_i) - \int_{t-\overline{\tau}_i}^{t-\tau_i(t)} \dot{\tilde{\delta}}_2(s)ds] = 0,$$

$$\sum_{i=1}^{m} 2\xi_i^T(t)\mathbf{S}_i[\tilde{\delta}_2(t) - \tilde{\delta}_2(t - \overline{\tau}_i) - \int_{t-\overline{\tau}_i}^{t} \dot{\tilde{\delta}}_2(s)ds] = 0, \tag{14}$$

where $\xi_i(t) = [\tilde{\delta}_2^T(t), \tilde{\delta}_2^T(t - \tau_i(t)), \tilde{\delta}_2^T(t - \overline{\tau}_i)]^T$.

Calculate the derivative of $V(t, \tilde{\delta}_2)$

$$\dot{V}(t,\tilde{\delta}_2) = 2\tilde{\delta}_2^T(t)\mathbf{P}\dot{\tilde{\delta}}_2(t) + \sum_{i=1}^{m}\tilde{\delta}_2^T(t)\mathbf{Q}_i^{(1)}\tilde{\delta}_2(t)$$
$$-\sum_{i=1}^{m}(1-\dot{\tau}_i(t))\tilde{\delta}_2^T(t-\tau_i(t))\mathbf{Q}_i^{(1)}\tilde{\delta}_2(t-\tau_i(t))$$
$$+\sum_{i=1}^{m}[\tilde{\delta}_2^T(t)\mathbf{Q}_i^{(2)}\tilde{\delta}_2(t) - \tilde{\delta}_2^T(t-\bar{\tau}_i)\mathbf{Q}_i^{(2)}\tilde{\delta}_2(t-\bar{\tau}_i)]$$
$$+\sum_{i=1}^{m}\bar{\tau}_i\dot{\tilde{\delta}}_2^T(t)(\mathbf{R}_i^{(1)}+\mathbf{R}_i^{(2)})\dot{\tilde{\delta}}_2(t)$$
$$-\sum_{i=1}^{m}\int_{t-\bar{\tau}_i}^{t}\dot{\tilde{\delta}}_2^T(s)(\mathbf{R}_i^{(1)}+\mathbf{R}_i^{(2)})\dot{\tilde{\delta}}_2(s)ds.$$

By (4) and (14), we get

$$\dot{V}(t,\tilde{\delta}_2) \le 2\tilde{\delta}_2^T(t)\mathbf{P}\dot{\tilde{\delta}}_2(t) + \sum_{i=1}^{m}\tilde{\delta}_2^T(t)(\mathbf{Q}_i^{(1)}+\mathbf{Q}_i^{(2)})\tilde{\delta}_2(t)$$
$$-\sum_{i=1}^{m}(1-\bar{\mu}_i)\tilde{\delta}_2^T(t-\tau_i(t))\mathbf{Q}_i^{(1)}\tilde{\delta}_2(t-\tau_i(t))$$
$$-\sum_{i=1}^{m}\tilde{\delta}_2^T(t-\bar{\tau}_i)\mathbf{Q}_i^{(2)}\tilde{\delta}_2(t-\bar{\tau}_i)$$
$$+\sum_{i=1}^{m}\bar{\tau}_i\dot{\tilde{\delta}}_2^T(t)(\mathbf{R}_i^{(1)}+\mathbf{R}_i^{(2)})\dot{\tilde{\delta}}_2(t)$$
$$-\sum_{i=1}^{m}\int_{t-\tau_i(t)}^{t}\dot{\tilde{\delta}}_2^T(s)\mathbf{R}_i^{(1)}\dot{\tilde{\delta}}_2(s)ds$$
$$-\sum_{i=1}^{m}\int_{t-\bar{\tau}_i}^{t-\tau_i(t)}\dot{\tilde{\delta}}_2^T(s)\mathbf{R}_i^{(1)}\dot{\tilde{\delta}}_2(s)ds$$
$$-\sum_{i=1}^{m}\int_{t-\bar{\tau}_i}^{t}\dot{\tilde{\delta}}_2^T(s)\mathbf{R}_i^{(2)}\dot{\tilde{\delta}}_2(s)ds$$
$$+\sum_{i=1}^{m}2\xi_i^T(t)\mathbf{N}_i[\tilde{\delta}_2(t)-\tilde{\delta}_2(t-\tau_i(t))$$
$$-\int_{t-\tau_i(t)}^{t}\dot{\tilde{\delta}}_2(s)ds]$$
$$+\sum_{i=1}^{m}2\xi_i^T(t)\mathbf{M}_i[\tilde{\delta}_2(t-\tau_i(t))-\tilde{\delta}_2(t-\bar{\tau}_i)$$
$$-\int_{t-\bar{\tau}_i}^{t-\tau_i(t)}\dot{\tilde{\delta}}_2(s)ds]$$
$$+\sum_{i=1}^{m}2\xi_i^T(t)\mathbf{S}_i[\tilde{\delta}_2(t)-\delta(t-\bar{\tau}_i)-\int_{t-\bar{\tau}_i}^{t}\dot{\tilde{\delta}}_2(s)ds].$$

