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Exponential P-Stability of Singularly Perturbed Impulsive Stochastic Delay Differential Systems

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Abstract: In this paper, we study singularly perturbed impulsive stochastic delay differential systems (SPISDDSs). By establishing an *L*-operator delay differential inequality and using the stochastic analysis technique, we obtain some sufficient conditions ensuring the exponential *p*-stability of any solution of SPISDDSs for sufficiently small $\varepsilon > 0$. The results extend and improve the earlier publications. An example is also discussed to illustrate the efficiency of the obtained results.

Keywords: Impulsive, *L*-operator delay differential inequality, p-stability, singularly perturbed, stochastic.

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1. INTRODUCTION

The last two decades witnessed considerable research efforts aimed at the investigation of singularly perturbed delay differential systems, which arise in the study of an "optically bistable device" [1] and in a variety of models for physiological processes or diseases [2]. And many good results on the stability of singularly perturbed delay differential systems have been reported, see e.g., [3-6]. On the other hand, many evolution processes exhibit abrupt changes of their states at certain moments, such as threshold phenomena in biology, bursting rhythm models in medicine, optimal control models in economics, circuit networks and frequency modulated systems, etc [7]. These abrupt changes are of the short duration and may be described by impulsive differential systems [7-10]. There are many papers have devoted to the stability of singularly perturbed impulsive delay differential systems, see e.g., [11-13].

However, besides delay and impulsive effects and singularly perturbed, stochastic effects likewise exist in real systems. A lot of dynamical systems have variable structures subject to stochastic abrupt changes, which may result from abrupt phenomena such as stochastic failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, etc [14]. In recent years, the stability investigation of stochastic delay differential systems with or without impulses is interesting to many investigators, and a large number of stability criteria of these systems have been reported [14-19].

Unfortunately, up to now, the exponential *p*-stability analysis problem of SPISDDSs is still an open problem that has not been properly studied although Socha [20], El-Ansary [21] and El-Ansary and Khalil [22] have discussed the stability of singularly perturbed stochastic systems without impulses. It is, therefore, our intention in this paper is to investigate the exponential *p*-stability analysis problem of SPISDDSs. By establishing an *L*-operator delay differential inequality and using the stochastic analysis technique, we obtain some sufficient conditions ensuring the exponential *p*-stability of any solution of SPISDDSs for sufficiently small $\varepsilon > 0$. The results extend and improve the earlier publications. An example is also discussed to illustrate the efficiency of the obtained results.

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2. MODEL AND PRELIMINARIES

To begin with, we introduce some notation and recall some basic definitions. Let I denote the n-dimensional unit matrix, $|\cdot|$ denote the Euclidean norm, $\mathcal{N} \triangleq \{1, 2,\dots,n\}$, $\mathbb{Z}_+ \triangleq \{1,2,\cdots,\}$, $\mathbb{R}_+ = [0,\infty)$. For $A, B \in \mathbb{R}^{m\times n}$ or $A, B \in \mathbb{R}^n$, $A \geq B (A \leq B, A > B, A < B)$ means that each pair of corresponding elements of A and B satisfies the inequality "$\geq (\leq, >, <)$". Especially, A is called a nonnegative matrix if $A \geq 0$, and z is called a positive vector if $z > 0$.

$C[X, Y]$ denotes the space of continuous mappings from the topological space X to the topological space Y. In particular, let $C \triangleq C[[-\tau, 0], \mathbb{R}^n]$.

$PC[\mathbb{J}, \mathbb{R}^n] = \{\psi : \mathbb{J} \to \mathbb{R}^n \mid \psi(s)$ is continuous for all but at most countable points $s \in \mathbb{J}$ and at these points $s \in \mathbb{J}$, $\psi(s^+)$ and $\psi(s^-)$ exist, $\psi(s) = \psi(s^+)\}$, where $\mathbb{J} \subset \mathbb{R}$ is an interval, $\psi(s^+)$ and $\psi(s^-)$ denote the right-hand and left-hand limits of the function $\psi(s)$, respectively. Especially, let $PC \triangleq PC[[-\tau, 0], \mathbb{R}^n]$.

For $x \in \mathbb{R}^n$, $\varphi \in C$ or $\varphi \in PC$, $p > 0$, we define

$[x]_+^p = (|x_1|^p, \cdots, |x_n|^p)^T$, especially $[x]_+ = (|x_1|, \cdots, |x_n|)^T$, $[\varphi(t)]_\tau = ([\varphi_1(t)]_\tau, \cdots, [\varphi_n(t)]_\tau)^T$, $[\varphi_i(t)]_\tau = \sup_{-\tau \le s \le 0}\{\varphi_i(t+s)\}, i \in \mathcal{N}$, and $D^+\varphi(t)$ denotes the upper right derivative of $\varphi(t)$ at time t.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $PC^b_{\mathcal{F}_0}[[-\tau,0],\mathbb{R}^n]$ denote the family of all bounded $\mathcal{F}_0$-measurable, $PC[[-\tau,0],\mathbb{R}^n]$-valued random variables ϕ, satisfying $\|\phi\|^p_{L^p} = \sup_{-\tau\le\theta\le 0} \mathbf{E}|\phi(\theta)|^p < \infty$, where $\mathbf{E}$ denotes the expectation of stochastic process.

