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On the H_∞ Deconvolution Fixed-lag Smoothing

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Abstract: The paper deals with the H_∞ deconvolution fixed-lag smoothing problem for a linear time-invariant discrete-time system with both known and unknown input series. The H_∞ fixed-lag smoother is derived by proposing a new approach termed as re-organized innovation analysis in Krein space. Under the new approach, it is clearly shown that the central deconvolution smoother in an H_∞ setting is the same as the one in an H_2 setting associated with one self-constructed stochastic state-space model. This insight allows us to calculate the complicated H_∞ deconvolution smoother in an intuitive and simple way. The deconvolution smoother is calculated by performing Riccati equation with the same order as the original system.

Keywords: Deconvolution, H_∞ smoothing, innovation analysis, Krein space, Riccati equation.

1. INTRODUCTION

Deconvolution has wide application in areas such as oil exploration, image restoration, fault detection, signal processing, communication and so on. In the past decades, much attention has been paid to the deconvolution problem, see [1,2,5,10] and references therein.

There are mainly two performances under which deconvolution problem has been considered, one is the H_2 index, the other is H_∞ performance. With the H_2 index, [5] consider the deconvolution estimator by using traditional Kalman filtering formulation, the estimator is usually related with the solution to the Riccati equation. [7] present the optimal deconvolution by using the polynomial approach where the solution is given by solving one spectral factorization and two polynomial equations.

In contrast to the H_2 index, since the design for an H_∞ estimator does not require the knowledge of the statistics of the system and observation noises and posses the robustness to the systems and noise uncertainties, the H_∞ deconvolution has been received much attention in recent years [4,6,9]. In particular, for the case where the input signal is ARMA process and the observation system is represented by ARMAX sequence, [4,6] investigate the H_∞ deconvolution problem by using polynomial method. The infinite horizon H_∞ deconvolution estimators including filter, predictor and smoother are given by the solutions from J-spectral factorization and polynomial equations. It should be noted that the J-spectral factorization for the H_∞ fixed-lag smoothing and prediction are more complicated than for the filtering as they are usually related with system augmentation [6]. Under the state space model, [3,10] studied the H_∞ finite-horizon deconvolution filtering. The deconvolution filter is calculated in terms of the solution to one standard H_∞ Riccati equation in [3], and Krein-space is used in [10]. With the help of LMI (Linear Matrix Inequality), [8] proposed an H_∞ deconvolution filter and [9] obtained a mixed H_2/H_∞ filter. All the calculation of the deconvolution filter is based on the available solution to the LMI. The deconvolution fixed-lag smoothing has been discussed in [4] and some works, however, it seems for the approach to be difficult. Different from the above approaches, a simple and efficient approach termed as *re-organized innovation analysis* will be proposed in this paper.

Re-organized innovation analysis approach is presented in our previous works [11-13] which considered the estimation problem of system state. Particularly, [12] considered the H_∞ fixed-lag smoothing for continuous-time systems for state estimation. [11] considered Kalman filtering for discrete-time systems for state estimation. In all, the above references mainly focus on the state estimation, which can be computed recursively. Different from our previous works, in this paper, we consider the H_∞ deconvolution estimation mainly fixe-lag smoothing problem for linear time-

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invariant discrete-time system [13]. Re-organized innovation analysis approach is proposed to deal with such a problem, and a new H_∞ deconvolution fixed-lag smoother is derived. The new approach shows that the H_∞ deconvolution fixed-lag smoothing is equivalent to an H_2 deconvolution fixed-lag smoothing associated with measurement-delayed system.

The rest of the paper is organized as follows. The system under consideration and the problem statement are given in Section 2. Some preliminaries about the H_∞ deconvolution fixed-lag smoother are given in Section 3. An H_∞ deconvolution fixed-lag smoother design will be given by re-organized innovation analysis in Krein space in Section 4. A numerical example is proposed to show the efficiency of the approach in Section 5. Some concluding remarks are made in Section 6.

2. PROBLEM STATEMENT

We consider the linear discrete-time system described as

$$x(k+1) = Ax(k) + Bu(k) + D_1 u_u(k),\ x(0), \tag{1}$$

$$y(k) = Cx(k) + D_2 u_u(k) + v(k), \tag{2}$$

$$s(k) = Lu_u(k), \tag{3}$$

where k is an integer, $x(k) \in \mathscr{R}^n$ is the state, $y(k) \in \mathscr{R}^p$ is measurement, $s(k) \in \mathscr{R}^r$ is the signal to be estimated. $u(k) \in \mathscr{R}^m$ is known input series, $u_u(k) \in \mathscr{R}^q$ is unknown input, $v(k) \in \mathscr{R}^p$ is measurement noise. A, C, B, D_1, D_2 and L are known constant matrices with appropriate dimensions. $u_u(k)$ and $v(k)$ are assumed to have bounded energy over the interval $[0, N]$.

