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Robust H_∞ Consensus Control of Uncertain Multi-Agent Systems with Time Delays

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Abstract: This paper is devoted to the robust H_∞ consensus control of multi-agent systems with model parameter uncertainties and external disturbances. In particular, switching networks of multiple agents with general linear dynamics are considered, and uncertain communication delays are also taken into account. It shows that a sufficient condition in terms of linear matrix inequalities (LMIs) is derived for the robust consensus performance with a given H_∞ disturbance attenuation level, and meanwhile the unknown feedback matrix of the proposed distributed state feedback protocol is also determined. A numerical example is included to validate the theoretical results.

Keywords: Consensus, communication delays, external disturbances, model uncertainties, multi-agent systems, robust H_∞ control.

1. INTRODUCTION

During the past decade, the decentralized cooperative control of multi-agent systems has been widely studied, such as flocking or swarming behaviors [1,2], formation control [3], and path planning [4], in which the consensus problem is commonly accepted as one of the most important and fundamental issues. For a multi-agent system, consensus means that the states of all agents are driven to a common value by implementing distributed protocols, based on the communication networks.

Recently, some researchers have solved the consensus problem of multi-agent systems with various external disturbances and random communication noises [11-18], on the basis of prior results for the determined systems [5-10]. For example, Li *et al*. considered the disturbance rejection problem arising in the coordination control of multi-agent systems with external disturbances in [13], and proved that this problem could be solved by analyzing the H_∞ problem of a set of independent subsystems. In [14], Lin *et al.* studied the consensus problem of first-order multi-agent systems with external disturbances and communication uncertainties for directed networks with zero and nonzero time delays. It turned out that this problem could be transformed into a robust H_∞ control problem. Furthermore, in [16], the consensus problem was solved for switching networks of multiple agents with linear coupling dynamics and subject to external disturbances, and a distributed dynamic output feedback protocol was proposed. However, the unavoidable model and parameter uncertainties in the agents' dynamic equations, resulting from modeling errors and varying environmental parameters, are not considered in the existing work. This motivates us to investigate the consensus problem of multi-agent systems with both model uncertainties and external disturbances.

In this paper, the consensus control problem is considered for switching networks of multiple agents modeled by general linear differential equations with both model uncertainties and external disturbances, and time delays arising from communication among agents are also taken into account. By reformulating the consensus control problem as a robust H_∞ control problem, a distributed protocol using the local delayed state information is proposed with an undetermined feedback matrix, and then sufficient conditions in terms of LMIs are given to ensure the consensus performance with a given H_∞ index and meanwhile determine the system matrix of the proposed protocol. Finally, simulation results show that under the proposed protocol, an uncertain multi-agent system with switching topology can reach the desired consensus performance in the presence of communication delays.

2. PROBLEM REFORMULATION AND PROTOCOL DESIGN

2.1. Problem statement and preliminaries

Consider a multi-agent system consisting of n identical agents with the ith one modeled by the following linear dynamics

$$\dot{x}_i(t) = Ax_i(t) + B_1\omega_i(t) + B_2u_i(t), \tag{1}$$

where $x_i(t) \in \mathbb{R}^m$ is the state, $u_i(t) \in \mathbb{R}^{m_2}$ is the control input or protocol, and $\omega_i(t) \in \mathbb{R}^{m_1}$ is the external disturbance that belongs to $L_2[0,\infty)$, the space of square-

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integrable vector functions over $[0,\infty)$. If system matrices A, B_1, B_2 are uncertain, they are assumed to take the following forms:

$$\begin{aligned} &A = A_0 + \Delta A(t), \quad B_1 = B_{10} + \Delta B_1(t), \\ &B_2 = B_{20} + \Delta B_2(t), \end{aligned} \tag{2}$$

where A_0, B_{10}, B_{20} are constant matrices, and $\Delta A(t)$, $\Delta B_1(t)$, $\Delta B_2(t)$ are time-varying uncertain matrices satisfying

$$[\Delta A(t) \;\; \Delta B_1(t) \;\; \Delta B_2(t)] = E\Sigma(t)[F_1 \quad F_2 \quad F_3]. \tag{3}$$

In (3), E and $F_i \, (i=1,2,3)$ are constant matrices of appropriate dimensions, and $\Sigma(t)$ is an unknown time-varying matrix that satisfies $\Sigma^{\mathrm{T}}(t)\Sigma(t) \le I$. It is also assumed that (A_0, B_{20}) is stabilized. A protocol $u_i(t)$ is said to asymptotically solve the consensus problem, if and only if the states of agents satisfy

$$\lim_{t\to\infty}(x_i(t) - x_j(t)) = \mathbf{0}, \quad \forall i,j \in \{1,\cdots,n\} \triangleq N. \tag{4}$$

Undirected graphs are used to model the interaction topologies among agents. Let $G=(V,E,\Gamma)$ be an undirected weighted graph of order n with the set of nodes $V=\{v_1,\cdots,v_n\}$, the set of undirected edges $E \subseteq V\times V$, and a symmetric adjacency matrix $\Gamma=[a_{ij}]$ with weighting factors $a_{ij}\ge 0$. It is stipulated that the adjacency elements associated with edges are positive, i.e., $(v_i,v_j)\in E$ or $(v_j,v_i)\in E$ if and only if $a_{ij}=a_{ji} > 0$. In particular, it is assumed that $a_{ii}=0$ for $\forall i \in N$. In graph G, node v_i represents the ith agent, and edge (v_i, v_j) represents that information is exchanged between agents i and j. Then the set of neighbors of v_i is denoted by $N_i=\{v_j \in V : (v_i,v_j)\in E\}$. The Laplacian matrix of a weighted graph G is defined as $L = D - \Gamma$, where diagonal matrix $D=\mathrm{diag}\{d_1,\cdots,d_n\}$ is named the degree matrix of G, whose diagonal elements are $d_i = \Sigma_{j=1}^n a_{ij}$, $i=1,\cdots,n$. An undirected path is a sequence of ordered edges of the form (v_{i_1}, v_{i_2}), $(v_{i_2}, v_{i_3}),\cdots$ in an undirected graph, where $v_{i_j}\in V$. If there is an undirected path from every node to every other node, the graph is said to be connected.

To describe the variable topologies, a piecewise-constant switching signal function $\sigma(t):[0,\infty)\mapsto \{1,2,\cdots,M\}\triangleq \Omega$ is defined, where $M\in\mathbb{Z}^+$ denotes the total number of all possible interaction undirected graphs. Then the interaction graph at time instant t is denoted by $G_{\sigma(t)}$, and the corresponding Laplacian matrix is $L_{\sigma(t)}$. In this paper, the switching graph $G_{\sigma(t)}$ is assumed to be always connected for $\forall \sigma(t)\in\Omega$.

Lemma 1 [6]**:** Let L be the Laplacian matrix associated with an undirected graph G. Then L has at least one zero eigenvalue and all of the nonzero eigenvalues are positive. Furthermore, matrix L has exactly one zero eigenvalue if and only if the undirected graph G is connected, and the eigenvector associated with zero is $\mathbf{1}$.

