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ORIGINAL ARTICLE

Similarity Relations, Invertibility and Eigenvalues of Intuitoinistic Fuzzy Matrix

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Abstract In this article, the similarity relations are studied, together with invertibility conditions and eigenvalues of intuitionistic fuzzy matrices (IFMs). Besides, idempotent, regularity, permutation matrix and spectral radius of IFMs are considered here with some properties and results for IFMs investigated.

Keywords Relation · Regular matrix · Permutation matrix · Eigenvalue · Spectral radius

1. Introduction

In 1971, Zadeh [24] introduced the similarity relations and fuzzy ordering. After that many authors [4, 5, 16] developed the theory of fuzzy relations and relational compositions along with their applications.

The use of fuzzy relations originated from the observation that the real-life objects can be related each other to a certain degree. In real-life situations, one person is either related with another or not. That is, there is no scope to mention about the degree or strength of relationship. But, using the concept of fuzzy set theory we can assign the degree/strength of relationship between two persons/objects. If there is no doubt or hesitation to determine the degree/strength of relationship, then the fuzzy set theory is enough to represent relationship. But, in general, it is very difficult to assign the degree/strength of relationship, there may be hesitation/uncertainty. To overcome the hesitation, intuitionistic fuzzy set (IFS) is successfully used. IFSs, defined by Atanassov in 1983 [1] give us the possibility to model hesitation and uncertainty by using an additional degree.

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The intuitionistic fuzzy relations (IFRs) was introduced by Burillo and Bustince [10-12]. In 2003, Deschrijver and Kerre [16] present an intuitionistic fuzzy version of the triangular compositions, theory and application to policy analysis and information systems and the variants of their compositions.

In this paper, the similarity relations, invertibility conditions and eigenvalues of IFM are introduced. Idempotent, regularity, permutation matrix and spectral radius of IFMs are considered here. Also some properties and results for IFMs are investigated.

2. Preliminaries

In this section, some basic notions related to this topics are recalled.

Definition 2.1 (Instuitionistic fuzzy set) *An IFS A in X (universe of discourse) is defined as an object of the following form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\},$$

where the functions $\mu_A : X \to [0,1]$ *and* $\nu_A : X \to [0,1]$ *is defined as the degree of membership and the degree of non-membership of the element* $x \in X$ *in A, respectively and for every* $x \in X$,

$$0 \le \mu_A(x) + \nu_A(x) \le 1.$$

Let $\mathcal{I}$ be the set of all real numbers lying between 0 and 1, i.e., $\mathcal{I} = \{x : 0 \le x \le 1\}$. Also let $\langle F\rangle$ be the set of tuples $\langle a, b\rangle$, where $a, b \in \mathcal{I}$ and $0 \le a + b \le 1$, i.e.,

$$\langle F\rangle = \{\langle a, b\rangle :\ 0 \le a + b \le 1;\ a, b \in \mathcal{I}\}.$$

The addition and multiplication between any two elements of $\langle F\rangle$ are defined bellow.

Definition 2.2 *Let* $x = \langle x_\mu, x_\nu\rangle$ *and* $y = \langle y_\mu, y_\nu\rangle$ *be any two elements of* $\langle F\rangle$. *The addition* $(+)$ *and multiplication* $(\cdot)$ *between x and y are defined as*

$$\begin{aligned} x + y &= \langle x_\mu, x_\nu\rangle + \langle y_\mu, y_\nu\rangle \\ &= \langle \max(x_\mu, y_\mu), \min(x_\nu, y_\nu)\rangle \\ &= \langle x_\mu \vee y_\mu, x_\nu \wedge y_\nu\rangle,\ \textit{where}\ x_\mu \vee y_\mu = \max(x_\mu, y_\mu), \end{aligned}$$

and

$$\begin{aligned} x \cdot y &= \langle x_\mu, x_\nu\rangle \cdot \langle y_\mu, y_\nu\rangle \\ &= \langle \min(x_\mu, y_\mu), \max(x_\nu, y_\nu)\rangle \\ &= \langle x_\mu \wedge y_\mu, x_\nu \vee y_\nu\rangle,\ \textit{where}\ x_\mu \wedge y_\mu = \min(x_\mu, y_\mu). \end{aligned}$$

