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Exponential Synchronization for Arrays of Coupled Neural Networks with Time-delay Couplings

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Abstract: This paper deals with global exponential synchronization in arrays of coupled delayed neural networks with both delayed coupling and one single delayed one. Through employing Kronecker product and convex combination technique, two novel synchronization criteria are presented in terms of linear matrix inequalities (LMIs), and these conditions are dependent on the bounds of both time-delay and its derivative. Through employing Matlab LMI Toolbox and adjusting some matrix parameters in the derived results, we can realize the design and applications of the addressed systems, which shows that our methods improve and extend those reported methods. The efficiency and applicability of the proposed results can be demonstrated by three numerical examples with simulations.

Keywords: Coupled neural networks, exponential synchronization, LMI approach, Lyapunov-Krasovskii functional, time-varying delay.

1. INTRODUCTION

In past decades, synchronization of various chaotic systems has gained considerable attention since the pioneering works of Pecora and Carroll [1,2]. Presently, it is widely known that many benefits of having synchronization or chaos synchronization can be existent in various engineering fields. Also, the existence of synchronization in language emergence and development results can help come up with the common vocabulary and agents' synchronization in organization management can improve their work efficiency. Recently, the problem on synchronization has been extensively investigated in chaotic systems owing to the potential applications in various engineering areas. Especially, since chaos synchronization in arrays of linearly coupled dynamical systems was firstly considered by [3], arrays of coupled systems including coupled delayed chaotic ones have attracted the researchers' attention as they can exhibit some interesting phenomena [4,5], and many elegant results have been derived in [6-21].

As one typical complex systems, delayed neural networks (DNNs) have been verified to exhibit some complex and unpredictable behaviors such as stable equilibria, periodic oscillations, bifurcation, and chaotic attractors. Thus chaos synchronization for arrays of coupled DNNs have been discussed by the researchers, and many elegant results have been proposed in [7-21]. In [7], by applying adaptive feedback controllers, the paper has studied the global synchronization of coupled complex networks with delayed coupling based on pinning control. The stability of synchronized state has been studied in arbitrarily coupled delayed complex networks in [8], where coupling configurations are not to be symmetric and irreducible. The synchronization of linearly coupled DNNs was investigated in [9], in which the dynamical behavior of the uncoupled system can be chaotic or others and the coupling configuration is variable. The authors in [10] have considered the robust synchronization of coupled DNNs under general impulsive control. In [11], this paper has proposed an adaptive procedure to the synchronization for coupled identical Yang-Yang type fuzzy DNNs based on one simple adaptive controller. In [12], with all parameters unknown, the authors studied the robust synchronization between two coupled DNNs that were linearly and unidirectionally coupled. Yet, those above-mentioned results were presented via some kind of complicated inequalities, which makes them uneasily checked and applied to real ceases by the most recently developed algorithms. Though employing Lyapunov functional and Kronecker product, the global synchronization and cluster one have been studied for DNNs including robust ones and discrete-time ones with delayed coupling or one single delayed coupling via LMIs in [13-21], and some easy-to-test sufficient conditions have been obtained. Yet,

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the system forms in [13-21] seemed simple and the most improved techniques in [22,23] weren't utilized to achieve the criteria, which make these results inapplicable to deal with DNNs of more general forms. Thus it is important and challenging to derive some less conservative results ensuring the global synchronization of coupled DNNs.

In this paper, the global exponential synchronization of N identical DNNs with delayed couplings is considered and two novel LMI-based conditions are derived by utilizing Kronecker product technique. It shows that the chaos synchronization of coupled DNNs is ensured by a suitable design of inner coupled linking matrix and inner delayed coupled linking ones. The addressed systems can include some present models as its special cases and some effective mathematical techniques are employed to reduce the conservatism. Finally, the efficiency of the synchronization criteria can be demonstrated by utilizing three numerical examples.

Notations: $\mathbf{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbf{R}^{n\times m}$ is the set of all $n\times m$ real matrices. For the symmetric matrices X, Y, $X>Y$ (respectively, $X\geq Y$) means that $X-Y>0$ $(X-Y\geq 0)$ is a positive-definite (respectively, positive-semidefinite) matrix; A^T represents the transpose of the matrix A; $\lambda_{\max}(A)$, $\lambda_{\min}(A)$ denote the maximum eigenvalue and minimum one of matrix A, respectively; I_n represents the $n\times n$ identity matrix; the symmetric term in a symmetric matrix is denoted by $*$, i.e., $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ * & Z \end{bmatrix}$.

2. PROBLEM FORMULATIONS

Suppose the nodes are coupled with states $x_i(t)$, $i\in\{1,\cdots,N\}$, then the DNNs of general form can be described by

$$\begin{aligned}\dot{x}_i(t) = &-C\beta(x_i(t)) + Af(x_i(t)) + Bf(x_i(t-\tau(t)))\\ &+I(t) + \sum_{j=1,j\neq i}^{N} l_{ij}F[x_j(t)-x_i(t)]\\ &+\sum_{j=1,j\neq i}^{N} l_{ij}K[x_j(t-\tau(t))-x_i(t)]\\ &+\sum_{j=1,j\neq i}^{N} l_{ij}J[x_j(t-\tau(t))-x_i(t-\tau(t))],\end{aligned} \quad (1)$$

in which $x_i(t)=[x_{i1}(t),\cdots,x_{in}(t)]^T\in\mathbf{R}^n$ is the state vector of the i-th network at time t, $\beta(x_i)=[\beta_1(x_{i1}),\cdots,\beta_n(x_{in})]^T$ stands for the behaved function, $f(x_i(\cdot))=[f_1(x_{i1}(\cdot)),\cdots,f_n(x_{in}(\cdot))]^T$, and $I(t)=[I_1(t),\cdots,I_n(t)]^T\in\mathbf{R}^n$ is the external input vector; $C=\text{diag}\{c_1,\cdots,c_n\}>0$, $A=[a_{ij}]_{n\times n}$, $B=[b_{ij}]_{n\times n}$; here $F=[f_{ij}]_{n\times n}$, $K=[k_{ij}]_{n\times n}$, $J=[j_{ij}]_{n\times n}$ are respectively the inner coupling matrices between the connected nodes i and j at time t and $t-\tau(t)$.

For the network (1), the following assumptions are adopted throughout this paper.

Assumption 1: $\tau(t)$ are the interval time-varying delay satisfying

$$0\leq\tau_0\leq\tau(t)\leq\tau_m,\ \dot{\tau}(t)\leq\mu<+\infty. \quad (2)$$

Here we set $\bar{\tau}_m=\tau_m-\tau_0$.

Assumtion 2: $L=[l_{ij}]_{N\times N}$ is the configuration matrix that is irreducible and satisfies

$$l_{ij}=l_{ji},\ i\neq j,\ l_{ii}=-\sum_{j=1,j\neq i}^{N} l_{ij}.$$

Here $l_{ij}>0$ if there is a connection between node i and the one j and otherwise, l_{ij}=0.