The H_∞ deconvolution estimation problem that is to be investigated for the systems model (1)-(3) can be respectively stated as

Problem S: Given a scalar $\gamma > 0$, known input series $\{u(0), \cdots, u(k)\}$, an integer $l > 0$ and the observation $\{y(0), \cdots, y(k)\}$, we find the fixed-lag smoothing estimate of $s(k_l)$, denoted by $\check{s}(k_l \mid k)$, such that

$$\sup_{(x(0), u_u, v) \neq 0} \frac{\Phi}{\Gamma} < \gamma^2, \tag{4}$$

where $\Phi = \sum_{k=l}^{N} [\check{s}(k_l \mid k) - s(k_l)]^T [\check{s}(k_l \mid k) - s(k_l)]$, $\Gamma = x^T(0)\Pi_0^{-1}x(0) + \sum_{k=0}^{N} u_u^T(k)u_u(k) + \sum_{k=0}^{N} v^T(k)v(k)$ $x(0)$ is unknown and Π_0 is a given positive definite matrix which reflects the relative uncertainty of the initial state $x(0)$ to the estimate $\check{x}(0) = 0$, and $k_l \triangleq k - l$ which will be used in the full paper.

Remark 1: When the integer $l < 0$, it is obvious from (1)-(3) that $s(k_l)$ is uncorrelated with $\{y(0), \cdots, y(k)\}$, which implies that $\check{s}(k_l \mid k) = 0$ for $l < 0$. When $l = 0$, $\check{s}(k \mid k)$ is termed as filter, which can be dealt with easily. While $\check{s}(k_l \mid k)$ for $l > 0$ is termed as fixed-lag smoother. Since the fixed-lag smoothing problem is more difficult, we shall study the case in the paper.

3. PRELIMINARIES

In this section we will give some preliminaries about the deconvolution fixed-lag smoothing.

Note that the denominator of the left side of (4) is positive, it is obvious that (4) is satisfied if and only if the following inequality holds

$$\begin{aligned} &\mathscr{J}_{l,N}(x(0), u(0), \cdots, u(N); u_u(0), \cdots, u_u(N); y(0), \cdots, \\ &y(N)) \triangleq x^T(0)\Pi_0^{-1}x(0) + \sum_{k=0}^{N} u_u^T(k)u_u(k) \\ &\quad + \sum_{k=0}^{N} v^T(k)v(k) - \gamma^{-2}\sum_{k=l}^{N} v_s^T(k)v_s(k) > 0, \end{aligned} \tag{5}$$

where

$$v_s(k) \triangleq \check{s}(k_l \mid k) - s(k_l). \tag{6}$$

Thus **Problem S** is equivalent to:

- $\mathscr{J}_{l,N}(\cdot)$ of (5) has minimum with respective to $x(0), u_u(0), \cdots, u_u(N)$;
- $\check{s}(k_l \mid k)$ can be chosen such that the value of $\mathscr{J}_{l,N}(\cdot)$ at its minimum is positive.

Note (5) can be written as the centralized form as

$$\mathscr{J}_{l,N}(\cdot) = \begin{bmatrix} x(0) \\ u_u \\ v_z \end{bmatrix}^T \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_{u_u} & 0 \\ 0 & 0 & Q_{v_z} \end{bmatrix}^{-1} \begin{bmatrix} x(0) \\ u_u \\ v_z \end{bmatrix}, \tag{7}$$

where $u_u = col\{u_u(0), \cdots, u_u(N)\}$, $v_z = col\{v_z(0), \cdots, v_z(N)\}$, $Q_{u_u} = diag\{Q_{u_u}(0), \cdots, Q_{u_u}(N)\}$ and $Q_{v_z} = diag\{Q_{v_z}(0), \cdots, Q_{v_z}(N)\}$, where

$$v_z(k) \triangleq \begin{cases} v(k), & 0 \le k < l, \\ \begin{bmatrix} v(k) \\ v_s(k) \end{bmatrix}, & k \ge l, \end{cases} \tag{8}$$

$$Q_{v_z}(k) \triangleq \begin{cases} I_p, & 0 \le k < l, \\ diag\{I_p, -\gamma^2 I_r\}, & k \ge l. \end{cases} \tag{9}$$

From (2)-(3), (6) and (8) we have

$$y_z(k) = \begin{cases} Cx(k) + D_2 u_u(k) + v_z(k), & 0 \le k < l, \\ \begin{bmatrix} C \\ 0_{r\times n} \end{bmatrix} x(k) + \begin{bmatrix} D_2 & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} u_u(k) \\ u_u(k_l) \end{bmatrix} + v_z(k), \\ & k \ge l, \end{cases} \tag{10}$$

where

$$y_z(k)=\begin{cases} y(k), & 0\le k<l,\\ \begin{bmatrix} y(k)\\ \check{s}(k_l\mid k)\end{bmatrix}, & k\ge l.\end{cases} \quad (11)$$

Same as [13], we introduce the following Krein-space stochastic system model associated with (1), (10) and $\mathscr{J}_{l,N}(\cdot)$ as

$$\mathbf{x}(k+1)=A\mathbf{x}(k)+B\mathbf{u}(k)+D_1\mathbf{u}_u(k),\ \ \mathbf{x}(0), \quad (12)$$

$$\mathbf{y}_z(k)=\begin{cases} C\mathbf{x}(k)+D_2\mathbf{u}_u(k)+\mathbf{v}_z(k), & 0\le k<l,\\ \begin{bmatrix} C\\ 0_{r\times n}\end{bmatrix}\mathbf{x}(k)+\begin{bmatrix} D_2 & 0\\ 0 & L\end{bmatrix}\begin{bmatrix}\mathbf{u}_u(k)\\ \mathbf{u}_u(k_l)\end{bmatrix}+\mathbf{v}_z(k), & k\ge l\end{cases} \quad (13)$$

then, we have the following results.

