International Journal of Control, Automation, and Systems (2011) 9(3):606-610 http://www.springer.com/12555
DOI 10.1007/s12555-011-0323-z

Design of Generalized Terminal Sliding Mode Control for Second-Order Systems

Young-Hun Jo, Yong-Hwa Lee, and Kang-Bak Park*

Abstract: In this paper, a generalized terminal sliding surface is proposed for second-order systems. It is shown that the proposed scheme guarantees that the system state gets to zero in a finite time, and the proposed generalized terminal sliding surface is a superset of the conventional terminal sliding surfaces. The experimental results are given to show the validity of the main result.

Keywords: Nonlinear systems, sliding mode control, state space method, terminal sliding surface.

1. INTRODUCTION

In most of previous works on systems and control field, it has been shown that the closed-loop system is asymptotically stable in the sense of Lyapunov. Although the asymptotic stability guarantees that the system state converges to zero as time approaches infinity, it does not specify when the system states will get to zero. That is, asymptotic stability does not imply finite time convergence. However, finite time stabilization is very important in many industrial applications such as motor systems, power systems, robot manipulators, spacecraft systems, and so on; thus, there have been many recent studies on finite time stabilization [1-6].

Sliding mode control (SMC) is one of the robust control schemes [7-11]. It is well known that the SMC has the invariance property to parameter uncertainties and external disturbances. Recently, to obtain finite time convergence, terminal sliding mode control schemes have been studied actively [3-5]. The conventional terminal sliding surfaces have been designed with a power function whose exponent is a rational number with positive odd numerator and denominator. However, there are lots of different kinds of sliding surfaces that guarantee finite time convergence to the origin in the state space. Thus, in this paper, we propose a generalized terminal sliding surface for second-order systems. It is shown that the proposed method guarantees that the system state gets to zero in a finite time, and the proposed generalized terminal sliding surface is a superset of the conventional terminal sliding surfaces. Some of the experimental results are given to show the validity of the main results.

Manuscript received December 31, 2010; revised February 9, 2011; accepted February 23, 2011. Recommended by Editor Myotaeg Lim. This work was supported by a Korea University Grant.

Young-Hun Jo, Yong-Hwa Lee, and Kang-Bak Park are with the Department of Control and Instrumentation Engineering, Korea University, 208 Seochang-dong, Yeongi-gun, Chungnam 339-700, Korea (e-mails: {heyguy84, now1217, kbpark}@korea.ac.kr).
* Corresponding author.

2. MAIN RESULTS

Consider a second-order nonlinear system with the following form:

$$\ddot{x} = f(x,\dot{x}) + b(x,\dot{x})u, \tag{1}$$

where x and $\dot{x}$ are state variables, $f(x,\dot{x})$ is a nonlinear term, u is a scalar input, and $b(x,\dot{x}) \neq 0$.

In the previous works on terminal sliding mode control systems, the conventional terminal sliding surfaces have been designed as

$$s = \dot{x} + cx^{p/q}, \tag{2}$$

where $c > 0$, $0 < \frac{p}{q} < 1$, and p and q are positive odd integers [3-5]. Although, the conventional terminal sliding surface (2) ensures finite time convergence, it is not the only one solution. Thus, we propose a generalized terminal sliding surface as follows:

$$s = \dot{x} + g(x), \tag{3}$$

where $g(\cdot) \in C^1$ is an odd function, i.e., $g(0) = 0$ and $0 < x \cdot g(x)$ if $x \neq 0$, and it holds the following condition:

$$\frac{dg(x)}{dx} \to \infty \quad \text{as} \quad x \to 0. \tag{4}$$

The above condition (4) implies that the proposed sliding surface has a slope of $-\infty$ at $x = 0$ in the state space (x vs. $\dot{x}$). Since a sliding surface should be designed such that the system is stable in the sliding mode, we derive the stability of the proposed sliding surface (3) in the following theorem.

Theorem 1: The proposed sliding surface (3) is stable.

Proof: Consider the Lyapunov function candidate:

$$V(x) = \frac{1}{2}x^2. \tag{5}$$

In the sliding mode, $s = 0$, and it means that

$\dot{x} = -g(x)$. Thus, $\dot{V}$ can be derived as

$$\dot{V}(x) = x\dot{x} = -x \cdot g(x). \tag{6}$$

It is clear that above $\dot{V}$ is a negative definite function since $g(x)$ is an odd function. It implies that the system is stable provided that the system is in the sliding mode, i.e., the proposed sliding surface (3) is stable.