Assumption 3: There exist two positive scalars π_i, γ_i such that $\beta_i(\cdot)$ satisfies $0<\gamma_i\leq\frac{\beta_i(x)-\beta_i(y)}{x-y}\leq\pi_i$, and

$$[\dot{\beta}_i(x)-\dot{\beta}_i(y)-\lambda_i(x-y)][\dot{\beta}_i v(x)-\dot{\beta}_i(y)-\psi_i(x-y)]\leq 0\ \forall\ x,y\in\mathbf{R}, i=1,\cdots,n.$$

Here we set $\Pi=\text{diag}\{\pi_1,\cdots,\pi_n\}$, $\Gamma=\text{diag}\{\gamma_1,\cdots,\gamma_n\}$, $\Pi_1=\Pi\Gamma$, $\Pi_2=\frac{1}{2}(\Pi+\Gamma)$, and

$$\Lambda_1=\text{diag}\{\lambda_1\psi_1,\cdots,\lambda_n\psi_n\},$$
$$\Lambda_2=\text{diag}\left\{\frac{\lambda_1+\psi_1}{2},\cdots,\frac{\lambda_n+\psi_n}{2}\right\}.$$

Assumption 4: For any $x,y\in\mathbf{R}$, and constants σ_i^+, σ_i^-, and $i=1,\cdots,n$, the activation function $f_i(\cdot)$ in (1) satisfies the condition

$$[f_i(x)-f_i(y)-\sigma_i^+(x-y)][f_i(x)-f_i(y)-\sigma_i^-(x-y)]\leq 0.$$

Here we denote $\Sigma_1=\text{diag}\{\sigma_1^+\sigma_1^-,\cdots,\sigma_n^+\sigma_n^-\}$ and $\Sigma_2=\text{diag}\left\{\frac{\sigma_1^++\sigma_1^-}{2},\cdots,\frac{\sigma_n^++\sigma_n^-}{2}\right\}$.

Based on Assumption 2, system (1) can be rewritten as the following form:

$$\begin{aligned}\dot{x}_i(t) = &-C\beta(x_i(t)) + Af(x_i(t)) + Bf(x_i(t-\tau(t)))\\ &+I(t)+\sum_{j=1}^{N} l_{ij}Fx_j(t)+\sum_{j=1}^{N} l_{ij}(K+J)\\ &\times x_j(t-\tau(t))-l_{ii}K[x_i(t-\tau(t))-x_i(t)].\end{aligned} \quad (3)$$

To address the problem, we denote the set $\mathscr{S}=\{x(s)=[x_1^T(s),\cdots,x_N^T(s)]: x_i(s)\in\mathscr{C}([t_0-\tau_m,\tau_0],\mathbf{R}^n),\ x_i(s)=x_j(s),\ i,j=1,2,\cdots,N\}$ as the synchronization manifold for system (3). In the case, system (3) reaches synchronization, i.e., $x_1(t)=x_2(t)=\cdots=x_N(t)=s(t)$, we can deduce the following synchronized state equation

$$\begin{aligned}\dot{s}(t)&=-C\beta(s(t))+Af(s(t))+Bf(s(t-\tau(t)))+I(t)\\&\quad-l_{ii}K[s(t-\tau(t))-s(t)],\end{aligned}\tag{4}$$

where $i=1,2,\cdots,N$. Obviously, the synchronization is invariant for the coupled system (4). Therefore, to realize complete synchronization, the assumption $l_{11}=\cdots=l_{NN}=l$ has to be imposed on system (4).

Remark 1: In Assumption 3, the assumption on $\dot{\beta}(x_i)$ is reasonable and does not result in the conservatism in many cases such as that, for the appropriate scalars *a*, *b*, *c*, $\beta(x_i)$ can be expressed by $ax_i, ax_i+b\sin(x_i)$, $ax_i+b\sin^2(x_i)$, $ax_i+c\cos(x_i)$, $ax_i+c\cos^3(x_i)$, $ax_i+c\tanh(x_i)$, respectively, which means that system (1) can include those addressed forms in [12-20] as its special cases.

As illustrated in [19], due to the communication delay, the array of coupled nodes cannot be decoupled, the synchronized state is always not the trajectory of an isolated node but a modified one as (4). Furthermore, delayed coupling matrix and the degree of the node play the important roles in the synchronized state. In the paper, we give an improved discussion for such synchronization.

Together with the Kronecker product in [13-16], we can reformulate the system (3) as follows:

$$\begin{aligned}\dot{\mathbf{x}}(t)&=-(I_N\otimes C)\mathbf{b}(\mathbf{x}(t))+(I_N\otimes A)\mathbf{f}(\mathbf{x}(t))\\&\quad+(I_N\otimes B)\mathbf{f}(\mathbf{x}(t-\tau(t)))+\mathbf{I}(t)\\&\quad+(L\otimes F)\mathbf{x}(t)+(L\otimes(K+J))\mathbf{x}(t-\tau(t))\\&\quad-l(I_N\otimes K)[\mathbf{x}(t-\tau(t))-\mathbf{x}(t)]\end{aligned}\tag{5}$$

with $\mathbf{x}(t)=[x_1^T(t),\cdots,x_N^T(t)]^T$, $\mathbf{b}(\mathbf{x}(t))=[\beta^T(x_1(t)),\cdots,\beta^T(x_N(t))]^T$, $\mathbf{f}(\mathbf{x}(\cdot))=[f^T(x_1(\cdot)),\cdots,f^T(x_N(\cdot))]^T$, and $\mathbf{I}(t)=[I^T(t),\cdots,I^T(t)]^T$.

The following definition and lemmas are adopted.

Definition 1 [14]**:** Dynamical networks (3) is said to achieve global exponential synchronization, if for any initial conditions $\phi_i(s),\ \phi_j(s)\in\mathscr{C}([t_0-\tau_m,t_0],\mathbf{R}^n)$, $i,j=1,\cdots,N$, there exist $T>t_0$ and $\varepsilon>0$ such that $\|x_i(t)-x_j(t)\|\le Me^{-\varepsilon t}$, in which $t>T$ and $\|\cdot\|$ denotes the Euclidean norm.

Lemma 1 [22]**:** For any constant matrix $X\in\mathbf{R}^{n\times n}$, $X=X^T\ge 0$, a scalar functional $h:=h(t)\ge 0$, and a vector function $\dot{x}:[-h,0]\to\mathbf{R}^n$ such that the following integration is well defined, then $-h\int_{t-h}^{h}\dot{x}^T(s)X\dot{x}(s)ds\le[x(t)-x(t-h)]^TX[x(t)-x(t-h)]$.

Lemma 2 [16]**:** Let $U=[u_{ij}]_{N\times N}$, $P\in\mathbf{R}^{\mathbf{n\times n}}$, $x=[x_1^T,\cdots,x_N^T]^T$, and $y=[y_1^T,\cdots,y_N^T]^T$ with $x_i,y_i\in\mathbf{R}^n$, i=1, $\cdots,N$. If $U=U^T$ and each row sum of U is 0, then $x^T(U\otimes P)y=-\sum_{1\le i<j\le N}u_{ij}(x_i-x_j)^TP(y_i-y_j)$.

3. DELAY-DEPENDENT SYNCHRONIZATION CRITERIA

In the section, by utilizing the most improved techniques utilized in [23], we state and investigate the exponential synchronization for the system (5).