In this paper, we consider the following singularly perturbed *Itô* impulsive stochastic delay differential systems:

$$\begin{cases} \varepsilon dx(t) = [A(t)x(t) + f(t, x(t-\tau(t)))]dt \\ \qquad +\sqrt{\varepsilon} g(t, x(t),\ x(t-\tau(t)))d\omega(t), \\ \qquad t \ge t_0,\ t \ne t_k, \\ x(t_k) = J_k(t_k, x(t_k^-)),\ k \in \mathbb{Z}_+, \\ x(t_0+s) = \phi(s),\ s \in [-\tau, 0], \end{cases} \tag{1}$$

where $0 \le \tau(t) \le \tau$, τ is constant, $\varepsilon \in (0, \varepsilon_0]$ is a small parameter, $A(t) = (a_{ij}(t))_{n\times n} \in PC[\mathbb{R}, \mathbb{R}^{n\times n}]$, $f: \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$, $g: \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$, i.e., $g(t,x,y) = g_{ij}(t,x,y))_{n\times m}$. Let $g_i(t,x,y) = (g_{i1}(t,x,y), \cdots, g_{im}(t,x,y))$ be i th row vector of $g(t,x,y)$, $i \in \mathcal{N}$. $\omega(t) = (\omega_1(t), \cdots, \omega_m(t))^T$ is an m-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, P)$. And the initial function $\phi(s) = (\phi_1(s), \cdots, \phi_n(s))^T \in PC^b_{\mathcal{F}_0}[[-\tau,0],\mathbb{R}^n]$, the impulsive function $J_k = (J_{1k}, \cdots, J_{nk})^T \in C[\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n]$, and the fixed impulsive moments t_k satisfy $t_1 < t_2 < \cdots, \lim_{k\to\infty} t_k = \infty$, $k \in \mathbb{Z}_+$.

Throughout this paper, we assume that for any $\phi \in PC^b_{\mathcal{F}_0}[[-\tau,0],\mathbb{R}^n]$, there exists at least one solution of (1), which is denoted by $x(t, t_0, \phi)$, or, $x(t)$, if no confusion occur.

Definition 1: The solution of (1) is said to be exponentially p-stable for sufficiently small ε if there exist finite constant vectors $K > 0$ and $\sigma > 0$, which are independent of $\varepsilon \in (0, \varepsilon_0]$ for some ε_0, and a constant $\lambda > 0$ such that $\mathbf{E}[x(t) - y(t)]_+^p \le Ke^{-\lambda(t-t_0)}$ for $t \ge t_0$ and for any initial perturbation satisfying $\sup_{s\in[-\tau,0]} \mathbf{E}[\phi(s) - \varphi(s)]_+^p < \sigma$. Here $y(t)$ is the solution of (1) corresponding to the initial condition φ. Especially, it is said to be exponentially stable in mean square when $p = 2$.

Lemma 1 (Arithmetic-mean-geometric-mean inequality [23]):

For $x_i \ge 0$, $\alpha_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$,

$$\prod_{i=1}^n x_i^{\alpha_i} \le \sum_{i=1}^n \alpha_i x_i,$$

the sign of equality holds if and only if $x_i = x_j$ for all $i, j \in \mathcal{N}$.

Lemma 2 [14]**:** For $x_i \ge 0$, $a_i \ge 0$, $i \in \mathcal{N}$ and $p \in \mathbb{Z}_+$, the following inequality holds

$$\left(\sum_{i=1}^n a_i x_i\right)^p \le \left(\sum_{i=1}^n a_i\right)^{(p-1)} \sum_{i=1}^n a_i x_i^p. \tag{2}$$

Lemma 3 [13]**:** Assume that $0 \le u(t) = (u_1(t), \cdots, u_n(t))^T \in \mathbb{R}^n$, $t \ge t_0$ satisfy

$$\begin{cases} D^+u(t) \le P(t)u(t) + Q(t)[u(t)]_\tau, & t \ge t_0, \\ u(t) = \varphi(t), & t \in [t_0-\tau, t_0],\ \varphi(t) \in PC, \end{cases} \tag{3}$$

where $P(t) = (p_{ij}(t))_{n\times n} \ge 0$ for $t \ge t_0$ and $i \ne j$, $Q(t) = (q_{ij}(t))_{n\times n} \ge 0$ for $t \ge t_0$. If there exist a positive vector $z = (z_1, \cdots, z_n)^T \in \mathbb{R}^n$ and two positive diagonal matrices $L = diag\{L_1, \cdots, L_n\}$, $H = diag\{h_1, \cdots, h_n\}$ with $0 < h_i < 1$ such that

$$(Q(t) + HP(t) + L)z \le 0, \quad t \ge t_0. \tag{4}$$

Then we have

$$u(t) \le ze^{-\lambda(t-t_0)},\ t \ge t_0, \tag{5}$$

where the positive constant λ is defined as

$$0 < \lambda < \lambda_0 = \min_{1\le i\le n}\left\{\inf_{t\ge t_0} \lambda_i(t): \lambda_i(t)z_i + \sum_{j=1}^n (p_{ij}(t) + q_{ij}(t)e^{\lambda_i(t)\tau})z_j = 0\right\}, \tag{6}$$

for the given z.

3. MAIN RESULTS

Let $C^{1,2}[\mathbb{R}_+\times\mathbb{R}^n, \mathbb{R}_+]$ denote the family of all nonnegative functions $\mathcal{V}(t,x)$ on $\mathbb{R}_+\times\mathbb{R}^n$ which are twice continuously differentiable in x and once in t. For each $\mathcal{V}(t,x) \in C^{1,2}[\mathbb{R}_+\times\mathbb{R}^n, \mathbb{R}_+]$, we define an operator L on $\mathcal{V}$, associated with the system (1), by

$$L\mathcal{V}(t,x) = \mathcal{V}_t(t,x) + \mathcal{V}_x(t,x) \left[\frac{1}{\varepsilon}\big(A(t)x(t) + f(t, x(t-\tau(t)))\big)\right]$$

$$+\frac{1}{2}\text{trace}\left[\left(\frac{1}{\sqrt{\varepsilon}}g(t,x,x(t-\tau(t)))\right)^T \mathcal{V}_{xx}\left(\frac{1}{\sqrt{\varepsilon}}g(t,x,x(t-\tau(t)))\right)\right],$$

$$\mathcal{V}_t(t,x)=\frac{\partial\mathcal{V}(t,x)}{\partial t},\ \mathcal{V}_x(t,x)=\left(\frac{\partial\mathcal{V}(t,x)}{\partial x_1},\cdots,\frac{\partial\mathcal{V}(t,x)}{\partial x_n}\right),$$

$$\mathcal{V}_{xx}=\left(\frac{\partial^2\mathcal{V}(t,x)}{\partial x_i\partial x_j}\right)_{n\times n}.$$