Lemma 2 [8]**:** Let $L_c = [L_{c_{ij}}]\in\mathbb{R}^{n\times n}$ be a symmetric matrix with

$$L_{c_{ij}} = \begin{cases} (n-1)/n, & i=j \\ -1/n, & i\ne j, \end{cases} \tag{5}$$

then the following statements hold:

1) The eigenvalues of L_c are 1 with multiplicity $n-1$ and 0 with multiplicity 1. The vectors $\mathbf{1}^{\mathrm{T}}$ and $\mathbf{1}$ are the left and the right eigenvectors of L_c associated with the zero eigenvalue, respectively.

2) There exists an orthogonal matrix $U\in\mathbb{R}^{n\times n}$, whose last column is $\mathbf{1}/\sqrt{n}$, such that $U^{\mathrm{T}}L_cU = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0\end{bmatrix}$ holds. Let $X\in\mathbb{R}^{n\times n}$ be the Laplacian matrix of any given undirected graph, then $U^{\mathrm{T}}XU = \begin{bmatrix} X_1 & 0 \\ 0 & 0\end{bmatrix}$ holds, where $X_1\in\mathbb{R}^{(n-1)\times(n-1)}$ is positive definite if and only if the graph is connected.

2.2. Problem reformulation

According to [16], we define controlled output functions

$$z_i(t) = x_i(t) - \frac{1}{n}\sum_{j=1}^n x_j(t), \quad i=1,\cdots,n, \tag{6}$$

to reformulate the consensus control problem of multi-agent system (1) as the following H_∞ control problem:

$$\begin{aligned} &\dot{x}(t) = (I_n\otimes A)x(t) + (I_n\otimes B_1)\omega(t) + (I_n\otimes B_2)u(t), \\ &z(t) = (L_c\otimes I_m)x(t), \end{aligned} \tag{7}$$

where $x(t)=[x_1^{\mathrm{T}}(t)\cdots x_n^{\mathrm{T}}(t)]^{\mathrm{T}}\in\mathbb{R}^{mn}$, $\omega(t)=[\omega_1^{\mathrm{T}}(t)\cdots \omega_n^{\mathrm{T}}(t)]^{\mathrm{T}}\in\mathbb{R}^{m_1 n}$, $u(t)=[u_1^{\mathrm{T}}(t)\cdots u_n^{\mathrm{T}}(t)]^{\mathrm{T}}\in\mathbb{R}^{m_2 n}$, $z(t)=[z_1^{\mathrm{T}}(t)\cdots z_n^{\mathrm{T}}(t)]^{\mathrm{T}}\in\mathbb{R}^{mn}$, and L_c is defined in (5). Then, the objective is to design distributed protocols $u_i(t)$ $(i=1,\cdots,n)$ such that

$$\|T_{z\omega}(s)\|_\infty = \sup_{v\in\mathbb{R}}\bar{\sigma}(T_{z\omega}(\mathrm{j}v)) = \sup_{\mathbf{0}\ne\omega(t)\in L_2[0,\infty)}\frac{\|z(t)\|_2}{\|\omega(t)\|_2} < \gamma,$$

where $T_{z\omega}(s)$ represents the closed-loop transfer function matrix from $\omega(t)$ to $z(t)$, $\gamma > 0$ is a given H_∞ performance index, and $\bar{\sigma}(\bullet)$ denotes the largest singular value. Equivalently, the closed-loop system satisfies the following dissipation inequality

$$\int_0^\infty \|z(t)\|^2 \,\mathrm{d}t < \gamma^2\int_0^\infty \|\omega(t)\|^2 \,\mathrm{d}t, \quad \forall\omega(t)\in L_2[0,\infty).$$

2.3. Protocol design

Using the neighbors' local information, the protocol of agent i is designed as

$$u_i(t)=K\sum_{j\in N_i(t)}a_{ij}(t)[x_i(t-d(t))-x_j(t-d(t))], \quad (8)$$

where $d(t)\geq 0$ is the time-varying communication delay and $d(t)\equiv 0$ holds in non-delayed networks, $N_i(t)$ is the neighbor set of agent i at time instant t, $a_{ij}(t)$ are adjacency elements of the corresponding interaction graph, and K is an undetermined feedback matrix. Substituting protocol (8) into the system (7) results in the following closed-loop system

$$\begin{aligned}\dot{x}(t)&=(I_n\otimes A)x(t)+(L_{\sigma(t)}\otimes B_2K)x(t-d(t))\\&\quad+(I_n\otimes B_1)\omega(t),\\ z(t)&=(L_c\otimes I_m)x(t).\end{aligned} \quad (9)$$

3. CONDITIONS OF ROBUST H_∞ CONSENSUS

In this section, the robust H_∞ theory is employed to investigate the robust H_∞ performance of switched system (9), and sufficient conditions in terms of LMIs are derived to ensure the desired consensus performance of multi-agent system (1) with switching topology. In the following, we assume that the time-varying communication delay in (8) satisfies

(C1) $0\leq d(t)\leq \bar{d}$, $\dot{d}(t)\leq h$ for $t\geq 0$, where $\bar{d}>0$ and $h\geq 0$, or

(C2) $0\leq d(t)\leq \bar{d}$ for $t\geq 0$, where $\bar{d}>0$.

Note that the concerned state solution of system (9), implying the strict consensus of all agents, is $x(t)=\mathbf{1}\otimes c$ with $c\in\mathbb{R}^m$. In order to apply the existing robust H_∞ theory, the obove nonzero solution must be transformed to be zero by some equivalent model transformations, which will be presented in the first subsection.

3.1. Model transformation

By Lemma 2, there exists an orthogonal matrix $U\in\mathbb{R}^{n\times n}$ such that

$$\begin{aligned}U^{\mathrm{T}}L_cU&=\begin{bmatrix}I_{n-1}&0\\0&0\end{bmatrix}\triangleq\bar{L}_c,\\ U^{\mathrm{T}}L_{\sigma(t)}U&=\begin{bmatrix}L_{1\sigma(t)}&0\\0&0\end{bmatrix}\triangleq\bar{L}_{\sigma(t)},\end{aligned} \quad (10)$$

where $L_{1\sigma(t)}$ is positive definite since the graph $G_{\sigma(t)}$ is connected. For the convenience of discussion, denote $U=[U_1\ \ U_2]$ with $U_2=\mathbf{1}/\sqrt{n}$ being its last column. Let