In arithmetic operations (such as addition, multiplication, etc.), only the values of the membership and the nonmembership are needed. So from now, we denote IFS as

$$A = \{x = \langle x_\mu, x_\nu\rangle \mid x \in X\}.$$

Definition 2.3 (Cartesion product of IFSs) *Let X_1 and X_2 be two universes and let $A = \{x = \langle x_\mu, x_\nu \rangle \mid x \in X_1\}$, $B = \{y = \langle y_\mu, y_\nu \rangle \mid y \in X_2\}$ be two IFSs. The Cartesian product of A and B is defined as follows*

$$A \times B = \{(x, y) \mid x \in X_1 \text{ and } y \in X_2\}.$$

Definition 2.4 (Intuitionistic fuzzy relation (IFR)) *An IFR between two IFSs A and B is defined as an IFS in $A \times B$. If R is a relation between A and B, $x \in A$ and $y \in B$, then $\mu_R(x, y)$ denotes the membership degree to which x is in relation R with y and $\nu_R(x, y)$ denotes the non-membership degree to which x is in relation R with y. Also, if $\pi_R(x, y)$ denotes the uncertainty degree to which x and y are in relation R with each other, then the real degree to which x is in relation R with y lies somewhere between $\mu_R(x, y)$ and $\mu_R(x, y) + \pi_R(x, y) = 1 - \nu_R(x, y)$.*

3. Similarity Relation on IFS

In this section, IFMs are introduced satisfying the properties of IFRs such as reflexive, symmetric and transitive.

Let $R(A, A)$ be an IFR on a set A. Let $\mu_R : A \to [0, 1]$ be the membership function and $\nu_R : A \to [0, 1]$ be the non-membership function and M_R be the IFM with respect to the relation R.

Definition 3.1 (Reflexive relation) *The relation $R(A, A)$ is reflexive if the diagonal entries of M_R are all $\langle 1, 0 \rangle$, i.e.,*

$$\mu_R(x, x) = 1 \text{ and } \nu_R(x, x) = 0 \text{ for all } x \in A.$$

Definition 3.2 (Symmetric relation) *The relation $R(A, A)$ is symmetric if $M_R = M_R^T$, where M_R^T is the transpose of M_R, i.e.,*

$$\mu_R(x, y) = \mu_R(y, x) \text{ and } \nu_R(x, y) = \nu_R(y, x) \text{ for all } x, y \in A.$$

Definition 3.3 (Transitive relation) *The relation $R(A, A)$ is transitive if $M_R \geq M_R^2$, i.e., $\mu_R(x, z) \geq \max_{y \in X}\{\min\{\mu_R(x, y), \mu_R(y, z)\}\}$ and $\nu_R(x, z) \leq \min_{y \in X}\{\max\{\nu_R(x, y), \nu_R(y, z)\}\}$ for all pair $(x, z) \in A \times A$.*

Definition 3.4 (Similarity relation) *The relation $R(A, A)$ is a similarity relation if and only if $R(A, A)$ is reflexive, symmetric and transitive.*

The set of all IFMs of order $m \times n$ over an IFS A in X is denoted by $F_{mn}(A)$ or simply F_{mn}. If $m = n$, then it is denoted by F_n. The zero matrix $O_n = [\langle 0, 1 \rangle]$ and the identity matrix I_n of order $n \times n$ is define as, I_n is the IFM whose principle diagonal elements are all $I = \langle 1, 0 \rangle$ and other elements are all $\phi = \langle 0, 1 \rangle$.

Proposition 3.1 *For an IFM $P \in F_n$, P is reflexive if $P \geq I_n$.*

Proof Since $P \geq I_n$, therefore all diagonal entries of P are $\langle 1, 0 \rangle$. Therefore P is a reflexive matrix.

Definition 3.5 For an IFM $P = [p_{ij}] = [\langle p_{ij\mu}, p_{ij\nu}\rangle] \in F_n$, we define the following IFMs:

Type of P	Definition	
Reflexive	$P \geq I_n$.	
Weakly reflexive	$p_{ii} \geq p_{ij}$	for all $i, j \in \{1, 2, 3, \cdots, n\}$.
Symmetric	$P = P^T$.	
Idempotent	$P = P^2$.	
Transitive	$P^2 \leq P$.	

Proposition 3.2 *Let $P \in F_n$ be a reflexive IFM. Then the following holds.*

1) *P^T is a reflexive IFM,*

2) *P^k is a reflexive IFM for some positive integer k,*

3) *$PQ \geq Q$ for $Q \in F_n$,*

4) *$QP \geq Q$ for $Q \in F_n$,*

5) *PQ and QP are reflexive IFMs if Q is reflexive,*

6) *PP^T and P^TP are reflexive IFMs.*

Proof 1) Since P is reflexive, its diagonal entries are all $\langle 1, 0\rangle$. Therefore, the diagonal entries of P^T are also $\langle 1, 0\rangle$. Hence P^T is reflexive.