Lemma 4: Assume that there exist functions $V_i(x)\in C^2[\mathbb{R}^n,\mathbb{R}_+]$ such that

$$LV_i(x)\le\sum_{j=1}^n\left(p_{ij}(t)V_j(x)+q_{ij}(t)[V_j(x)]_\tau\right),\ t\ge t_0, i\in\mathcal{N}, \tag{7}$$

where $P(t)=(p_{ij}(t))_{n\times n}\ge 0$ for $t\ge t_0$ and $i\ne j$, $Q(t)=(q_{ij}(t))_{n\times n}\ge 0$ for $t\ge t_0$. If there exist a positive vector $z=(z_1,\cdots,z_n)^T\in\mathbb{R}^n$ and two positive diagonal matrices L=diag $\{L_1,\cdots,L_n\}$, H=diag $\{h_1,\cdots,h_n\}$ with $0<h_i<1$ such that

$$(Q(t)+HP(t)+L)z\le 0,\quad t\ge t_0. \tag{8}$$

Then we have

$$\mathbf{E}V_i(x(t))\le z_i e^{-\lambda(t-t_0)},\quad t\ge t_0, \tag{9}$$

where the positive constant λ is defined as

$$0<\lambda<\lambda_0=\min_{1\le i\le n}\left\{\inf_{t\ge t_0}\lambda_i(t):\ \lambda_i(t)z_i+\sum_{j=1}^n(p_{ij}(t)+q_{ij}(t)e^{\lambda_i(t)\tau})z_j=0\right\}, \tag{10}$$

for the given z.

Proof: Since $x(t)$ is the solution process of (1) and $V_i(x)\in C^2[\mathbb{R}^n,\mathbb{R}_+]$, by the *Itô* formula, we can get (For convenience, throughout this proof, we assume $t\ge t_0$, $i\in\mathcal{N}$)

$$V_i(x(t))=V_i(x(t_0))+\int_{t_0}^t LV_i(x(s))ds+\int_{t_0}^t\frac{\partial V_i(x(s))}{\partial x}\frac{1}{\sqrt{\varepsilon}}g(s,x(s),x(s-\tau(s))d\omega(s).$$

Then we have

$$\mathbf{E}V_i(x(t))=\mathbf{E}V_i(x(t_0))+\int_{t_0}^t\mathbf{E}LV_i(x(s))ds. \tag{11}$$

So, for small enough $\Delta t>0$, we have

$$\mathbf{E}V_i(x(t+\Delta t))=\mathbf{E}V_i(x(t_0))+\int_{t_0}^{t+\Delta t}\mathbf{E}LV_i(x(s))ds. \tag{12}$$

Thus from (7), (11) and (12), we have

$$\begin{aligned}\mathbf{E}V_i(x(t+\Delta t))-\mathbf{E}V_i(x(t))&=\int_t^{t+\Delta t}\mathbf{E}LV_i(x(s))ds\\&\le\int_t^{t+\Delta t}\left\{\sum_{j=1}^n[p_{ij}(t)\mathbf{E}V_j(x(s))+q_{ij}(t)[\mathbf{E}V_j(x(s))]_\tau]\right\}ds.\end{aligned} \tag{13}$$

From (13), we obtain that

$$D^+\mathbf{E}V_i(x(t))\le\sum_{j=1}^n\left(p_{ij}(t)\mathbf{E}V_j(x(t))+q_{ij}(t)[\mathbf{E}V_j(x(t))]_\tau\right). \tag{14}$$

By a similar argument with the proof of Lemma 1 in [13], one can know that that (10) has at least one positive solution $\lambda<\lambda_0$. Thus from Lemma 3, we know Lemma 4 is true.

In the following, we will obtain several sufficient conditions ensuring the exponential p-stability of (1) by employing Lemma 4. Here, we firstly introduce the following assumptions.

(A_1) For any $x,y\in\mathbb{R}^n$, there exists nonnegative matrix $U(t)=(u_{ij}(t))_{n\times n}$, $t\ge t_0$, such that

$$[f(t,x)-f(t,y)]_+\le U(t)[x-y]_+,\ t\ge t_0.$$

(A_2) For any $x,\bar{x},y,\bar{y}\in\mathbb{R}^n$, there exist nonnegative matrices $C(t)=(c_{ij}(t))_{n\times n}$ and $D(t)=(d_{ij}(t))_{n\times n}$, $t\ge t_0$, such that

$$\begin{aligned}&(g_i(t,x,y)-g_i(t,\bar{x},\bar{y}))(g_i(t,x,y)-g_i(t,\bar{x},\bar{y}))^T\\&\le\sum_{j=1}^n c_{ij}(t)|x_j-\bar{x}_j|^2+\sum_{j=1}^n d_{ij}(t)|y_j-\bar{y}_j|^2,\ i\in\mathcal{N}.\end{aligned} \tag{15}$$