Consequently, substituting (11) yield

$$\begin{aligned}
&\le \sum_{i=1}^{m}\xi_i^T(t)\Theta_i\xi_i(t)\\
&-\sum_{i=1}^{m}\int_{t-\tau_i(t)}^{t}[\xi_i^T(t)\mathbf{N}_i+\dot{\tilde{\delta}}_2^T(s)\mathbf{R}_i^{(1)}][\mathbf{R}_i^{(1)}]^{-1}\\
&\qquad\times[\mathbf{N}_i^T\xi_i(t)+\mathbf{R}_i^{(1)}\dot{\tilde{\delta}}_2(s)]ds\\
&-\sum_{i=1}^{m}\int_{t-\bar{\tau}_i}^{t-\tau_i(t)}[\xi_i^T(t)\mathbf{S}_i+\dot{\tilde{\delta}}_2^T(s)\mathbf{R}_i^{(1)}][\mathbf{R}_i^{(1)}]^{-1}\\
&\qquad\times[\mathbf{S}_i^T\xi_i(t)+\mathbf{R}_i^{(1)}\dot{\tilde{\delta}}_2(s)]ds\\
&-\sum_{i=1}^{m}\int_{t-\bar{\tau}_i}^{t}[\xi_i^T(t)\mathbf{M}_i+\dot{\tilde{\delta}}_2^T(s)\mathbf{R}_i^{(2)}][\mathbf{R}_i^{(2)}]^{-1}\\
&\qquad\times[\mathbf{M}_i^T\xi_i(t)+\mathbf{R}_i^{(2)}\dot{\tilde{\delta}}_2(s)]ds,
\end{aligned} \tag{15}$$

where $\Theta_i(i=1,\cdots,m)$ follows the definition in LMI (7). Note that Θ_i expressed by LMI is obtained by applying Lemma 3. Obviously, the last three items of (15) are less than zero. Therefore, we can conclude that if exists matrices $\mathbf{P}$, $\mathbf{Q}_i^{(1)}$, $\mathbf{Q}_i^{(2)}$, $\mathbf{R}_i^{(1)}$, $\mathbf{R}_i^{(2)}$, $i=1,2,...m$ and free-weighting matrices $\mathbf{N}_i$, $\mathbf{M}_i$, $\mathbf{S}_i$, $i=1,2,...m$ with proper dimension such that LMI (7) holds, a positive constant $c_0>0$ can be found that satisfying $\dot{V}(t,\tilde{\delta}_2)\le -c_0\,\|\tilde{\delta}_2(t)\|^2$. So, the subsystem (11) is asymptotically stable. Further, the zero equilibrium of (9) is asymptotically stable. As a result, the multi-agent systems asymptotically achieve average consensus. The proof of Theorem 1 is complete.

4.2. Strongly-connected switching topologies

Theorem 2: Consider the multi-agent systems with m communication delays under the assumption that the topology determined by any switching signal $\sigma:[0,\infty)\to\{1,2...N_{ss}\}$ satisfying Assumptions 1, 2 and 3. Given the upper bound on communication delays $\bar{\tau}_i$ and the upper bound on their derivative $\bar{\mu}_i$, $i=1,2,...,m,$ the protocol (2) globally asymptotically solves the average consensus of multi-agent systems (6) if there exist common matrices with proper dimensions