2) Since P is reflexive, $P \geq I_n$. Then

$P^2 \geq P \geq I_n$ (Multiplying by P in both side).

Proceeding in this way, we get $P^k \geq P^{k-1} \geq \cdots \geq P^2 \geq P \geq I_n$ for any positive integer k.

Hence P^k is reflexive.

3) $P \geq I_n$, then $PQ \geq I_nQ$, this implies that $PQ \geq Q$.

4) Also $QP \geq QI_n$ or $QP \geq Q$.

5) Since Q is reflexive $Q \geq I_n$.Then

$PQ \geq Q \geq I_n$ and $QP \geq Q \geq I_n$.

Hence PQ and QP are also reflexive.

6) From 1) and 5), it follows that PP^T and P^TP are reflexive.

Proposition 3.3 *If $P \in F_n$ be transitive and reflexive, then P is idempotent.*

Proof Since P is reflexive, $P \geq I_n$. Therefore

$$P^2 \geq P \geq I_n. \tag{1}$$

Also, P is transitive,

$$P^2 \leq P. \tag{2}$$

Combining (1) and (2), we get $P^2 = P$.

Hence P is idempotent.

The condition is not sufficient which can be shown by the following example.

Example 3.1 Let

$$P = \begin{bmatrix} \langle 0.8, 0.1\rangle & \langle 0.8, 0.1\rangle \\ \langle 0.8, 0.1\rangle & \langle 0.8, 0.1\rangle \end{bmatrix} \ngeq I_2.$$

Hence P is not reflexive. But

$$\begin{aligned} P^2 = P \cdot P &= \begin{bmatrix} \langle 0.8, 0.1\rangle & \langle 0.8, 0.1\rangle \\ \langle 0.8, 0.1\rangle & \langle 0.8, 0.1\rangle \end{bmatrix} \begin{bmatrix} \langle 0.8, 0.1\rangle & \langle 0.8, 0.1\rangle \\ \langle 0.8, 0.1\rangle & \langle 0.8, 0.1\rangle \end{bmatrix} \\ &= \begin{bmatrix} \langle 0.8, 0.1\rangle & \langle 0.8, 0.1\rangle \\ \langle 0.8, 0.1\rangle & \langle 0.8, 0.1\rangle \end{bmatrix} \\ &= P. \end{aligned}$$

That is, P is idempotent.

The proof of the following result is straight forward.

Proposition 3.4 *If P and Q are two symmetric IFMs in F_n such that $PQ = QP$, then PQ is symmetric.*

Remark 3.1 If P is a symmetric IFM in F_n, then P^k is symmetric for any positive integer k.

Proposition 3.5 *If P and Q are transitive IFMs in F_n such that $PQ = QP$, then PQ is transitive.*

Proof Since P and Q are transitive, $P^2 \leq P$ and $Q^2 \leq Q$. Now

$$\begin{aligned} (PQ)^2 = (PQ)(PQ) &= P(QP)Q \quad \text{(by associative property)} \\ &= P(PQ)Q \quad \text{(Since } PQ = QP\text{)} \\ &= (PP)(QQ) \\ &= P^2Q^2. \end{aligned}$$

That is, $(PQ)^2 \leq PQ$.

Hence PQ is transitive.

Remark 3.2 If P is transitive in F_n, then P^k is transitive for any positive integer k.

Proposition 3.6 *If $P = [p_{ij}] = [\langle p_{ij\mu}, p_{ij\nu}\rangle] \in F_n$ is symmetric and transitive, then $p_{ij} \leq p_{ii}$ for $i, j \in \{1, 2, 3, \cdots, n\}$.*

Proof Since P is symmetric, $p_{ij} = p_{ji}$ for all $i, j \in \{1, 2, 3, \cdots, n\}$.

Also since P is transitive, $P^2 \leq P$ i.e., $P \geq P^2$. Thus

$$p_{ij} \geq \max_k\{\min(p_{ik}, p_{kj})\} \quad \text{for all } i, j.$$

That is

$$\begin{aligned} p_{ii} &\geq \max_k\{\min(p_{ik}, p_{kj})\} \quad \text{for } i = j \text{ and } k \in \{1, 2, 3, \cdots, n\} \\ &\geq \min(p_{ij}, p_{ji}) \quad \text{for } k = j \text{ for each } i. \end{aligned}$$

This gives $p_{ii} \geq p_{ij}$ (Since $p_{ij} = p_{ji}$).