(A_3) There exist a positive vector $z=(z_1,\cdots,z_n)^T\in\mathbb{R}^n$ and two positive diagonal matrices W=diag$\{w_1,\cdots,w_n\}$, $S=\text{diag}\{s_1,\cdots,s_n\}$, with $0<s_i<1$, $i\in\mathcal{N}$ such that

$$(\hat{Q}(t)+S\hat{P}(t)+W)z\le 0,\ t\ge t_0, \tag{16}$$

where

$$\hat{P}(t)=(\hat{p}_{ij}(t))_{n\times n},\ \hat{p}_{ij}(t)=a_{ij}(t)+(p-1)c_{ij}(t),\ i\ne j,$$

$$\hat{p}_{ii}(t)=pa_{ii}(t)+(p-1)(\sum_{j\ne i}^n a_{ij}(t)+\sum_{j=1}^n u_{ij}(t))+\frac{1}{2}(p-1)(p-2)\sum_{j=1}^n(c_{ij}(t)+d_{ij}(t))+(p-1)c_{ii}(t),$$

$$\hat{Q}(t)=(\hat{q}_{ij}(t))_{n\times n},$$

$$\hat{q}_{ii}(t)=u_{ij}(t)+(p-1)d_{ij}(t),\ p\in\mathbb{Z}_+,i,j\in\mathcal{N}.$$

(A_4) For any $x,y\in\mathbb{R}^n$, there exist constant matrices $M_k=(M_{k_{ij}})_{n\times n}\ge 0$ such that

$$[J_k(t,x)-J_k(t,y)]_+\le M_k[x-y]_+,\ t\ge t_0. \tag{17}$$

(A_5) There exists a positive constant η satisfying

$$\frac{\ln \eta_k}{t_k - t_{k-1}} \le \eta < \lambda(\varepsilon),\ k \in \mathbb{Z}_+, \tag{18}$$

where $\eta_k \ge 1$ satisfies

$$\eta_k z \ge \hat{M}_k z, \quad \hat{M}_k = (\hat{M}_{k_{ij}})_{n\times n},$$
$$\hat{M}_{k_{ij}} \ge M_{k_{ij}} \left(\sum_{j=1}^{n} M_{k_{ij}} \right)^{(p-1)}, \tag{19}$$

and $\lambda(\varepsilon)$ is defined as

$$0 < \lambda(\varepsilon) < \lambda_0(\varepsilon) = \min_{1\le i\le n} \Bigg\{ \inf_{t\ge t_0} \lambda_i(t,\varepsilon) : \lambda_i z_i + \sum_{j=1}^{n} \left(\frac{\hat{p}_{ij}(t)}{\varepsilon} + \frac{\hat{q}_{ij}(t)}{\varepsilon} e^{\lambda_i \tau} \right) z_j = 0 \Bigg\}, \tag{20}$$

for the given z.

Theorem 1: Assume that $A(t) = (a_{ij}(t))_{n\times n} \ge 0$ for $t \ge t_0$ and $i \ne j$, further suppose that (A_1)-(A_5) hold. Then there exists a small $\varepsilon_0 > 0$ such that the solution of (1) is exponentially p-stable for sufficiently small $\varepsilon \in (0, \varepsilon_0)]$.

Proof: By a similar argument with the proof of Lemma 1 in [13], one can know that the $\lambda(\varepsilon)$ defined by (20) is reasonable. For any $\phi, \varphi \in PC^b_{F_0}[[-\tau,0],\mathbb{R}^n]$, let $x(t), y(t)$ be two solutions processes of (1) through (t_0, ϕ), (t_0, φ) respectively. Since $\phi, \varphi \in PC^b_{\mathcal{F}_0}[[-\tau,0],\mathbb{R}^n]$ are bounded and (16) holds, we can always choose a positive vector z such that

$$\mathbf{E}\,|x_i(t) - y_i(t)|^p \le z_i e^{-\lambda(\varepsilon)(t-t_0)}, \quad t \in [t_0 - \tau, t_0],\ i \in \mathcal{N}. \tag{21}$$

Let $V_i(\tilde{x}(t)) = |\tilde{x}_i(t)|^p$, $p \in \mathbb{Z}_+$, $i \in \mathcal{N}$, where $\tilde{x}(t) = x(t) - y(t)$. Then, by conditions (A_1), (A_2) and Lemma 2, we have

