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# Spectral Factorization for Multiple Input Delayed Discrete-Time Systems with Applications to Control

**Hong-Guo Zhao, Huan-Shui Zhang, Peng Cui, and Xiao Lu**

**Abstract:** This paper deals with the linear quadratic regulation problems for the linear discrete-time systems with $l$ input delays. The design of the optimal control law is transformed into solving one Diophantine equation and one spectral factorization with delays. A new and simple approach for the spectral factorization is proposed based on reorganized innovation analysis. The calculation of spectral factor comes down to solving $l+1$ Riccati equations with the same dimension as the original systems.

**Keywords:** Backwards stochastic systems, reorganized innovation, spectral factorization.

## 1. INTRODUCTION

The linear quadratic regulation (LQR) problems for time-delay systems have attracted much attention as the time-delays occur naturally in many engineering fields such as chemical processes and communication systems. For continuous-time systems, a number of different techniques have been introduced to cope with the LQR problems [1-3]. Among them, the infinite-dimensional approach in [2,3] leads to the solutions in terms of operator Riccati equations, which are difficult to be calculated. Another solutions to the LQR with multiple input delays were given in [1], where the LQR problems have been considered as a limiting case of $H_\infty$ control for systems with input delays. In the discrete-time context, when there is only a single input delay, the optimal tracking problem has been addressed in [4]. For systems with multiple input delays, earlier approach is to convert a delay problem into a delay-free one by state augmentation [5]. However, the solutions depend on solving higher dimension Riccati equation, which leads to a much expensive computational cost, especially when the delays are large.

In this paper, we shall study the steady-state LQR for the discrete-time systems with $l$ input delays by applying spectral factorization approach [6,7]. The optimal control law will be given in terms of the solutions to one Diophantine equation and one spectral factorization with delays. Different from the state-augmentation approach as in traditional [5], we propose a new and simple approach to deal with the spectral factorization with delays by applying the reorganized innovation approach [8-10]. It is to be shown that the calculation of spectral factorization only requires solving $l+1$ standard Riccati equations with same dimension as the original systems.

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Hong-Guo Zhao is with the Department of Information Science and Technology, Taishan University, Taian 271021, P. R. China (e-mail: h_g_zhao@126. com).

Huan-Shui Zhang and Peng Cui are with the School of Control Science and Engineering, Shandong University, Jinan 250061, P. R. China (e-mails: {hszhang, cuipeng}@sdu.edu.cn).

Xiao Lu is with the Key Laboratory for Robot and Intelligent Technology, Shandong University of Science and Technology, Qingdao 266510, P. R. China (e-mail: luxiao98@163.com).

## 2. PROBLEM FORMULATION

Consider the discrete-time systems with $l$ input delays described by

$$x(t+1) = \Phi^T x(t) + \sum_{i=0}^{l} \Gamma_{(i)}^T u_i(t-h_i), \tag{1}$$

where $x(t) \in R^n$ is the state, $u_i(t) \in R^{p_i}$ are control input. $T$ stands for the transpose. The time-delays $h_i$ satisfy $0 = h_0 < h_1 < \cdots < h_l$.

The quadratic cost function associated with (1) is

$$J = \sum_{i=0}^{l} \sum_{t=h_i}^{\infty} u_i^T(t-h_i) R_i u_i(t-h_i) + \sum_{t=0}^{\infty} x^T(t) Q x(t), \tag{2}$$

where the matrices $R_i$ are positive definite and the matrix $Q$ is non-negative definite and bounded.

The LQR problem is stated as: Find the control input sequence $\{u_i(t),\ i = 0,1,\cdots,l,\ 0 \le t < \infty\}$, which can make the resultant systems asymptotically stable and minimize the cost function (2).