4. Invertible Matrices

Von Neumann [21] was first who introduced the regularity for rings. We know that $axa = a$ holds for all $x \geq a$ in the max-min intuitionistic fuzzy algebra (IFA). So, every element in IFA is regular, although all IFMs are not regular. Regular matrices are very interesting because of their close relationship with inverses.

Cen [13] introduced T-ordering in fuzzy matrices and discussed the relationship between the T-ordering and the g-inverses. Khan and Pal [18] introduced the concept of g-inverses for IFMs, minus partial ordering and studies several properties of it. Sriram and Murugadas [22] study the relation between the minus-ordering and the various g-inverses of IFM.

Definition 4.1 (Regular IFM) *An IFM $A \in F_{mn}$ is said to be regular if there exists $X \in F_{nm}$, such that $AXA = A$. In this case, X is called a generalized inverse (g-inverse) of A and it is denoted by A^-. The set of all g-inverses of A is denoted by $A\{1\}$.*

Definition 4.2 (Invertible matrix) *An IFM $A \in F_n$ is said to be invertible if and only if there exists $B \in F_n$, such that $AB = BA = I_n$.*

Definition 4.3 (Permutation matrix) *An IFM $A \in F_n$ is called permutation matrix if it has exactly one entry $I = \langle 1, 0 \rangle$ in each row and each column and all the other entries are $\phi = \langle 0, 1 \rangle$.*

The intuitionistic fuzzy permutation matrices (IFPMs) play an important role in mathematics specially in matrix theory. Here we investigate that if an IFM is a permutation matrix, then it is invertible.

Proposition 4.1 *Let $A \in F_n$ be an intuitionistic fuzzy permutation matrix (IFPM). Then $AA^T = A^T A = I_n$.*

Proof Let $A = [a_{ij}] = [\langle a_{ij\mu}, a_{ij\nu} \rangle]$. Then $A^T = (a_{ji}) = [\langle a_{ji\mu}, a_{ji\nu} \rangle] = [b_{ij}]$ (say).

Now, the ijth entries of AA^T is

$$\sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{n} a_{ik} a_{jk} = \begin{cases} \phi, \text{ if } i \neq j, \\ I, \text{ if } i = j. \end{cases}$$

(Since A is an IFPM $\sum_{k=1}^{n} a_{ik} a_{ik} = I$.)

Hence AA^T is an identity matrix of order n.

Similarly, it can be proved that $A^T A = I_n$. Thus, $AA^T = A^T A = I_n$.

Proposition 4.2 *Let $A \in F_n$, A is invertible if and only if A is an IFPM.*

Proof Condition is necessary: Let A be a permutation matrix. Then

$AA^T = A^T A = I_n$ (By Proposition 4.1).

Hence A is invertible and A^T is the inverse of A, i.e., $A^- = A^T$.

Condition is sufficient: Let $A = [a_{ij}]$ be invertible and $B = [b_{ij}]$ be the inverse of A. Then $AB = BA = I_n$ follows that,

$$\sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{n} b_{ik} a_{kj} = \phi \text{ for } i \neq j \text{ and } \sum_{k=1}^{n} a_{ik} b_{ki} = \sum_{k=1}^{n} b_{ik} a_{ki} = I.$$

Then by max-min algebra, we get each

$$a_{ik}b_{kj} = b_{ik}a_{kj} = \phi \text{ for } i \neq j \text{ and } k \in \{1, 2, 3, \cdots, n\}, \tag{3}$$

and

$$a_{ik}b_{ki} = b_{ik}a_{ki} = I \text{ for at least one } k \in \{1, 2, \cdots, n\} \text{ and for } i \in \{1, 2, \cdots, n\}. \tag{4}$$

Now (3) implies that

$$a_{ik} = \phi \text{ or } b_{kj} = \phi \text{ or both } a_{ik} = b_{kj} = \phi \text{ for } i \neq j, k \in \{1, 2, 3, \cdots, n\}, \tag{5}$$

and

$$b_{ik} = \phi \text{ or } a_{kj} = \phi \text{ or both } b_{ik} = a_{kj} = \phi \text{ for } i \neq j, k \in \{1, 2, 3, \cdots, n\}. \tag{6}$$

Also (4) implies that

$$a_{ik} = b_{ki} = I \text{ and } a_{ki} = b_{ik} = I \tag{7}$$

for at least one $k \in \{1, 2, 3, \cdots, n\}$ and for each $i \in \{1, 2, 3, \cdots, n\}$.