In the following theorem, finite time convergence is derived for the proposed sliding surface.

Theorem 2: The proposed sliding surface (3) guarantees that the system state gets to zero in a finite time if the system is in the sliding mode.

Proof: If the system is in the sliding mode,

$$s = \dot{x} + g(x) = 0. \tag{7}$$

Since $g(x)$ is an odd function, assume that $g(x)$ is in the sector $(0, k]$, where k is a finite positive number, i.e.,

$$0 < xg(x) \le kx^2 \quad \forall x \ne 0. \tag{8}$$

Applying (7) to (8), one can derive the following inequalities:

$$xg(x) \le kx^2 \;\Rightarrow\; -x\dot{x} \le kx^2 \;\Rightarrow\; -\frac{\dot{x}}{x} \le k. \tag{9}$$

Integrating both sides of (9), one can easily obtain the following inequality:

$$k \cdot t\Big|_{t=0}^{t=t_r} \ge -\ln|x|\Big|_{x=x(0)}^{x=0} \Rightarrow k \cdot t_r \ge \ln|x(0)| + \infty, \tag{10}$$

where t_r represents the relaxation time, at which the system state gets to zero. Clearly, the last inequality (10) should hold for any $|x(0)| \ne 0$. However, it is obvious that there is no finite positive number k such that t_r to be finite for any $|x(0)| \ne 0$. That is, no matter how small $|x(0)| \ne 0$ is, there is no finite positive number k satisfying the inequality (10). It implies that the sliding surface guaranteeing finite time convergence should satisfy the condition (4).

Remark 1: It is clear that the proposed generalized terminal sliding surface (3) is a superset of the conventional terminal sliding surface (2), that is, $g(x)=cx^{p/q}$ satisfies the condition (4).

Remark 2: From Theorem 2, it is possible to design a variety of another terminal sliding surfaces guaranteeing finite time convergence as that shown in Fig. 1. In this figure, the solid line represents the following circular terminal sliding surface (CTSS):

$$s = \begin{cases} \dot{x} + r \cdot \operatorname{sgn}(x) & \text{if } |x| > r, \\ \dot{x} + \sqrt{-x^2 + 2r|x|}\operatorname{sgn}(x) & \text{otherwise,} \end{cases} \tag{11}$$

where $x_1 = x,\ x_2 = \dot{x},\ r = 1$ and $\operatorname{sgn}(x)$ represents a signum function. The dotted line shows the following extended terminal sliding surface (ETSS):

$$s = \dot{x} + c|x|^{p/q}\operatorname{sgn}(x), \tag{12}$$

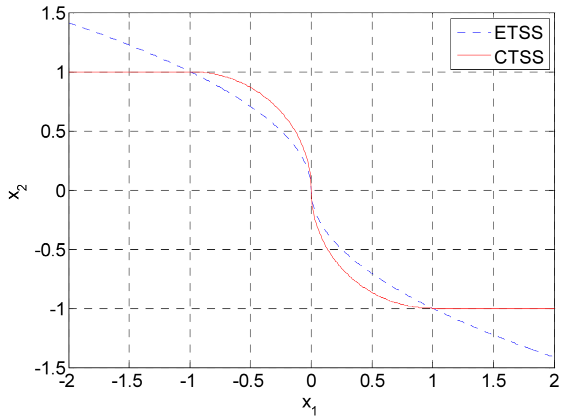

Fig. 1. Examples of terminal sliding surfaces.

where $c = 1$ and $\frac{p}{q} = \frac{1}{2}$. Obviously, both of the two sliding surfaces (11) and (12) hold the condition (4), i.e., they have a slope of $-\infty$ at $x = 0$ in the state space (x vs. $\dot{x}$).

Remark 3: For the sliding surface (12), note that the exponent of the power function can be any real number in (0, 1). In the conventional terminal sliding surface (2), it should be a rational number with positive odd numerator and denominator. However, it is shown that it can be extended to a real number by using the proposed sliding surface.

Remark 4: In some cases, the sliding surface (11) can be useful because the physical limits of hardware systems can be applied to designing the sliding surface. For example, if the maximum speed, $|\dot{x}|$, of the plant is limited by 10, one can design a terminal sliding surface (11) with $r = 9 < 10$.

Since all of the above results have been derived under the condition that the system is in the sliding mode, we give the following theorem to show the existence of the sliding mode for the stability of the overall system.