Lemma 1: Consider the system model (1)-(3), given a scalar $\gamma>0$ and an integer $l>0$, then $\mathscr{J}_{l,N}(\cdot)$ has the minimum over $\{x(0),u_u,v_z\}$ if and only if $Q_{w_z}(k)$ and $Q_{v_z}(k)$ have the same inertia, where $Q_{v_z}(k)$ is as (9) and $Q_{w_z}(k)\triangleq<\mathbf{w}_z(k),\mathbf{w}_z(k)>$, is the covariance matrix of innovation $\mathbf{w}_z(k)$ which is given by

$$\mathbf{w}_z(k)=\mathbf{y}_z(k)-\hat{\mathbf{y}}_z(k), \quad (14)$$

where $\hat{\mathbf{y}}_z(k)$ is the projection of $\mathbf{y}_z(k)$ onto $\mathscr{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{k-1}\}$. In this case the minimum value of $\mathscr{J}_{l,N}(\cdot)$ is

$$\begin{aligned}\mathscr{J}_{l,N}^{m}(\cdot)&=\sum_{k=0}^{l-1}[y(k)-C\hat{x}(k)]^TQ_{w_z}^{-1}(k)[y(k)-C\hat{x}(k)]\\ &+\sum_{k=l}^{N}\begin{bmatrix} y(k)-C\hat{x}(k)\\ \check{s}(k_l\mid k)-L\hat{u}_u(k_l\mid k-1)\end{bmatrix}^TQ_{w_z}^{-1}(k)\\ &\times\begin{bmatrix} y(k)-C\hat{x}(k)\\ \check{s}(k_l\mid k)-L\hat{u}_u(k_l\mid k-1)\end{bmatrix},\end{aligned} \quad (15)$$

where $\hat{x}(k)$ and $\hat{u}_u(k_l\mid k-1)$ are obtained from the Krein space projection of $\mathbf{x}(k)$ and $\mathbf{u}_u(k_l)$ onto $\mathscr{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{k-1}\}$, respectively.

Proof: In accordance with [12], $\mathscr{J}_{l,N}(\cdot)$ of (5) has the minimum over $\{x(0),u_u,v_z\}$ if and only if $Q_{w_z}(k)$ and $Q_{v_z}(k)$ have the same inertia, and the minimum, if exists, is given as

$$\begin{aligned}\mathscr{J}_{l,N}^{m}(\cdot)&=\sum_{k=0}^{N}w_z^T(k)Q_{w_z}^{-1}(k)w_z(k)\\ &=\sum_{k=0}^{N}\left[y_z(k)-\hat{y}_z(k)\right]^TQ_{w_z}^{-1}(k)\left[y_z(k)-\hat{y}_z(k)\right],\end{aligned} \quad (16)$$

In view of (11), then (15) follows directly. This completes the proof of Lemma.

Now it is clear that an H_∞ deconvolution smoother $\check{s}(k_l\mid k)$ that achieves (4) can be obtained from (15) such that $\mathscr{J}_{l,N}^{m}(\cdot)>0$. To this end, we need to calculate the covariance $Q_{w_z}(k)$ and the projections $\hat{\mathbf{u}}_u(k_l\mid k-1)$ and $\hat{\mathbf{x}}(k)$ associated with the Krein-space state space model (12)-(13). Note (13) is with time delay l to which the standard Kalman filtering formulation is not applicable. We shall apply the approach termed as *re-organized innovation analysis* [11,12] to convert the time delay into a delay-free system.

3.1. Kalman filtering in Krein space

In this subsection, we first re-organize the observations from (13) as delay free observations and then define the associated innovation sequence. Secondly, we derive the Kalman filtering formulation for the system (12)-(13) based on the re-organized innovation. Finally, the projections $\hat{\mathbf{x}}(k)$ and $\hat{\mathbf{u}}_u(k_l\mid k-1)$ and the covariance $Q_{w_z}(k)$ can be calculated.

3.1.1 Re-organized innovation sequence

Note (13), the observation $\mathbf{y}_z(k)$ can be decomposed as

$$\mathbf{y}_z(k)=\begin{cases}\mathbf{y}(k), & 0\le k<l,\\ \begin{bmatrix}\mathbf{y}(k)\\ \check{\mathbf{s}}(k_l\mid k)\end{bmatrix}, & k\ge l,\end{cases} \quad (17)$$

denote

$$\mathscr{Y}_2(i)\triangleq\begin{bmatrix}\mathbf{y}(i)\\ \check{\mathbf{s}}(i\mid i+l)\end{bmatrix}, \quad (18)$$

$$\mathscr{Y}_1(i)\triangleq\mathbf{y}(i). \quad (19)$$

We will give the following lemmas, some proofs which are omitted in this paper and some definitions can be referred to [13].

Lemma 2: $\{\mathbf{w}_2(0),\cdots,\mathbf{w}_2(k_l-1);\mathbf{w}_1(k_l),\cdots,\mathbf{w}_1(k-1)\}$ is the innovation sequence which spans the same linear space as $\mathscr{L}\{\mathscr{Y}_2(0),\cdots,\mathscr{Y}_2(k_l-1);\mathscr{Y}_1(k_l),\cdots,\mathscr{Y}_1(k-1)\}$ or equivalently $\mathscr{L}\{\mathbf{y}_z(0),\cdots\mathbf{y}_z(k-1)\}$.