Let the results of (7) exist for $k = p$ (say), that is $a_{ip} = b_{pi} = I = \langle 1, 0\rangle$. Then from (5), we get $b_{pj} = \phi = \langle 0, 1\rangle$ for all $i \neq j$ and $a_{jp} = \phi = \langle 0, 1\rangle$ for all $i \neq j$. Therefore, the pth row of B has exactly one I and the remaining entries are all ϕ and pth column of A has exactly one I and the remaining entries are all ϕ.

Similarly, by using (6) the pth row of A has exactly one I and the remaining entries are all ϕ and pth column of B has exactly one I and the remaining entries are all ϕ.

Thus, A and B both are intuitionistic fuzzy permutation matrices.

Remark 4.1 An intuitionistic fuzzy permutation matrix $A \in F_n$ is invertible and A^T is the inverse of it.

Remark 4.2 The permutation matrices are only the invertible matrices in F_n.

Example 4.1 Let

$$A = \begin{bmatrix} \langle 0,1\rangle & \langle 1,0\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,0\rangle \\ \langle 1,0\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{bmatrix}.$$

Then

$$A^T = \begin{bmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,0\rangle \\ \langle 1,0\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 1,0\rangle & \langle 0,1\rangle \end{bmatrix}.$$

Therefore,

$$AA^T = \begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 1,0\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,0\rangle \end{bmatrix} = I_3.$$

Similarly, $A^TA = I_3$.

Hence, A is invertible and $A^{-} = A^{T}$.

5. Eigenvalues of IFMs

Eigenvalue problems are very important in many fields. These are formulated when modeling real cases into mathematical models. For example, the natural frequencies and mode shapes in vibration problems, the principal axes in elasticity and dynamics, the Markov chain in stochastic modeling and queueing theory, and the analytical hierarchy process for decision making, etc. all come up with eigenvalue problems.

Many authors [9, 14, 15] studied the eigenvalues of fuzzy matrices. Here we introduced the eigenvalues of IFM.

Definition 5.1 *Let $A \in F_n$ and a scalar $\lambda = \langle \lambda_\mu, \lambda_\nu \rangle \in F$ be called an eigenvalue of A and a non-zero vector X be called a row (column) eigenvector associated with the eigenvalue λ of A if $XA = \lambda X$ ($AX = \lambda X$).*

Theorem 5.1 *If $A = [a_{ij}] = [\langle a_{ij\mu}, a_{ij\nu} \rangle]$ is an IFM of order $n \times n$, such that $a_{1i} = a_{2i} = \cdots = a_{i-1,i} = a_{i+1,i} = \cdots = a_{ni} = \phi$ (say) where $i \in \{1, 2, 3, \cdots, n\}$, then a_{ii} is an eigenvalue corresponding to the column eigenvector $[\phi, \phi, \phi, \cdots, I, \cdots, \phi]^T$, where $I = \langle 1, 0 \rangle$ is the ith entry.*

Proof Here $X = [\phi, \phi, \phi, \cdots, I, \cdots, \phi]^T = (x_{i1})$ (say). Then

$$AX = \begin{bmatrix} \sum_{k=1}^{n} a_{1k}x_{k1} \\ \sum_{k=1}^{n} a_{2k}x_{k2} \\ \sum_{k=1}^{n} a_{3k}x_{k3} \\ \vdots \\ \sum_{k=1}^{n} a_{nk}x_{kn} \end{bmatrix} = \begin{bmatrix} \phi \\ \phi \\ \vdots \\ a_{ii} \\ \vdots \\ \phi \end{bmatrix} = a_{ii} \begin{bmatrix} \phi \\ \phi \\ \vdots \\ I \\ \vdots \\ \phi \end{bmatrix}.$$

(Since the ith entry $\sum_{k=1}^{n} a_{ik}x_{ki} = a_{i1}\phi + a_{i2}\phi + \cdots + a_{ii}I + \cdots + a_{in}\phi = a_{ii}$.)
Therefore, $AX = a_{ii}X$.