Theorem 3: For the system (1) with the proposed sliding surface (3), the following controller guarantees that the system state gets to zero in a finite time.

$$u = \frac{1}{b(x,\dot{x})}\left(-f(x,\dot{x}) - \frac{dg(x)}{dx}\dot{x} - k_1 s - k_2 \operatorname{sgn}(s)\right). \tag{13}$$

Proof: Let the Lyapunov function candidate be

$$V(s) = \frac{1}{2}s^2. \tag{14}$$

Applying (1) and (3) to $\dot{V}$, the following inequality can be obtained.

$$\begin{aligned} \dot{V}(s) = s\dot{s} &= s\left(\ddot{x} + \frac{dg(x)}{dx}\dot{x}\right) \\ &= s\left(f + bu + \frac{dg(x)}{dx}\dot{x}\right) \\ &= s\left(-k_1 s - k_2 \operatorname{sgn}(s)\right) \\ &= -k_1 s^2 - k_2|s|. \end{aligned} \tag{15}$$

From (15), it is clear that the reaching time is shorter than $\frac{|s(0)|}{k_2}$, i.e., the system gets in the sliding mode in a finite time. In addition, from Theorem 2, if the system is in the sliding mode, then the system state goes to zero in a finite time. Thus, the system state gets to zero in a finite time.

Remark 5: The proposed sliding surface (3) can be a guideline for designing a terminal sliding surface for second-order systems. That is, to get the finite time convergence, a terminal sliding surface should be designed in the state space such that the condition (4) holds, i.e., it should have a slope of $-\infty$ at $x=0$ in the state space (x vs. $\dot{x}$).

Remark 6: From condition (4), it can be known that one should check out whether or not controller (13) suffers from the singularity problem. If it does, then the singularity problem can be avoided by several methods as in the previous works [3,4]. The following switching control law is one of them:

$$u = \begin{cases} 0 & \text{at singular points,} \\ (13) & \text{otherwise.} \end{cases} \tag{16}$$

Remark 7: If the system is in the sliding mode, then it is shown that $\dot{x} = -g(x)$ since $s=0$. Thus, using a similar method to those of previous works, the controller (13) can be modified as

$$u = \frac{1}{b(x,\dot{x})}\left(-f(x,\dot{x}) + \frac{dg(x)}{dx}g(x) - k_2 \operatorname{sgn}(s)\right). \tag{17}$$

Note that the function $g(x)$ can be designed such that $\frac{dg(x)}{dx}g(x)=0$ at $x=0$. Actually, terminal sliding surfaces (11) and (12) satisfy the above condition so that the modified controller (17) does not suffer from singularity problem if the system is in the sliding mode.

3. EXPERIMENTAL RESULTS

To show the validity of the proposed scheme, the experimental results for the actual DC motor system are given. In the experimental setup, a TMS320F2812 DSP processor was used, and sampling time was set to 1msec. The following model is used for the DC Motor: $\ddot{x} = -40.65\dot{x} + 46.67u$. Two examples of the proposed generalized terminal sliding surface (4) have been applied to the system: the circular terminal sliding surface (11) and the extended terminal sliding surface (12).

Figures 2-6 show the experimental results of the circular terminal sliding surface when) $r=2$, k_1 =70, k_2 =100, and $x(0) = -\pi$(rad.). The output angle and angular velocity are shown in Figs. 2-3. Figure 2 represents that the output gets to zero in a finite time. As can be seen in Fig. 4, the reaching phase is very short so that the relaxation time is almost the same as the sum of the elapsed time for the signum function part and circular part. For the signum function part, i.e., the constant velocity part, it is easy to obtain the elapsed time since $|x|$ decreases at a constant rate r. The approximated elapsed time can be calculated by $t_{r\ \text{const}} \cong \frac{|x(0)|-r}{r} = \frac{|-\pi|-2}{2} \cong$ 0.571 seconds. For the circular part, $t_{r\ \text{circular}} = \frac{\pi}{2}$ (sec.) can be easily derived from (11). Thus, the approximated total relaxation time can be derived as $t_{r1} \cong 0.571 + \frac{\pi}{2} \cong 2.142$ seconds. It is clear from Fig. 2 that the

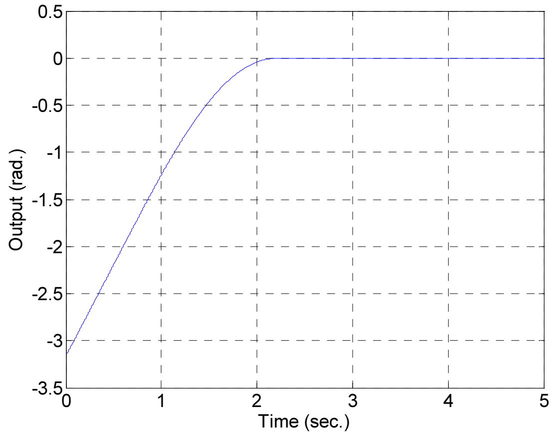

Fig. 2. Output (x).