$$\mathbf{P}=\mathbf{P}^T>0,\ \mathbf{Q}_i^{(1)}=\left[\mathbf{Q}_i^{(1)}\right]^T\ge 0,\ \mathbf{Q}_i^{(2)}=\left[\mathbf{Q}_i^{(2)}\right]^T\ge 0,$$
$$\mathbf{R}_i^{(1)}=\left[\mathbf{R}_i^{(1)}\right]^T\ge 0,\ \mathbf{R}_i^{(2)}=\left[\mathbf{R}_i^{(2)}\right]^T\ge 0,\ i=1,2,...,m,$$

and common free-weighting matrices

$$\mathbf{N}_i=\left[\mathbf{N}_i^{(1)},\mathbf{N}_i^{(2)},\mathbf{N}_i^{(3)}\right]^T,\quad \mathbf{M}_i=\left[\mathbf{M}_i^{(1)},\mathbf{M}_i^{(2)},\mathbf{M}_i^{(3)}\right]^T,$$
$$\mathbf{S}_i=\left[\mathbf{S}_i^{(1)},\mathbf{S}_i^{(2)},\mathbf{S}_i^{(3)}\right]^T,\quad i=1,2,...,m$$

such that the following LMI hold:

$$\sum_{i=1}^{m}\Theta_{\sigma i}<0,\forall\sigma:[0,\infty]\to\{1,2,...,N_{ss}\}, \tag{16}$$

where

$$\Theta_{\sigma i}=\begin{bmatrix}\Theta_{\sigma i}^{(1)}+\Theta_{\sigma i}^{(2)} & \Theta_{\sigma i}^{(3)} & \bar{\tau}_i\mathbf{N}_i & \bar{\tau}_i\mathbf{S}_i & \bar{\tau}_i\mathbf{M}_i\\ \star & \Theta_{\sigma i}^{(4)} & 0 & 0 & 0\\ \star & \star & -\bar{\tau}_i\mathbf{R}_i^{(1)} & 0 & 0\\ \star & \star & \star & -\bar{\tau}_i\mathbf{R}_i^{(1)} & 0\\ \star & \star & \star & \star & -\bar{\tau}_i\mathbf{R}_i^{(2)}\end{bmatrix},$$

$$\Theta_{\sigma i}^{(1)} = \begin{bmatrix} \mathbf{P}\widetilde{\mathbf{H}} + \widetilde{\mathbf{H}}^T\mathbf{P} + \mathbf{Q}_i^{(1)} + \mathbf{Q}_i^{(2)} & \mathbf{P}\tilde{\Gamma}_{\sigma i} & 0 \\ \star & -(1-\bar{\mu}_i)\mathbf{Q}_i^{(1)} & 0 \\ \star & \star & -\mathbf{Q}_i^{(2)} \end{bmatrix},$$

$$\Theta_{\sigma i}^{(2)} = \left[\mathbf{N}_i + \mathbf{M}_i, -\mathbf{N}_i + \mathbf{S}_i, -\mathbf{S}_i - \mathbf{M}_i\right] + \left[\mathbf{N}_i + \mathbf{M}_i, -\mathbf{N}_i + \mathbf{S}_i, -\mathbf{S}_i - \mathbf{M}_i\right]^T,$$

$$\Theta_{\sigma i}^{(3)} = \bar{\tau}_i[\widetilde{\mathbf{H}}, \tilde{\Gamma}_{\sigma i}, 0]^T(\mathbf{R}_i^{(1)} + R_i^{(2)}),$$

$$\Theta_{\sigma i}^{(4)} = -\bar{\tau}_i(\mathbf{R}_i^{(1)} + \mathbf{R}_i^{(2)}),$$

$$\widetilde{\mathbf{H}} = \mathbf{I}_{n-1} \otimes \mathbf{H}, \quad \tilde{\Gamma}_{\sigma i} = (\mathbf{E}_{c1}^T \mathbf{L}_{\sigma i} \mathbf{E}_{c1}) \otimes \Gamma,$$

and $\mathbf{E}_{c1}$ is the matrix of the eigenvectors corresponding to the non-zero eigenvalues of the Laplacian of complete graph with dimension $n \times n$.