Theorem 1: Supposing that Assumptions 1-4 hold, then the dynamical system (5) is globally exponentially synchronized, if there exist $n\times n$ matrices $P>0$, $S>0$, $Z>0$, $L_i(i=1,2)$, $n\times n$ matrices $P_l>0$, $Q_l>0$, R_l (l=1,2,3) making $\begin{bmatrix}P_l & R_l\\ R_l^T & Q_l\end{bmatrix}\ge 0$, and $n\times n$ diagonal matrices $U>0$, $V>0$, $W>0$, $H>0$, $R>0$, $Q>0, G>0$, $T_i>0\ (i=1,2)$ such that the LMIs in (6) hold

$$\begin{aligned}&\Omega_{ij}-I_1^TZI_1<0,\ \Omega_{ij}-I_2^TZI_2<0,\\&\forall\,1\le i<j\le N,\end{aligned}\tag{6}$$

where $I_1=[0_{n\cdot 2n}\ -I_n\ \ 0_{n\cdot 4n}\ \ I_n\ \ 0_{n\cdot 3n}]$, $I_2=[0_n\ \ I_n\ \ 0_{n\cdot 5n}\ -I_n\ \ 0_{n\cdot 3n}]$, and Ω_{ij} is at the top of next page with

$$\begin{aligned}\Xi_{11}&=-S+P_2+l(L_1^TK+K^TL_1)-l_{ij}N(L_1^TF+F^TL_1)\\&\quad-\Gamma^TG-G^T\Gamma-U\Sigma_1-T_1\Pi_1-T_2\Lambda_1,\\\Xi_{14}&=R_2+L_1^TA+U\Sigma_2,\\\Xi_{17}&=P-L_1^T+2(\Pi^TR-\Gamma^TQ)+lK^TL_2-l_{ij}NF^TL_2,\\\Xi_{18}&=-lL_1^TK-l_{ij}NL_1^T(K+J),\\\Xi_{1,10}&=lK^TQ^T-L_1^TC-l_{ij}NF^TQ^T+G^T+T_1\Pi_2,\\\Xi_{1,11}&=T_2\Lambda_2+Q^T-R^T,\\\Xi_{22}&=-P_2+P_1+P_3-S-Z-W\Sigma_1,\\\Xi_{25}&=-R_2+R_1+R_3+W\Sigma_2,\\\Xi_{33}&=-P_3-Z-H\Sigma_1,\\\Xi_{55}&=-Q_2+Q_1+Q_3-W,\\\Xi_{77}&=-L_2^T-L_2+\tau_0^2S+\bar{\tau}_m^2Z,\\\Xi_{78}&=-lL_2^TK-l_{ij}NL_2^T(K+J),\\\Xi_{88}&=-(1-\mu)P_1-2Z-V\Sigma_1,\\\Xi_{89}&=V\Sigma_2-(1-\mu)R_1,\\\Xi_{8,10}&=-lK^TQ^T-l_{ij}N(K+J)^TQ^T,\\\Xi_{99}&=-(1-\mu)Q_1-V.\end{aligned}$$

Proof: Based on Assumption 3 and matrix $U=[u_{ij}]_{N\cdot N}=\begin{bmatrix}N-1 & \cdots & -1\\ \vdots & \ddots & \vdots\\ -1 & \cdots & N-1\end{bmatrix}$, we construct the following Lyapunov-Krasovskii functional:

$$\Omega_{ij}=\begin{bmatrix}\Xi_{11} & S & 0 & \Xi_{14} & 0 & 0 & \Xi_{17} & \Xi_{18} & L_1^TB & \Xi_{1,10} & \Xi_{1,11}\\ * & \Xi_{22} & 0 & 0 & \Xi_{25} & 0 & 0 & Z & 0 & 0 & 0\\ * & * & \Xi_{33} & 0 & 0 & H\Sigma_2-R_3 & 0 & Z & 0 & 0 & 0\\ * & * & * & -U+Q_2 & 0 & 0 & A^TL_2 & 0 & 0 & A^TQ^T & 0\\ * & * & * & * & \Xi_{55} & 0 & 0 & 0 & 0 & 0 & 0\\ * & * & * & * & * & -Q_3-H & 0 & 0 & 0 & 0 & 0\\ * & * & * & * & * & * & \Xi_{77} & \Xi_{78} & L_2^TB & -L_2^TC-R^T & 0\\ * & * & * & * & * & * & * & \Xi_{88} & \Xi_{89} & \Xi_{8,10} & 0\\ * & * & * & * & * & * & * & * & \Xi_{99} & B^TQ^T & 0\\ * & * & * & * & * & * & * & * & * & -QC-C^TQ^T-T_1 & 0\\ * & * & * & * & * & * & * & * & * & * & -T_2\end{bmatrix}.$$

$$V(\mathbf{x}(t))=\sum_{i=1}^{3}V_i(\mathbf{x}(t)), \tag{7}$$

where

$$\begin{aligned}V_1(\mathbf{x}(t))&=\mathbf{x}^T(t)(U\otimes P)\mathbf{x}(t)+2[\Theta\mathbf{x}(t)-\mathbf{b}(\mathbf{x}(t))]^T\\&\quad\times(U\otimes R)\mathbf{x}(t)+2[\mathbf{b}(\mathbf{x}(t))-\Upsilon\mathbf{x}(t)]^T\\&\quad\times(U\otimes Q)\mathbf{x}(t),\\ V_2(\mathbf{x}(t))&=\int_{t-\tau(t)}^{t-\tau_0}\begin{bmatrix}\mathbf{x}(s)\\ \mathbf{f}(\mathbf{x}(s))\end{bmatrix}^T\left(U\otimes\begin{bmatrix}P_1 & R_1\\ * & Q_1\end{bmatrix}\right)\\&\quad\times\begin{bmatrix}\mathbf{x}(s)\\ \mathbf{f}(\mathbf{x}(s))\end{bmatrix}ds+\int_{t-\tau_0}^{t}\begin{bmatrix}\mathbf{x}(s)\\ \mathbf{f}(\mathbf{x}(s))\end{bmatrix}^T\\&\quad\times\left(U\otimes\begin{bmatrix}P_2 & R_2\\ * & Q_2\end{bmatrix}\right)\begin{bmatrix}\mathbf{x}(s)\\ \mathbf{f}(\mathbf{x}(s))\end{bmatrix}ds\\&\quad+\int_{t-\tau_m}^{t-\tau_0}\begin{bmatrix}\mathbf{x}(s)\\ \mathbf{f}(\mathbf{x}(s))\end{bmatrix}^T\\&\quad\times\left(U\otimes\begin{bmatrix}P_3 & R_3\\ * & Q_3\end{bmatrix}\right)\begin{bmatrix}\mathbf{x}(s)\\ \mathbf{f}(\mathbf{x}(s))\end{bmatrix}ds,\\ V_3(\mathbf{x}(t))&=\int_{-\tau_0}^{0}\int_{t+\theta}^{t}\tau_0\dot{\mathbf{x}}^T(s)(U\otimes S)\dot{\mathbf{x}}(s)dsd\theta\\&\quad+\int_{-\tau_m}^{-\tau_0}\int_{t+\theta}^{t}\overline{\tau}_m\dot{\mathbf{x}}^T(s)(U\otimes Z)\dot{\mathbf{x}}(s)dsd\theta\end{aligned}$$

with $\Theta=\text{diag}\{\underbrace{\Pi,\Pi,\cdots,\Pi}_{N}\}$, $\Upsilon=\text{diag}\{\underbrace{\Gamma,\Gamma,\cdots,\Gamma}_{N}\}$, $R=\text{diag}\{r_1,\cdots,r_n\}$, and $Q=\text{diag}\{q_1,\cdots,q_n\}$. Now by directly computing $\dot{V}_i(\mathbf{x}(t))(i=1,2,3)$ along the trajectory of system (5), it can be deduced that

