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Robust H_∞ Fuzzy Control for Discrete-time Nonlinear Systems

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Abstract: This paper studies the problem of robust H_∞ control for discrete-time nonlinear systems presented as Takagi--Sugeno's fuzzy models. The generalized non-parallel distributed compensation (non-PDC) law and non-quadratic Lyapunov function is constructed by the proposed homogeneouspo-lynomially basis-dependent matrix function (HPB-MF for abbreviation). Based on the generalized non-PDC law and non-quadratic Lyapunov function, some linear matrix inequalities (LMIs) are obtained by exploiting the possible combinations of the basis functions. These LMIs ensure the asymptotic stability of the closed-loop system and guarantee a norm bound constraint on disturbance attenuation. In addition, it is shown that the LMIs become less conservative as the degree of HPB-MF increases. The merit of the methods presented in this paper lies in their less conservatism than other methods, as shown by a numerical example borrowed from the literature.

Keywords: Homogeneous polynomially basis-dependent matrix function, robust control, linear matrix inequality, non-quadratic Lyapunov function, Takagi--Sugeno's fuzzy model.

1. INTRODUCTION

Over the past few years, there have significant research efforts devoted to the analysis and control design of Takagi--Sugeno's(T-S) fuzzy systems (see [1-3] and the references therein). The main motivation for its development was its applications on the stability and performance of many practical nonlinear systems [4-7]. Undoubtedly, the Lyapunov theory is one of the approaches to deal with this issue and the quadratic Lyapunov function is the main technique for testing the stability [8-11]. To overcome the conservatism arisen from the use of a single Lyapunov matrix in quadratic stability methods, more effective Lyapunov methods have been presented. See, for example, piecewise quadratic Lyapunov functions [12,13], the weighting-dependent Lyapunov functions [14,15], etc. Many important issues have been studied for T-S fuzzy control systems, such as, H_∞ performance [16,17], robustness [18,19], reliability [20,21], time-delay [22,23] and adaptive control [24,25]. It is noted that all of the aforementioned research efforts have been focused on PDC law, on the other hand, using the non-PDC law design methods along with non-quadratic Lyapunov functions, some conditions were proposed in [26]. More recently, based on an extended non-quadratic Lyapunov function with more variables, some conditions were provided in [27-30] which were less conservative than those in [26].

Generally speaking, in order to reduce the conservatism, the main technique is to introduce more slack variables. For example in [8,27,28], Q_{ij}, Q_{ij}^{kl} and Θ_{kl} are introduced to obtain less conservative results. But for robust H_∞ control problem, since the computa-tional effort has to be considered, it is impractical to introduce so large number of variables Q_{ij}^{kl}, Θ_{kl} etc. except for [16] in which the slack variables Q_{ijl} are introduced partially. In order to obtain less conservative results and consider the computational burden, in this paper, we give another method to introduce variables.

This paper focuses on the robust H_∞ controller design of discrete-time T-S fuzzy models. First, the HPB-MF is proposed. Then, the generalized non-PDC law and non-quadratic Lyapunov function are obtained by the application of the HPB-MF. Some new conditions to stabilize the fuzzy system are obtained by using the generalized non-PDC law and non-quadratic Lyapunov function. These conditions are expressed as LMIs by exploiting the possible combinations of fuzzy basis functions. It is shown that these conditions become less conservative as the degree of HPB-MF increases since more free variables are generated leading to less conservative results. Although the number of the LMIs also increases, each LMI is easy to be fulfilled. Moreover, if the conditions are fulfilled for a certain degree, then a feasible solution exists for all larger ones.

The paper is organized as follows: The problems to be treated are formally stated in Section 2 and the HPB-MF is also introduced in this section. Section 3 is devoted to obtain theoretical results. Some comparisons to show the effectiveness of our methods are available in Section 4.

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Section 5 concludes the paper.

In this paper, for two symmetric matrices A and B, $A > B$ means that $A - B$ is positive definite. A^T denotes the transpose of A. A star $(*)$ in a symmetric matrix denotes the transposed element in the symmetric position. The symbol I_n stands for the identity matrix in $R^{n\times n}$. $l_2[0,\infty)$ refers to the space of square summable infinite vector sequences. $\|\bullet\|_2$ stands for the l_2 norm.

2. PROBLEM STATEMENT AND PRELIMINARIES

The discrete-time T--S fuzzy system under investigation is described as follows:

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{r} h_i(\xi(k))(A_i + \Delta A_i(k))x(k) \\ &+ \sum_{i=1}^{r} h_i(\xi(k))(B_i + \Delta B_i(k))u(k) \\ &+ \sum_{i=1}^{r} h_i(\xi(k))E_i\omega(k), \end{aligned} \quad (1)$$

$$z(k) = \sum_{i=1}^{r} h_i(\xi(k))(C_i x(k) + D_i u(k) + M_i\omega(k), \quad (2)$$

where $x(k)$ is the state, $u(k)\in R^m$ is the control input, $z(k)\in R^p$ is the controlled output, $\omega(k)\in R^s$ is an exoge-nous disturbance input. Matrices $A_i \in R^{n\times n}$ $B_i \in R^{n\times m}$, $E_i \in R^{n\times s}$, $C_i \in R^{p\times n}$, $D_i \in R^{p\times m}$, $M_i \in R^{p\times s}$ are system matrices. $i = 1,2,\cdots,r$ and r is the number of IF--THEN rules. $\Delta A_i(k)$ and $\Delta B_i(k)$ represent the time-varying uncertainties which have the following structure:

$$[\Delta A_i(k)\ \ \Delta B_i(k)] = HF(k)[E_{1i}\ E_{2i}],\ i = 1,2,\cdots r, \quad (3)$$

where $F(k)\in R^{\alpha\times\beta}$ is an unknown matrix function with Lebesgue measurable elements and satisfying

$$F^T(k)F(k) \le I_\beta. \quad (4)$$

$H \in R^{n\times\alpha}$, $E_{1i} \in R^{\beta\times n}$ and $E_{1i} \in R^{\beta\times m}$ are known constant matrices with appropriate dimensions that specify how the uncertain parameters in $F(k)$ enter the nominal matrices A_i and B_i. $h_i(\xi(k))$ is the basis functions which satisfies

$$h_i(\xi(k)) \ge 0,\ \sum_{i=1}^{r} h_i(\xi(k)) = 1. \quad (5)$$

The aim of this paper is to construct the control law $u(k)$ such that the closed-loop system (3), (4) has the property that for any nonzero $\omega(k)\in l_2[0,\infty)$ and all admissible uncertainties, $\|z(k)\|_2 < \gamma\|\omega(k)\|$ under zero-initial condition.

For simplicity, the basis functions $h_i(\xi(k))$ in (1) are represented as h_i and $h_i(\xi(k+1)) = h_{+i}$.

Let

$$\begin{aligned} H^{\{q,r\}} &= \{h_{i_1}h_{i_2}\cdots h_{i_q} \,|\, i_1 \le i_2\cdots i_q, \\ &\quad i_1, i_2\cdots i_q \in \{1,2,\cdots,r\}\}, \end{aligned} \quad (6)$$

$$\begin{aligned} H_+^{\{q,r\}} &= \{h_{+i_1}h_{+i_2}\cdots h_{+i_q} \,|\, i_1 \le i_2\cdots i_q, \\ &\quad i_1, i_2\cdots i_q \in \{1,2,\cdots,r\}\} \end{aligned} \quad (7)$$

and the corresponding subscript is

$$\wp^{\{q,r\}} = \left\{ i_1 i_2\cdots i_q \,|\, i_1 \le i_2\cdots i_q, i_1, i_2,\cdots i_q \in \{1,2,\cdots,r\}\right\}$$

There are $\dfrac{(q+r-1)!}{q!(r-1)!}$ elements in $H\{q,r\}$. For example, as $r=3, q=3$

$$H^{\{3,3\}} =$$
$$\{h_1^3, h_1^2h_2, h_1^2h_3, h_1h_2^2, h_1h_2h_3, h_1h_3^2, h_2^3, h_2^2h_3, h_2h_3^2, h_3^3\},$$
$$\wp^{\{3,3\}} = \{111,112,113,122,123,133,222,223,233,333\}.$$

Definition 1: The HPB-MF is defined as follows

$$f\left(H^{\{q,r\}}, S_{i_1i_2\cdots i_q}\right) = \sum_{h_{i_1}h_{i_2}\cdots h_{i_q}\in H^{\{q,r\}}} h_{i_1}h_{i_2}\cdots h_{i_q} S_{i_1i_2\cdots i_q}, \quad (8)$$

$$f\left(H_+^{\{q,r\}}, S_{i_1i_2\cdots i_q}\right) = \sum_{h_{+i_1}h_{+i_2}\cdots h_{+i_q}\in H^{\{q,r\}}} h_{+i_1}h_{+i_2}\cdots h_{+i_q} S_{i_1i_2\cdots i_q}, \quad (9)$$

where $S_{i_1i_2\cdots i_q}$ are matrix variables and q is the degree of HPB-MF.

For example as $r = 3,\ q = 3$ one has

$$\begin{aligned} f\left(H^{\{3,3\}}, S_{i_1i_2i_3}\right) &= h_1^3S_{111} + h_1^2h_2S_{112} + h_1^2h_3S_{113} \\ &+ h_1h_2^2S_{122} + h_1h_2h_3S_{123} + h_2^3S_{222} \\ &+ h_2^2h_3S_{223} + h_3^3S_{333} \end{aligned}$$

and for the special cases $q = 1$ and $q = 0$, one gets

$$f\left(H^{\{1,r\}}, S_{i_1}\right) = S(H) = h_1S_1 + h_2S_2 + \cdots + h_rS_r. \quad (10)$$

Definition 2: For each $h_{i_1}h_{i_2}\cdots h_{i_q} \in H^{\{q,r\}}$

$$h_{i_1}h_{i_2}\cdots h_{i_q} = h_1^{d_1}h_2^{d_2}\cdots h_r^{d_r}, d_1 + d_2 + \cdots d_r = q. \quad (11)$$

The function $g(i_1i_2\cdots i_q)$ dependent on the value of $i_1,\ i_2,\cdots i_q$ is defined as follows

$$g\left(i_1i_2\cdots i_q\right) = \frac{q!}{d_1!d_2!\cdots d_r}. \quad (12)$$

For example, as $r=3,\ q=3,\ i_1=i_2=1,\ i_3=2,$ one has $h_1h_1h_2=h_1^2h_2,\ d_1=2,\ d_2=1,\ d_3=0,$ then

$$g(112)=\frac{3!}{2!1!0!}=3.$$

Definition 3: The $f\left(H^{\{q,r\}},S_{i_1i_2\cdots i_q}\right)$ is said to be positive definite (semi-definite) in $H^{\{q,r\}}$ if there exist some matrices $S_{i_1i_2\cdots i_q}$ such that $f\left(H^{\{q,r\}},S_{i_1i_2\cdots i_q}\right)>0$ ($f\left(H^{\{q,r\}},S_{i_1i_2\cdots i_q}\right)\geq 0$) for all $h_{i_1}h_{i_2}\cdots h_{i_q}\in H^{\{q,r\}}$.