$$\begin{aligned}\hat{x}(t)&=(U^{\mathrm{T}}\otimes I_m)\bar{x}(t)=\begin{bmatrix}(U_1^{\mathrm{T}}\otimes I_m)\bar{x}(t)\\(U_2^{\mathrm{T}}\otimes I_m)\bar{x}(t)\end{bmatrix}\triangleq\begin{bmatrix}\hat{x}^1(t)\\\hat{x}^2(t)\end{bmatrix},\\ \hat{\omega}(t)&=(U^{\mathrm{T}}\otimes I_{m_1})\omega(t)=\begin{bmatrix}(U_1^{\mathrm{T}}\otimes I_{m_1})\omega(t)\\(U_2^{\mathrm{T}}\otimes I_{m_1})\omega(t)\end{bmatrix}\triangleq\begin{bmatrix}\hat{\omega}^1(t)\\\hat{\omega}^2(t)\end{bmatrix},\\ \hat{z}(t)&=(U^{\mathrm{T}}\otimes I_m)z(t)=\begin{bmatrix}(U_1^{\mathrm{T}}\otimes I_m)z(t)\\(U_2^{\mathrm{T}}\otimes I_m)z(t)\end{bmatrix}\triangleq\begin{bmatrix}\hat{z}^1(t)\\\hat{z}^2(t)\end{bmatrix},\end{aligned} \quad (11)$$

where

$$\bar{x}(t)=x(t)-\frac{\mathbf{1}}{n}\otimes\Big(\sum_{j=1}^{n}x_j(t)\Big)=(L_c\otimes I_m)x(t). \quad (12)$$

From (9), (10), (11), and (12), we have

$$\begin{aligned}\dot{\hat{x}}(t)&=(\bar{L}_c\otimes A)\hat{x}(t)+(\bar{L}_c\bar{L}_{\sigma(t)}\otimes B_2K)\hat{x}(t-d(t))\\&\quad+(\bar{L}_c\otimes B_1)\hat{\omega}(t),\\ \hat{z}(t)&=(\bar{L}_c\otimes I_m)\hat{x}(t),\end{aligned} \quad (13)$$

which can be divided into the following two subsystems:

$$\begin{aligned}\dot{\hat{x}}^1(t)&=(I_{n-1}\otimes A)\hat{x}^1(t)+(L_{1\sigma(t)}\otimes B_2K)\hat{x}^1(t-d(t))\\&\quad+(I_{n-1}\otimes B_1)\hat{\omega}^1(t)\\&\triangleq E\hat{x}^1(t)+F_{\sigma(t)}\hat{x}^1(t-d(t))+G\hat{\omega}^1(t),\\ \hat{z}^1(t)&=\hat{x}^1(t),\end{aligned} \quad (14)$$

and $\dot{\hat{x}}^2(t)=0$, $\hat{z}^2(t)=0$. Obviously, $\|T_{z\omega}(s)\|_\infty=\|T_{\hat{z}\hat{\omega}}(s)\|_\infty=\|T_{\hat{z}^1\hat{\omega}^1}(s)\|_\infty$ holds by the definition of H_∞ norm. Therefore, we can analyze the H_∞ performance of the reduced-order system (14) instead of (9). To summarize, the robust consensus performance of the closed-loop multi-agent system (9) is achieved with H_∞ disturbance attenuation index γ, if the system (14) is asymptotically stable and satisfies the H_∞ performance level γ.

3.2. Condition establishment

Rewrite the equivalent form of (14) as

$$\begin{aligned}\dot{\hat{x}}^1(t)&=(I_{n-1}\otimes A+L_{1\sigma(t)}\otimes B_2K)\hat{x}^1(t)+(L_{1\sigma(t)}\otimes B_2K)\\&\quad\cdot(\hat{x}^1(t-d(t))-\hat{x}^1(t))+(I_{n-1}\otimes B_1)\hat{\omega}^1(t)\\&\triangleq H_{\sigma(t)}\hat{x}^1(t)+F_{\sigma(t)}\eta(t)+G\hat{\omega}^1(t),\\ \hat{z}^1(t)&=\hat{x}^1(t),\end{aligned} \quad (15)$$

i.e., $\hat{x}^1(t-d(t))-\hat{x}^1(t)\triangleq\eta(t)$. To analyze the H_∞ performance of the delayed system (15), we introduce the following lemmas.

Lemma 3 (Schur Complement Formula)**:** Let $F=[F_{ij}]_{i,j=1}^2\in\mathbb{R}^{n\times n}$ be a symmetric matrix in the partitioned form, where $F_{11}\in\mathbb{R}^{r\times r}$, $F_{12}\in\mathbb{R}^{r\times(n-r)}$ and $F_{22}\in\mathbb{R}^{(n-r)\times(n-r)}$. Then $F<0$ if and only if $F_{11}<0$, $F_{22}-F_{21}F_{11}^{-1}F_{12}<0$ or equivalently $F_{22}<0$, $F_{11}-F_{12}F_{22}^{-1}F_{21}<0$.

Lemma 4 [9]**:** For any real differentiable vector function $x(t)\in\mathbb{R}^n$ and any $n\times n$ constant matrix $W=W^{\mathrm{T}}>0$, we have

$$\bar{d}^{-1}[x(t)-x(t-d(t))]^{\mathrm{T}}W[x(t)-x(t-d(t))] \le \int_{t-d(t)}^{t}\dot{x}^{\mathrm{T}}(s)W\dot{x}(s)\mathrm{d}s, \quad \forall t\ge 0, \tag{16}$$

where $d(t)$ satisfies (C1) or (C2).

Lemma 5: For any given index $\gamma>0$, the system (15) is asymptotically stable and $\|T_{\hat{z}^1\hat{\omega}^1}(s)\|_\infty<\gamma$ holds for any $d(t)$ satisfying (C1) or (C2), if there exist positive definite matrices $R,S\in\mathbb{R}^{(n-1)m\times(n-1)m}$ such that

$$\begin{bmatrix} RH_{\sigma(t)}+H_{\sigma(t)}^{\mathrm{T}}R & RF_{\sigma(t)} & RG & \bar{d}H_{\sigma(t)}^{\mathrm{T}}S & I \\ F_{\sigma(t)}^{\mathrm{T}}R & -S & 0 & \bar{d}F_{\sigma(t)}^{\mathrm{T}}S & 0 \\ G^{\mathrm{T}}R & 0 & -\gamma^2 I & \bar{d}G^{\mathrm{T}}S & 0 \\ \bar{d}SH_{\sigma(t)} & \bar{d}SF_{\sigma(t)} & \bar{d}SG & -S & 0 \\ I & 0 & 0 & 0 & -I \end{bmatrix}<0 \tag{17}$$

holds for $\forall\sigma(t)\in\Omega$.