$$\begin{aligned}\dot{V}_1(\mathbf{x}(t))&=2\mathbf{x}^T(t)(U\otimes P)\dot{\mathbf{x}}(t)+2[\Theta\mathbf{x}(t)-\mathbf{b}(\mathbf{x}(t))]^T\\&\quad\times(U\otimes R)\dot{\mathbf{x}}(t)+2\mathbf{b}^T(\mathbf{x}(t))(U\otimes Q)\\&\quad\times[-(I_N\otimes C)\mathbf{b}(\mathbf{x}(t))+(I_N\otimes A)\mathbf{f}(\mathbf{x}(t))\\&\quad+(I_N\otimes B)\mathbf{f}(\mathbf{x}(t-\tau(t)))+\mathbf{I}(t)\\&\quad+(L\otimes F)\mathbf{x}(t)+(L\otimes(K+J))\mathbf{x}(t-\tau(t))\\&\quad-l(I_N\otimes K)[\mathbf{x}(t-\tau(t))-\mathbf{x}(t)]]\\&\quad-2\mathbf{x}^T(t)(U\otimes\Gamma Q)\dot{\mathbf{x}}(t)\\&\quad+2\dot{\mathbf{x}}^T(t)(U\otimes(\Pi^TR-\Gamma^TQ))\mathbf{x}(t)\\&\quad+2\dot{\mathbf{b}}^T(\mathbf{x}(t))(U\otimes(Q-R))\mathbf{x}(t),\end{aligned} \tag{8}$$

$$\begin{aligned}\dot{V}_2(\mathbf{x}(t))&\le[\mathbf{x}^T(t-\tau_0)(U\otimes(P_1-P_2+P_3))\mathbf{x}(t-\tau_0)\\&\quad+2\mathbf{x}^T(t-\tau_0)(U\otimes(R_1-R_2+R_3))\\&\quad\times\mathbf{f}(\mathbf{x}(t-\tau_0))+\mathbf{f}^T(\mathbf{x}(t-\tau_0))\\&\quad\times(U\otimes(Q_1-Q_2+Q_3))\mathbf{f}(\mathbf{x}(t-\tau_0))]\\&\quad-(1-\mu)[\mathbf{x}^T(t-\tau(t))(U\otimes P_1)\mathbf{x}(t-\tau(t))\\&\quad+2\mathbf{x}^T(t-\tau(t))(U\otimes R_1)\mathbf{f}(\mathbf{x}(t-\tau(t)))\\&\quad+\mathbf{f}^T(\mathbf{x}(t-\tau(t)))(U\otimes Q_1)\mathbf{f}(\mathbf{x}(t-\tau(t)))]\\&\quad+[\mathbf{x}^T(t)(U\otimes P_2)\mathbf{x}(t)\\&\quad+2\mathbf{x}^T(t)(U\otimes R_2)\mathbf{f}(\mathbf{x}(t))\\&\quad+\mathbf{f}^T(\mathbf{x}(t))(U\otimes Q_2)\mathbf{f}(\mathbf{x}(t))]\\&\quad-[\mathbf{x}^T(t-\tau_m)(U\otimes P_3)\mathbf{x}(t-\tau_m)\\&\quad+2\mathbf{x}^T(t-\tau_m)(U\otimes R_3)\mathbf{f}(\mathbf{x}(t-\tau_m))\\&\quad+\mathbf{f}^T(\mathbf{x}(t-\tau_m))(U\otimes Q_3)\mathbf{f}(\mathbf{x}(t-\tau_m))],\end{aligned} \tag{9}$$

$$\begin{aligned}\dot{V}_3(\mathbf{x}(t))&=\dot{\mathbf{x}}^T(t)[\tau_0^2(U\otimes S)+\overline{\tau}_m^2(U\otimes Z)]\dot{\mathbf{x}}(t)\\&\quad-\int_{t-\tau_0}^{t}\tau_0\dot{\mathbf{x}}^T(s)(U\otimes S)\dot{\mathbf{x}}(s)ds\\&\quad-\int_{t-\tau_m}^{t-\tau_0}\overline{\tau}_m\dot{\mathbf{x}}^T((s)(U\otimes Z)\dot{\mathbf{x}}(s)ds.\end{aligned} \tag{10}$$

Based on the methods in [22], it follows from $\overline{\tau}_m=[\tau_m-\tau(t)]+[\tau(t)-\tau_0]$ and Lemma 1 that

$$\begin{aligned}&-\int_{t-\tau_m}^{t-\tau(t)}[\tau(t)-\tau_0]\dot{\mathbf{x}}^T(s)(U\otimes Z)\dot{\mathbf{x}}(s)ds\\&\qquad\le\nu[\mathbf{x}(t-\tau(t))-\mathbf{x}(t-\tau_m)]^T(U\otimes Z)\\&\qquad\quad\times[\mathbf{x}(t-\tau(t))-\mathbf{x}(t-\tau_m)],\end{aligned} \tag{11}$$

$$\begin{aligned}&-\int_{t-\tau(t)}^{t-\tau_0}[\tau_m-\tau(t)]\dot{\mathbf{x}}^T(s)(U\otimes Z)\dot{\mathbf{x}}(s)ds\\&\qquad\le\omega[\mathbf{x}(t-\tau_0)-\mathbf{x}(t-\tau(t))]^T(U\otimes Z)\\&\qquad\quad\times[\mathbf{x}(t-\tau_0)-\mathbf{x}(t-\tau(t))],\end{aligned} \tag{12}$$

in which $\nu = \frac{\tau(t)-\tau_0}{\overline{\tau}_m}$ and $\omega = \frac{\tau_m - \tau(t)}{\overline{\tau}_m}$. Together with the terms in (10)-(12), we can estimate $\dot{V}_3(\mathbf{x}(t))$ as

$$\begin{aligned}\dot{V}_3(\mathbf{x}(t)) \le\, & \dot{\mathbf{x}}^T(t)[\tau_0^2(U\otimes S)+\overline{\tau}_m^2(U\otimes Z)]\dot{\mathbf{x}}(t)-[\mathbf{x}(t)\\ & -\mathbf{x}(t-\tau_0)]^T(U\otimes S)[\mathbf{x}(t)-\mathbf{x}(t-\tau_0)]\\ & -(1+\nu)[\mathbf{x}(t-\tau(t))-\mathbf{x}(t-\tau_m)]^T\\ & \times(U\otimes Z)[\mathbf{x}(t-\tau(t))-\mathbf{x}(t-\tau_m)]\\ & -(1+\omega)[\mathbf{x}(t-\tau_0)-\mathbf{x}(t-\tau(t))]^T\\ & \times(U\otimes Z)[\mathbf{x}(t-\tau_0)-\mathbf{x}(t-\tau(t))].\end{aligned} \tag{13}$$