3.1.2 Riccati equations

Definition 1:

$$\begin{aligned}&\mathscr{P}_2(i)\triangleq<\mathbf{e}_2(i),\mathbf{e}_2(i)>,\quad 0\le i\le k_l,\\ &\mathscr{P}_1(k_l+i)\triangleq<\mathbf{e}_1(k_l+i),\mathbf{e}_1(k_l+i)>,\quad 0\le i\le l,\end{aligned} \quad (20)$$

where $\mathbf{e}_2(i)$ and $\mathbf{e}_1(k_l+i)$ are the state estimation errors.

In view of Definition 1, it is easy to observe that $\mathscr{P}_2(i)$ is the solution to the standard Riccati equation for the system (12) and (18), i.e.,

$$\mathcal{P}_2(i+1) = A\mathcal{P}_2(i)A^T - \mathcal{K}_2(i)Q_2(i)\mathcal{K}_2^T(i) + D_1D_1^T, \quad \mathcal{P}_2(0) = \Pi_0, \tag{21}$$

where

$$\mathcal{K}_2(i) = \begin{bmatrix} A\mathcal{P}_2(i)C^T + D_1D_2^T & D_1L^T \end{bmatrix} Q_2^{-1}(i),$$
$$Q_2(i) \triangleq \langle \mathbf{w}_2(i), \mathbf{w}_2(i) \rangle = \begin{bmatrix} C\mathcal{P}_2(i)C^T + D_2D_2^T + I_p & D_2L^T \\ LD_2^T & -\gamma^2 I_r + LL^T \end{bmatrix}, \quad 0 \le i \le k_l - 1. \tag{22}$$

The matrices of $\mathcal{P}_1(k_l+i)$ $(i = 1, 2, \cdots, l)$ are calculated by the following lemma.

Lemma 3:

$$\mathcal{P}_1(k_l+i+1) = A\mathcal{P}_1(k_l+i)A^T - \mathcal{K}_1(k_l+i)Q_1(k_l+i)\mathcal{K}_1^T(k_l+i) + D_1D_1^T, \quad \mathcal{P}_1(k_l) = \mathcal{P}_2(k_l), \quad i = 0, \cdots, l-1, \tag{23}$$

where

$$\mathcal{K}_1(k_l+i) = \begin{bmatrix} A\mathcal{P}_1(k_l+i)C^T + D_1D_2^T \end{bmatrix} \times Q_1^{-1}(k_l+i), \tag{24}$$

and

$$Q_1(k_l+i) \triangleq \langle \mathbf{w}_1(k_l+i), \mathbf{w}_1(k_l+i) \rangle = C\mathcal{P}_1(k_l+i)C^T + D_2D_2^T + I_p. \tag{25}$$

3.1.3 Calculation of estimate $\hat{\mathbf{x}}(k)$

Note that $\hat{\mathbf{x}}(k)$ is the projection of $\mathbf{x}(k)$ onto $\mathcal{L}\{\mathbf{y}_z(0), \cdots, \mathbf{y}_z(k-1)\}$, or equivalently onto $\mathcal{L}\{\mathcal{Y}_2(0), \cdots, \mathcal{Y}_2(k_l-1); \mathcal{Y}_1(k_l), \cdots, \mathcal{Y}_1(k-1)\}$. Thus, $\hat{\mathbf{x}}(k)$ can be written as $\hat{\mathbf{x}}(k,1)$ which is calculated in the following lemma.

Lemma 4: Consider the systems (12) and (13) in Krein space, $\hat{\mathbf{x}}(k) = \hat{\mathbf{x}}(k,1)$ is computed as

Step 1: Calculate $\hat{\mathbf{x}}(k_l,2)$ recursively for $k = l+1, l+2 \cdots$ as

$$\hat{\mathbf{x}}(k_l,2) = A\hat{\mathbf{x}}(k_l-1,2) + B\mathbf{u}(k_l-1) + \mathcal{K}_2(k_l-1) \times \left(\mathcal{Y}_2(k_l-1) - \begin{bmatrix} C \\ 0_{r\times n} \end{bmatrix} \hat{\mathbf{x}}(k_l-1,2) \right), \quad \hat{\mathbf{x}}(0,2) = 0, \tag{26}$$

where

$$\mathcal{K}_2(k_l-1) = \begin{bmatrix} A\mathcal{P}_2(k_l-1)C^T + D_1D_2^T & D_1L^T \end{bmatrix} Q_2^{-1}(k_l-1),$$
$$Q_2(k_l-1) = \begin{bmatrix} C\mathcal{P}_2(k_l-1)C^T + D_2D_2^T + I_p & D_2L^T \\ LD_2^T & -\gamma^2 I_r + LL^T \end{bmatrix}, \tag{27}$$

and $\mathcal{P}_2(k_l-1)$ is computed by (21), with $\mathcal{P}_2(0) = P_0$.

Step 2: Calculate $\hat{\mathbf{x}}(k_l+i,1)$ $(i = 1, \cdots, l)$ with the initial value of $\hat{\mathbf{x}}(k_l,1) = \hat{\mathbf{x}}(k_l,2)$ as

$$\hat{\mathbf{x}}(k_l+i+1,1) = A\hat{\mathbf{x}}(k_l+i,1) + B\mathbf{u}(k_l+i) + \mathcal{K}_1(k_l+i)\left[\mathcal{Y}_1(k_l+i) - C\hat{\mathbf{x}}(k_l+i,1)\right], \quad i = 0, 1, \cdots, l-1, \tag{28}$$

where

$$\mathcal{K}_1(k_l+i) = \begin{bmatrix} A\mathcal{P}_1(k_l+i)C^T + D_1D_2^T \end{bmatrix} \times Q_1^{-1}(k_l+i),$$
$$Q_1(k_l+i) = C\mathcal{P}_1(k_l+i)C^T + D_2D_2^T + I_p, \tag{29}$$

and $\mathcal{P}_1(k_l+i)$ is calculated by (23).