Proof: Firstly, we study the stability of system (15) without external disturbances. Define a common Lyapunov function as

$$V(t)=\hat{x}^{1\mathrm{T}}(t)R\hat{x}^1(t)+\bar{d}\int_{t-\bar{d}}^{t}(\tau-t+\bar{d})\dot{\hat{x}}^{1\mathrm{T}}(\tau)S\dot{\hat{x}}^1(\tau)\mathrm{d}\tau,$$

where $R,S\in\mathbb{R}^{(n-1)m\times(n-1)m}$ are positive definite. The time derivative of $V(t)$ is

$$\begin{aligned}\dot{V}(t)&\le \hat{x}^{1\mathrm{T}}(t)(RH_{\sigma(t)}+H_{\sigma(t)}^{\mathrm{T}}R)\hat{x}^1(t)+2\hat{x}^{1\mathrm{T}}(t)RF_{\sigma(t)}\eta(t)\\&\quad+\bar{d}^2\dot{\hat{x}}^{1\mathrm{T}}(t)S\dot{\hat{x}}^1(t)-\bar{d}\int_{t-d(t)}^{t}\dot{\hat{x}}^{1\mathrm{T}}(\tau)S\dot{\hat{x}}^1(\tau)\mathrm{d}\tau\\&\le\hat{x}^{1\mathrm{T}}(t)(RH_{\sigma(t)}+H_{\sigma(t)}^{\mathrm{T}}R)\hat{x}^1(t)+2\hat{x}^{1\mathrm{T}}(t)RF_{\sigma(t)}\eta(t)\\&\quad+\bar{d}^2\dot{\hat{x}}^{1\mathrm{T}}(t)S\dot{\hat{x}}^1(t)-\eta^{\mathrm{T}}(t)S\eta(t)\\&=\hat{x}^{1\mathrm{T}}(t)(RH_{\sigma(t)}+H_{\sigma(t)}^{\mathrm{T}}R)\hat{x}^1(t)+2\hat{x}^{1\mathrm{T}}(t)RF_{\sigma(t)}\eta(t)\\&\quad+\bar{d}^2[H_{\sigma(t)}\hat{x}^1(t)+F_{\sigma(t)}\eta(t)]^{\mathrm{T}}S[H_{\sigma(t)}\hat{x}^1(t)\\&\quad+F_{\sigma(t)}\eta(t)]-\eta^{\mathrm{T}}(t)S\eta(t)\\&=[\hat{x}^{1\mathrm{T}}(t)\ \ \eta^{\mathrm{T}}(t)]\left(\begin{bmatrix} RH_{\sigma(t)}+H_{\sigma(t)}^{\mathrm{T}}R & RF_{\sigma(t)}\\ F_{\sigma(t)}^{\mathrm{T}}R & -S\end{bmatrix}\right.\\&\quad\left.+\begin{bmatrix}\bar{d}H_{\sigma(t)}^{\mathrm{T}}S\\ \bar{d}F_{\sigma(t)}^{\mathrm{T}}S\end{bmatrix}S^{-1}[\bar{d}SH_{\sigma(t)}\ \ \bar{d}SF_{\sigma(t)}]\right)\begin{bmatrix}\hat{x}^1(t)\\ \eta(t)\end{bmatrix},\end{aligned}$$

where Lemma 4 has been applied in the second step. Then by Lemma 3, if

$$\begin{bmatrix} RH_{\sigma(t)}+H_{\sigma(t)}^{\mathrm{T}}R & RF_{\sigma(t)} & \bar{d}H_{\sigma(t)}^{\mathrm{T}}S \\ F_{\sigma(t)}^{\mathrm{T}}R & -S & \bar{d}F_{\sigma(t)}^{\mathrm{T}}S \\ \bar{d}SH_{\sigma(t)} & \bar{d}SF_{\sigma(t)} & -S\end{bmatrix}<0, \tag{18}$$

then $\dot{V}(t)<0$ holds. Therefore, the system (15) is asymptotically stable when $\hat{\omega}^1(t)\equiv 0$, since (18) can be guaranteed by the condition (17) due to Lemma 3.

Subsequently, we discuss the performance of system (15) with nonzero disturbance $\hat{\omega}^1(t)$. By the similar method in the stability analysis, we find that the time derivative of $V(t)$ along the solution to (15) satisfies

$$\begin{aligned}\dot{V}(t)&\le\begin{bmatrix}\hat{x}^1(t)\\ \eta(t)\\ \hat{\omega}^1(t)\end{bmatrix}^{\mathrm{T}}\left(\begin{bmatrix} RH_{\sigma(t)}+H_{\sigma(t)}^{\mathrm{T}}R & RF_{\sigma(t)} & RG\\ F_{\sigma(t)}^{\mathrm{T}}R & -S & 0\\ G^{\mathrm{T}}R & 0 & 0\end{bmatrix}\right.\\&\quad\left.+\begin{bmatrix}\bar{d}H_{\sigma(t)}^{\mathrm{T}}S\\ \bar{d}F_{\sigma(t)}^{\mathrm{T}}S\\ \bar{d}G^{\mathrm{T}}S\end{bmatrix}S^{-1}[\bar{d}SH_{\sigma(t)}\ \ \bar{d}SF_{\sigma(t)}\ \ \bar{d}SG]\right)\begin{bmatrix}\hat{x}^1(t)\\ \eta(t)\\ \hat{\omega}^1(t)\end{bmatrix}\\&\triangleq\varsigma^{\mathrm{T}}(t)\Theta_{\sigma(t)}\varsigma(t),\end{aligned}$$

where $\varsigma(t)=[\hat{x}^{1\mathrm{T}}(t)\ \ \eta^{\mathrm{T}}(t)\ \ \hat{\omega}^{1\mathrm{T}}(t)]^{\mathrm{T}}$. For any $T>0$, consider the following cost function

$$J_T=\int_0^T\hat{z}^{1\mathrm{T}}(t)\hat{z}^1(t)\mathrm{d}t-\gamma^2\int_0^T\hat{\omega}^{1\mathrm{T}}(t)\hat{\omega}^1(t)\,\mathrm{d}t.$$

Under the zero initial condition $(V(0)=0)$, we have

$$\begin{aligned}J_T&=\int_0^T[\hat{x}^{1\mathrm{T}}(t)\hat{x}^1(t)-\gamma^2\hat{\omega}^{1\mathrm{T}}(t)\hat{\omega}^1(t)+\dot{V}(t)]\mathrm{d}t-V(T)\\&\le\int_0^T[\varsigma^{\mathrm{T}}(t)(\Theta_{\sigma(t)}+\mathrm{diag}\{I,0,-\gamma^2 I\})\varsigma(t)]\mathrm{d}t-V(T)\\&\triangleq\int_0^T\varsigma^{\mathrm{T}}(t)\ \bar{\Theta}_{\sigma(t)}\varsigma(t)\mathrm{d}t-V(T).\end{aligned}$$

According to Lemma 3, the condition (17) is equivalent to $\bar{\Theta}_{\sigma(t)}<0$, from which $J_T<0$ follows. That is,

$$\int_0^T\|\hat{z}^1(t)\|^2\mathrm{d}t<\gamma^2\int_0^T\|\hat{\omega}^1(t)\|^2\mathrm{d}t.$$

Let $T\to\infty$, we have $\int_0^\infty\|\hat{z}^1(t)\|^2\mathrm{d}t<\gamma^2\int_0^\infty\|\hat{\omega}^1(t)\|^2$ dt, which completes the proof.