For any $n\times n$ matrices $L_i(i=1,2)$, noting that $UL=NL$ and $(U\otimes L_i^T)(L\otimes F)=(NL)\otimes(L_i^TF)$, $(U\otimes L_i^T)(L\otimes K)=(NL)\otimes(L_i^TK)$ for $i=1,2$, it follows from (5) that

$$\begin{aligned}0 =\, & 2[\mathbf{x}^T(t)(U\otimes L_1^T)+\dot{\mathbf{x}}^T(t)(U\otimes L_2^T)]\\ & \times\{-\dot{\mathbf{x}}(t)-(I_N\otimes C)\mathbf{b}(\mathbf{x}(t))+(I_N\otimes A)\mathbf{f}(\mathbf{x}(t))\\ & +(I_N\otimes B)\mathbf{f}(\mathbf{x}(t-\tau(t)))+\mathbf{I}(t)\\ & -l(I_N\otimes K)[\mathbf{x}(t-\tau(t))-\mathbf{x}(t)]\}\\ & +2[\mathbf{x}^T(t)(NL\otimes L_1^TF)+\dot{\mathbf{x}}^T(t)(NL\otimes L_2^TF)]\mathbf{x}(t)\\ & +2[\mathbf{x}^T(t)(NL\otimes L_1^T(K+J))\\ & +\dot{\mathbf{x}}^T(t)(NL\otimes L_2^T(K+J))]\mathbf{x}(t-\tau(t)).\end{aligned} \tag{14}$$

Here we can employ the following terms to simplify the subsequent proof

$$x_{ij}=x_i-x_j,\quad \dot{x}_{ij}=\dot{x}_i-\dot{x}_j,\quad f(x_{ij})=f(x_i)-f(x_j),$$
$$\beta(x_{ij})=\beta(x_i)-\beta(x_j),\quad \dot{\beta}(x_{ij})=\dot{\beta}(x_i)-\dot{\beta}(x_j).$$

Based on $(U\otimes Q)\mathbf{I}(t)=0$, $(U\otimes L_i^T)\mathbf{I}(t)=0$ for $i=1,2$, and Lemma 2, combining (8), (9), (13), and (14) yields

$$\begin{aligned}\dot{V}(\mathbf{x}(t)) \le -\sum_{1\le i<j\le N} & \Big\{\mathbf{u}_{ij}[2x_{ij}^T(t)P\dot{x}_{ij}(t)+2[\Pi x_{ij}(t)\\ & -\beta(x_{ij}(t))]^TR\dot{x}_{ij}(t)+2\beta^T(x_{ij}(t))\\ & \times Q[-C\beta(x_{ij}(t))+Af(x_{ij}(t))\\ & +Bf(x_{ij}(t-\tau(t)))-lK[x_{ij}(t-\tau(t))\\ & -x_{ij}(t)]]-2x_{ij}^T(t)\Gamma Q\dot{x}_{ij}(t)\\ & +2\dot{x}_{ij}^T(t)(\Pi R-\Gamma Q)x_{ij}(t)\\ & +2\dot{\beta}^T(x_{ij}(t))(Q-R)x_{ij}(t)\\ & +[x_{ij}^T(t-\tau_0)(P_1-P_2+P_3)x_{ij}(t-\tau_0)\\ & +2x_{ij}^T(t-\tau_0)(R_1-R_2+R_3)f(x_{ij}(t-\tau_0))\\ & +f^T(x_{ij}(t-\tau_0))(Q_1-Q_2+Q_3)\\ & \times f(x_{ij}(t-\tau_0))]-(1-\mu)\\ & \times[x_{ij}^T(t-\tau(t))P_1x_{ij}(t-\tau(t))\\ & +2x_{ij}^T(t-\tau(t))R_1f(x_{ij}(t-\tau(t)))\\ & +f^T(x_{ij}(t-\tau(t)))Q_1f(x_{ij}(t-\tau(t)))]\\ & +[x_{ij}^T(t)P_2x_{ij}(t)+2x_{ij}^T(t)R_2f(x_{ij}(t))\\ & +f^T(x_{ij}(t))Q_2f(x_{ij}(t))]\\ & -[x_{ij}^T(t-\tau_m)P_3x_{ij}(t-\tau_m)\\ & +2x_{ij}^T(t-\tau_m)R_3f(x_{ij}(t-\tau_m))\\ & +f^T(x_{ij}(t-\tau_m))Q_3f(x_{ij}(t-\tau_m))]\\ & +\dot{x}_{ij}^T(t)[\tau_0^2S+\overline{\tau}_m^2Z]\dot{x}_{ij}(t)-[x_{ij}(t)\\ & -x_{ij}(t-\tau_0)]^TS[x_{ij}(t)-x_{ij}(t-\tau_0)]\\ & -(1+\nu)[x_{ij}(t-\tau(t))-x_{ij}(t-\tau_m)]^T\\ & \times Z[x_{ij}(t-\tau(t))-x_{ij}(t-\tau_m)]\\ & -(1+\omega)[x_{ij}(t-\tau_0)-x_{ij}(t-\tau(t))]^T\\ & \times Z[x_{ij}(t-\tau_0)-x_{ij}(t-\tau(t))]\\ & +2[x_{ij}^T(t)L_1^T+\dot{x}_{ij}^T(t)L_2^T]\{-\dot{x}_{ij}(t)\\ & -C\beta(x_{ij}(t))+Af(x_{ij}(t))\\ & +Bf(x_{ij}(t-\tau(t)))-lK[x_{ij}(t-\tau(t))-x_{ij}(t)]\}]\\ & +2Nl_{ij}[[\beta^T(x_{ij}(t))Q+x_{ij}^T(t)L_1^T\\ & +\dot{x}_{ij}^T(t)L_2^T]Fx_{ij}(t)+[\beta^T(x_{ij}(t))Q+x_{ij}^T(t)L_1^T\\ & +\dot{x}_{ij}^T(t)L_2^T](K+J)x_{ij}(t-\tau(t))]\Big\}.\end{aligned} \tag{15}$$

By utilizing Asumption 3 for any $n\times n$ diagonal matrix $G>0$, the following inequality holds

$$0\le \sum_{1\le i<j\le N}\Big\{2[\beta^T(x_{ij}(t))-x_{ij}^T(t)\Gamma^T]Gx_{ij}(t)\Big\}. \tag{16}$$

For any $n\times n$ diagonal matrices $U>0$, $V>0$, $W>0$, $H>0$, $T_i>0(i=1,2)$ and $\Sigma_i, \Pi_i, \Lambda_i(i=1,2)$ in Asumptions 3 and 4, it follows that

$$\begin{aligned}0\le \sum_{1\le i<j\le N} & \Big\{-[x_{ij}^T(t)U\Sigma_1x_{ij}(t)\\ & -2x_{ij}^T(t)U\Sigma_2f(x_{ij}(t))+f^T(x_{ij}(t))Uf(x_{ij}(t))]\\ & -[x_{ij}^T(t-\tau(t))V\Sigma_1x_{ij}(t-\tau(t))\\ & -2x_{ij}^T(t-\tau(t))V\Sigma_2f(x_{ij}(t-\tau(t)))\\ & +f^T(x_{ij}(t-\tau(t)))Vf(x_{ij}(t-\tau(t)))]\\ & -[x_{ij}^T(t-\tau_0)W\Sigma_1x_{ij}(t-\tau_0)\\ & -2x_{ij}^T(t-\tau_0)W\Sigma_2f(x_{ij}(t-\tau_0))\\ & +f^T(x_{ij}(t-\tau_0))Wf(x_{ij}(t-\tau_0))]\\ & -[x_{ij}^T(t-\tau_m)H\Sigma_1x_{ij}(t-\tau_m)\\ & -2x_{ij}^T(t-\tau_m)H\Sigma_2f(x_{ij}(t-\tau_m))\\ & +f^T(x_{ij}(t-\tau_m))Hf(x_{ij}(t-\tau_m))]\\ & -[x_{ij}^T(t)T_1\Pi_1x_{ij}(t)-2x_{ij}^T(t)T_1\Pi_2\beta(x_{ij}(t))\end{aligned} \tag{17}$$

$$+\beta^T(x_{ij}(t))T_1\beta(x_{ij}(t))]$$
$$-[x_{ij}^T(t)T_2\Lambda_1 x_{ij}(t)-2x_{ij}^T(t)T_2\Lambda_2\dot{\beta}(x_{ij}(t))$$
$$+\dot{\beta}^T(x_{ij}(t))T_2\dot{\beta}(x_{ij}(t))]\Big\}.$$