Remark 2: The calculation procedure for $\hat{\mathbf{x}}(k) = \hat{\mathbf{x}}(k,1)$ is as

- Compute $\hat{\mathbf{x}}(k_l,2)$ with $\hat{\mathbf{x}}(k_l-1,2)$ by (26);
- Compute $\hat{\mathbf{x}}(k_l+i,1)$ for $i = 1, 2, \cdots, l$ with $\hat{\mathbf{x}}(k_l,1) = \hat{\mathbf{x}}(k_l,2)$ by (28).

Then $\hat{\mathbf{x}}(k)$ is obtained by $\hat{\mathbf{x}}(k_l+i,1)$ at $i = l$.

3.1.4 Calculation of $\hat{\mathbf{u}}_u(k_l \mid k-1)$

In this part, we shall calculate $\hat{\mathbf{u}}_u(k_l \mid k-1)$. For the convenience of discussion, we denote

$$R_{k_l+i,1}^{k_l} \triangleq \langle \mathbf{u}_u(k_l), \mathbf{e}_1(k_l+i) \rangle, \quad i = 1, \cdots, l-1. \tag{30}$$

Lemma 5: $\hat{\mathbf{u}}_u(k_l \mid k-1)$ is calculated by

$$\hat{\mathbf{u}}_u(k_l \mid k-1) = D_2^T Q_1^{-1}(k_l)\left[\mathcal{Y}_1(k_l) - C\hat{\mathbf{x}}(k_l,1)\right] + \sum_{i=1}^{l-1} R_{k_l+i,1}^{k_l} C^T Q_1^{-1}(k_l+i) \times \left[\mathcal{Y}_1(k_l+i) - C\hat{\mathbf{x}}(k_l+i,1)\right], \tag{31}$$

where $\hat{\mathbf{x}}(k_l+i,1), i = 1, \cdots, l-1$ can be computed in (28), $\hat{\mathbf{x}}(k_l,1) = \hat{\mathbf{x}}(k_l,2)$ is calculated by (26), and $R_{k_l+i,1}^{k_l}$ is calculated recursively as

$$R_{k_l+i,1}^{k_l} = R_{k_l+i-1,1}^{k_l}\left[A - K_1(k_l+i-1)C\right]^T, \quad i = 2, \cdots, l-1 \tag{32}$$

with

$$R_{k_l+1,1}^{k_l} = \left[D_1 - \mathcal{K}_1(k_l)D_2\right]^T. \tag{33}$$

In addition $\mathcal{K}_1(k_l+i)$ and $Q_1(k_l+i)$ are respectively as in (24) and (25), and $\mathcal{P}_1(k_l+i)$ is the solution to (23).

3.2. Calculation of covariance $Q_{w_z}(k)$

In this part, we shall calculate the covariance matrix $Q_{w_z}(k) = \langle \mathbf{w}_z(k), \mathbf{w}_z(k) \rangle$.

Lemma 6: The covariance matrix $Q_{w_z}(k)$ is computed by

$$Q_{w_z}(k) = \begin{cases} C\mathcal{P}_1(k)C^T + D_2D_2^T + I_p, & 0 \le k < l, \\ \begin{bmatrix} \Delta_1(k) & C\left(R_{k,1}^{k_l}\right)^T L^T \\ LR_{k,1}^{k_l}C^T & L\Delta_2(k)L^T - \gamma^2 I_r \end{bmatrix}, & k \ge l, \end{cases} \tag{34}$$

where

$$\begin{aligned} \Delta_1(k) &= C\mathcal{P}_1(k)C^T + D_2D_2^T + I_p \\ \Delta_2(k) &= I_q - D_2^T\left[C\mathcal{P}_1(k_l)C^T + D_2D_2^T + I_p\right]D_2 \\ &\quad - \sum_{i=1}^{l-1} R_{k_l+i,1}^{k_l} C^T Q_1^{-1}(k_l+i) C\left(R_{k_l+i,1}^{k_l}\right)^T, \end{aligned} \tag{35}$$

while $\mathcal{P}_1(k_l) = \mathcal{P}_2(k_l)$ is the solution to Riccati recursion (21) and $R_{k_l+i,1}^{k_l}$ is as in (32).

Proof: For $0 \le k < l$, the proof is straightforward from (13) and (14).