First of all, we consider the multi-agent system (1) by neglecting the model uncertainties in (2), i.e., matrices A, B_1, B_2 are known constant. Denote $\lambda_{\sigma(t)i}$ as the ith positive real eigenvalue of matrix $L_{\sigma(t)}$, $i=1,\cdots,n-1$. Let $\sigma_* i_*$ and $\sigma^* i^*$ be the subscripts associated with the minimum and the maximum nonzero eigenvalues of all matrices $L_{\sigma(t)}$, respectively.

Theorem 1: Under protocol (8), the multi-agent system (1) achieves consensus with a given H_∞ disturbance attenuation index γ, if there exists a scalar $\alpha>0$, a positive definite matrix $P\in\mathbb{R}^{m\times m}$ and a matrix $Q\in\mathbb{R}^{m_2\times m}$ such that linear matrix inequality (LMI)

$$\begin{bmatrix} \Psi_{\sigma(t)i}+\Psi_{\sigma(t)i}^{\mathrm{T}} & \alpha\lambda_{\sigma(t)i}B_2Q & B_1 & \bar{d}\Psi_{\sigma(t)i}^{\mathrm{T}} & P \\ \alpha\lambda_{\sigma(t)i}Q^{\mathrm{T}}B_2^{\mathrm{T}} & -\alpha P & 0 & \bar{d}\alpha\lambda_{\sigma(t)i}Q^{\mathrm{T}}B_2^{\mathrm{T}} & 0 \\ B_1^{\mathrm{T}} & 0 & -\gamma^2 I & \bar{d}B_1^{\mathrm{T}} & 0 \\ \bar{d}\Psi_{\sigma(t)i} & \bar{d}\alpha\lambda_{\sigma(t)i}B_2Q & \bar{d}B_1 & -\alpha P & 0 \\ P & 0 & 0 & 0 & -I \end{bmatrix} < 0 \quad (19)$$

is satisfied for $\sigma(t)i=\sigma_* i_*$ and $\sigma^* i^*$, where $\Psi_{\sigma(t)i} = AP+\lambda_{\sigma(t)i}B_2Q$. If the above two LMIs are feasible, then the feedback matrix of the consensus protocol is given by

$$K = QP^{-1}. \quad (20)$$

Proof: By Lemma 5, the multi-agent system (1) achieves consensus with the H_∞ index γ under protocol (8), if there exist positive definite matrices $R, S \in \mathbb{R}^{(n-1)m\times(n-1)m}$ satisfying (17) for $\forall\sigma(t)\in\Omega$. To simplify the consensus condition analysis, let the undetermined matrices R and S in inequality (17) take the special form: $R = I_{n-1}\otimes\bar{R}$ and $S = I_{n-1}\otimes\bar{S}$.

Since matrix $L_{1\sigma(t)}$ is positive definite, there exists an orthogonal matrix $U_{1\sigma(t)}\in\mathbb{R}^{(n-1)\times(n-1)}$ such that $U_{1\sigma(t)}^{\mathrm{T}}L_{1\sigma(t)}U_{1\sigma(t)} = \mathrm{diag}\{\lambda_{\sigma(t)1},\lambda_{\sigma(t)2},\cdots,\lambda_{\sigma(t)(n-1)}\}$, where $\lambda_{\sigma(t)i}$ $(i=1,\cdots,n-1)$ are the positive eigenvalues of $L_{\sigma(t)}$. Let $\bar{U}_{1\sigma(t)} = U_{1\sigma(t)}\otimes I_m$. According to the proof of theorem 3.3 in [16], we pre- and post-multiply the matrix inequality (17) with $\Upsilon = \mathrm{diag}\{\bar{U}_{1\sigma(t)},\bar{U}_{1\sigma(t)},\bar{U}_{1\sigma(t)},\bar{U}_{1\sigma(t)},\bar{U}_{1\sigma(t)}\}^{\mathrm{T}}$ and Υ^{T}, respectively, and conduct the elementary transformation of matrices to obtain a series of matrix inequalities

$$\Phi_{\sigma(t)i} = \begin{bmatrix} \bar{R}H_{\sigma(t)i}+H_{\sigma(t)i}^{\mathrm{T}}\bar{R} & \bar{R}F_{\sigma(t)i} & \bar{R}B_1 & \bar{d}H_{\sigma(t)i}^{\mathrm{T}}\bar{S} & I \\ F_{\sigma(t)i}^{\mathrm{T}}\bar{R} & -\bar{S} & 0 & \bar{d}F_{\sigma(t)i}^{\mathrm{T}}\bar{S} & 0 \\ B_1^{\mathrm{T}}\bar{R} & 0 & -\gamma^2 I & \bar{d}B_1^{\mathrm{T}}\bar{S} & 0 \\ \bar{d}\bar{S}H_{\sigma(t)i} & \bar{d}\bar{S}F_{\sigma(t)i} & \bar{d}\bar{S}B_1 & -\bar{S} & 0 \\ I & 0 & 0 & 0 & -I \end{bmatrix} < 0, \quad i=1,\cdots,n-1, \quad (21)$$

that are equivalent to (17), where $H_{\sigma(t)i} = A+\lambda_{\sigma(t)i}B_2K$ and $F_{\sigma(t)i} = \lambda_{\sigma(t)i}B_2K$.

To determine the feedback matrix K, let $\bar{S} = \alpha^{-1}\bar{R}$, $\alpha > 0$. Then pre- and post-multiplying (21) with $\mathrm{diag}\{\bar{R}^{-1},\bar{S}^{-1},I,\bar{S}^{-1},I\}$ leads to the LMI (19) with $\bar{R}^{-1}=P$ and $K\bar{R}^{-1}=Q$. Note that (19) is an LMI in terms of matrices P and Q. Due to the convex property of LMIs, if (19) holds when $\lambda_{\sigma(t)i}$ takes its extreme values $\lambda_{\sigma_* i_*}$ and $\lambda_{\sigma^* i^*}$, then the LMI (19) holds for $\forall\sigma(t)i$. Consequently, the multi-agent system (1) achieves consensus with H_∞ index γ if (19) is satisfied for $\sigma(t)i=\sigma_* i_*$ and $\sigma^* i^*$. Further, if the above two LMIs are feasible, then (20) is derived from $KP = Q$.

According to the previous development, the consensus condition is further given for the multi-agent system (1) with model uncertainties (2).

Lemma 6: Given symmetric matrices $X, Y, Z\in\mathbb{R}^{n\times n}$ satisfying $X\geq 0$, $Y<0$, $Z\geq 0$, the inequality

$$(\zeta^{\mathrm{T}}Y\zeta)^2 - 4\zeta^{\mathrm{T}}X\zeta\zeta^{\mathrm{T}}Z\zeta > 0 \quad (22)$$

holds for any vector $\mathbf{0}\neq\zeta\in\mathbb{R}^n$, if and only if there exists a scalar $\lambda>0$ such that

$$\lambda^2 X+\lambda Y+Z<0. \quad (23)$$

Proof: The necessity is obtained from Lemma 4 of chapter 5 in [19]. And the sufficiency can be easily derived by pre- and post-multiplying the matrix inequality (23) with nonzero vectors ζ^{T} and ζ.