$$\begin{aligned}
&LV_i(\tilde{x}(t))\\
&= p\,|x_i(t) - y_i(t)|^{(p-2)} (x_i(t) - y_i(t))\\
&\quad \times \frac{1}{\varepsilon}\Bigg[\sum_{j=1}^{n} a_{ij}(t)(x_j(t) - y_j(t)) + (f_i(t, x(t-\tau(t))) - f_i(t, y(t-\tau(t)))) \Bigg]\\
&\quad + \frac{1}{2} p(p-1)\,|x_i(t) - y_i(t)|^{p-2}\ \mathrm{sgn}(x_i(t) - y_i(t))\\
&\quad \times \Big(\frac{1}{\sqrt{\varepsilon}}(g_i(t, x(t), x(t-\tau(t))) - g_i(t, y(t), y(t-\tau(t))))\Big)\\
&\quad \times \Big(\frac{1}{\sqrt{\varepsilon}}(g_i(t, x(t), x(t-\tau(t))) - g_i(t, y(t), y(t-\tau(t))))\Big)^T\\
&\le \frac{p a_{ii}(t)}{\varepsilon} |\tilde{x}_i(t)|^p + \frac{p}{\varepsilon} |\tilde{x}_i(t)|^{(p-1)} \sum_{j\ne i} a_{ij}(t)\,|\tilde{x}_j(t)|\\
&\quad + \frac{p}{\varepsilon} |\tilde{x}_i(t)|^{p-1} \Big(\sum_{j=1}^{n} u_{ij}(t)\,|\tilde{x}_j(t-\tau(t))|\Big)\\
&\quad + \frac{1}{2\varepsilon} p(p-1)\,|\tilde{x}_i(t)|^{p-2} \left(\sum_{j=1}^{n} c_{ij}(t)\,|\tilde{x}_j(t)|^2 + \sum_{j=1}^{n} d_{ij}(t)\,|\tilde{x}_j(t-\tau(t))|^2 \right)\\
&\le \frac{p a_{ii}(t)}{\varepsilon} |\tilde{x}_i(t)|^p + \frac{1}{\varepsilon} \sum_{j\ne i} a_{ij}(t)[(p-1)\,|\tilde{x}_i(t)|^p + |\tilde{x}_j(t)|^p]\\
&\quad + \frac{1}{\varepsilon} \sum_{j=1}^{n} u_{ij}(t)[(p-1)\,|\tilde{x}_i(t)|^p + |\tilde{x}_j(t-\tau(t))|^p]\\
&\quad + \frac{p-1}{2\varepsilon} \Bigg[\sum_{j=1}^{n} c_{ij}(t)((p-2)\,|\tilde{x}_i(t)|^p + 2\,|\tilde{x}_j(t)|^p)\\
&\quad + \sum_{j=1}^{n} d_{ij}(t)((p-2)\,|\tilde{x}_i(t)|^p + 2\,|\tilde{x}_j(t-\tau(t))|^p \Bigg]\\
&= \frac{1}{\varepsilon} \Bigg[p a_{ii}(t) + (p-1)\Big(\sum_{j\ne i}^{n} a_{ij}(t) + \sum_{j=1}^{n} u_{ij}(t)\Big)\\
&\quad + \frac{1}{2}(p-1)(p-2) \sum_{j=1}^{n} (c_{ij}(t) + d_{ij}(t)) \Bigg] |\tilde{x}_i(t)|^p\\
&\quad + \frac{1}{\varepsilon} \Big(\sum_{j\ne i}^{n} a_{ij}(t) + (p-1) \sum_{j=1}^{n} c_{ij}(t) \Big) |\tilde{x}_j(t)|^p\\
&\quad + \frac{1}{\varepsilon} \sum_{j=1}^{n} (u_{ij}(t) + (p-1) d_{ij}(t))\,|\tilde{x}_j(t-\tau(t))|^p\\
&\le \frac{1}{\varepsilon} \sum_{j=1}^{n} \hat{p}_{ij}(t) V_j(\tilde{x}) + \frac{1}{\varepsilon} \sum_{j=1}^{n} \hat{q}_{ij}(t) [V_j(\tilde{x})]_\tau.
\end{aligned} \tag{22}$$

From condition (A_3), we have

$$\left(\frac{\hat{Q}(t)}{\varepsilon} + S\frac{\hat{P}(t)}{\varepsilon} + \frac{W}{\varepsilon} \right) z \le 0, \quad t \ge t_0. \tag{23}$$

Then, all assumptions of Lemma 4 are satisfied by (22), (23) and (A_3), so

$$\mathbf{E} V_i(\tilde{x}(t)) = \mathbf{E}\,|x_i(t) - y_i(t)|^p \le z_i e^{-\lambda(\varepsilon)(t-t_0)}, \quad t \in [t_0, t_1),\ i \in \mathcal{N}, \tag{24}$$

where $\lambda(\varepsilon)$ is determined by (20) and the positive constant vector z is determined by (16).

Using the discrete part of (1), condition (A_4), (A_5), (24) and Lemma 2, we can obtain that

$$\begin{aligned}
&\mathbf{E}\,|x_i(t_1) - y_i(t_1)|^p\\
&\le \mathbf{E}\Big[\Big(\sum_{j=1}^{n} M_{1_{ij}}\Big)^{(p-1)} \sum_{j=1}^{n} M_{1_{ij}}\ |x_j(t_1^-) - y_j(t_1^-)|^p\Big]\\
&\le \sum_{j=1}^{n} \hat{M}_{1_{ij}} \mathbf{E}\,|x_j(t_1^-) - y_j(t_1^-)|^p\\
&\le \sum_{j=1}^{n} \hat{M}_{1_{ij}} z_j e^{-\lambda(\varepsilon)(t-t_0)}\\
&\le \eta_1 z_i e^{-\lambda(\varepsilon)(t-t_0)}, \quad i \in \mathcal{N}.
\end{aligned} \tag{25}$$

This, together with (24), lead to

$$\mathbf{E}\,|\,x_i(t)-y_i(t)\,|^p \le \eta_1 z_i e^{-\lambda(\varepsilon)(t-t_0)},\quad t\in[t_1-\tau,t_1],\ i\in\mathcal{N}. \tag{26}$$

By a similar argument with (24), we can use (26) to derive that

$$\mathbf{E}\,|\,x_i(t)-y_i(t)\,|^p \le \eta_1 z_i e^{-\lambda(\varepsilon)(t-t_0)},\quad t\in[t_1,t_2),\ i\in\mathcal{N}. \tag{27}$$

Therefore, by simple induction, we have

$$\mathbf{E}\,|\,x_i(t)-y_i(t)\,|^p \le \eta_1\cdots\eta_{k-1} z_i e^{-\lambda(\varepsilon)(t-t_0)},\quad t\in[t_{k-1},t_k),\ k\in\mathbb{Z}_+,\ i\in\mathcal{N}. \tag{28}$$

In term of (18), we have $\eta_k \le e^{\eta(t_k-t_{k-1})},\ k\in\mathbb{Z}_+$, and then

$$\eta_1\cdots\eta_{k-1} \le e^{\eta(t_{k-1}-t_0)} \le e^{\eta(t-t_0)},\quad t\in[t_{k-1},t_k),\ k\in\mathbb{Z}_+,\ i\in\mathcal{N}. \tag{29}$$