Proof: Similar to the procedure of Theorem 1, two subsystems can be obtained as follow

$$\dot{\tilde{\delta}}_1(t) = \mathbf{H}\tilde{\delta}_1(t), \tag{17}$$

$$\dot{\tilde{\delta}}_2(t) = (\mathbf{I}_{n-1} \otimes \mathbf{H})\tilde{\delta}_2(t) + \sum_{q=1}^{m}(\tilde{\mathbf{L}}_{\sigma \mathbf{q}} \otimes \Gamma)\tilde{\delta}_2(t-\tau_q), \tag{18}$$

where $\tilde{\mathbf{L}}_{\sigma \mathbf{q}} = \mathbf{E}_{c1}^T \mathbf{L}_{\sigma \mathbf{q}} \mathbf{E}_{c1}$, $q = 1, \cdots, m$, $\mathbf{H}$ and Γ follow the definitions in (6).

It is clear that subsystem (17) is asymptotically stable with Assumption 3 for any switching signal σ. Therefore, if common quadratic Lyapunov-Krasovskii (CQLK) functional like (12) can be found for subsystem (18), then the stability criterion can be obtained for strongly-connected switching topologies. We omit the detailed procedure because it is similar to that in Theorem 1.

4.3. Extension to jointly-connected topologies

The above conclusion derived from Theorem 2 is also valid for the jointly-connected switching topologies case with the assumption as follows

Assumption 4: the switching topologies of multi-agent systems are jointly connected;

Assumption 5: for all jointly-connected topologies determined by switching signal σ, the sum of the in-degree or out-degree of all nodes equals with each other.

Corollary 1: Consider the multi-agent systems with m communication delays under the assumption that the switching topologies determined by all switching signal $\sigma : [0, \infty) \to \{1, 2 ... N_{ss}\}$ satisfying Assumptions 2 to 5. Given the upper bound on communication delays $\bar{\tau}_i$ and the upper bound on their derivative $\bar{\mu}_i$, $i = 1, 2, ..., m$, the protocol (2) globally asymptotically solves the average consensus of multi-agent systems (6) if there exist common matrices with proper dimensions

$$\mathbf{P} = \mathbf{P}^T > 0, \quad \mathbf{Q}_i^{(1)} = [\mathbf{Q}_i^{(1)}]^T \geq 0, \quad \mathbf{Q}_i^{(2)} = [\mathbf{Q}_i^{(2)}]^T \geq 0,$$
$$\mathbf{R}_i^{(1)} = [\mathbf{R}_i^{(1)}]^T \geq 0, \quad \mathbf{R}_i^{(2)} = [\mathbf{R}_i^{(2)}]^T \geq 0, \quad i = 1, 2, ..., m$$

and common free-weighting matrices

$$\mathbf{N}_i = \left[\mathbf{N}_i^{(1)}, \mathbf{N}_i^{(2)}, \mathbf{N}_i^{(3)}\right]^T, \quad \mathbf{M}_i = \left[\mathbf{M}_i^{(1)}, \mathbf{M}_i^{(2)}, \mathbf{M}_i^{(3)}\right]^T,$$
$$\mathbf{S}_i = \left[\mathbf{S}_i^{(1)}, \mathbf{S}_i^{(2)}, \mathbf{S}_i^{(3)}\right]^T, \quad i = 1, 2, ..., m$$

such that the following LMI hold:

$$\sum_{i=1}^{m} \Theta_{\sigma i} < 0, \forall \sigma : [0, \infty] \to \{1, 2, ..., N_{ss}\}, \tag{19}$$

where $\Theta_{\sigma i}, i = 1, \cdots, m$ follows the definition in (16) except that $\widetilde{\mathbf{H}} = \mathbf{I}_\kappa \otimes \mathbf{H}$ and $\tilde{\Gamma}_{\sigma i} = (\mathbf{E}_{c\kappa}^T \mathbf{L}_{\sigma i} \mathbf{E}_{c\kappa}) \otimes \Gamma$. Here, $\kappa = rank(\sum_{i=1}^{m} \mathbf{L}_{\sigma i})$, $\mathbf{E}_{c\kappa}$ is the matrix composed of κ columns of matrix $\mathbf{E}_{c1}$.