Theorem 2: Under protocol (8), the multi-agent system (1) with model uncertainties (2) achieves robust consensus with a given H_∞ disturbance attenuation index γ, if for scalars α, $\lambda>0$, there exists a positive definite matrix $P\in\mathbb{R}^{m\times m}$ and a matrix $Q\in\mathbb{R}^{m_2\times m}$ such that LMI

$$\begin{bmatrix} \bar{\Psi}_{\sigma(t)i}+\bar{\Psi}_{\sigma(t)i}^{\mathrm{T}} & \alpha\lambda_{\sigma(t)i}B_{20}Q & B_{10} & \bar{d}\bar{\Psi}_{\sigma(t)i}^{\mathrm{T}} & P & \lambda E & \lambda^{-1}\chi_{\sigma(t)i}^{\mathrm{T}} \\ \alpha\lambda_{\sigma(t)i}Q^{\mathrm{T}}B_{20}^{\mathrm{T}} & -\alpha P & 0 & \bar{d}\alpha\lambda_{\sigma(t)i}Q^{\mathrm{T}}B_{20}^{\mathrm{T}} & 0 & 0 & \alpha\lambda^{-1}\lambda_{\sigma(t)i}Q^{\mathrm{T}}F_3^{\mathrm{T}} \\ B_{10}^{\mathrm{T}} & 0 & -\gamma^2 I & \bar{d}B_{10}^{\mathrm{T}} & 0 & 0 & \lambda^{-1}F_2^{\mathrm{T}} \\ \bar{d}\bar{\Psi}_{\sigma(t)i} & \bar{d}\alpha\lambda_{\sigma(t)i}B_{20}Q & \bar{d}B_{10} & -\alpha P & 0 & \lambda\bar{d}E & 0 \\ P & 0 & 0 & 0 & -I & 0 & 0 \\ \lambda E^{\mathrm{T}} & 0 & 0 & \lambda\bar{d}E^{\mathrm{T}} & 0 & -I & 0 \\ \lambda^{-1}\chi_{\sigma(t)i} & \alpha\lambda^{-1}\lambda_{\sigma(t)i}F_3Q & \lambda^{-1}F_2 & 0 & 0 & 0 & -I \end{bmatrix} < 0,$$

$$\begin{aligned} \bar{\Psi}_{\sigma(t)i} &= A_0P+\lambda_{\sigma(t)i}B_{20}Q, \\ \chi_{\sigma(t)i} &= F_1P+\lambda_{\sigma(t)i}F_3Q \end{aligned} \quad (24)$$

is satisfied for $\sigma(t)i=\sigma_* i_*$ and $\sigma^* i^*$. If the above two

LMIs are feasible, then the feedback matrix of the consensus protocol can be determined by (20).

Proof: Combining the proof of Theorem 1 with the definition of negative definite matrices, the multi-agent system (1) reaches the desired robust H_∞ consensus if $\xi^{\mathrm{T}}\Phi_{\sigma(t)i}\xi<0$ holds for any nonzero vector ξ and $\forall\sigma(t)i$. From (2) and (3), we know that $\Phi_{\sigma(t)i}=\bar{\Phi}_{\sigma(t)i}+\Delta\bar{\Phi}_{\sigma(t)i}$, where

$$\bar{\Phi}_{\sigma(t)i}=\begin{bmatrix}\bar{R}\bar{H}_{\sigma(t)i}+\bar{H}_{\sigma(t)i}^{\mathrm{T}}\bar{R} & \bar{R}\bar{F}_{\sigma(t)i} & \bar{R}B_{10} & \bar{d}\bar{H}_{\sigma(t)i}^{\mathrm{T}}\bar{S} & I\\ \bar{F}_{\sigma(t)i}^{\mathrm{T}}\bar{R} & -\bar{S} & 0 & \bar{d}\bar{F}_{\sigma(t)i}^{\mathrm{T}}\bar{S} & 0\\ B_{10}^{\mathrm{T}}\bar{R} & 0 & -\gamma^2 I & \bar{d}B_{10}^{\mathrm{T}}\bar{S} & 0\\ \bar{d}\bar{S}\bar{H}_{\sigma(t)i} & \bar{d}\bar{S}\bar{F}_{\sigma(t)i} & \bar{d}\bar{S}B_{10} & -\bar{S} & 0\\ I & 0 & 0 & 0 & -I\end{bmatrix},$$

$\bar{H}_{\sigma(t)i}=A_0+\lambda_{\sigma(t)i}B_{20}K$, $\bar{F}_{\sigma(t)i}=\lambda_{\sigma(t)i}B_{20}K$, and

$$\Delta\bar{\Phi}_{\sigma(t)i}=\begin{bmatrix}(\bar{R}\Delta\bar{H}_{\sigma(t)i}+\Delta\bar{H}_{\sigma(t)i}^{\mathrm{T}}\bar{R}) & \bar{R}\Delta\bar{F}_{\sigma(t)i} & \bar{R}E\Sigma(t)F_2 & \bar{d}\Delta\bar{H}_{\sigma(t)i}^{\mathrm{T}}\bar{S} & 0\\ \Delta\bar{F}_{\sigma(t)i}^{\mathrm{T}}\bar{R} & 0 & 0 & \bar{d}\Delta\bar{F}_{\sigma(t)i}^{\mathrm{T}}\bar{S} & 0\\ F_2^{\mathrm{T}}\Sigma^{\mathrm{T}}(t)E^{\mathrm{T}}\bar{R} & 0 & 0 & \bar{d}F_2^{\mathrm{T}}\Sigma^{\mathrm{T}}(t)E^{\mathrm{T}}\bar{S} & 0\\ \bar{d}\bar{S}\Delta\bar{H}_{\sigma(t)i} & \bar{d}\bar{S}\Delta\bar{F}_{\sigma(t)i} & \bar{d}\bar{S}E\Sigma(t)F_2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$\Delta\bar{H}_{\sigma(t)i}=E\Sigma(t)F_1+\lambda_{\sigma(t)i}E\Sigma(t)F_3K$,

$\Delta\bar{F}_{\sigma(t)i}=\lambda_{\sigma(t)i}E\Sigma(t)F_3K$.