Hence, a_{ii} is the eigenvalue corresponding to the column eigenvector

$$X = [\phi, \phi, \cdots, I, \cdots, \phi]^T.$$

Example 5.1 Let

$$A = \begin{bmatrix} \langle 0.5, 0.4 \rangle & \langle 0, 1 \rangle & \langle 0.6, 0.2 \rangle \\ \langle 0.6, 0.3 \rangle & \langle 0.4, 0.3 \rangle & \langle 0.7, 0.2 \rangle \\ \langle 0.7, 0.3 \rangle & \langle 0, 1 \rangle & \langle 0.8, 0.2 \rangle \end{bmatrix} \text{ and } X = [\langle 0, 1 \rangle \; \langle 1, 0 \rangle \; \langle 0, 1 \rangle]^T.$$

Then

$$\begin{aligned} AX &= \begin{bmatrix} \langle 0.5, 0.4\rangle & \langle 0, 1\rangle & \langle 0.6, 0.2\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.4, 0.3\rangle & \langle 0.7, 0.2\rangle \\ \langle 0.7, 0.3\rangle & \langle 0, 1\rangle & \langle 0.8, 0.2\rangle \end{bmatrix} \begin{bmatrix} \langle 0, 1\rangle \\ \langle 1, 0\rangle \\ \langle 0, 1\rangle \end{bmatrix} \\ &= \begin{bmatrix} \langle 0, 1\rangle \\ \langle 0.4, 0.3\rangle \\ \langle 0, 1\rangle \end{bmatrix} = \langle 0.4, 0.3\rangle \begin{bmatrix} \langle 0, 1\rangle \\ \langle 1, 0\rangle \\ \langle 0, 1\rangle \end{bmatrix} \\ &= \langle 0.4, 0.3\rangle X. \end{aligned}$$

Thus, $\langle 0.4, 0.3\rangle$ is the eigenvalue of A corresponding to the column eigenvector X.

Theorem 5.2 *Let $A = [a_{ij}] = [\langle a_{ij\mu}, a_{ij\nu}\rangle] \in F_n$. If $a_{i1} = a_{i2} = \cdots = a_{i,i-1} = a_{i,i+1} = \cdots = a_{in} = \phi$ (say) where $i \in \{1, 2, 3, \cdots, n\}$. Then, a_{ii} is an eigenvalue corresponding to the row eigenvector $(\phi, \phi, \phi, \cdots, I, \cdots, \phi)$, where I is the ith entry.*

Proof The proof is similar to Theorem 5.1.

Example 5.2 Let

$$A = \begin{bmatrix} \langle 0.6, 0.3\rangle & \langle 0.8, 0.2\rangle & \langle 0.7, 0.3\rangle \\ \langle 0, 1\rangle & \langle 0.6, 0.4\rangle & \langle 0, 1\rangle \\ \langle 0.7, 0.2\rangle & \langle 0.5, 0.4\rangle & \langle 0.8, 0.1\rangle \end{bmatrix} \text{ and } X = (\langle 0, 1\rangle\ \langle 1, 0\rangle\ \langle 0, 1\rangle).$$

Then

$$\begin{aligned} XA &= (\langle 0, 1\rangle\ \langle 1, 0\rangle\ \langle 0, 1\rangle) \begin{bmatrix} \langle 0.6, 0.3\rangle & \langle 0.8, 0.2\rangle & \langle 0.7, 0.3\rangle \\ \langle 0, 1\rangle & \langle 0.6, 0.4\rangle & \langle 0, 1\rangle \\ \langle 0.7, 0.2\rangle & \langle 0.5, 0.4\rangle & \langle 0.8, 0.1\rangle \end{bmatrix} \\ &= (\langle 0, 1\rangle\ \langle 0.6, 0.4\rangle\ \langle 0, 1\rangle) \\ &= \langle 0.6, 0.4\rangle(\langle 0, 1\rangle\ \langle 1, 0\rangle\ \langle 0, 1\rangle). \end{aligned}$$

Therefore, $XA = \langle 0.6, 0.4\rangle X$.

Hence, $\langle 0.6, 0.4\rangle$ is the eigenvalue of A corresponding to the row eigenvector X.

Theorem 5.3 *If $A = [a_{ij}] = [\langle a_{ij\mu}, a_{ij\nu}\rangle] \in F_n$ such that $a_{1i} = a_{2i} = a_{3i} = \cdots = a_{ni} = \lambda \geq a_{ij}$ for all $i, j \in \{1, 2, 3, \cdots, n\}$, then λ is an eigenvalue corresponding to the column eigenvector $[I, I, I, \cdots, I]^T$.*

Proof Since $a_{1i} = a_{2i} = a_{3i} = \cdots = a_{ni} = \lambda \geq a_{ij}$ for all $i, j \in \{1, 2, 3, \cdots, n\}$.
Therefore, $\sum\limits_{j=1}^{n} a_{ij} = \lambda$. Also $X = [I, I, I, \cdots, I]^T$. Then

$$AX = \begin{bmatrix} \sum\limits_{j=1}^{n} a_{1j}I \\ \sum\limits_{j=1}^{n} a_{2j}I \\ \vdots \\ \sum\limits_{j=1}^{n} a_{nj}I \end{bmatrix} = \begin{bmatrix} \sum\limits_{j=1}^{n} a_{1j} \\ \sum\limits_{j=1}^{n} a_{2j} \\ \vdots \\ \sum\limits_{j=1}^{n} a_{nj} \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ \vdots \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} = \lambda X.$$

This shows λ is an eigenvalue of A corresponding to the column eigenvector X.