Proof: Similar to the procedure of state decomposition aforementioned, the disagree system can be obtained as follows

$$\dot{\delta}(t) = (\mathbf{I}_n \otimes \mathbf{H})\delta(t) + \sum_{q=1}^{m}(\mathbf{L}_{\sigma \mathbf{q}} \otimes \Gamma)\delta(t - \tau_q). \tag{20}$$

According to Definition 4, jointly-connected switching topologies do not require each topology having a spanning tree structure. With the assumption that (A5) holds, we have

$$rank(\sum_{i=1}^{m} \mathbf{L}_{\sigma i}) = \kappa, \forall \sigma : [0, \infty] \to \{1, 2, ..., N_{ss}\}. \tag{21}$$

Different from the case of strongly-connected fixed and switching topologies, define new $\tilde{\delta}^{\dagger} = [(\tilde{\delta}_1^{\dagger})^T, (\tilde{\delta}_2^{\dagger})^T]^T$ as follows

$$\tilde{\delta}_1^{\dagger} = [\tilde{\delta}_1^{(0)}, \cdots, \tilde{\delta}_1^{(l-1)}, \cdots, \tilde{\delta}_{n-\kappa}^{(0)}, \cdots, \tilde{\delta}_{n-\kappa}^{(l-1)}]^T,$$
$$\tilde{\delta}_2^{\dagger} = [\tilde{\delta}_{n-\kappa+1}^{(0)}, \cdots, \tilde{\delta}_{n-\kappa+1}^{(l-1)}, \cdots, \tilde{\delta}_n^{(0)}, \cdots, \tilde{\delta}_n^{(l-1)}]^T.$$

Then the stability of (20) is equivalent to the stabilities of the following two subsystems

$$\dot{\tilde{\delta}}_1^{\dagger}(t) = (\mathbf{I}_{n-\kappa} \otimes \mathbf{H})\tilde{\delta}_1^{\dagger}(t), \tag{22}$$

$$\dot{\tilde{\delta}}_2^{\dagger}(t) = (\mathbf{I}_\kappa \otimes \mathbf{H})\tilde{\delta}_2^{\dagger}(t) + \sum_{q=1}^{m}(\tilde{\mathbf{L}}_{\sigma q} \otimes \Gamma)\tilde{\delta}_2^{\dagger}(t - \tau_q). \tag{23}$$

Here $\tilde{\mathbf{L}}_{\sigma \mathbf{q}} = \mathbf{E}_{c\kappa}^T \mathbf{L}_{\sigma \mathbf{q}} \mathbf{E}_{c\kappa}$. Analogously, assumption (A3) assures the Hurwitz stability of subsystem (22). Then, if there exist a common jointly quadratic Lyapunov-Krasovskii(CJQLK) functional for subsystem (23) determined by all switching signal $\sigma : [0, \infty) \to \{1, 2 ... N_{ss}\}$, the multi-agent systems with jointly-connected topologies achieve average consensus.

Remark 1: There is no assumption on the derivative of communication delays in Theorem 1, Theorem 2 and Corollary 1. The conclusions can be obtained for any $\bar{\mu}_i$, $i = 1, 2, ...m$. Therefore, the criteria derived from this paper generalize the work in [11-13]. If a new Lyapunov-Krasovskii functional is built includes the first, the third and the fourth items of (12), we can obtain the delay dependent/derivative independent criteria.

Remark 2: Stability criteria derived from Theorem 1, Theorem 2 and Corollary 1 are also valid for the agent with first-order or second-order dynamics when l=1 or l=2. We will illustrate the comparison results with the existing conclusions in Section 5. Moreover, the conclusions of Theorem 2 and Corollary 1 are valid for any switching signal σ because of the existence of CQLK or CJQLK functional for disagreement system.

5. EXAMPLES AND DISCUSSIONS

5.1. Illustrative example and simulation

We now offer an illustrative example and simulation to show the effectiveness of the proposed method. Select G_a in Fig. 1 as the communication interconnection of third-order multi-agent systems with double communication delays. For simplicity, suppose that their adjacency matrices are limited to 0,1 matrices.

Let

$$\mathbf{L}_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{L}_2 = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

denote the topology associated with two communication delays respectively. Assume that $\bar{\mu}_1 = 0.5$, $\bar{\mu}_2 = 0.9$ and $\bar{\tau}_1 = 0.01$. The coefficients in consensus protocol (2) are $\beta_0 = 10$, $\beta_1 = 10$ and $\beta_2 = 10$.

With the above parameters and by using Matlab LMI Toolbox, we can obtain $\bar{\tau}_2 = 0.084$ through solving the feasible solution of (7). Numerical simulation further confirms the obtained result. It is clear from Fig. 2 that multi-agent systems achieve average consensus asymptotically. Plenty of experiment results show that the protocol with bigger coefficients or the topology with higher connectivity allows larger communication delays emphatically.