In case of $k \ge l$, note that $\hat{\mathbf{u}}_u(k \mid k-1) = 0$, it follows from (13) and (14) that

$$\begin{aligned} \mathbf{w}_z(k) &= \mathbf{y}_z(k) - \hat{\mathbf{y}}_z(k) \\ &= \begin{bmatrix} C \\ 0_{r\times n} \end{bmatrix}\left[\mathbf{x}(k) - \hat{\mathbf{x}}(k \mid k-1)\right] \\ &\quad + \begin{bmatrix} D_2 & 0 \\ 0 & L \end{bmatrix}\begin{bmatrix} \mathbf{u}_u(k) \\ \mathbf{u}_u(k_l) - \hat{\mathbf{u}}_u(k_l \mid k-1) \end{bmatrix} + \mathbf{v}_z(k), \end{aligned} \tag{36}$$

where $\hat{\mathbf{x}}(k \mid k-1)$ and $\hat{\mathbf{u}}_u(k_l \mid k-1)$ are respectively the projections of $\mathbf{x}(k)$ and $\mathbf{u}_u(k_l)$ onto $\mathcal{L}\{\mathbf{y}_z(0), \cdots, \mathbf{y}_z(k-1)\}$, or equivalently $\mathcal{L}\{\mathbf{w}_2(0), \cdots, \mathbf{w}_2(k_l-1); \mathbf{w}_1(k_l), \cdots, \mathbf{w}_1(k-1)\}$. Thus $Q_{w_z}(k) = <\mathbf{w}_z(k), \mathbf{w}_z(k)>$ is given from (36) as

$$\begin{aligned} Q_{w_z}(k) &= \begin{bmatrix} C\mathcal{P}_1(k)C^T & 0 \\ 0 & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} D_2 & 0 \\ 0 & L \end{bmatrix}\left\langle \begin{bmatrix} \mathbf{u}_u(k) \\ \lambda(k_l) \end{bmatrix}, \begin{bmatrix} \mathbf{u}_u(k) \\ \lambda(k_l) \end{bmatrix} \right\rangle \begin{bmatrix} D_2 & 0 \\ 0 & L \end{bmatrix}^T \\ &\quad + \begin{bmatrix} C \\ 0 \end{bmatrix}\left\langle \mathbf{e}(k), \begin{bmatrix} \mathbf{u}_u(k) \\ \lambda(k_l) \end{bmatrix} \right\rangle \begin{bmatrix} D_2 & 0 \\ 0 & L \end{bmatrix}^T \\ &\quad + \begin{bmatrix} D_2 & 0 \\ 0 & L \end{bmatrix}\left\langle \begin{bmatrix} \mathbf{u}_u(k) \\ \lambda(k_l) \end{bmatrix}, \mathbf{e}(k) \right\rangle \begin{bmatrix} C \\ 0 \end{bmatrix}^T + \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_r \end{bmatrix}, \end{aligned} \tag{37}$$

where $\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k \mid k-1)$, $\mathcal{P}_1(k) = <\mathbf{e}(k), \mathbf{e}(k)>$ and $\lambda(k_l) = \mathbf{u}_u(k_l) - \hat{\mathbf{u}}_u(k_l \mid k-1)$. From (12) and (13), it is easy to observe that $\mathbf{u}_u(k)$ is uncorrelated with $\lambda(k_l)$ and $\mathbf{e}(k)$. Thus (37) follows that

$$\begin{aligned} Q_{w_z}(k) &= \begin{bmatrix} C\mathcal{P}_1(k)C^T & 0 \\ 0 & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} D_2 & 0 \\ 0 & L \end{bmatrix}\begin{bmatrix} I_q & 0 \\ 0 & <\lambda(k_l), \lambda(k_l)> \end{bmatrix}\begin{bmatrix} D_2 & 0 \\ 0 & L \end{bmatrix}^T \\ &\quad + \begin{bmatrix} C \\ 0 \end{bmatrix}\begin{bmatrix} 0 & <\mathbf{e}(k), \lambda(k_l)> \end{bmatrix}\begin{bmatrix} D_2 & 0 \\ 0 & L \end{bmatrix}^T \\ &\quad + \begin{bmatrix} D_2 & 0 \\ 0 & L \end{bmatrix}\begin{bmatrix} 0 \\ <\lambda(k_l), \mathbf{e}(k)> \end{bmatrix}\begin{bmatrix} C \\ 0 \end{bmatrix}^T + \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_r \end{bmatrix}. \end{aligned} \tag{38}$$

From (31), $\lambda(k_l)$ is given by

$$\begin{aligned} \lambda(k_l) &= \mathbf{u}_u(k_l) - D_2^T Q_1^{-1}(k_l)\mathbf{w}_1(k_l) \\ &\quad - \sum_{i=1}^{l-1} R_{k_l+i,1}^{k_l} C^T Q_1^{-1}(k_l+i)\mathbf{w}_1(k_l+i), \end{aligned} \tag{39}$$

it is easy to know that $\lambda(k_l)$ is uncorrelated with $\mathbf{w}_1(k_l), \mathbf{w}_1(k_l+1), \cdots, \mathbf{w}_1(k-1)$, and thus (39) follows that

$$\begin{aligned} &\left\langle \lambda(k_l), \lambda(k_l) \right\rangle + D_2^T Q_1^{-1}(k_l) D_2 \\ &\quad + \sum_{i=1}^{l-1} R_{k_l+i,1}^{k_l} C^T Q_1^{-1}(k_l+i) C\left[R_{k_l+i,1}^{k_l}\right]^T = I_q. \end{aligned} \tag{40}$$

Similarly we can derive that

$$\left\langle \lambda(k_l), \mathbf{e}(k) \right\rangle = R_{k,1}^{k_l}. \tag{41}$$

Thus $Q_{w_z}(k)$ is computed from (38) as (34).

4. H_∞ DECONVOLUTION FIXED-LAG SMOOTHER

Having calculated $\hat{\mathbf{x}}(k)$, $\hat{\mathbf{u}}_u(k_l \mid k-1)$ and $Q_{w_z}(k)$, we are now in the position to present the H_∞ deconvolution fixed-lag smoother.