## 3. SOLUTIONS TO STEADY-STATE LQR

Denote

$$u(t) = col\{u_0(t), u_1(t), \cdots, u_l(t)\}, \tag{3}$$

the equations (1)- (2) can be respectively rewritten as

$$x(t+1)=\Phi^T x(t)+\bar{\Gamma}^T(q^{-1})u(t), \tag{4}$$

$$J=\sum_{t=0}^{\infty}u^T(t)Ru(t)+\sum_{t=0}^{\infty}x^T(t)Qx(t), \tag{5}$$

where

$$\begin{aligned}&\bar{\Gamma}^T(q^{-1})=\begin{bmatrix}\Gamma_{(0)}^T & \Gamma_{(1)}^T q^{-h_1} & \cdots & \Gamma_{(l)}^T q^{-h_l}\end{bmatrix},\\ &R=R_0\oplus R_1\oplus\cdots\oplus R_l,\end{aligned} \tag{6}$$

and $q^{-1}$ is the backward shift operator, i.e., $q^{-1}s(t)=s(t-1)$, the symbol $\oplus$ denotes a block diagonal matrix and $col\{\cdot\}$ denotes column vector.

Let us introduce a right coprime matrix-fraction description (MFD) of the transfer matrix for (4)

$$(I_n-\Phi^T q^{-1})^{-1}\bar{\Gamma}^T(q^{-1})q^{-1}=C^T(q^{-1})[A^T(q^{-1})]^{-1}, \tag{7}$$

where $C^T(q^{-1})$ and $A^T(q^{-1})$ are polynomial matrices.

By applying the results in [6], the optimal control law, denoted by $u^*(t)$, associated with (4) and (5) is given as

$$u^*(t)=-\alpha^{-1}(q^{-1})\beta(q^{-1})x(t), \tag{8}$$

where

$$\alpha(q^{-1})A^T(q^{-1})+\beta(q^{-1})C^T(q^{-1})=D^T(q^{-1}), \tag{9}$$

while $D^T(q^{-1})$ is a polynomial matrix and satisfies

$$D(q)D^T(q^{-1})=C(q)QC^T(q^{-1})+A(q)RA^T(q^{-1}), \tag{10}$$

where $D^T(q^{-1})$ exists if and only if [7]

$$rank\begin{bmatrix}RA^T(q^{-1})\\ QC^T(q^{-1})\end{bmatrix}=p_0+\cdots+p_l. \tag{11}$$

Now we have the following results.

**Theorem 1:** Consider the system (1)-(2). The optimal control $u_i^*(t), i=0,1,\cdots,l, 0\le t<\infty$ are computed by

$$u_i^*(t)=\overbrace{[\underbrace{0\ \cdots\ 0\ I_{p_i}}_{i+1\ blocks}\ 0\ \cdots\ 0]}^{l+1\ blocks}u^*(t). \tag{12}$$

**Proof:** By making use of (3) and (8), (12) is derectly obtained.

## 4. SPECTRAL FACTORIZATION

### 4.1. ARMA innovation model

To give the solutions to spectral factor $D(q)$, we introduce discrete-time backwards stochastic models

$$X(t)=\Phi X(t+1)+e(t), \tag{13}$$

$$\begin{aligned}&Y_{(i)}(t)=\Gamma_{(i)}X(t_{h_i}+1)+v_{(i)}(t),\\ &t_{h_i}=t+h_i,\ i=0,1,\cdots l,\end{aligned} \tag{14}$$

where $e(t)\in R^n$ and $v_{(i)}(t)\in R^{p_i}$ are mutually uncorrelated zero means white noises with $E[e(k)e^T(j)]=Q\delta_{kj}$ and $E[v_{(i)}(k)v_{(i)}^T(j)]=R_i\delta_{kj}$ respectively. $\delta_{kj}$ is Kronecker delta function, and $E[\cdot]$ denotes the mathematical expectation.

Denote

$$Y(t)=col\{Y_{(0)}(t),Y_{(1)}(t),\cdots,Y_{(l)}(t)\}. \tag{15}$$

Using (15), we define

$$W(t)\equiv Y(t)-\hat{Y}(t\,|\,t+1), \tag{16}$$

where $\hat{Y}(t\,|\,t+1)$ is the one-step prediction.