Now together with the terms (15)-(17) and $u_{ij}=-1$, we can obtain

$$\dot{V}(\mathbf{x}(t))\le\sum_{1\le i<j\le N}\zeta_{ij}^T(t)[\Omega_{ij}-\nu I_1^T ZI_1-\omega I_2^T ZI_2]\zeta_{ij}(t)$$
$$:=\sum_{1\le i<j\le N}\zeta_{ij}^T(t)\Delta_{ij}(t)\zeta_{ij}(t),$$

where Ω_{ij} is presented in (6), and

$$\zeta_{ij}^T(t)=[x_{ij}^T(t)\;\; x_{ij}^T(t-\tau_0)\;\; x_{ij}^T(t-\tau_m)\;\; f^T(x_{ij}(t))$$
$$f^T(x_{ij}(t-\tau_0))\;\; f^T(x_{ij}(t-\tau_m))\;\; \dot{x}_{ij}^T(t)$$
$$x_{ij}^T(t-\tau(t))\;\; f^T(x_{ij}(t-\tau(t)))\;\; \beta^T(x_{ij}(t))$$
$$\dot{\beta}^T(x_{ij}(t))].$$

Through using Schur-complement and convex combination, the LMIs in (6) can guarantee $\Delta_{ij}(t)<0$ to be true and thus, there must exist a positive scalar $\chi>0$ such that $\Delta_{ij}(t)\le-\chi I<0$. Therefore, one can deduce

$$\dot{V}(\mathbf{x}(t))\le\sum_{1\le i<j\le N}\zeta_{ij}^T(t)\Delta_{ij}(t)\zeta_{ij}(t)$$
$$\le-\chi\sum_{1\le i<j\le N}[\|x_{ij}(t)\|^2+\|x_{ij}(t-\tau(t))\|^2],$$

which indicates that the system (5) can reach the asymptotical synchronization.

Furthermore, based on the proof in [14], there must exist two positive scalars $\varpi>0,\ k>0$ such that

$$\|x_{ij}(t)\|\le\varpi\sum_{1\le i<j\le N}\sup_{-2\tau_m\le s\le 0}\|\phi_i(s)-\phi_j(s)\|\cdot e^{-kt}$$

for $t\ge t_0$. By Definition 1, the dynamical system (5) can achieve the exponential synchronization, and it completes the proof.

Remark 2: Theorem 1 presents one novel delay-dependent criterion guaranteeing the system (5) to be exponentially synchronized. In [19], the authors considered global synchronization for arrays of coupled DNNs of simple forms and in the paper, we derive a more general DNNs and extended constant delay to time variable one, which extends the present methods. Moreover, by using LMI in Matlab Toolbox, it is straightforward and convenient to check the feasibility of the proposed results without tuning any parameters.

Remark 3: During estimating

$$-\int_{t-\tau_m}^{t-\tau(t)}\overline{\tau}_m\dot{x}^T(s)(U\otimes Z)\dot{x}(s)ds,$$
$$-\int_{t-\tau(t)}^{t-\tau_0}\overline{\tau}_m\dot{x}^T(s)(U\otimes Z)\dot{x}(s)ds,$$

the previous ignored terms

$$-\int_{t-\tau_m}^{t-\tau(t)}[\tau(t)-\tau_0]\dot{x}^T(s)(U\otimes Z)\dot{x}(s)ds,\quad\text{and}$$
$$-\int_{t-\tau(t)}^{t-\tau_0}[\tau_m-\tau(t)]\dot{x}^T(s)(U\otimes Z)\dot{x}(s)ds$$

have been considered based on convex combination in Theorem 1, which can help reduce considerable conservatism. Moreover, Theorem 1 has not utilized the free-weighting matrix variables widely employed in present literature, which can result in computational simplicity in a mathematical point of view.

By utilizing the proof of Theorem 1, we try to address the following systems of more general form

$$\begin{aligned}\dot{x}_i(t)&=-C\beta(x_i(t))+Af(x_i(t))+Bg(x_i(t-\tau(t)))\\&+I(t)+\sum_{j=1,j\ne i}^N l_{ij}F[x_j(t)-x_i(t)]\\&+\sum_{j=1,j\ne i}^N l_{ij}K[x_j(t-\tau(t))-x_i(t)]\\&+\sum_{j=1,j\ne i}^N l_{ij}J[x_j(t-\tau(t))-x_i(t-\tau(t))]\end{aligned}\tag{18}$$

with the matrices C, A, B, F, K, J similar to relevant ones in system (1), and $g(x_i(\cdot))=[g_1(x_{i1}(\cdot)),\cdots,g_n(x_{in}(\cdot))]^T$ satisfying

$$\begin{aligned}&[g_i(\alpha)-g_i(\beta)-\rho_i^+(\alpha-\beta)][g_i(\alpha)-g_i(\beta)\\&\quad-\rho_i^-(\alpha-\beta)]\le 0,\ \forall\ i=1,\cdots,n,\end{aligned}\tag{19}$$

in which $\rho_i^-,\ \rho_i^+$ are given constants. Here we set

$$\begin{aligned}\Sigma_3&=\mathrm{diag}\{\rho_1^+\rho_1^-,\cdots,\rho_n^+\rho_n^-\},\\\Sigma_4&=\mathrm{diag}\left\{\frac{\rho_1^++\rho_1^-}{2},\cdots,\frac{\rho_n^++\rho_n^-}{2}\right\}.\end{aligned}\tag{20}$$

Theorem 2: Supposing that Assumptions 1-4 and (19) hold, then the dynamical system (18) is globally exponentially synchronized, if there exist $n\times n$ matrices $P>0,\ S>0,\ Z>0,\ L_i(i=1,2)$, $n\times n$ matrices $P_l>0,\ Q_l>0,\ R_l(l=1,2,3)$ making $\begin{bmatrix}P_l & R_l\\ R_l^T & Q_l\end{bmatrix}\ge 0$, and $n\times n$ diagonal matrices $U_i>0(i=1,2),\ V>0,\ W>0,\ H>0,\ R>0,\ Q>0,\ G>0,\ T_i>0(i=1,2)$ such that, for $1\le i<j\le N$,

$$\Omega_{ij}-I_1^T ZI_1<0,\ \Omega_{ij}-I_2^T ZI_2<0,\tag{21}$$

where $I_1=[0_{n\cdot 2n}\ -I_n\ 0_{n\cdot 5n}\ I_n\ 0_{n\cdot 3n}]$, $I_2=[0_n\ I_n\ 0_{n\cdot 6n}\ -I_n\ 0_{n\cdot 3n}]$, and Ω_{ij} is expressed in next page with

$$\Xi_{11}=P_2-S+l(L_1^T K+K^T L_1)-l_{ij}N(L_1^T F+F^T L_1)$$
$$-2\Gamma^T G-U_1\Sigma_1-U_2\Sigma_3-T_1\Pi_1-T_2\Lambda_1,$$