The following well known lemma is useful in this paper.

Lemma 1 [31]**:** Let, $A,\ H,\ E$ and F be real matrices of appropriate dimensions with $F^TF\leq I.$ For any matrix $P>0$ and scalar $\varepsilon>0$ such that $P-\varepsilon HH^T>0,$ then we have

$$\left(A+HEF\right)^T P^{-1}\left(A+HEF\right)\leq A^T(P-\varepsilon HH^T)A+\varepsilon^{-1}E^TE.$$

3. MAIN RESULTS

Theorem 1: Consider the following control law

$$u(k)=f\left(H^{\{q-1,r\}},Y_{i_1i_2\cdots i_{q-1}}\right)f^{-1}\left(H^{\{q-1,r\}},G_{i_1i_2\cdots i_{q-1}}\right)x(k). \tag{13}$$

The closed-loop system (1), (2) is asymptotically stable with γ disturbance attenuation, if there exist matrices $Y_{i_1i_2\cdots i_{q-1}}\in R^{m\times n},\ G_{i_1i_2\cdots i_{q-1}}\in R^{n\times n},\ \varepsilon>0$ and positive matrices $P_{i_1i_2\cdots i_q}\in R^{n\times n},\ i_1i_2\cdots i_q\in\wp^{\{q,r\}}$ satisfying $\Psi^1_{j_1j_2\cdots j_q}<0,\Psi^2_{j_1j_2\cdots j_q}<0.$

$$\Psi^1_{j_1j_2\cdots j_q}=\begin{bmatrix} -G_{i_1i_2\cdots i_{q-1}}-G^T_{i_1i_2\cdots i_{q-1}}+P_{i_1i_2\cdots i_q} & * & * & * & * & * & * \\ 0 & J_1 & * & * & * & * & * \\ A_{i_q}G_{i_1i_2\cdots i_{q-1}}+B_{i_q}Y_{i_1i_2\cdots i_{q-1}} & \mathrm{E} & J_2 & * & * & * & * \\ 0 & 0 & \varepsilon\tilde{H}^T & J_3 & * & * & * \\ E_{1i_q}G_{i_1i_2\cdots i_{q-1}}+E_{2i_q}Y_{i_1i_2\cdots i_{q-1}} & 0 & 0 & 0 & J_4 & * & * \\ 0 & 0 & 0 & 0 & 0 & J_5 & * \\ C_{i_q}G_{i_1i_2\cdots i_{q-1}}+D_{i_q}Y_{i_1i_2\cdots i_{q-1}} & \mathrm{M} & 0 & 0 & 0 & 0 & J_6 \end{bmatrix},$$

$i_1i_2\cdots i_q\in\wp^{\{q,r\}},\ i_1=i_2=\cdots=i_q,\ j_1j_2\cdots j_q\in\wp^{\{q,r\}},$

$$J_1=-g\left(i_1i_2\cdots i_q\right)\gamma^2,\ J_2=-\frac{g\left(i_1i_2\cdots i_q\right)}{g\left(j_1j_2\cdots j_q\right)}P_{j_1j_2\cdots j_q},$$

$$J_3=-g\left(i_1i_2\cdots i_q\right)\varepsilon I_{\alpha+s},\quad J_4=-g\left(i_1i_2\cdots i_q\right)\varepsilon I_{\beta},$$

$$J_5=-g\left(i_1i_2\cdots i_q\right)\varepsilon I_s,\quad J_6=-g\left(i_1i_2\cdots i_q\right)\varepsilon I_p,$$

$$\tilde{H}=[H\ \ 0],\quad \mathrm{E}=\frac{(q-1)!}{d_1!d_2!\cdots d_r}\left(E_{i1}+E_{i2}+\cdots E_{iq}\right),$$

$$\mathrm{M}=\frac{(q-1)!}{d_1!d_2!\cdots d_r}\left(M_{i1}+M_{i2}+\cdots M_{iq}\right),$$