It follows from (16) that

$$\begin{aligned}&Y(t)=\bar{\Gamma}\hat{\chi}(t)+W(t),\\ &\hat{\chi}(t)=col\{\hat{X}(t+1\,|\,t+1),\cdots,\hat{X}(t_{h_l}+1\,|\,t+1)\},\end{aligned} \tag{17}$$

where

$$\bar{\Gamma}=\Gamma_{(0)}\oplus\cdots\oplus\Gamma_{(i)}\oplus\cdots\oplus\Gamma_{(l)}, \tag{18}$$

$$\hat{X}(t_{h_i}+1\,|\,t+1)=q^{h_i}\hat{X}(t+1\,|\,t+1)+K_i(q)W(t), \tag{19}$$

$$K_i(q)=\sum_{j=1}^{h_i}K_{ij}q^j, \tag{20}$$

$$K_{ij}=E[X(t_{h_i}+1)W^T(t+j)]Q_W^{-1}, \tag{21}$$

$$Q_W=E[W(t)W^T(t)]. \tag{22}$$

In the above, $q$ is the forward shift operator, i.e., $qs(t)=s(t+1)$.

By substituting (19) into (17), it follows that

$$Y(t)=\bar{\Gamma}(q)\hat{X}(t+1\,|\,t+1)+K(q)W(t)+W(t), \tag{23}$$

where

$$\bar{\Gamma}(q)=\begin{bmatrix}\Gamma_{(0)}\\ \Gamma_{(1)}q^{h_1}\\ \vdots\\ \Gamma_{(l)}q^{h_l}\end{bmatrix},\quad K(q)=\begin{bmatrix}0\\ \Gamma_{(1)}K_1(q)\\ \vdots\\ \Gamma_{(l)}K_l(q)\end{bmatrix}. \tag{24}$$

On the other hand, we have

$$\hat{X}(t\,|\,t)=\Phi\hat{X}(t+1\,|\,t+1)+K_0W(t), \tag{25}$$

where $K_0$ is defined as

$$K_0=E[X(t)W^T(t)]Q_W^{-1}. \tag{26}$$

Substituting (25) into (23) and using (7) yields

$$\begin{aligned}A(q)Y(t)=&\ C(q)K_0W(t)+A(q)K(q)W(t)\\ &+A(q)W(t).\end{aligned} \tag{27}$$

We now present the results for spectral factorization.

**Theorem 2:** The spectral factor $D(q)$ in (10) is given as

$$D(q)=\{C(q)K_0+A(q)K(q)+A(q)\}Q_W^{1/2}. \tag{28}$$

**Proof:** By making use of (15), it follows from (14) that

$$\begin{aligned}&Y(t)=\bar{\Gamma}\chi(t)+v(t),\\&\chi(t)=col\{X(t+1),\cdots,X(t_{h_i}+1),\cdots,X(t_{h_l}+1)\},\end{aligned} \tag{29}$$

where

$$v(t)=col\{v_{(0)}(t),v_{(1)}(t),\cdots,v_{(l)}(t)\}, \tag{30}$$

is a white noise with zero mean and covariance matrix $R$ as in (6).

Noting that (29) can be further rewritten as

$$Y(t)=\bar{\Gamma}(q)X(t+1)+v(t), \tag{31}$$

where $\bar{\Gamma}(q)$ is as in (24).

From (13), we have

$$X(t+1)=(I_n-\Phi q)^{-1}qe(t). \tag{32}$$

Substituting (32) into (31) and using (7) yields

$$A(q)Y(t)=C(q)e(t)+A(q)v(t). \tag{33}$$

By comparing (33) with (27), (28) follows derectly.