$\Xi_{14} = L_1^T A + U_1\Sigma_2, \quad \Xi_{15} = R_2 + U_2\Sigma_4,$

$\Xi_{18} = P - L_1^T + 2(\Pi^T R - \Gamma^T Q) + lK^T L_2 - l_{ij} NF^T L_2,$

$\Xi_{19} = -lL_1^T K - l_{ij} NL_1^T (K+J),$

$\Xi_{11} = P_2 - S + l(L_1^T K + K^T L_1) - l_{ij} N(L_1^T F + F^T L_1)$
$\quad - 2\Gamma^T G - U_1\Sigma_1 - U_2\Sigma_3 - T_1\Pi_1 - T_2\Lambda_1,$

$\Xi_{14} = L_1^T A + U_1\Sigma_2, \ \Xi_{15} = R_2 + U_2\Sigma_4,$

$\Xi_{18} = P - L_1^T + 2(\Pi^T R - \Gamma^T Q) + lK^T L_2 - l_{ij} NF^T L_2,$

$\Xi_{19} = -lL_1^T K - l_{ij} NL_1^T (K+J),$

$\Xi_{1,11} = lK^T Q^T - L_1^T C - l_{ij} NF^T Q^T + G^T + T_1\Pi_2,$

$\Xi_{1,12} = T_2\Lambda_2 + (Q-R)^T,$

$\Xi_{22} = -P_2 + P_1 + P_3 - S - Z - W\Sigma_3,$

$\Xi_{26} = -R_2 + R_1 + R_3 + W\Sigma_4, \quad \Xi_{33} = -P_3 - Z - H\Sigma_3,$

$\Xi_{66} = -Q_2 + Q_1 + Q_3 - W, \ \Xi_{88} = -L_2^T - L_2 + \tau_0^2 S + \overline{\tau}_m^2 Z,$

$\Xi_{89} = -lL_2^T K - l_{ij} NL_2^T (K+J), \ \Xi_{8,11} = -L_2^T C - R^T,$

$\Xi_{99} = -(1-\mu)P_1 - 2Z - V\Sigma_3,$

$\Xi_{9,10} = -(1-\mu)R_1 + V\Sigma_4,$

$\Xi_{9,11} = -lK^T Q^T - l_{ij} N(K+J)^T Q^T,$

$\Xi_{10,10} = -(1-\mu)Q_1 - V,$

$$\Omega_{ij} = \begin{bmatrix} \Xi_{11} & S & 0 & \Xi_{14} & \Xi_{15} & 0 & 0 & \Xi_{18} & \Xi_{19} & L_1^T B & \Xi_{1,11} & \Xi_{1,12} \\ * & \Xi_{22} & 0 & 0 & 0 & \Xi_{26} & 0 & 0 & Z & 0 & 0 & 0 \\ * & * & \Xi_{33} & 0 & 0 & 0 & H\Sigma_4 - R_3 & 0 & Z & 0 & 0 & 0 \\ * & * & * & -U_1 & 0 & 0 & 0 & A^T L_2 & 0 & 0 & A^T Q^T & 0 \\ * & * & * & * & -U_2 + Q_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Xi_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -Q_3 - H & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Xi_{88} & \Xi_{89} & L_2^T B & \Xi_{8,11} & 0 \\ * & * & * & * & * & * & * & * & \Xi_{99} & \Xi_{9,10} & \Xi_{9,11} & 0 \\ * & * & * & * & * & * & * & * & * & \Xi_{10,10} & B^T Q^T & 0 \\ * & * & * & * & * & * & * & * & * & * & -2QC - T_1 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & -T_2 \end{bmatrix}.$$

Proof: Based on Theorem 1, one can easily derive the theorem and the detailed proof is omitted here.

Remark 4: If there does not exist one single delayed coupling in systems (1) and (18), i.e., K=0, we can easily derive the relevant results without the restriction on $l_{11} = l_{22} = \cdots = l_{NN}$ in $L = [l_{ij}]_{N\times N}$.

Remark 5: Theorems 1-2 require the upper bound μ of time-delay $\tau(t)$ to be known. If μ is unknown, by setting $P_1 = R_1 = Q_1$ in (7), we can derive the delay-dependent and delay-derivative-independent criteria for the global synchronization based on Theorems 1-2.

4. NUMERICAL EXAMPLES

In the section, three examples are provided to illustrate the effectiveness of the proposed results.

Example 1: We consider the following DNNs

$$\dot{x}(t) = -C\beta(x(t)) + Af(x(t)) + Bf(x(t-\tau(t))) + I(t)$$

with $C = \text{diag}\{1.5, 1.5, 1.5\}$, and

$$A = \begin{bmatrix} -10 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -12 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & -2 \\ 2 & -3 & 2 \\ -2 & 2 & 3 \end{bmatrix},$$

$$I(t) = \begin{bmatrix} 0.1 \\ -0.2 \\ 0.3 \end{bmatrix}, \quad \beta(x) = \begin{bmatrix} 0.8x_1 + 0.1\sin(2x_1) \\ 0.8x_2 + 0.1\cos(2x_2) \\ 0.8x_3 + 0.1\sin(2x_3) \end{bmatrix},$$

$$\tau(t) = 0.5 + 0.2\sin(40t) + 0.05\cos^2(80t),$$
$$f_i(x_i) = 0.3(|x_i + 1| - |x_i - 1|) + 0.1\tanh(x_i).$$

One can get $\tau_0 = 0.3$, $\tau_m = 0.75$, $\mu = 16$, and the activation functions $f_i(x_i)$ satisfy Assumption 4. Now setting the inner linking matrix $L = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$, and the inner coupling matrices $F = \text{diag}\{7,7,7\}$, $K = \text{diag}\{0.05, 0.05, 0.05\}$, $J = \text{diag}\{0.1, 0.1, 0.1\}$, we consider a dynamic networks consisting of three linearly coupled identical DNNs with couplings as

$$\begin{aligned} \dot{x}_i(t) &= -C\beta(x_i(t)) + Af(x_i(t)) + Bf(x_i(t-\tau(t))) \\ &\quad + I(t) + \sum_{j=1}^{3} l_{ij} Fx_j(t) + \sum_{j=1}^{3} l_{ij}(K+J) \\ &\quad \times x_j(t-\tau(t)) + 2K[x_i(t-\tau(t)) - x_i(t)] \end{aligned} \tag{22}$$

for i=1,2,3. Fig. 1 shows that system (22) has a chaotic attractor. Together with Theorem 1 and LMI in Matlab Toolbox, there exist the feasible solutions to the LMIs in (6), which can guarantee the array of system (22) to achieve the exponential synchronization. The total error is defined by

$$\text{error}(t) = \sum_{i=1}^{3} \sqrt{[x_{1i}(t) - x_{2i}(t)]^2 + [x_{2i}(t) - x_{3i}(t)]^2},$$