$$\Psi^2_{j_1j_2\cdots j_q}=\begin{bmatrix} \left\{\begin{array}{l}-G_{i_1i_2\cdots i_{q-1}}-G^T_{i_1i_2\cdots i_{q-1}}\\ -G_{i_2i_3\cdots i_q}-G^T_{i_2i_3\cdots i_q}+P_{i_1i_2\cdots i_q}\end{array}\right\} & * & * & * & * & * & * \\ 0 & J_1 & * & * & * & * & * \\ \left\{\begin{array}{l}A_{i_q}G_{i_1i_2\cdots i_{q-1}}+B_{i_q}Y_{i_1i_2\cdots i_{q-1}}\\ +A_{i_1}G_{i_2i_3\cdots i_q}+B_{i_1}Y_{i_2i_3\cdots i_q}\end{array}\right\} & \mathrm{E} & J_2 & * & * & * & * \\ 0 & 0 & g\left(i_1i_2\cdots i_q\right)\varepsilon\tilde{H}^T & J_3 & * & * & * \\ \left\{\begin{array}{l}E_{1i_q}G_{i_1i_2\cdots i_{q-1}}+E_{2i_q}Y_{i_1i_2\cdots i_{q-1}}\\ +E_{1i_1}G_{i_2i_3\cdots i_q}+E_{2i_1}Y_{i_2i_3\cdots i_q}\end{array}\right\} & 0 & 0 & 0 & J_4 & * & * \\ 0 & 0 & 0 & 0 & 0 & J_5 & * \\ \left\{\begin{array}{l}C_{i_q}G_{i_1i_2\cdots i_{q-1}}+D_{i_q}Y_{i_1i_2\cdots i_{q-1}}\\ C_{i_1}G_{i_2i_3\cdots i_q}+D_{i_1}Y_{i_2i_3\cdots i_q}\end{array}\right\} & \mathrm{M} & 0 & 0 & 0 & 0 & J_6 \end{bmatrix},$$

$i_1i_2\cdots i_q\in\wp^{\{q,r\}}\setminus\{i_1=i_2=\cdots=i_q\},\ j_1j_2\cdots j_q\in\wp^{\{q,r\}}.$

Proof: Applying the generalized control law (13), it follows from (1)-(3) and (10) that the closed-loop system can be described as

$$x\left(k+1\right)=\left(A_c+HF\left(k\right)E_c\right)x\left(k\right)+E\left(H\right)\omega\left(k\right),$$
$$z\left(k\right)=C_cx\left(k\right)+M\left(H\right)\omega\left(k\right),$$

where

$$A_c=A\left(H\right)+B\left(H\right)f\left(H^{\{q-1,r\}},Y_{i_1i_2\cdots i_{q-1}}\right)\times f^{-1}\left(H^{\{q-1,r\}},G_{i_1i_2\cdots i_{q-1}}\right),$$

$$E_c=E_1\left(H\right)+E_2\left(H\right)f\left(H^{\{q-1,r\}},Y_{i_1i_2\cdots i_{q-1}}\right)\times f^{-1}\left(H^{\{q-1,r\}},G_{i_1i_2\cdots i_{q-1}}\right),$$

$$C_c = C(H) + D(H) f\left(H^{\{q-1,r\}}, Y_{i_1 i_2 \cdots i_{q-1}}\right) \times f^{-1}\left(H^{\{q-1,r\}}, G_{i_1 i_2 \cdots i_{q-1}}\right).$$

Choosing the parameter-dependent Lyapunov candidate function as

$$V(x(k)) = x(k)^T f^{-1}\left(H^{\{q,r\}}, P_{i_1 i_2 \cdots i_q}\right) x(k) \tag{16}$$

applying Schur's complement along with Lemma 1 and using the fact that $G^T P^{-1} G \geq G^T + G - P$ which holds for all square matrices G and P of compatible dimensions, one has $\Omega < 0$ if $\Psi < 0$, where

$$\Omega = z(k)^T z(k) - \gamma^2 \omega(k)^T \omega(k) + \Delta V(x(k)),$$

$$\mathrm{X} = -f^T\left(H^{\{q-1,r\}}, G_{i_1 i_2 \cdots i_{q-1}}\right) - f\left(H^{\{q-1,r\}}, G_{i_1 i_2 \cdots i_{q-1}}\right) + f\left(H^{\{q,r\}}, G_{i_1 i_2 \cdots i_q}\right),$$

$$\Psi = \begin{bmatrix} \mathrm{X} & * & * & * & * & * & * \\ 0 & -\gamma^2 I_s & * & * & * & * & * \\ \left\{\begin{matrix} A(H) f\left(H^{\{q-1,r\}}, G_{i_1 i_2 \cdots i_{q-1}}\right) \\ +B(H) f\left(H^{\{q-1,r\}}, Y_{i_1 i_2 \cdots i_{q-1}}\right) \end{matrix}\right\} & E(H) & -f\left(H_+^{\{q,r\}}, P_{j_1 j_2 \cdots j_q}\right) & * & * & * & * \\ 0 & 0 & \varepsilon \tilde{H}^T & -\varepsilon I_{\alpha+s} & * & * & * \\ \left\{\begin{matrix} E_1(H) f\left(H^{\{q-1,r\}}, G_{i_1 i_2 \cdots i_{q-1}}\right) \\ +E_2(H) f\left(H^{\{q-1,r\}}, Y_{i_1 i_2 \cdots i_{q-1}}\right) \end{matrix}\right\} & 0 & 0 & 0 & -\varepsilon I_\beta & * & * \\ 0 & 0 & 0 & 0 & 0 & -\varepsilon I_s & * \\ \left\{\begin{matrix} C(H) f\left(H^{\{q-1,r\}}, G_{i_1 i_2 \cdots i_{q-1}}\right) \\ +D(H) f\left(H^{\{q-1,r\}}, Y_{i_1 i_2 \cdots i_{q-1}}\right) \end{matrix}\right\} & M(H) & 0 & 0 & 0 & 0 & -I_p \end{bmatrix}, \tag{17}$$

note

$$\Psi = \sum_{j_1 j_2 \cdots j_q \in \wp^{\{q,r\}}} h_{+j_1} h_{+j_2} \cdots h_{+j_q} g\left(j_1 j_2 \cdots j_q\right) \Psi_{j_1 j_2 \cdots j_q},$$