4.2. Computation of $K_0, K_{ij}, j=1,\cdots,h_i, i=1,\cdots,l,$ and $Q_W$

Denote

$$\bar{Y}(t)=\begin{cases}col\{Y_{(0)}(t),Y_{(1)}(t),\cdots,Y_{(i-1)}(t)\},\\ \qquad\qquad N-h_i<t\le N-h_{i-1},\\ col\{Y_{(0)}(t),Y_{(1)}(t),\cdots,Y_{(l)}(t)\},\\ \qquad\qquad t\le N-h_l.\end{cases} \tag{34}$$

In view of (34), the linear space $L\{\bar{Y}(N),\bar{Y}(N-1),\cdots,\bar{Y}(t)\}$ spanned by observation $\{\bar{Y}(N),\bar{Y}(N-1),\cdots,\bar{Y}(t)\}$ is equivalent to

$$L\{Y_{l+1}(N),\cdots,Y_{l+1}(t_{h_l});\cdots;Y_1(t_{h_1}-1),\cdots,Y_1(t)\},$$

where

$$Y_s(t)=\Gamma_s X(t+1)+V_s(t),\ s=l+1,l,\cdots,1, \tag{35}$$

and

$$\begin{aligned}&Y_s(t)=col\{Y_{(0)}(t),Y_{(1)}(t-h_1),\cdots,Y_{(s-1)}(t-h_{s-1})\},\\&V_s(t)=col\{v_{(0)}(t),v_{(1)}(t-h_1),\cdots,v_{(s-1)}(t-h_{s-1})\},\\&\Gamma_s^T=\begin{bmatrix}\Gamma_{(0)}^T & \Gamma_{(1)}^T & \cdots & \Gamma_{(s-1)}^T\end{bmatrix}.\end{aligned}$$

It is obvious that $V_s(t)$ is a white noise with zero mean and covariance matrix $Q_{V_s}=R_0\oplus\cdots\oplus R_{s-1}$.

Based on the new observation $Y_s(t)$, we define

$$W_s(t)\equiv Y_s(t)-\hat{Y}_s(t), \tag{36}$$

where $\hat{Y}_s(t)$ is projection of $Y_s(t)$ onto linear space $L\{Y_{l+1}(N),\cdots,Y_{l+1}(t_{h_l});\cdots;Y_s(t_{h_s}-1),\cdots,Y_s(t+1)\}$.

Then it follows from (35) and (36) that

$$W_s(t)=\Gamma_s\tilde{X}(t+1\,|\,t+1,s)+V_s(t), \tag{37}$$

where

$$\tilde{X}(t+1\,|\,t+1,s)=X(t+1)-\hat{X}(t+1\,|\,t+1,s), \tag{38}$$

while $\hat{X}(t+1\,|\,t+1,s)$ is the projection of $X(t+1)$ onto the linear space $L\{W_{l+1}(N),\cdots,W_{l+1}(t_{h_l});\cdots;W_s(t_{h_s}-1),\cdots,W_s(t+1)\}$.

**Lemma 1:** The uncorrelated white noise sequence $\{W_{l+1}(N),\cdots,W_{l+1}(t_{h_l});\cdots;W_s(t_{h_s}-1),\cdots,W_s(t_{h_{s-1}});\cdots;W_1(t_{h_1}-1),\cdots,W_1(t)\}$ is called reorganized innovation sequence, which contains the same observation information as $\{\bar{Y}(N),\bar{Y}(N-1),\cdots,\bar{Y}(t)\}$.

**Proof:** The proof is similar to that of Lemma 2.1 in [9].

Next, we are to compute the reorganized innovation covariance matrices.

From (37), for $s=l+1$, the reorganized innovation covariance matrix is given by

$$Q_{W_{l+1}}\equiv E[W_{l+1}(t)W_{l+1}^T(t)]=\Gamma_{l+1}P\Gamma_{l+1}^T+Q_{V_l}, \tag{39}$$

where $P$ satisfies the following algebraic Riccati equation

$$P=\Phi P\Phi^T+Q-\Phi P\Gamma_{l+1}^T Q_{W_{l+1}}^{-1}\Gamma_{l+1}P\Phi^T. \tag{40}$$