Let $\xi=[\xi_1^{\mathrm{T}}\xi_2^{\mathrm{T}}\xi_3^{\mathrm{T}}\xi_4^{\mathrm{T}}\xi_5^{\mathrm{T}}]^{\mathrm{T}}$ be a nonzero vector. The inequality $\xi^{\mathrm{T}}\Phi_{\sigma(t)i}\xi<0$ holds if and only if

$$\xi^{\mathrm{T}}\Phi_{\sigma(t)i}\xi=\xi^{\mathrm{T}}\bar{\Phi}_{\sigma(t)i}\xi+2[\xi_1^{\mathrm{T}}\bar{R}E+\bar{d}\xi_4^{\mathrm{T}}\bar{S}E]\Sigma(t)[(F_1+\lambda_{\sigma(t)i}F_3K)\xi_1+\lambda_{\sigma(t)i}F_3K\xi_2+F_2\xi_3]<0 \tag{25}$$

is satisfied for any $\Sigma^{\mathrm{T}}(t)\Sigma(t)\le I$. Actually, if one takes

$$\Sigma(t)=\frac{\vartheta_0\vartheta_{\sigma(t)i}^{\mathrm{T}}}{\|\vartheta_0\|_2\|\vartheta_{\sigma(t)i}\|_2} \tag{26}$$

with $\vartheta_0=E^{\mathrm{T}}\bar{R}\xi_1+\bar{d}E^{\mathrm{T}}\bar{S}\xi_4$ and $\vartheta_{\sigma(t)i}=(F_1+\lambda_{\sigma(t)i}F_3K)\xi_1+\lambda_{\sigma(t)i}F_3K\xi_2+F_2\xi_3$, then $\xi^{\mathrm{T}}\Phi_{\sigma(t)i}\xi$ reaches its maximum value. Therefore, (25) holds for any $\Sigma(t)$ satisfying $\Sigma^{\mathrm{T}}(t)\Sigma(t)\le I$ if and only if it holds when $\Sigma(t)$ is taken as (26). instituting (26) into (25) leads to

$$\xi^{\mathrm{T}}\bar{\Phi}_{\sigma(t)i}\xi+2\sqrt{\xi^{\mathrm{T}}X\xi}\sqrt{\xi^{\mathrm{T}}Z_{\sigma(t)i}\xi}<0, \tag{27}$$

where

$$X=\begin{bmatrix}\bar{R}EE^{\mathrm{T}}\bar{R} & 0 & 0 & \bar{d}\bar{R}EE^{\mathrm{T}}\bar{S} & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ \bar{d}\bar{S}EE^{\mathrm{T}}\bar{R} & 0 & 0 & \bar{d}^2\bar{S}EE^{\mathrm{T}}\bar{S} & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix} \tag{28}$$

and

$$Z_{\sigma(t)i}=\begin{bmatrix}Y_{\sigma(t)i}^{\mathrm{T}}Y_{\sigma(t)i} & \lambda_{\sigma(t)i}Y_{\sigma(t)i}^{\mathrm{T}}F_3K & Y_{\sigma(t)i}^{\mathrm{T}}F_2 & 0 & 0\\ \lambda_{\sigma(t)i}K^{\mathrm{T}}F_3^{\mathrm{T}}Y_{\sigma(t)i} & \lambda_{\sigma(t)i}^2K^{\mathrm{T}}F_3^{\mathrm{T}}F_3K & \lambda_{\sigma(t)i}K^{\mathrm{T}}F_3^{\mathrm{T}}F_2 & 0 & 0\\ F_2^{\mathrm{T}}Y_{\sigma(t)i} & \lambda_{\sigma(t)i}F_2^{\mathrm{T}}F_3K & F_2^{\mathrm{T}}F_2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix} \tag{29}$$

with $Y_{\sigma(t)i}=F_1+\lambda_{\sigma(t)i}F_3K$. Obviously, (27) is satisfied if and only if $\bar{\Phi}_{\sigma(t)i}<0$ and

$$(\xi^{\mathrm{T}}\bar{\Phi}_{\sigma(t)i}\xi)^2-4\xi^{\mathrm{T}}X\xi\xi^{\mathrm{T}}Z_{\sigma(t)i}\xi>0. \tag{30}$$

Using Lemma 6, we obtain that (30) holds if and only if there exists a scalar $\lambda>0$ satisfying

$$\bar{\Phi}_{\sigma(t)i}+\lambda^2X+\lambda^{-2}Z_{\sigma(t)i}<0.$$

Inserting (28) and (29) into the above inequality results in

$$\bar{\Phi}_{\sigma(t)i}+W^{\mathrm{T}}W<0 \tag{31}$$

with

$$W=\begin{bmatrix}\lambda E^{\mathrm{T}}\bar{R} & 0 & 0 & \lambda\bar{d}E^{\mathrm{T}}\bar{S} & 0\\ \lambda^{-1}Y_{\sigma(t)i} & \lambda^{-1}\lambda_{\sigma(t)i}F_3K & \lambda^{-1}F_2 & 0 & 0\end{bmatrix}.$$

According to Lemma 3, matrix inequality (31) becomes

$$\begin{bmatrix}\bar{\Phi}_{\sigma(t)i} & W^{\mathrm{T}}\\ W & -I\end{bmatrix}<0. \tag{32}$$

Similar to Theorem 1, pre- and post-multiplying (32) with the symmetric matrix $\mathrm{diag}\{\bar{R}^{-1},\bar{S}^{-1},I,\bar{S}^{-1},I,I,I\}$ and letting $\bar{S}=\alpha^{-1}\bar{R}$ $(\alpha>0)$ lead to the LMI (24) with $\bar{R}^{-1}=P$ and $K\bar{R}^{-1}=Q$. Combining with the convex property of LMIs, we have that if for a scalar $\lambda>0$, there exists a positive definite matrix P and a matrix Q such that the LMI (24) holds for $\sigma(t)i=\sigma_*i_*$ and σ^*i^*, then the multi-agent system (1) with model uncertainties (2) achieves robust consensus with the H_∞ index γ under protocol (8). Further, if the above two LMIs are feasible, then $K=QP^{-1}$ is derived.

4. SIMULATION RESULTS

In this section, we provide a simulation example to illustrate the robust consensus performance of uncertain multi-agent systems under the proposed protocol. In particular, a network of four agents is considered, and the matrices in (1) and (2) are

$$A_0=\begin{bmatrix}0 & -1\\ 2 & 1\end{bmatrix},\ B_{10}=\begin{bmatrix}0\\ 1\end{bmatrix},\ B_{20}=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix},$$
$$E=\begin{bmatrix}0 & 0\\ 0.2 & 0.2\end{bmatrix},\ \Sigma(t)=\begin{bmatrix}\sin(10t) & 0\\ 0 & 1-e^{-t}\end{bmatrix},$$
$$F_1=\begin{bmatrix}0.4 & 0\\ 0 & 0.5\end{bmatrix},\ F_2=\begin{bmatrix}0.3\\ 0.3\end{bmatrix},\ F_3=\begin{bmatrix}0.2 & 0\\ 0 & 0.2\end{bmatrix}.$$

Here, the communication delay in networks is $d(t)=0.1\cos(t)$ that satisfies (c1). The external disturbance is supposed to be

$$\begin{aligned}\omega(t)&=[\omega_1(t)\ \ \omega_2(t)\ \ \omega_3(t)\ \ \omega_4(t)]^{\mathrm{T}}\\ &=[2w(t)\ \ -1.5w(t)\ \ 1.2w(t)\ \ 1.8w(t)]^{\mathrm{T}}\end{aligned}$$

with $w(t)$ shown in Fig. 1, which is band-limited white noise on the time interval [0,10]. The H_∞ performance index is chosen as $\gamma=1$. For simplicity, the possible interaction graphs are constrained to be within the set $\{G_1, G_2, G_3\}$ as given in Fig. 2, and it is assumed that all the nonzero weighting factors are 1. Then the maximum and the minimum nonzero eigenvalues of the corresponding three Laplacian matrices are 4 and 0.5858.