Therefore, combining (28) and (29), we obtain

$$\mathbf{E}\,|\,x_i(t)-y_i(t)\,|^p \le z_i e^{-(\lambda(\varepsilon)-\eta)(t-t_0)},\quad t\in[t_{k-1},t_k),\ k\in\mathbb{Z}_+,\ \forall\varepsilon>0,\ i\in\mathcal{N}. \tag{30}$$

That is

$$\mathbf{E}[x(t)-y(t)]_+^p \le z e^{-(\lambda(\varepsilon)-\eta)(t-t_0)},\quad t\in[t_{k-1},t_k),\ k\in\mathbb{Z}_+,\ \forall\varepsilon>0. \tag{31}$$

For any $t\ge t_0$, let $\lambda_i(t,\varepsilon)$ be defined as the unique positive zero of

$$\lambda_i z_i + \sum_{j=1}^{n}\left(\frac{\hat p_{ij}(t)}{\varepsilon}+\frac{\hat q_{ij}(t)}{\varepsilon}e^{\lambda_i\tau}\right)z_j = 0. \tag{32}$$

Differentiate both sides of (32) with respect to the variable ε, we have

$$\frac{d}{d\varepsilon}\lambda_i(t,\varepsilon) = \frac{-\lambda_i z_i}{\varepsilon z_i + \sum_{j=1}^{n}\hat q_{ij}(t)\tau e^{\lambda_i\tau}z_j} < 0, \tag{33}$$

so $\lambda_i(t,\varepsilon)$ is monotonically decreasing with respect to the variable ε, which implies that $\lambda_0(\varepsilon)$ is also monotonically decreasing with respect to the variable ε. So, by a similar argument with the proof of Lemma 1 in [13], one can deduce that there exists a small $\varepsilon_0>0$ such that the solution of (1) is exponentially p-stable for sufficiently small $\varepsilon\in(0,\varepsilon_0]$. The proof is completed.

Remark 1: For system (1), when $g(t,x(t),x(t-\tau))=0$, then it degenerates to the deterministic singularly perturbed impulsive delay differential systems:

$$\begin{cases}\varepsilon dx(t) = [A(t)x(t)+f(t,x(t-\tau(t)))]dt,\ t\ge t_0,\ t\ne t_k,\\ x(t_k)=J_k(t_k,x(t_k^-)),\ k\in\mathbb{Z}_+,\\ x(t_0+s)=\varphi(s),\ s\in[-\tau,0],\end{cases} \tag{34}$$

the same as the models discussed in [13], for (34), if noting that:

1) When $g(t,x(t),x(t-\tau))=0$, (A_2) is satisfies by taking $C(t)=D(t)=0$.
2) When p=1, (A_3) degenerates to
   (A'_3) There exist a positive vector $z=(z_1,\cdots,z_n)^T\in\mathbb{R}^n$ and two positive diagonal matrices W=diag$\{w_1,\cdots,w_n\}$, $S=\mathrm{diag}\{s_1,\cdots,s_n\}$, with $0<s_i<1,\ i\in\mathcal{N}$ such that
   $$(U(t)+SA(t)+W)z\le 0,\ t\ge t_0. \tag{35}$$
3) When p=1 and $\hat M_k = M_k$, (A_5) degenerates to
   (A'_5) There exists a positive constant η satisfying
   $$\frac{\ln\eta_k}{t_k-t_{k-1}}\le\eta<\lambda(\varepsilon),\ k\in\mathbb{Z}_+, \tag{36}$$
   where η_k satisfies
   $$\eta_k\ge 1\quad\text{and}\quad \eta_k z\ge M_k z, \tag{37}$$
   and $\lambda(\varepsilon)$ is defined as
   $$0<\lambda(\varepsilon)<\lambda_0(\varepsilon)=\min_{1\le i\le n}\left\{\inf_{t\ge t_0}\lambda_i(t,\varepsilon):\ \lambda_i z_i+\sum_{j=1}^{n}\left(\frac{a_{ij}(t)}{\varepsilon}+\frac{u_{ij}(t)}{\varepsilon}e^{\lambda_i\tau}\right)z_j=0\right\}, \tag{38}$$
   for the given z.

Then we can easily obtain the following Corollary.

Corollary 1: Assume that $A(t)=(a_{ij}(t))_{n\times n}\ge 0$ for $t\ge t_0$ and $i\ne j$, further suppose that (A_1), (A'_3), (A_4) and (A'_5) hold. Then there exists a small $\varepsilon_0>0$ such that the solution of (34) is exponentially stable for sufficiently small $\varepsilon\in(0,\varepsilon_0]$.

Remark 2: From Corollary 1, it is easy to obtain Theorem 1 in [13]. In fact, "$\eta_k \triangleq \max\{\|M_k\|,1\}$" of condition (H_4) in Theorem 1 in [13] ensure that the above (37) holds.

Remark 3: If $J_k(t,x)=x,\ t\ge t_0$, that is there have no impulses in (1), then by Theorem 3.1, we can obtain the following result.

Corollary 2: Assume that $A(t)=(a_{ij}(t))_{n\times n}\ge 0$ for $t\ge t_0$ and $i\ne j$, further suppose that (A_1)-(A_3) hold. Then there exists a small $\varepsilon_0>0$ such that the solution of (1) is exponentially p-stable for sufficiently small $\varepsilon\in(0,\varepsilon_0]$.

Proof: When $J_k(t,x)=x,\ k\in\mathbb{Z}_+$, the conditions (A_4) and (A_5) is obviously satisfied on choosing $M_k=I$ in (17), $\hat M_k = I$ and η_k=1 in (19) and $\eta=0$ in (18).