Similarly, for $i=l,l-1,\cdots,1,$ the reorganized innovation covariance matrices are given by

$$\begin{aligned}&Q_{W_i}(k)\equiv E[W_i(t-k)W_i^T(t-k)]=\Gamma_i P_i(k)\Gamma_i^T+Q_{V_{i-1}},\\&k=h_l-h_i+1,\,h_l-h_i+2,\cdots,\,h_l-h_{i-1},\end{aligned} \tag{41}$$

where $P_i(k)$ satisfies the following Riccati equation

$$\begin{aligned}P_i(k)=&\,\Phi P_i(k-1)\Phi^T+Q-\Phi P_i(k-1)\Gamma_i^T\\&\times Q_{W_i}^{-1}(k-1)\Gamma_i P_i(k-1)\Phi^T\end{aligned} \tag{42}$$

with $P_i(h_l-h_i)=P_{i+1}(h_l-h_i)$ and $P_l(0)=P.$

Further, for the sake of convenience to discuss, we define

$$\begin{aligned}&U_{l+1}(d,0)\equiv E[X(t+1-d)\tilde{X}^T(t+1\,|\,t+1,l+1)],\\&U_i(d,k)\equiv E[X(t+1-d)\tilde{X}^T(t+1-k\,|\,t+1-k,i)],\end{aligned}$$

where $\tilde{X}(t+1\,|\,t+1,l+1)$ and $\tilde{X}(t+1-k\,|\,t+1-k,i)$ are as in (38).

The $U_{l+1}(d,0)$ and $U_i(d,k)$ can be calculated in the lemma below

**Lemma 2:**

$$U_{l+1}(d,0)=\begin{cases}P[\mathrm{A}^T]^{-d}, & d\le 0,\\ \Phi^d P, & d>0,\end{cases} \tag{43}$$

$$U_i(d,k)=\begin{cases}P_i(d)\mathrm{A}_i^T(d)\cdots \mathrm{A}_i^T(k-1), & k\ge d,\\ \Phi^{d-k}P_i(k), & k<d,\end{cases} \tag{44}$$

where

$$\mathrm{A}=\Phi-\Phi P\Gamma_{l+1}^T Q_{W_{l+1}}^{-1}\Gamma_{l+1},$$
$$\mathrm{A}_i(k)=\Phi-\Phi P_i(k)\Gamma_i^T Q_{W_i}^{-1}\Gamma_i.$$

**Proof:** The proof is similar to Lemma 3 in [11].

Next, the innovation covariance matrix $Q_W$ is calculated by using reorganized innovation analysis approach in the following theorem.

**Theorem 3:** The innovation covariance matrix $Q_W$ is given as

$$Q_W=\begin{bmatrix}\Gamma_{(0)}Q_{\tilde X}\Gamma_{(0)}^T+R_0 & \cdots & Q_{\varsigma_l\tilde X}^T\\ \cdots & \cdots & \cdots\\ Q_{\varsigma_l\tilde X} & \cdots & \Gamma_{(l)}Q_{\varsigma_l}\Gamma_{(l)}^T+R_l\end{bmatrix}, \tag{45}$$

where

$$Q_{\tilde X}=U_1(h_l,h_l),$$
$$Q_{\varsigma_i\tilde X}=U_1(h_l-h_i,h_l),$$
$$Q_{\varsigma_{i'}\varsigma_i}=U_{i+1}(h_l-h_{i'},h_l-h_i),\ i'\ne i,\ i',i=l,l-1,\cdots,1$$

and

$$\begin{aligned}Q_{\varsigma_i}=&U_{i+1}(h_l-h_i,h_l-h_i)(I_n-\Gamma_{i+1}^T Q_{W_{i+1}}^{-1}(h_l-h_i)\\ &\times\Gamma_{i+1}U_{i+1}^T(h_l-h_i,h_l-h_i))\\ &-\sum_{m=2}^{i}\sum_{n=1}^{h_m-h_{m-1}}U_m(h_l-h_i,h_l-h_m+n)\Gamma_m^T\\ &\times Q_{W_m}^{-1}(h_l-h_m+n)\Gamma_m U_m^T(h_l-h_i,h_l-h_m+n)\\ &-\sum_{n=1}^{h_1-1}U_1(h_l-h_i,h_l-h_1+n)\Gamma_1^T\\ &\times Q_{W_1}^{-1}(h_l-h_1+n)\Gamma_1U_1^T(h_l-h_i,h_l-h_1+n)\end{aligned}$$