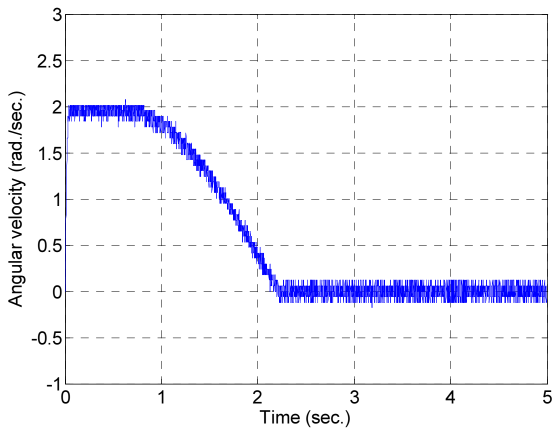

Fig. 3. Angular velocity ($\dot{x}$).

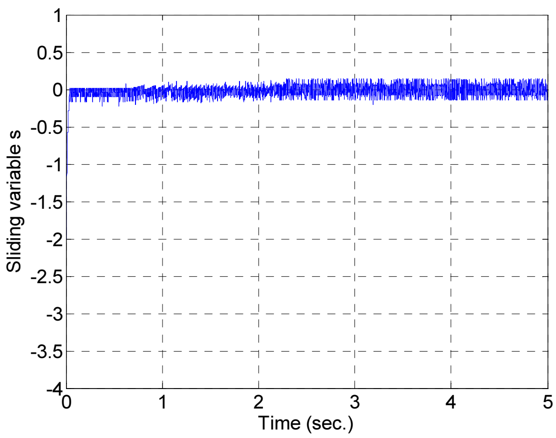

Fig. 4. Sliding variable (s).

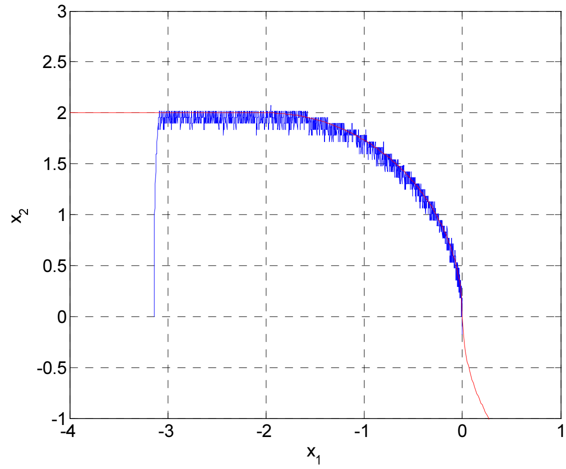

Fig. 5. Phase portrait (x vs. $\dot{x}$).

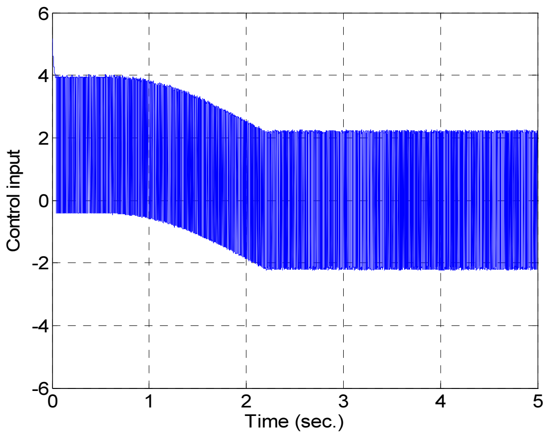

Fig. 6. Control input (u).

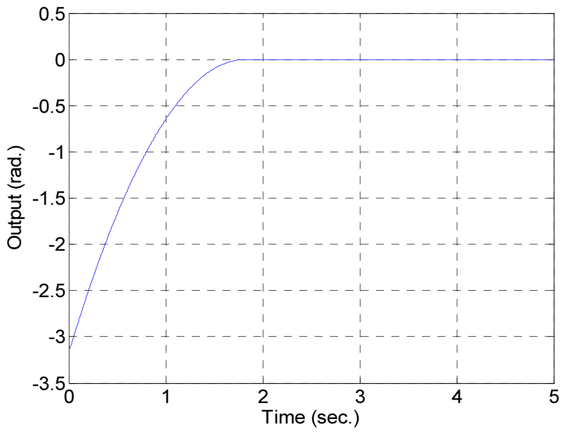

Fig. 7. Output (x).