Theorem 1: Consider the system model (1)-(3), given a scalar $\gamma > 0$ and an integer $l > 0$. Suppose $\mathcal{P}_2(k)$ is the bounded solution to Riccati equation (21) and $\mathcal{P}_1(k)$ is calculated by (23). Then the H_∞ deconvolution fixed-lag smoother that achieves (4) exists if and only if

$$Q_{v_z}(k) = \begin{cases} I_p, & 0 \le k < l, \\ diag\{I_p, -\gamma^2 I_r\}, & k \ge l \end{cases}$$

and $Q_{w_z}(k)$ have the same inertia for $k = 0, \cdots, N$, where $Q_{w_z}(k)$ is as (34).

If this is the case, one possible level-γ H_∞ white noise fixed-lag smoother is given by

$$\check{s}(k_l \mid k) = L\hat{u}_u(k_l \mid k), \tag{42}$$

where $\hat{u}_u(k_l \mid k)$ is obtained from the projection of $\mathbf{u}_u(k_l)$ onto $\mathcal{L}\{\mathbf{y}_z(0), \cdots, \mathbf{y}_z(k)\}$, which is given by

$$\hat{u}_u(k_l \mid k) = D_2^T Q_1^{-1}(k_l)\left[y(k_l) - C\hat{x}(k_l, 1)\right] \tag{43}$$

$$+\sum_{i=1}^{l} R_{k_l+i,1}^{k_l} C^T Q_1^{-1}(k_l+i) \times\left[y(k_l+i)-C\hat{x}(k_l+i,1)\right],$$

where $\hat{x}(k_l,1)=\hat{x}(k_l,2)$ and $\hat{x}(k_l+i,1)$ for $i=1,\cdots,l$ are calculated by (26) and (28), respectively. In addition, $R_{k_l+i,1}^{k_l}$ is calculated by (32)-(33), $Q_1(k_l+i)$ is as (25).

Proof: From Lemma 1, it is easy to know that the H_∞ deconvolution fixed-lag smoother that achieves (4) exists if and only if $Q_{w_z}(k)$ and $Q_{v_z}(k)$ have the same inertia, where $Q_{v_z}(k)$ is as (9) and $Q_{w_z}(k)$ is as (34). If this is the case, the minimum value of $\mathscr{J}_{l,N}(\cdot)$ is given by (15), where $\hat{x}(k)$ and $\hat{u}_u(k_l\mid k-1)$ in (15) are obtained from $\hat{\mathbf{x}}(k)$ and $\hat{\mathbf{u}}_u(k_l\mid k-1)$ which have been calculated in last section. Note (34), it is obvious that $Q_{w_z}(k)>0$ for $0\le k<l$. For $k\ge l$, using matrix LDU, $Q_{w_z}(k)$ is given by

$$Q_{w_z}(k)=\begin{bmatrix} I & 0\\ R_{21}(k)R_{11}^{-1}(k) & I\end{bmatrix}\begin{bmatrix} R_{11}(k) & 0\\ 0 & \Delta_3(k)\end{bmatrix} \times\begin{bmatrix} I & R_{11}^{-1}(k)R_{21}(k)\\ 0 & I\end{bmatrix}, \tag{44}$$

where

$$\Delta_3(k)=R_{22}(k)-R_{21}(k)R_{11}^{-1}(k)R_{12}(k), \tag{45}$$

while $R_{11}(k)=C\mathscr{P}_1(k)C^T+D_2D_2^T+I_p$, $R_{21}(k)=LR_{k,1}^{k_l}C^T$, $R_{12}(k)=\left(R_{21}(k)\right)$ and $R_{22}(k)=L\Delta_2(k)L^T-\gamma^2 I_r$ and $\Delta_2(k)$ is as in (35). Thus, from (44), $Q_{w_z}(k)$ and $Q_{v_z}(k)$ have the same inertia if and only if $R_{11}(k)>0$ and $\Delta_3(k)<0$. Substituting (44) into (15) yields

$$\begin{aligned}\mathscr{J}_{l,N}^{m}(\cdot)&=\sum_{k=0}^{N}\left[y(k)-C\hat{x}(k)\right]^T R_{11}^{-1}(k)\left[y(k)-C\hat{x}(k)\right]\\&+\sum_{k=l}^{N}\left(\check{s}(k_l\mid k)-L\hat{u}_u(k_l\mid k-1)\right.\\&\left.-R_{21}(k)R_{11}^{-1}(k)\left[y(k)-C\hat{x}(k)\right]\right)^T\\&\times\Delta_3^{-1}(k)\left(\check{s}(k_l\mid k)-L\hat{u}_u(k_l\mid k-1)\right.\\&\left.-R_{21}(k)R_{11}^{-1}(k)\left[y(k)-C\hat{x}(k)\right]\right).\end{aligned} \tag{46}$$

Since $R_{11}(k)>0$ and $\Delta_3(k)<0$, to achieve $\mathscr{J}_{l,N}^{m}(\cdot)>0$, one natural choice is to set

$$\check{s}(k_l\mid k)-L\hat{u}_u(k_l\mid k-1) -R_{21}(k)R_{11}^{-1}(k)\left[y(k)-C\hat{x}(k)\right]=0,$$

Thus the H_∞ deconvolution smoother $\check{s}(k_l\mid k)$ can be given by

$$\begin{aligned}&\check{s}(k_l\mid k)\\&=R_{21}(k)R_{11}^{-1}(k)\left[y(k)-C\hat{x}(k)\right]+L\hat{u}_u(k_l\mid k-1)\\&=LR_{k,1}^{k_l}C^TQ_1^{-1}(k)\left[y(k)-C\hat{x}(k)\right]+L\hat{u}_u(k_l\mid k-1).\end{aligned} \tag{47}$$

Note $\hat{u}_u(k_l\mid k-1)$ in (31), therefore, substituting (31) into (47), it follows that

$$\check{s}(k_l\mid k)=L\hat{u}_u(k_l\mid k), \tag{48}$$

where $\hat{u}_u(k_l\mid k)$ is as (43).