5.2. Conservativeness comparisons

In order to show the benefit of our results, we will compare the conservativeness of criteria with the existing

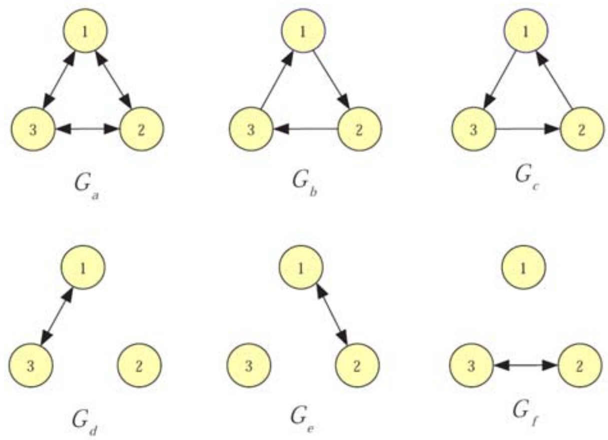

Fig. 1. Six balanced digraphs (topology G_a, G_b, G_c is strongly connected, and the union of topology G_d, G_e and G_f is jointly connected).

Table 1. Allowable upper bound on communication delay for first-order multi-agent systems with fixed topology G_a.

	$\bar{\mu}=0$	$\bar{\mu}=0.5$	$\bar{\mu}=0.9$	any $\bar{\mu}$
[11] Theorem 1	0.471	0.408	0.349	-
[11] Corollary 2	-	-	-	0.333
Theorem 1	0.471	0.419	0.408	0.408

Table 2. Allowable upper bound on communication delay for first-order multi-agent systems with strongly-connected switching topologies {G_b, G_c}.

	$\bar{\mu}=0$	$\bar{\mu}=0.5$	$\bar{\mu}=0.9$	any $\bar{\mu}$
[12] Theorem 1	0.499	-	-	-
[12] Theorem 2	0.499	0.249	0.049	-
Theorem 2	0.577	0.540	0.540	0.540

Table 3. Allowable upper bound on communication delay for first-order multi-agent systems with jointly-connected switching topologies {G_d, G_e, G_f}.

	$\bar{\mu}=0$	$\bar{\mu}=0.5$	$\bar{\mu}=0.9$	any $\bar{\mu}$
[13] Theorem 1	0.499	0.249	0.049	-
Corollary 1	0.625	0.590	0.584	0.584

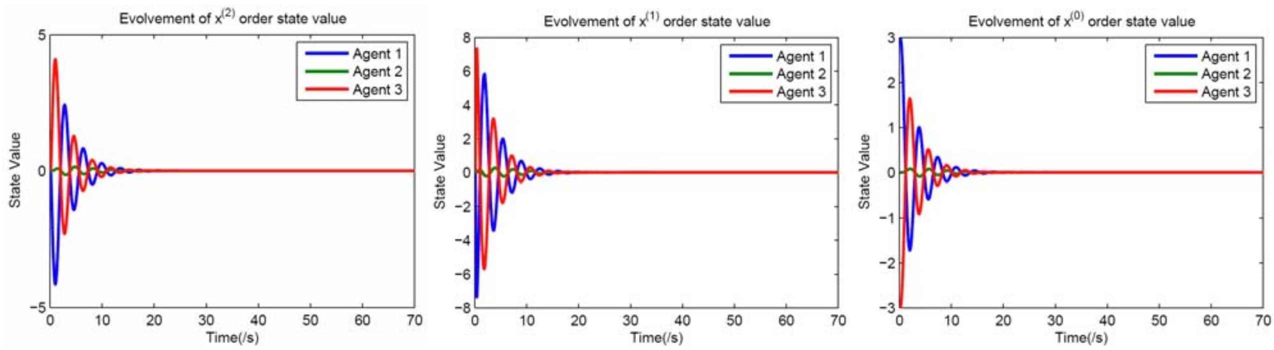

Fig. 2. State evolvement of multi-agent systems with double time-varying communication delays $\tau_1(t) = 0.01 \cdot |\sin(\frac{0.5}{0.01}t)|$ and $\tau_2(t) = 0.084|\sin(\frac{0.9}{0.084}t)|$ when $\beta_0 = \beta_1 = \beta_2 = 10$.