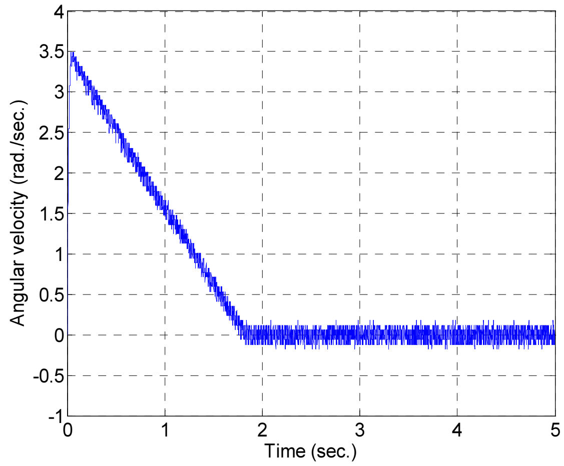

Fig. 8. Angular velocity ($\dot{x}$).

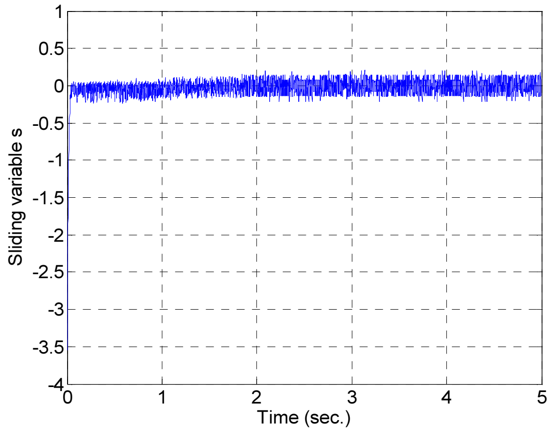

Fig. 9. Sliding variable (s).

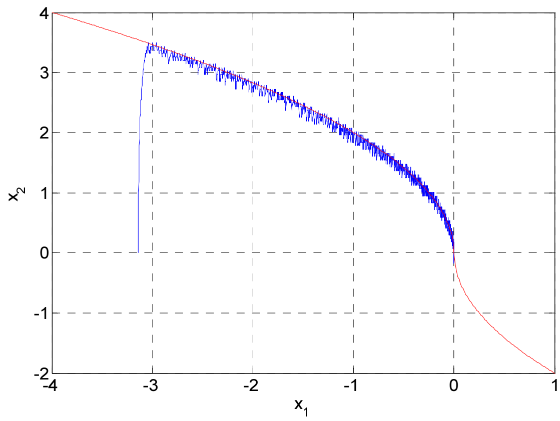

Fig. 10. Phase portrait (x vs. $\dot{x}$).

experimental result, 2.218 seconds, is almost the same as t_{r1}. In the phase portrait (see Fig. 5), it is obvious that the system state reaches the sliding surface, the red line, and the overall system is in the sliding mode thereafter. Fig. 6 shows that the control input shows the chattering phenomena almost all the time since the reaching phase is very short (see Fig. 4). The actual reaching time was 0.040 seconds.

The results of the extended terminal sliding surface (12) are shown in Figs. 7-11. The parameters of the sliding mode controller (13) were set to $c=2$, $\frac{p}{q}=\frac{1}{2}$, $k_1=70$, $k_2=100$, and $x(0)=-\pi$(rad.). The output angle and angular velocity curves are given in Figs. 7 and 8. Figure 7 shows that the output gets to zero in a finite time. Since the system gets in the sliding mode very rapidly (0.041 seconds) as can be seen in Fig. 9, the relaxa-

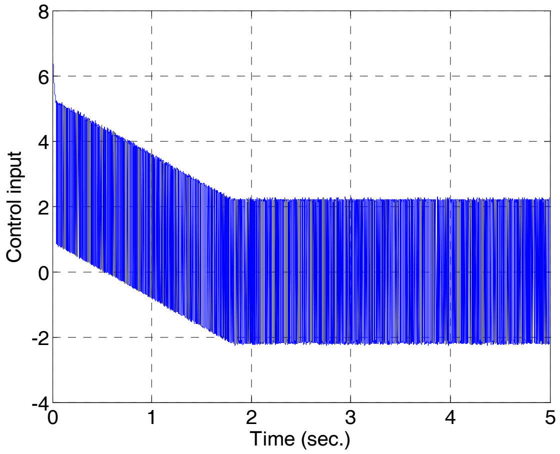

Fig. 11. Control input (u).