Remark 3: The design of the H_∞ deconvolution fixed-lag smoother in Hilbert space is equivalent to the design of the H_2 deconvolution fixed-lag smoother in Krein space. However, the H_∞ deconvolution fixed-lag smoother needs the existence condition, i.e., the first part of Theorem 1.

Remark 4: Consider the fixed-lag ($l>0$) smoothing of the system model (1)-(3), note (42) and (43), the fixed-lag smoother can be composed of $l+1$ parts, one of which is the H_∞ deconvolution filtering $\check{s}(k_l\mid k_l)=L\hat{u}_u(k_l\mid k_l)$, and the other l parts are the projections of $L\mathbf{u}_u(k_l)$ onto $\mathbf{w}_l(k_l+1),\cdots,\mathbf{w}_1(k)$, respectively.

5. NUMERICAL EXAMPLE

In this section, we will give an example to show the proposed approach.

Consider the system with $A(k)=\begin{bmatrix}0.9 & 0.5\\ 0 & 0.5\end{bmatrix}$, $B(k)=\begin{bmatrix}1\\ 0.5\end{bmatrix}$, $C(k)=\begin{bmatrix}2 & 1\end{bmatrix}$, $D_1(k)=\begin{bmatrix}1\\ 2\end{bmatrix}$, $D_2(k)=4$, $L(k)=1$, and the input sequences are $u(k)\sim\mathscr{N}(0,1)$, the initial value $\check{x}(0\mid l)=\begin{bmatrix}0\\0\end{bmatrix}$, $P(0)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $\Pi_0=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Take $N=100$, $l=20$ and $\gamma^2=1.05$.

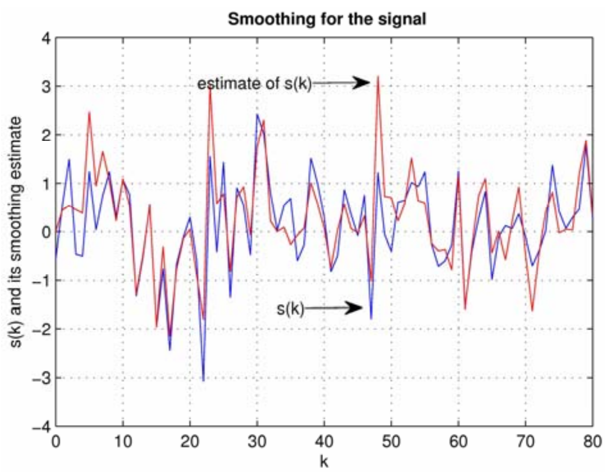

Fig. 1. The smoothing estimate for signal $s(k)$ with $\gamma^2=1.05$.

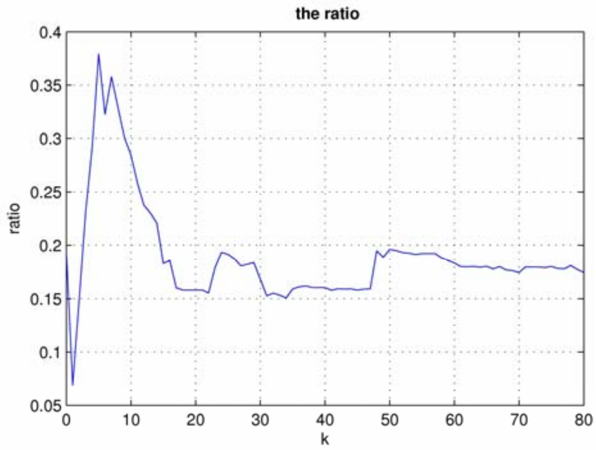

Fig. 2. The ratio between the error covariance and the energy of unknown noises with $\gamma^2 = 1.05$.

In according to formulations (42) and (43) in Theorem 1, we can give the smoothing estimate for $s(k)$ as in Figs. 1 and 2. From the below two figures, it can be easily seen that the proposed approach can give a good smoothing estimation to the unknown input.

6. CONCLUSIONS

The H_∞ deconvolution fixed-lag smoothing has been studied for the linear time-invariant discrete-time system. A new approach to the problem has been proposed by using the innovation analysis in Krein space. It has been clearly shown that the calculation for the H_∞ deconvolution estimate is equivalent to the one for the H_2 estimate in a ceratin Krein-space. The optimal deconvolution fixed-lag smoother is calculated in terms of the solutions to Riccati equations with the same order as the original system [13].

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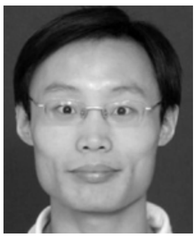

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