where

$$\Psi_{j_1 j_2 \cdots j_q} = \begin{bmatrix} \mathrm{X} & * & * & * & * & * & * \\ 0 & -\gamma^2 I_s & * & * & * & * & * \\ \left\{\begin{matrix} A(H) f\left(H^{\{q-1,r\}}, G_{i_1 i_2 \cdots i_{q-1}}\right) \\ +B(H) f\left(H^{\{q-1,r\}}, Y_{i_1 i_2 \cdots i_{q-1}}\right) \end{matrix}\right\} & E(H) & \dfrac{-P_{j_1 j_2 \cdots j_q}}{g\left(P_{j_1 j_2 \cdots j_q}\right)} & * & * & * & * \\ 0 & 0 & \varepsilon \tilde{H}^T & -\varepsilon I_{\alpha+s} & * & * & * \\ \left\{\begin{matrix} E_1(H) f\left(H^{\{q-1,r\}}, G_{i_1 i_2 \cdots i_{q-1}}\right) \\ +E_2(H) f\left(H^{\{q-1,r\}}, Y_{i_1 i_2 \cdots i_{q-1}}\right) \end{matrix}\right\} & 0 & 0 & 0 & -\varepsilon I_\beta & * & * \\ 0 & 0 & 0 & 0 & 0 & -\varepsilon I_s & * \\ \left\{\begin{matrix} C(H) f\left(H^{\{q-1,r\}}, G_{i_1 i_2 \cdots i_{q-1}}\right) \\ +D(H) f\left(H^{\{q-1,r\}}, Y_{i_1 i_2 \cdots i_{q-1}}\right) \end{matrix}\right\} & M(H) & 0 & 0 & 0 & 0 & -I_p \end{bmatrix},$$

since $g\left(j_1 j_2 \cdots j_q\right) > 0$, one has $\Psi < 0$ if $\Psi_{j_1 j_2 \cdots j_q} < 0$. Exploiting the possible combinations in $H^{\{q,r\}}$ and note $g\left(i_1 i_2 \cdots i_q\right) = 1$ if $i_1 = i_2 = \cdots i_q$ one gets

$$\Psi_{j_1 j_2 \cdots j_q} = \sum_{\substack{i_1 i_2 \cdots i_q \in \wp^{\{q,r\}} \\ i_1 = i_2 = \cdots = i_q}} h_{i_1} h_{i_2} \cdots h_{i_q} \Psi^1_{j_1 j_2 \cdots j_q} + \sum_{\substack{i_1 i_2 \cdots i_q \in \wp^{\{q,r\}} \\ \setminus \{i_1 = i_2 = \cdots = i_q\}}} h_{i_1} h_{i_2} \cdots h_{i_q} \Psi^2_{j_1 j_2 \cdots j_q}$$

with

$$\mathrm{E} = g\left(i_2 i_3 \cdots i_q\right) E_{i_1} + g\left(i_1 i_3 \cdots i_q\right) E_{i_2} + \cdots g\left(i_1 i_2 \cdots i_{q-1}\right) E_{i_q} = \frac{(q-1)!}{d_1! d_2! \cdots d_r}\left(E_{i_1} + E_{i_2} + \cdots E_{i_q}\right),$$

$$\mathrm{M} = g\left(i_2 i_3 \cdots i_q\right) M_{i_1} + g\left(i_1 i_3 \cdots i_q\right) M_{i_2} + \cdots g\left(i_1 i_2 \cdots i_{q-1}\right) M_{i_q} = \frac{(q-1)!}{d_1! d_2! \cdots d_r}\left(M_{i_1} + M_{i_2} + \cdots M_{i_q}\right).$$

It is easy to verify that if the LMIs in Theorem 1 hold, then $\Omega < 0$.

On the other hand, let

$$J_N = \sum_{k=0}^{N-1}\left(z(k)^T z(k) - \gamma^2 \omega(k)^T \omega(k)\right).$$

For any nonzero $\omega(k) \in l_2[0\ \infty)$ and zero initial condition $x(0) = 0$, one gets

$$J_N = \sum_{k=0}^{N-1}\left(z(k)^T z(k) - \gamma^2 \omega(k)^T \omega(k) + \Delta V(x(k))\right) - V(x(N))$$

and hence, $\Omega < 0$ implies $J_N < 0$ which guarantees that the closed-loop system (1), (2) is asymptotically stable with γ disturbance attenuation for all admissible uncertainties.

The following Theorem 2 shows that lower disturbance attenuation can be obtained by increasing q.

Theorem 2: For some given value of q, the corresponding theorem obtained from Theorem 1 is expressed as LMIs and denoted as $T_\Delta^{(q-1,q-1,q)}$. For any integer $\hat{q}$ satisfying $\hat{q} > q$, $T_\Delta^{(\hat{q}-1,\hat{q}-1,\hat{q})}$ is feasible if $T_\Delta^{(q-1,q-1,q)}$ is feasible.