Example 5.3 Let

$$A = \begin{bmatrix} \langle 0.5, 0.4\rangle & \langle 0.8, 0.2\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.7, 0.3\rangle & \langle 0.8, 0.2\rangle & \langle 0.5, 0.3\rangle \\ \langle 0.6, 0.2\rangle & \langle 0.8, 0.2\rangle & \langle 0.7, 0.2\rangle \end{bmatrix} \text{ and } X = [\langle 1, 0\rangle\ \langle 1, 0\rangle\ \langle 1, 0\rangle]^T.$$

Then

$$AX = \begin{bmatrix} \langle 0.5, 0.4\rangle & \langle 0.8, 0.2\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.7, 0.3\rangle & \langle 0.8, 0.2\rangle & \langle 0.5, 0.3\rangle \\ \langle 0.6, 0.2\rangle & \langle 0.8, 0.2\rangle & \langle 0.7, 0.2\rangle \end{bmatrix} \begin{bmatrix} \langle 1, 0\rangle \\ \langle 1, 0\rangle \\ \langle 1, 0\rangle \end{bmatrix} = \begin{bmatrix} \langle 0.8, 0.2\rangle \\ \langle 0.8, 0.2\rangle \\ \langle 0.8, 0.2\rangle \end{bmatrix}$$

$$= \langle 0.8, 0.2\rangle \begin{bmatrix} \langle 1, 0\rangle \\ \langle 1, 0\rangle \\ \langle 1, 0\rangle \end{bmatrix}.$$

Hence, $AX = \langle 0.8, 0.2\rangle X$.

Thus, $\langle 0.8, 0.2\rangle$ is the column eigenvalue of A corresponding to the eigenvector X.

Theorem 5.4 *If $A = [a_{ij}] = [\langle a_{ij\mu}, a_{ij\nu}\rangle] \in F_n$ such that $a_{i1} = a_{i2} = a_{i3} = \cdots = a_{in} = \lambda \geq a_{ij}$ for all $i, j \in \{1, 2, 3, \cdots, n\}$, then λ is an eigenvalue of A corresponding to the row eigenvector $(I, I, I, \cdots, I)$.*

Proof The proof is similar to Theorem 5.3.

Example 5.4 Let

$$A = \begin{bmatrix} \langle 0.5, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 0.7, 0.1\rangle \\ \langle 0.8, 0.1\rangle & \langle 0.5, 0.3\rangle & \langle 0.6, 0.2\rangle \\ \langle 0.9, 0.1\rangle & \langle 0.9, 0.1\rangle & \langle 0.9, 0.1\rangle \end{bmatrix} \text{ and } X = (\langle 1, 0\rangle\ \langle 1, 0\rangle\ \langle 1, 0\rangle).$$

Then

$$XA = (\langle 1, 0\rangle\ \langle 1, 0\rangle\ \langle 1, 0\rangle) \begin{bmatrix} \langle 0.5, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 0.7, 0.1\rangle \\ \langle 0.8, 0.1\rangle & \langle 0.5, 0.3\rangle & \langle 0.6, 0.2\rangle \\ \langle 0.9, 0.1\rangle & \langle 0.9, 0.1\rangle & \langle 0.9, 0.1\rangle \end{bmatrix}$$

$$= (\langle 0.9, 0.1\rangle\ \langle 0.9, 0.1\rangle\ \langle 0.9, 0.1\rangle)$$

$$= \langle 0.9, 0.1\rangle(\langle 1, 0\rangle\ \langle 1, 0\rangle\ \langle 1, 0\rangle).$$

Therefore, $XA = \langle 0.9, 0.1\rangle X$.

Hence, $\langle 0.9, 0.1\rangle$ is the eigenvalue of A corresponding to the row eigenvector X.