wth $U_s(\cdot,\cdot)$ and $Q_{W_s}(\cdot)$, $s=l+1,\cdots,1$, calculated by (43), (44) and (39), (41), respectively.

**Proof:** Note that (16), the innovation $W(t)$ allows us to be rewritten as

$$\begin{aligned}&W(t)=\bar\Gamma\tilde\chi(t)+v(t),\\ &\tilde\chi(t)=col\{\tilde X(t+1\,|\,t+1,1),\cdots\varsigma_i(t),\cdots,\varsigma_l(t)\},\end{aligned} \tag{46}$$

where

$$\tilde X(t+1\,|\,t+1,1)=X(t+1)-\hat X(t+1\,|\,t+1,1),$$

and

$$\begin{aligned}\varsigma_i(t)=&X(t_{h_i}+1)-\hat X(t_{h_i}+1\,|\,t+1)\\ =&\tilde X(t_{h_i}+1\,|\,t_{h_i}+1,i+1)-U_{i+1}(h_l-h_i,h_l-h_i)\\ &\times\Gamma_{i+1}^T Q_{W_{i+1}}^{-1}(h_l-h_i)W_{i+1}(t_{h_i})\\ &-\sum_{m=2}^{i}\sum_{n=1}^{h_m-h_{m-1}}U_m(h_l-h_i,h_l-h_m+n)\Gamma_m^T\\ &\times Q_{W_m}^{-1}(h_l-h_m+n)W_m(t_{h_m}-n)\\ &-\sum_{n=1}^{h_1-1}U_1(h_l-h_i,h_l-h_1+n)\Gamma_1^T\\ &\times Q_{W_1}^{-1}(h_l-h_1+n)W_1(t_{h_1}-n).\end{aligned}$$

Substituting (46) into (22) and using Lemma 2, we prove (45).

What comes on next is to calculate $K_0$ and $K_{ij}$ in the following theorem, respectively.

**Theorem 4:** The $K_0$ is calculated as

$$K_0=[\Pi_X\ \ \Pi_{X\varsigma_1}\ \cdots\ \Pi_{X\varsigma_i}\ \cdots\ \Pi_{X\varsigma_l}]\bar\Gamma^T Q_W^{-1}, \tag{47}$$

where

$$\Pi_X=U_1(h_l+1,h_l),$$

and

$$\begin{aligned}\Pi_{X\varsigma_i}=&U_{i+1}(h_l+1,h_l-h_i)-U_{i+1}(h_l+1,h_l-h_i)\Gamma_{i+1}^T\\ &\times Q_{W_{i+1}}^{-1}(h_l-h_i)\Gamma_{i+1}U_{i+1}^T(h_l-h_i,h_l-h_i))\\ &-\sum_{m=2}^{i}\sum_{n=1}^{h_m-h_{m-1}}U_m(h_l+1,h_l-h_m+n)\Gamma_m^T\\ &\times Q_{W_m}^{-1}(h_l-h_m+n)\Gamma_m U_m^T(h_l-h_i,h_l-h_m+n)\\ &-\sum_{n=1}^{h_1-1}U_1(h_l+1,h_l-h_1+n)\Gamma_1^T\\ &\times Q_{W_1}^{-1}(h_l-h_1+n)\Gamma_1U_1^T(h_l-h_i,h_l-h_1+n).\end{aligned}$$