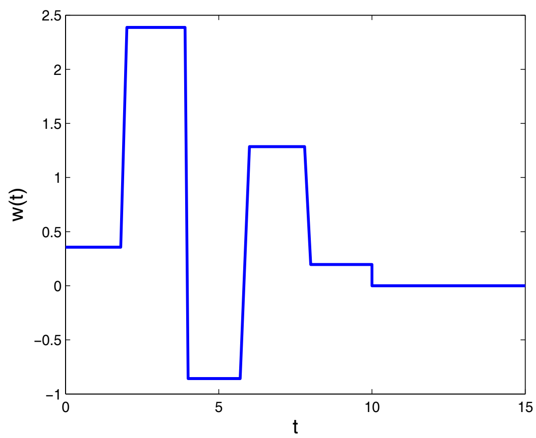
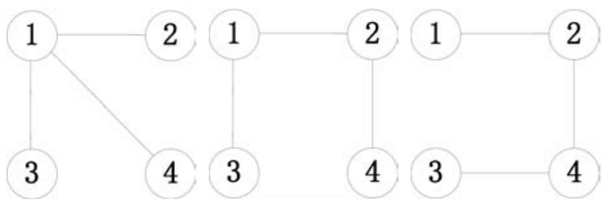


Fig. 1. Trajectory of $w(t)$.



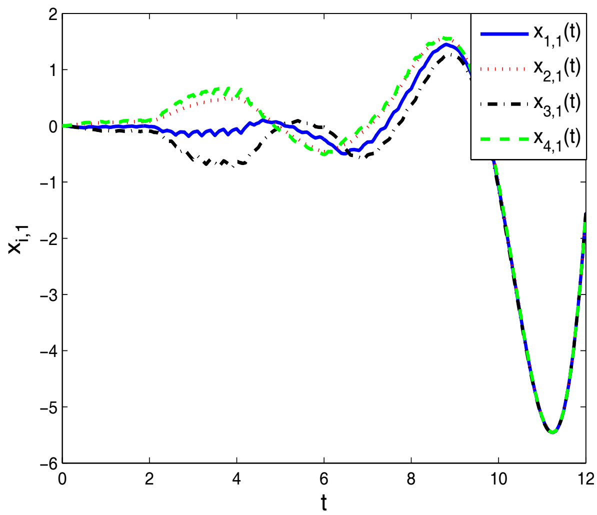
(a) G_1. (b) G_2. (c) G_3.

Fig. 2. Undirected interaction graphs.

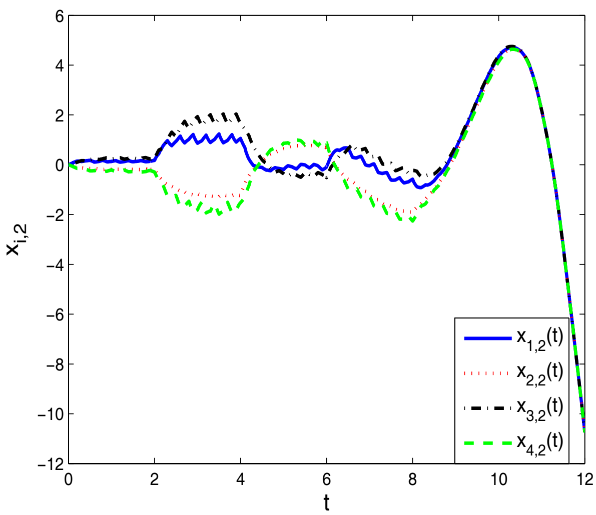
According to Theorem 2, the feedback matrix of protocol (8) can be determined as

$$K=\begin{bmatrix}-2.2684 & -0.2213\\ -0.1565 & -1.0674\end{bmatrix}$$

by solving two LMIs related to $\lambda_{\sigma(t)i}=4$ and 0.5858. Fig. 3 depicts the state trajectories of four agents $x_i(t)=[x_{i,1}(t)\ \ x_{i,2}(t)]^{\mathrm{T}}$ under the designed protocol. It can be seen that the disagreements among agents are small compared with the intensity of external disturbance given in Fig. 1, and all agents reach consensus again within 12s. Fig. 4 exhibits the energy relationship between the controlled output $z(t)$ and the external disturbance $\omega(t)$. Obviously, the consensus is achieved with H_∞ disturbance attenuation index 1 in the presence of delay $d(t)$, which validates the effectiveness of the proposed protocol and demonstrates the correctness of the obtained theoretical results.



(a) The first element: $x_{i,1}(t)$.



(b) The second element: $x_{i,2}(t)$.

Fig. 3. State trajectories of four agents.

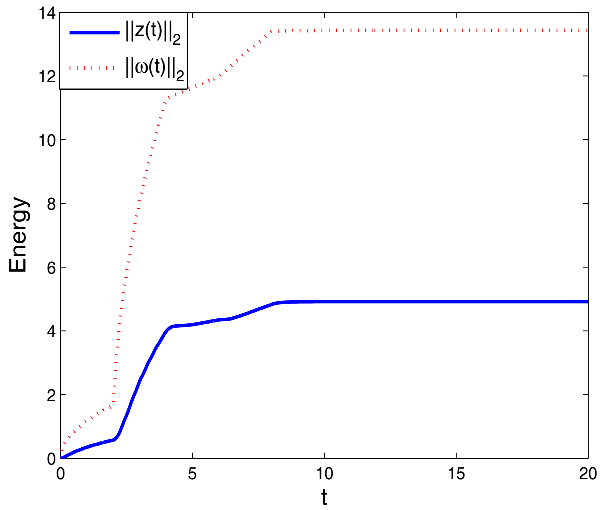


Fig. 4. Energy trajectories of $z(t)$ and $\omega(t)$.

5. CONCLUSIONS

By applying the robust H_∞ theory, we have addressed the consensus control problem for switching networks of autonomous agents with model uncertainties, subject to external disturbances and communication delays. Sufficient conditions are given to ensure the consensus performance with a given H_∞ disturbance attenuation level for the disturbed multi-agent system without and with model uncertainties respectively, and meanwhile determine the feedback matrix of the proposed distributed state-feedback protocol accordingly. Simulation results illustrate the satisfactory robust consensus performance of uncertain multi-agent systems under the designed protocol.

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