Proof: Suppose the LMIs in $T_\Delta^{(q-1,q-1,q)}$ are feasible, let

$$f\left(H^{\{q,r\}}, G_{i_1 i_2 \cdots i_q}\right) = \left(\sum_{i=1}^{r} h_i\right) f\left(H^{\{q-1,r\}}, G_{i_1 i_2 \cdots i_{q-1}}\right), \quad (18)$$

$$f\left(H^{\{q,r\}}, Y_{i_1 i_2 \cdots i_q}\right) = \left(\sum_{i=1}^{r} h_i\right) f\left(H^{\{q-1,r\}}, Y_{i_1 i_2 \cdots i_{q-1}}\right), \quad (19)$$

$$f\left(H^{\{q+1,r\}}, G_{i_1 i_2 \cdots i_{q+1}}\right) = \left(\sum_{i=1}^{r} h_i\right) f\left(H^{\{q,r\}}, P_{i_1 i_2 \cdots i_q}\right), \quad (20)$$

the LMIs in $T_\Delta^{(q,q,q+1)}$ can be obtained by linear combination of those in $T_\Delta^{(q-1,q-1,q)}$. That is, for any solutions satisfying $T_\Delta^{(q-1,q-1,q)}$ will be bound to satisfy $T_\Delta^{(q,q,q+1)}$. With recursion, For any integer $\hat{q}$ satisfying $\hat{q} > q$, $T_\Delta^{(\hat{q}-1,\hat{q}-1,\hat{q})}$ is also feasible.

Remark 1: A similar approach can be found in [32] for linear time-invariant (LTI) systems where the upper bound for the complexity parameter is computed, however, the results are focused only on LTI systems. By utilizing the Polya's theorems on positive forms on the standard simplex, a method is developed in [33] for continuous-time T-S fuzzy systems. The method proposed in [33] exploits the possible combinations of LMIs or introduce more decision variables X_{ijkl}. To solve the robust control problem, however, these variables X_{ijkl} may increase the computational burden. The reference [34] provide another method to introduce more slack variables by increasing the degree of polynomial matrix function, however, similar to [32], they are applied to LTI systems.

Note, [13] utilizes the non-quadratic Lyapunov function

$$V(x) = x(k)^T\left(\sum_{i=1}^{r} h_i S_i\right)x(k)$$

and non-PDC law

$$u(k) = -\left(\sum_{i=1}^{r} h_i Y_i\right)\left(\sum_{i=1}^{r} h_i G_i\right)^{-1} x(k)$$

to cope with the stability problem of fuzzy system. The extension of the method in [26] to deal with the robust H_∞ control problem is straightforward. Using the Lyapunov function and control law given in [26] and following the same line in Theorem 1 one gets the following Theorem 3.

Theorem 3: The closed-loop system (1), (2) is asymptotically stable with γ disturbance attenuation, if there exist matrices $Y_i \in R^{m\times n}$, $G_i \in R^{n\times n}$, $\varepsilon > 0$ and positive matrices $S_i \in R^{n\times n}$, $i = 1, 2, \cdots r$ satisfying the following LMIs (21), (22).

Remark 2: The Lyapunov function utilized in Theorem 3 is linear-dependent on the basis function h_i, while in $T_\Delta^{(q-1,q-1,q)}$, the Lyapunov function can be quadratic, cubic or even higher degree on h_i. As the degree increases, more slack variables are generated leading to less conservative results than those by Theorem 3. The following example shows this point.

4. SIMULATION EXAMPLE

In this section, we compare our results with other methods using example borrowed from the literatures.

$$\begin{bmatrix} \begin{Bmatrix} -G_i - G_i^T + P_i \\ -G_j - G_j^T + P_j \end{Bmatrix} & * & * \\ 0 & -2\gamma^2 & * \\ \begin{Bmatrix} A_i G_j + B_i Y_j \\ +A_j G_i + B_j Y_i \end{Bmatrix} & E_i + E_j & -2P_k \\ 0 & 0 & 2\varepsilon \tilde{H}^T \\ \begin{Bmatrix} E_{1i} G_j + E_{2i} Y_j \\ +E_{1j} G_i + E_{2j} Y_i \end{Bmatrix} & 0 & 0 \\ 0 & 0 & 0 \\ \begin{Bmatrix} C_i G_j + D_i Y_j \\ +C_j G_i + D_j Y_i \end{Bmatrix} & M_i + M_j & 0 \end{bmatrix}$$

$$\begin{matrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ -2\varepsilon I_{\alpha+s} & * & * & * \\ 0 & -2\varepsilon I_\beta & * & * \\ 0 & 0 & -2\varepsilon I_s & * \\ 0 & 0 & 0 & -2I_p \end{matrix}\Bigg] < 0, \quad (21)$$

$i = 1,2,\cdots r-1,\ j = i+1,\cdots r,\ k = 1,2,\cdots r$

$$\begin{bmatrix} -G_i - G_i^T + P_i & * & * & * & * & * & * \\ 0 & -\gamma^2 & * & * & * & * & * \\ A_iG_i + B_iY_i & E_i & -P_j & * & * & * & * \\ 0 & 0 & \varepsilon\tilde{H}^T & -\varepsilon I_{\alpha+s} & * & * & * \\ E_{1i}G_i + E_{2i}Y_i & 0 & 0 & 0 & -\varepsilon I_\beta & * & * \\ 0 & 0 & 0 & 0 & 0 & -\varepsilon I_s & * \\ C_iG_i + D_iY_i & M_i & 0 & 0 & 0 & 0 & -I_p \end{bmatrix} < 0. \quad (22)$$

$i, j = 1,2,\cdots r$

All the experiments have been performed with a Celeron (R) 2.8 GHz, 512 MB RAM, using LMI solver and m-file of MATLAB 7.0.