The $K_{ij}$, $j=1,\cdots,h_i$, $i=1,\cdots,l$ are given as

$$K_{ij}=[\Omega_X\ \ \Omega_{X\varsigma_1}\ \cdots\ \Omega_{X\varsigma_i}\ \cdots\ \Omega_{X\varsigma_l}]\bar\Gamma^T Q_W^{-1}, \tag{48}$$

where

$$\Omega_X=U_1(h_l-h_i,h_l-j),$$

and

$$\begin{aligned}\Omega_{X\varsigma_i}=&U_{i+1}(h_l-h_i,h_l-h_i-j)\\ &-\sum_{n=0}^{j}U_{i+1}(h_l-h_i,h_l-h_i-n)\Gamma_{i+1}^T\\ &\times Q_{W_{i+1}}^{-1}(h_l-h_i-n)\Gamma_{i+1}U_{i+1}^T(h_l-h_i-j,h_l-h_i-n)\\ &-\sum_{m=2}^{i}\sum_{n=1}^{h_m-h_{m-1}}U_m(h_l-h_i,h_l-h_i+n)\Gamma_m^T\\ &\times Q_{W_m}^{-1}(h_l-h_m+n)\Gamma_m U_m^T(h_l-h_i-j,h_l-h_m+n)\\ &-\sum_{n=1}^{h_1-1}U_1(h_l-h_i,h_l-h_1+n)\Gamma_1^T\\ &\times Q_{W_1}^{-1}(h_l-h_1+n)\Gamma_1U_1^T(h_l-h_i-j,h_l-h_1+n).\end{aligned}$$

Table 1. Computational cost for various delays $h_1$, $h_2$.

| $h_1$ | 1 | 1 | 2 | 4 |
|---|---|---|---|---|
| $h_2$ | 2 | 3 | 4 | 5 |
| $MD_{aug}$ | 1848 | 2552 | 4824 | 11112 |
| $MD_{new}$ | 820 | 932 | 1044 | 1156 |
| $MD_{aug} - MD_{new}$ | 1028 | 1620 | 3780 | 9956 |

In the above, $U_{l+1}(\cdot,\cdot)$ and $Q_{W_{l+1}}(\cdot)$ are obtained by (43) and (39), respectively. $U_i(\cdot,\cdot)$ are given by (44), and $Q_{W_i}(\cdot)$ are computed by (41).

**Proof:** The proof is similar to the Theorem 3.

## 5. COMPARISON OF COMPUTATIONAL COST

This section is devoted to compare the computational cost for the spectral factorization via the presented approach and the state augmented method.

Consider the systems (13)-(14) with $n = p_0 = p_1 = p_2 = 2$ and $l = 2$. Let $MD_{aug}$ and $MD_{new}$ denote the number of multiplications and divisions for the augmented method and the new approach in one step, respectively. Then, the simulation results are shown in Table 1.

As is obvious from Table 1, the larger the time-delays $h_1$, $h_2$, the larger the difference of the computational cost between the state augmentation and the presented approach, which implies the presented approach in this paper is simpler.

## 6. CONCLUSIONS

In this paper, the optimal controllers are designed via one Diophantine equation and one spectral factorization with delays, where the key technique for deriving spectral factorization is the time-domain reorganized innovation approach. In contrast to the augmented method, the presented approach is simpler for derivation and calculation, especially when delays are larger.

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**Hong-Guo Zhao** received his Ph.D. degree in Control Theory and Control Engineering from Shandong University, P.R. China, in 2008. He joined Taishan University in 2008 and his research interests cover Kalman filtering, optimal control and time delay systems.

**Huan-Shui Zhang** received his Ph.D. degree in Northeast University, P.R. China, in 1997. His research interests include robust filtering, optimal control, and signal processing.

**Peng Cui** received his Ph.D. degree in Control Theory and Control Engineering from Shandong University, P.R. China, in 2008. He joined Shandong University in 2008 and his current research interests include state estimation, optimal control and networked control systems.

**Xiao Lu** received his Ph.D. degree from Dalian University of Technology, P.R. China, in 2008. Currently, he joined Shandong University of Science and Technology, and his research interests include robust filtering and control, and time delay systems.