tion time is almost the same as $t_{r2} \cong \dfrac{|x(0)|^{1-\frac{p}{q}}}{c\left(1-\dfrac{p}{q}\right)} \cong 1.772$ seconds. The experimental data, 1.846 second, shows almost the same result (see Fig. 7). Figure 10 shows that the system is in the sliding mode. The control input is given in Fig. 11. It can be seen that the control input does not suffer from the singularity problem.

5. CONCLUSIONS

In this paper, a generalized terminal sliding surface for the second-order systems has been proposed. It has been shown that a terminal sliding surface should be designed such that it has a slope of $-\infty$ at $x = 0$ in the state space in order to obtain finite time convergence. It has been also shown that the proposed generalized terminal sliding surface is a superset of the conventional terminal sliding surfaces, and several other terminal sliding surfaces have been introduced. The experimental results have shown the validity of the main result.

REFERENCES

[1] Y. Wu, B. Wang, and G. D. Zong, "Finite-time tracking controller design for nonholonomic systems with extended chained form," *IEEE Trans. Circuits Syst. II: Express Briefs*, vol. 52, no. 11, pp. 798-802, Nov. 2005.

[2] J. Resendiz, W. Yu, and L. Fridman, "Two-stage neural observer for mechanical systems," *IEEE Trans. Circuits Syst. II: Express Briefs*, vol. 55, no. 10, pp. 1076-1080, Oct. 2008.

[3] L. Wang, T. Chai, and L. Zhai, "Neural-network-based terminal sliding-mode control of robotic manipulators including actuator dynamics," *IEEE Trans. Ind. Electron.*, vol. 56, no. 9, pp. 3296-3304, Sep. 2009.

[4] L. Hui and J. Li, "Terminal sliding mode control for spacecraft formation flying," *IEEE Trans. Aerospace and Electron. Syst.*, vol. 45, no. 3, pp. 835-846, Jul. 2009.

[5] S. Janardhanan and B. Bandyopadhyay, "On discretization of continuous-time terminal sliding mode," *IEEE Trans. Automat. Control*, vol. 51, no. 9, pp. 1532-1536, Sep. 2006.

[6] G. Garcia, S. Tarbouriech, and J. Bernussou, "Finite-time stabilization of linear time-varying continuous systems," *IEEE Trans. Automat. Control*, vol. 54, no. 2, pp. 364-369, Feb. 2009.

[7] V. I. Utkin, "Variable structure systems with sliding modes," *IEEE Trans. Automat. Control*, vol. 22, no. 2, pp. 212-222, 1977.

[8] J. Y. Hung, W. Gao, and J. C. Hung, "Variable structure control: a survey," *IEEE Trans. Ind. Electron.*, vol. 40, no. 1, pp. 2-22, 1993.

[9] K. D. Young, V. I. Utkin, and U. Ozguner, "A control engineer's guide to sliding mode control," *IEEE Trans. Control Systems Technology*, vol. 7, no. 3, pp. 328-342, 1999.

[10] Y. Niu, B. Chen, and X. Wang, "sliding mode control for a class of nonlinear ito stochastic systems with state and input delays," *International Journal of Control, Automation, and Systems*, vol. 7, no. 3, pp. 365-370, 2009.

[11] M.-C. Pai, "Observer-based adaptive sliding mode control for nonlinear uncertain state-delayed systems," *International Journal of Control, Automation, and Systems*, vol. 7, no. 4, pp. 536-544, 2009.

Young-Hun Jo received his B.S. degree in Control and Instrumentation Engineering from Korea University in 2009. His research interests include nonlinear control, robot control, and digital control.

Yong-Hwa Lee received his B.S. degree in Control and Instrumentation Engineering from Korea University in 2010. His research interests include nonlinear control, robust control, and sliding mode control.

Kang-Bak Park received his Ph.D. degree in Electrical Engineering from Korea Advanced Institute of Science and Technology (KAIST) in 1997. He is a professor in the Department of Control and Instrumentation Engineering at Korea University. His research interests include nonlinear control, sliding mode control, robot control, and sampled-data system.