Example: In this example, the system under consideration is a nonlinear system modified from example 1 in [16]

$$\begin{aligned} x_1(k+1) &= x_1(k) + \Delta a_1(k)x_1(k) - x_1(k)x_2(k) \\ &\quad + 0.01\omega_2(k) - 0.03\omega_1(k) + (5 + x_1(k))u(k), \\ x_2(k+1) &= -x_1(k) - 0.5x_2(k) + 2x_1(k)u(k) + 0.01\omega_2(k), \\ z(k+1) &= -0.1x_1(k) - 0.5x_2(k) + 0.5u(k) + 0.01\omega_1(k) \\ &\quad + 0.01\omega_2(k), \end{aligned}$$

where $\Delta a_1(k)$ is the uncertain parameters satisfying $\Delta a_1(k) \in [-0.1,\ 0.1]$.

Under the assumption that, the nonlinear system is exactly represented by the T-S fuzzy system given by

Rule 1: If $x_1(k)$ is about h_1 then

$$\begin{aligned} x(k+1) &= (A_1 + \Delta A_1(k))x(k) + (B_1 + \Delta B_1(k))u(k) \\ &\quad + E_1\omega(k), \\ z(k+1) &= C_1x(k) + D_1u(k) + M_1\omega(k). \end{aligned}$$

Rule 2: If $x_1(k)$ is about h_2 then

$$\begin{aligned} x(k+1) &= (A_2 + \Delta A_2(k))x(k) + (B_2 + \Delta B_2(k))u(k) \\ &\quad + E_2\omega(k), \\ z(k+1) &= C_2x(k) + D_2u(k) + M_2\omega(k), \end{aligned}$$

$$h_1 = \frac{x_1(k) + \beta}{2\beta},\quad h_2 = 1 - h_1,\quad x_1(k) \in [-\beta\ \beta],$$

$$A_1 = \begin{bmatrix} 1 & -\beta \\ -1 & -0.5 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & \beta \\ -1 & -0.5 \end{bmatrix},\ D_1 = D_2 = 0.5,$$

$$B_1 = \begin{bmatrix} 5+\beta \\ 2\beta \end{bmatrix},\ B_2 = \begin{bmatrix} 5-\beta \\ -2\beta \end{bmatrix},\ M_1 = M_2 = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}^T,$$

$$E_1 = E_2 = \begin{bmatrix} -0.03 & 0.01 \\ 0.00 & 0.01 \end{bmatrix},\ C_1 = C_2 = \begin{bmatrix} -0.03 & 0.01 \\ 0.00 & 0.01 \end{bmatrix},$$

and $\Delta A_1(k)$, $\Delta A_2(k)$, $\Delta B_1(k)$, $\Delta B_2(k)$ can be represented in the form of (3) and (4) with

$$H = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\quad E_{11} = E_{12} = [0.1\ \ 0],\quad E_{21} = E_{22} = 0.$$

Since γ is related to the level of disturbance attenuation, the aim here is to compute the minimum value of γ for some given β and any admissible uncertainties $\Delta a_1(k) \in [-0.1,\ 0.1]$. The minimum value of γ is obtained by means of the following convex optimization problem:

$$\gamma_{\min} = \min \gamma$$
$$\text{s.t. } T_\Delta^{(q-1,q-1,q)} \text{ with different value of } q$$

Note the LMI solver utilized here is MINCX(LMIS, C, OPTIONS) where the OPTIONS is set to [1e-5 100 0 0 0].

Applying $T_\Delta^{(q-1,q-1,q)}$ with different values of q the results are shown in Table 1 ('Th.3'in Table 1 represents Theorem 3 and '×' represents infeasible). One can conclude from Table 1 that $T_\Delta^{(q-1,q-1,q)}$ guarantees a larger feasible area and achieves a smaller $\gamma_{\min}$ than Theorem 3. For example, Theorem 3 is infeasible for $\beta \geq 1.51$ while $T_\Delta^{(2,2,3)}$ is feasible even for $\beta = 1.67$.

Table 1. The value of $\gamma_{\min}$ obtained by different methods.

β	$\gamma_{\min}$ Th.3	$\gamma_{\min}$ $T_\Delta^{(0,0,1)}$	$\gamma_{\min}$ $T_\Delta^{(1,1,2)}$	$\gamma_{\min}$ $T_\Delta^{(2,2,3)}$
0.01	0.0187	0.0187	0.0187	0.0187
0.50	0.0197	0.0263	0.0197	0.0197
0.8835	0.0209	11.4070	0.0209	0.0209
1.5	0.5247	×	0.3330	0.0360
1.5069	22.5168	×	0.8580	0.0371
1.5109	×	×	15.2800	0.0378
1.67	×	×	×	0.7854

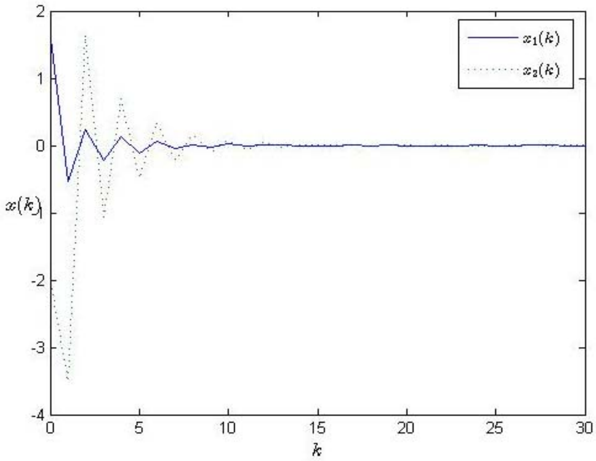

Fig. 1. The state responses.