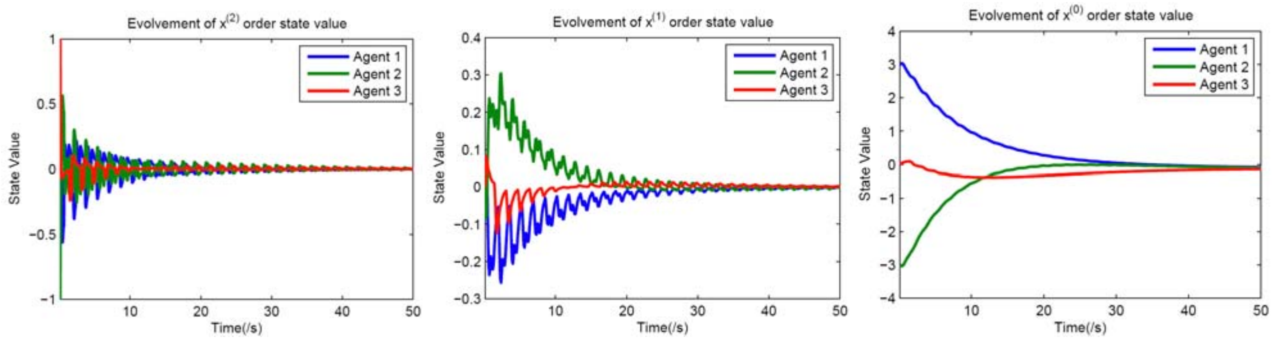

Fig. 3. State evolvement of multi-agent systems with uniform communication delay $\tau(t)=0.351$s when the switching interval is 1s for jointly-connected topologies G_d, G_e, G_f.

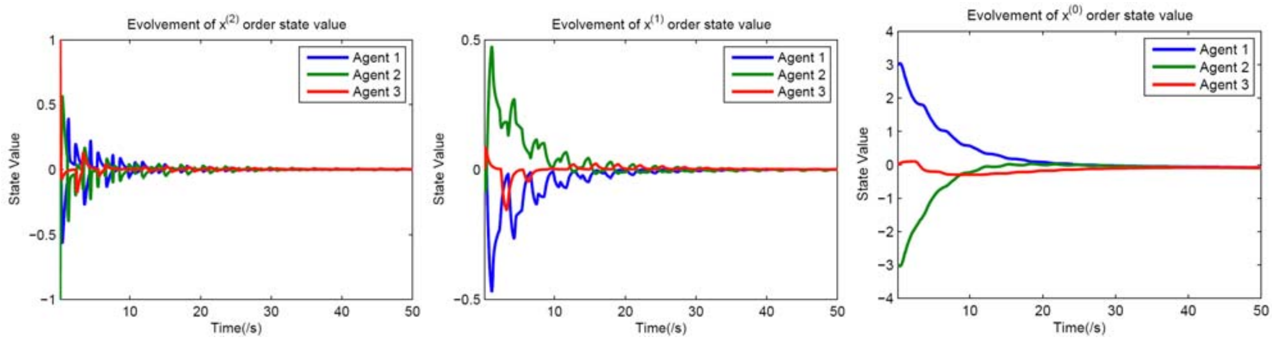

Fig. 4. State evolvement of multi-agent systems with uniform communication delay $\tau(t)=0.351$s when the switching interval is 2s for jointly-connected topologies G_d, G_e, G_f.