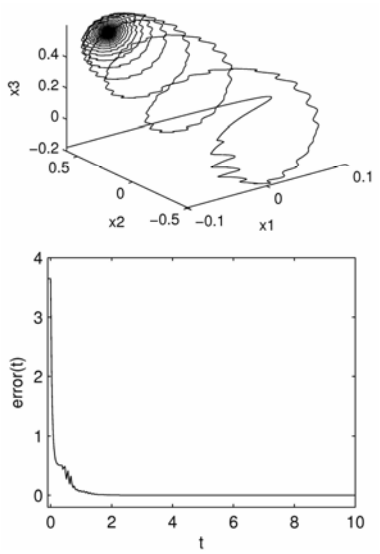

Fig. 1. Synchronized trajectory and total synchronous error of system (22).

and the synchronization error can be checked in Fig. 1. During the process of simulation, the initial conditions of nodes are selected as $x_1=[-0.5\ -0.3\ 0.1]^T$, $x_2=[0.7\ -0.5\ -0.2]^T$, and $x_3=[1\ 0.5\ 0.3]^T$.

Example 2: Consider one 2-dimensional delayed network system (18) as follows:

$$\dot{x}(t)=-C\beta(x(t))+Af(x(t))+Bg(x(t-\tau(t)))+I(t),$$

where

$$C=\begin{bmatrix}1.5 & 0\\ 0 & 1.5\end{bmatrix},\quad A=\begin{bmatrix}2 & -0.2\\ -0.3 & 3\end{bmatrix},$$

$$B=\begin{bmatrix}-1.5 & -0.1\\ -0.2 & -2.5\end{bmatrix},\quad \beta(x)=\begin{bmatrix}0.8x_1+0.2\dfrac{e^{x_1}}{1+e^{x_1}}\\ 0.8x_2+0.2\dfrac{e^{x_2}}{1+e^{x_2}}\end{bmatrix},$$

$$I(t)=\begin{bmatrix}0.10\\ -0.05\end{bmatrix},$$

$$\tau(t)=0.6+0.25\sin(6t)+0.1\cos^2(20t),$$
$$f_i(x_i)=\tanh(x_i),\ g_i(x_i)=0.5(|x_i+1|-|x_i-1|),$$
$$i=1,2.$$

Choosing the following inner linking matrix $L=\begin{bmatrix}-3 & 1 & 2\\ 1 & -2 & 1\\ 2 & 1 & -3\end{bmatrix}$ and the inner coupling matrices $F=\operatorname{diag}\{5,5\}$, K=0, $J=\operatorname{diag}\{0.15,0.15\}$, we still consider a dynamic networks consisting of three coupled identical networks with delayed coupling as

$$\begin{aligned}\dot{x}_i(t)&=-C\beta(x_i(t))+Af(x_i(t))+Bf(x_i(t-\tau(t)))\\ &\quad+I(t)+\sum_{j=1}^{3}l_{ij}[Fx_j(t)+Jx_j(t-\tau(t))]\end{aligned}\tag{23}$$

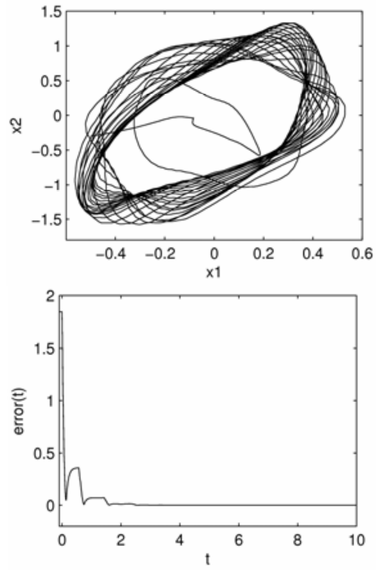

Fig. 2. Synchronized trajectory and total synchronous error of system (23).

for i=1,2,3. Fig. 2 shows that the system has a chaotic attractor. Based on Theorem 2, there dose exist the feasible solution to the LMIs in (21), which can guarantee to achieve the global exponential synchronization for the system (23). The total error of (23) is defined by

$$\text{error}(t)=\sum_{i=1}^{2}\sqrt{[x_{1i}(t)-x_{2i}(t)]^2+[x_{2i}(t)-x_{3i}(t)]^2},$$

and the total synchronous error can be depicted in Fig. 2 with the initial conditions $x_1=[-0.5\ -0.3]^T$, $x_2=[0.3\ 0.7]^T$, and $x_3=[-0.5\ -0.6]^T$.

Example 3: Consider one typical 2-dimensional delayed network system (1) described by

$$\dot{x}(t)=-Cx(t)+Af(x(t))+Bf(x(t-\tau(t))),$$

where

$$C=\begin{bmatrix}1.2 & 0\\ 0 & 1.2\end{bmatrix},\quad A=\begin{bmatrix}1.5 & -1.5\\ -1.4 & 1.2\end{bmatrix},$$

$$B=\begin{bmatrix}1.5 & 1.2\\ -2.5 & 2.5\end{bmatrix},\quad f(x)=0.5\begin{bmatrix}|x_1+1|-|x_1-1|\\ |x_2+1|-|x_2-1|\end{bmatrix}.$$

By setting $J=0$, we consider a dynamic networks consist-ing of three linearly coupled identical DNNs as

$$\begin{aligned}\dot{x}_i(t)&=-Cx_i(t)+Af(x_i(t))+Bf(x_i(t-\tau(t)))\\ &\quad+\sum_{j=1}^{3}l_{ij}Fx_j(t)+\sum_{j=1}^{3}l_{ij}Kx_j(t-\tau(t))\\ &\quad-l_{ij}K[x_i(t-\tau(t))-x_i(t)]\end{aligned}\tag{24}$$

for $i=1,2,3,$ and choose $L=\begin{bmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{bmatrix}$, $F=$

Table 1. Maximum allowable delay upper bound of τ_m for various τ_0 and μ in Example 3.

$\tau_0 \diagdown \mu$	methods	unknown μ	0.9	1.0
0	[19]	0.468	78.54	0.478
0	Theorem 1	1.263	110.1	1.267
10	Theorem 1	11.11	123.4	11.22

$\text{diag}\{5,5\}$, and $K = \text{diag}\{0.3, 0.3\}$. Together with Theorem 1 and Remark 5, we can verify that the system (24) is globally exponentially synchronized, and the corresponding τ_0, τ_m derived by the methods in the paper and the relevant ones in [19] are listed in Table 1, which shows that our methods can be more applicable and less conservative than the ones in [19].

5. CONCLUSIONS

This paper has studied the global exponential synchronization for arrays of coupled delayed neural networks. Two novel conditions have been established by employing Lyapunov-Krasovskii functional and convex combination techniques. It is worth pointing out that, the addressed systems can be of more general forms than the present ones and some good mathematical techniques have been employed. The derived synchronization criteria are presented in terms of LMIs, which can be checked easily by resorting to Matlab LMI Toolbox. Finally, three numerical examples are utilized to illustrate the efficiency of the derived methods by simulation results.

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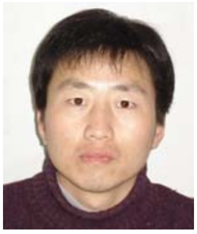

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