Remark 4: From Lemma 4 and the proof of Theorem 1, it is obvious that the results obtained in this paper still hold for $\varepsilon=1$. So our method in this paper can obviously be applied to the following more general impulsive stochastic delay differential systems:

$$\begin{cases} \mathscr{E}dx(t) = [A(t)x(t) + f(t, x(t-\tau(t)))]dt \\ \qquad +\sqrt{\mathscr{E}}g(t, x(t), x(t-\tau(t)))d\omega(t), \\ \qquad\qquad t \ge t_0, t \ne t_k, \\ x(t_k) = J_k(t_k, x(t_k^-)), k \in \mathbb{Z}_+, \\ x(t_0+s) = \phi(s), s \in [-\tau, 0], \end{cases} \tag{39}$$

where $\mathscr{E} \triangleq$ diag $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$, $\sqrt{\mathscr{E}} \triangleq$ diag $\{\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}, \cdots, \sqrt{\varepsilon_n}\}$, $\varepsilon_i \in (0, \varepsilon_0] \cup \{1\}$ is a small parameter.

Remark 5: In [20], Socha investigated the exponential stability of a similar system to (39) with $J_k(t,x)=x$ by employing suitable Lyapunov functions. Obviously, Socha's approach can also be used to study the stability of (39) with $J_k(t,x)=x$. However, it needs not only the continuous differentiability of the drift and diffusions coefficients but also the boundedness of the derivatives of the drift and diffusion coefficients. However, our approach in this paper does not require these conditions.

4. EXAMPLE

The following illustrative example will demonstrate the effectiveness of our results.

Example 1: Consider the following SPISDDSs:

$$\begin{cases} \varepsilon dx_1(t) = [(-9-\sin t)x_1(t) \\ \qquad +(2+\sin t)\arctan x_1(t-\tau(t)) \\ \qquad +(1+\cos^2(t))\arctan x_2(t-\tau(t))]dt \\ \qquad +\sqrt{\varepsilon}\sin t x_1(t)dw_1(t) \\ \qquad +\sqrt{\varepsilon}\cos t x_2(t-\tau(t))dw_2(t),\ t \ne t_k, \\ \varepsilon dx_2(t) = [(-10+\cos t)x_2(t) \\ \qquad +(\sin^2 t)\arctan x_1(t-\tau(t)) \\ \qquad +(2+\cos^2(t))\arctan x_2(t-\tau(t))]dt \\ \qquad +\sqrt{\varepsilon}\sqrt{2}x_1(t-\tau(t))dw_1(t) \\ \qquad +\sqrt{\varepsilon}\cos t x_2(t)dw_2(t),\ t \ne t_k, \end{cases} \tag{40}$$

with

$$\begin{cases} x_1(t_k) = \alpha_{1k}x_1(t_k^-) - \beta_{1k}x_2(t_k^-) \\ x_2(t_k) = \beta_{2k}x_1(t_k^-) + \alpha_{2k}x_2(t_k^-), \end{cases}$$

where α_{ik} and β_{ik} are nonnegative constants, $\tau(t)=e^{-t} \le 1 \triangleq \tau$; The impulsive moments $t_k (k \in \mathbb{Z}_+)$ satisfy: $t_1=0$, $t_1 < t_2 < \cdots$ and $\lim_{k\to+\infty} t_k = +\infty$.

Taking $p=2$, we can get

$$A(t) = \begin{bmatrix} -9-\sin t & 0 \\ 0 & -10+\cos t \end{bmatrix},$$

$$U(t) = \begin{bmatrix} 2+\sin t & 1+\cos^2 t \\ \sin^2 t & 2+\cos^2(t) \end{bmatrix},$$

$$C(t) = \begin{bmatrix} \sin^2 t & 0 \\ 0 & 1 \end{bmatrix},\quad D(t) = \begin{bmatrix} 0 & \cos^2 t \\ 2 & 0 \end{bmatrix},$$

$$\hat{P}(t) = \begin{bmatrix} -14-\sin t & 0 \\ 0 & -16+2\cos t \end{bmatrix},$$

$$\hat{Q}(t) = \begin{bmatrix} 2+\sin t & 1+\cos^2 t \\ 2+\sin^2 t & 2+\cos^2(t) \end{bmatrix},\quad R_k = \begin{bmatrix} \alpha_{1k} & \beta_{1k} \\ \beta_{2k} & \alpha_{2k} \end{bmatrix}.$$

So there exist $z=(1,1)^T$, $W=\text{diag}\{1,1\}$ and $S=\text{diag}\{0.5, 0.5\}$ such that

$$\begin{aligned} &(\hat{Q}(t)+S\hat{P}(t)+W)z \\ &= (-3+2\cos^2 t+0.5\sin t, -2+\cos t)^T \le 0,\ t \ge t_0. \end{aligned} \tag{41}$$

Let $\eta_k = \max\{(\alpha_{1k}+\beta_{1k})^2, (\alpha_{2k}+\beta_{2k})^2\}$ and $\hat{M}_{k_{ij}} = M_{k_{ij}}(\sum_{j=1}^2 M_{k_{ij}})$, $i,j=1,2$, then $\hat{M}_{k_{ij}}$ satisfies (19) and η_k satisfies $\eta_k z \ge \hat{M}_k z$, $k \in \mathbb{Z}_+$.

Case 1: Let $\alpha_{1k} = \alpha_{2k} = \frac{1}{3}e^{0.2k}$, $\beta_{1k} = \beta_{2k} = \frac{2}{3}e^{0.2k}$, and $t_k - t_{k-1} = 2k$, then we obtain that there exists an $\eta = 0.2 > 0$ such that

$$\eta_k = e^{0.4k} \ge 1 \text{ and } \frac{\ln \eta_k}{t_k - t_{k-1}} = \frac{\ln e^{0.4k}}{2k} = 0.2 = \eta.$$

And for any $\varepsilon > 0$, the positive constant $\lambda(\varepsilon)$ is determined by the following systems:

$$\begin{cases} \lambda_1(t) + \frac{1}{\varepsilon}(-14-\sin t+(3+2\cos^2 t+\sin t)e^{\lambda_1(t)}) = 0, \\ \lambda_2(t) + \frac{1}{\varepsilon}(-12+2\cos t+5e^{\lambda_2(t)}) = 0. \end{cases} \tag{42}$$

So for a given ε, we can obtain the corresponding λ by (42). By the proof of Theorem 1, we know that λ is monotonically decreasing with respect to the variable ε, then there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, we have $\lambda > \eta$. Therefore, all the conditions of Theorem 1 are satisfied, we conclude that the solution of (40) is exponentially stable in mean square for sufficiently small $\varepsilon > 0$.