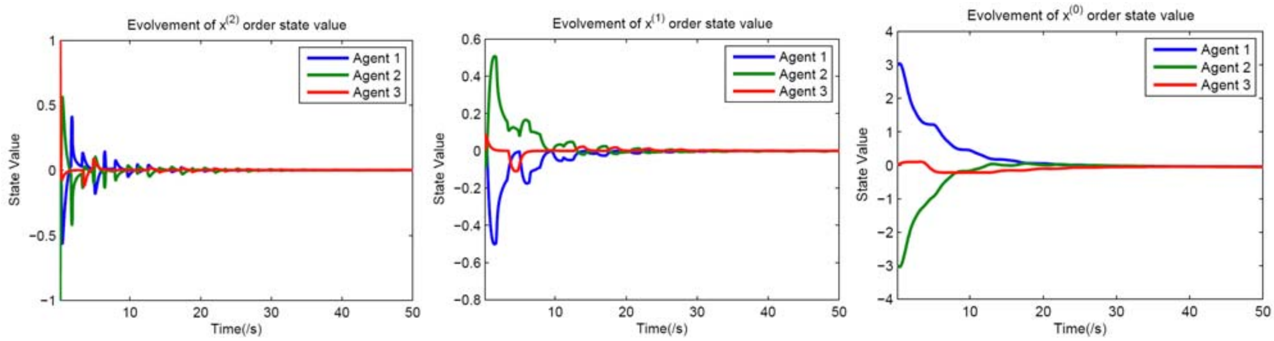

Fig. 5. State evolvement of multi-agent systems with uniform communication delay $\tau(t)=0.351$s when the switching interval is 3s for jointly-connected topologies G_d, G_e, G_f.

Table 4. Allowable upper bound on communication delay for third-order multi-agent systems with fixed topology G_a.

	$\bar{\mu}=0$	$\bar{\mu}=0.5$	$\bar{\mu}=0.9$	any $\bar{\mu}$
[14] Theorem 1	3.616	3.479	3.191	-
Theorem 1	4.129	3.635	3.613	3.613

results for first-order [11-13] and high-order [14] multi-agent systems with uniform communication delay. Tables 1 to 3 list the allowable upper bounds on communication delay for first-order multi-agent systems when β_0=1, and Table 4 gives the results for third-order multi-agent systems when β_0=1, β_1=10 and β_2=10.

Remark 3: The comparison results show that the criterion derived from this paper are better than [11-14]. Moreover, the upper bound on the delay for any $\bar{\mu}$ can be achieved. There are two reasons which contributed to this. One is the term $\sum_{i=1}^{m}\int_{t-\bar{\tau}_i}^{t}\delta^T(s)Q_i^{(2)}\delta(s)ds$ added in Lyapunov-Krasovskii functional comparing with [12,13], which widens the range of the integral for communication delay; The other one is the FWM introduced in this paper which did not magnify the derivative of $V(t,\delta(t))$ unnecessarily.

5.3. Relationship between switching interval and convergence time

As Theorem 2 and Corollary 1 mentioned, the criteria are valid for any switching signal σ because of the existence of CQLK or CJQLK functional for disagreement system. However, the switching interval for switching graphs is important to guarantee the convergence time of the consensus of multi-agent systems. Next, the simulation results will be given to illustrate the relationship between switching interval and convergence time for switching topologies. Select G_d, G_e, G_f in Fig. 1 as the jointly-connected communication interconnections of third-order multi-agent systems with uniform communication delay. Let $\beta_0 = \beta_1 = \beta_2 = 10$, the upper bound on communication equals to 0.351 by Corollary 1. The simulation results for different switching interval (1s, 2s and 3s) are given in Figs. 3 to 5. The conclusion can be obtained that higher ratio of the delay and the switching interval results in longer convergence time. Moreover, the vibration for high-order state is acute and getting worse.

6. CONCLUSIONS

The stability criteria for average consensus of high-order multi-agent systems with multiple time-varying communication delays were provided in this paper. It can be used to justify the convergence property of multi-agent systems with multiple time-varying communication delays and fixed/switching topologies. Mainly, the following contributions were concluded in this paper:

1) Through justifying the existence of Lyapunov-Krasovskii or common Lyapunov-Krasovskii functional, the stability criteria can be obtained to determine the average consensus for fixed and switching topologies. Further, the stability conclusion is extended to the case of jointly-connected interconnections.

2) The existence of Lyapunov-Krasovskii or common Lyapunov-Krasovskii functional is equivalent to solving feasible solution of LMI. Meanwhile, FWM method was employed to justify whether the Lyapunov-Krasovskii functional derivative is negative definite to assure lower conservativeness than the existing results [11-14].

3) Moreover, the conclusions derived from Theorem 1, Theorem 2 and Corollary 1 also characterized the relationship among communication delays, their derivative and the coefficient of the consensus protocol. It can be regarded as an important basis to depress the influence of limited communication conditions effectively.

Although the method proposed in this paper has less conservativeness, it is possible to improve the stability criteria and reduce the dimensions of the free weighting matrices to be determined. Suitable Lyapunov-Krasovskii functional may be a possible way to achieve that goal, which deserves further research.

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