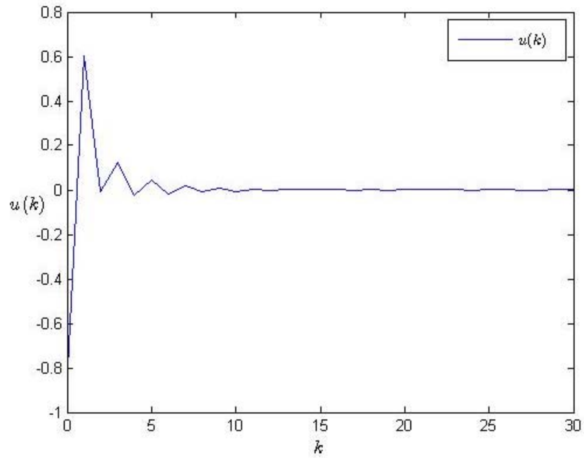

Fig. 2. The control input.

In addition, applying $T_{\Delta}^{(1,1,2)}$, $T_{\Delta}^{(2,2,3)}$ with $\beta = 1.5069$ one gets $\gamma_{\min} = 0.8580$ and $\gamma_{\min} = 0.0371$ respectively, which are smaller than the one obtained by Theorem 3. Note, as $\beta = 0.01$ one gets the same result $\gamma_{\min} = 0.0187$ for different theorems. This shows the bound of the H_{∞} performance under this condition is tight and need not continue the search for lower performance $\gamma_{\min}$ by increasing q.

Applying $T_{\Delta}^{(2,2,3)}$ with $\beta = 1.67$, one gets $\gamma_{\min} = 0.7854$ and the following results

$$G_{11} = \begin{bmatrix} 0.0398 & 0.1821 \\ -0.0382 & 0.2682 \end{bmatrix},\ G_{12} = \begin{bmatrix} 0.1106 & -0.1442 \\ -0.0585 & 0.4217 \end{bmatrix}$$

$$G_{22} = \begin{bmatrix} 0.0066 & 0.0250 \\ -0.0148 & 0.1323 \end{bmatrix},\ Y_{11} = \begin{bmatrix} -0.0178 & 0.0341 \end{bmatrix}$$

$$Y_{12} = \begin{bmatrix} -0.0220 & 0.0018 \end{bmatrix},\ Y_{22} = \begin{bmatrix} 0.0046 & -0.0665 \end{bmatrix}$$

$$P_{111} = \begin{bmatrix} 0.0191 & 0.0147 \\ 0.0147 & 0.1430 \end{bmatrix},\ P_{112} = \begin{bmatrix} 0.1403 & -0.0644 \\ -0.0644 & 0.5707 \end{bmatrix}$$

$$P_{122} = \begin{bmatrix} 0.1271 & -0.0472 \\ -0.0472 & 0.5482 \end{bmatrix},\ P_{222} = \begin{bmatrix} 0.0114 & 0.0247 \\ 0.0247 & 0.1299 \end{bmatrix}$$

The exogenous disturbance input $\omega(k) \in l_2[0, \infty)$ is

$$\omega(k) = \begin{bmatrix} (\text{rand}(\bullet) - 0.3)/(1 + 0.01k) \\ (\text{rand}(\bullet) - 0.3)/(1 + 0.01k) \end{bmatrix}$$

and $\Delta a_1(k) = 0.1\sin(k)$. Fig. 1 shows the states response with the initial condition $x(0) = [1.67 \quad -2]^T$, while Figs. 2 and 3 present the corresponding controlled input and controlled output with exogenous disturbance, respectively. From these simulations, it can be seen the designed fuzzy controller ensures the asymptotic stability of the closed-system and guarantees a prescribed H_{∞} performance level under the uncertain parameter $\Delta a_1(k)$.

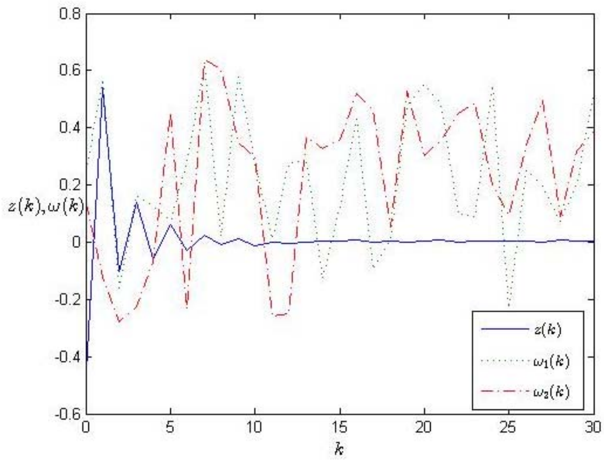

Fig. 3. The control output and exogenous disturbance.

5. CONCLUSIONS

The stability analysis for discrete-time fuzzy systems with T-S model has been studied in this paper. Some sufficient conditions for the existence of a generalized non-PDC law have been obtained. It has been shown that the proposed generalized non-PDC law can not only stabilize the system but also guarantee a prescribed level on the disturbance attenuation. In addition, the design approach has been applied to an example of nonlinear discrete-time systems with disturbance input, and the results have showed the effectiveness of the proposed approach.

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