Case 2: Let $\alpha_{1k} = \alpha_{2k} = 1$ and $\beta_{1k} = \beta_{2k} = 0$, then (40) becomes the singularly perturbed stochastic delay differential systems without impulses. So by Corollary 2, the solution of (40) is exponentially stable in mean square for sufficiently small $\varepsilon > 0$.

5. CONCLUSIONS

This paper is concerned with the stability analysis of SPISDDSs. By establishing an L-operator delay differential inequality and using the stochastic analysis technique, some suffcient conditions ensuring the exponential p-stability of SPISDDSs have been obtained. The derived criteria do not require the continuous differentiability of the drift and diffusions coefficients or the boundedness of the derivatives of the drift and diffusion coefficients.

REFERENCES

[1] M. W. Derstine, H. M. Gibbs, F. A. Hopf, and D. L. Kaplan, "Bifurcation gap in a hybrid optical system," *Phys. Rev. A*, vol. 26, no. 6, pp. 3720-3722, December 1982.

[2] M. C. Mackey and L. Glass, "Oscillation and chaos in physiological control systems," *Science*, vol. 197, no. 4300, pp. 287-289, July 1977.

[3] J. H. Cruz and P. Z. Táboas, "Periodic solutions and stability for a singularly perturbed linear delay differential equation," *Nonlinear Anal.*, vol. 67, no. 6, pp. 1657-1667, September 2007.

[4] X. Z. Liu, X. M. Shen, and Y. Zhang, "Exponential stability of singularly perturbed systems with time delay," *Appl. Anal*., vol. 82, no. 2, pp. 117-130, February 2003.

[5] H. J. Tian, "The exponential asymptotic stability of singularly perturbed delay differential equations with a bounded lag," *J. Math. Anal. Appl*., vol. 270, no. 1, pp. 143-149, June 2002.

[6] H. J. Tian, "Dissipativity and exponential stability of theta-method for singularly perturbed delay differential equations with a bounded lag," *J. Comput. Math.*, vol. 21, no. 6, pp. 715-726, November 2003.

[7] D. Y. Xu, Z. G. Yang, and Z. C. Yang, "Exponential stability of nonlinear impulsive neutral differential equations with delays," *Nonlinear Anal*., vol. 67, no. 5, pp. 1426-1439, September 2007.

[8] D. Y. Xu, W. Zhu, and S. J. Long, "Global exponential stability of impulsive integro-differential equation," *Nonlinear Anal*., vol. 64, no. 12, pp. 2805-2816, June 2006.

[9] D. D. Bainov and P. S. Simenov, *Systems with Impulse Effect: Stability Theory and Applications*, Ellis Horwood Limited, Chichester, 1989.

[10] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations,* World Scientific, Singapore, 1989.

[11] P. S. Simeonov and D. D. Bainov, "Stability of the solutions of singularly perturbed systems with impulse effect," *J. Math. Anal. Appl*., vol. 136, no. 2, pp. 575-588, December 1988.

[12] P. S. Simeonov and D. D. Bainov, "Exponential stability of the solutions of singularly perturbed systems with impulse effect," *J. Math. Anal. Appl.*, vol. 151, no. 2, pp. 462-487, September 1990.

[13] W. Zhu, D. Y. Xu, and C. D. Yang, "Exponential stability of singularly perturbed impulsive delay differential equations," *J. Math. Anal. Appl*., vol. 328, no. 2, pp. 1161-1172, April 2007.

[14] Z. G. Yang, D. Y. Xu, and L. Xiang, "Exponential *p*-stability of impulsive stochastic differential equations with delays," *Phys. Lett. A*, vol. 359, no. 2, pp. 129-137, November 2006.

[15] S.-E.A. Mohammed, *Stochastic Functional Differential Equations*, Longman Scientific and Technical, 1986.

[16] X. R. Mao, *Stochastic Differential Equations and Applications*, Horwood Publication, Chichester, 1997.

[17] X. R. Mao, "Attraction, stability and boundedness for stochastic differential delay equations," *Nonlinear Anal*., vol. 47, no. 7, pp. 4795-4806, August 2001.

[18] X. R. Mao, "Razumikihin-type theorems on exponential stability of stochastic functional differential equations," *Stochast. Proc. Appl.*, vol. 65, no. 2, pp. 233-250, December 1996.

[19] L. G. Xu and D. Y. Xu, "Mean square exponential stability of impulsive control stochastic systems with time-varying delay," *Phys. Lett*. A, vol. 373, no. 3, pp. 328-333, January 2009.

[20] L. Socha, "Exponential stability of singularly perturbed stochastic systems," *IEEE Trans. Automat. Contr.*, vol. 45, no. 3, pp. 576-580, March 2000.

[21] M. El-Ansary, "Stochastic feedback design for a class of nonlinear singularly perturbed systems," *Int. J. Syst. Sci*., vol. 22, no. 10, pp. 2013-2023, October 1991.

[22] M. El-Ansary and H. K. Khalil, "On the interplay of singular perturbations and wide-band stochastic fluctuations," *SIAM J. Contr. Optim.*, vol. 24, no. 1, pp. 83-94, January 1986.

[23] E. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, New York, 1961.

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