Definition 5.2 (Diagonally dominant) *Let $A = [a_{ij}] \in F_n$ be an IFM. A is called row diagonally dominant if $a_{ii} \geq \sum_{j \neq i, j=1}^{n} a_{ij}$. A is called column diagonally dominant if $a_{ii} \geq \sum_{i \neq j, i=1}^{n} a_{ij}$. A is called diagonally dominant if it is both row as well as column diagonally dominant.*

Theorem 5.5 *If $A = [a_{ij}] = [\langle a_{ij\mu}, a_{ij\nu}\rangle] \in F_n$ such that $a_{11} = a_{22} = a_{33} = \cdots = a_{nn} = c$ (say) and if A is diagonally dominant, then c is an eigenvalue corresponding to the row (column) eigenvector $(I, I, I, \cdots, I)\left([I, I, I, \cdots, I]^T\right)$.*

Proof Since the IFM $A = [a_{ij}]$ is diagonally dominant, therefore $\sum_{j=1}^{n} a_{ij} = a_{ii} = c$ and $\sum_{i=1}^{n} a_{ij} = a_{jj} = c$. Also $X = [I, I, I, \cdots, I]^T$. Then

$$AX = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}I \\ \sum_{j=1}^{n} a_{2j}I \\ \vdots \\ \sum_{j=1}^{n} a_{nj}I \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} \\ \sum_{j=1}^{n} a_{2j} \\ \vdots \\ \sum_{j=1}^{n} a_{nj} \end{bmatrix} = \begin{bmatrix} c \\ c \\ \vdots \\ c \end{bmatrix} = c \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} = cX.$$

Thus, c is an eigenvalue of the IFM A corresponding to the column eigenvector X. Similarly, we can proved the theorem for row eigenvector.

Example 5.5 Let

$$A = \begin{bmatrix} \langle 0.8, 0.1\rangle & \langle 0.5, 0.4\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.7, 0.2\rangle & \langle 0.8, 0.1\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.5, 0.4\rangle & \langle 0.7, 0.2\rangle & \langle 0.8, 0.1\rangle \end{bmatrix} \text{and } X = (\langle 1, 0\rangle\ \langle 1, 0\rangle\ \langle 1, 0\rangle).$$

Therefore,

$$\begin{aligned} XA &= (\langle 1, 0\rangle\ \langle 1, 0\rangle\ \langle 1, 0\rangle) \begin{bmatrix} \langle 0.8, 0.1\rangle & \langle 0.5, 0.4\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.7, 0.2\rangle & \langle 0.8, 0.1\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.5, 0.4\rangle & \langle 0.7, 0.2\rangle & \langle 0.8, 0.1\rangle \end{bmatrix} \\ &= (\langle 0.8, 0.1\rangle\ \langle 0.8, 0.1\rangle\ \langle 0.8, 0.1\rangle) \\ &= \langle 0.8, 0.1\rangle(\langle 1, 0\rangle\ \langle 1, 0\rangle\ \langle 1, 0\rangle). \end{aligned}$$

That is, $XA = \langle 0.8, 0.1\rangle X$.

Hence, $\langle 0.8, 0.1\rangle$ is the eigenvalue of A corresponding to the row eigenvector X.

Corollary 5.1 *Let $A = [a_{ij}] = [\langle a_{ij\mu}, a_{ij\nu}\rangle] \in F_n$. If $\sum_{j=1}^{n} a_{1j} = \sum_{j=1}^{n} a_{2j} = \cdots = \sum_{j=1}^{n} a_{nj} = c$ (say), then c is an eigenvalue of A corresponding to the column eigenvector $[I, I, \cdots, I]^T$.*

Corollary 5.2 *Let $A = [a_{ij}] = [\langle a_{ij\mu}, a_{ij\nu}\rangle] \in F_n$. If $\sum_{i=1}^{n} a_{i1} = \sum_{i=1}^{n} a_{i2} = \cdots = \sum_{i=1}^{n} a_{in} = c$ (say), then c is an eigenvalue of A corresponding to the row eigenvector $(I, I, \cdots, I)$.*

Theorem 5.6 *Let $A \in F_n$. Then A has a zero column if and only if $\phi \in \sigma(A)$ (set of all eigenvalues of A).*

Proof Condition is necessary: Let ith column of A be zero, we take $X = [\phi, \phi, \cdots, I, \cdots, \phi]^T$, where I is the ith entry. Then X is a non-zero vector satisfying the equation $AX = \phi X = \phi$. Hence, X is an column eigenvector corresponding to the eigenvalue ϕ.

Condition is sufficient: Let $X = [x_1, x_2, x_3, \cdots, x_n]^T$ be a column eigenvector corresponding to the eigenvalue ϕ. Then $AX = \phi$. We assume that $x_i \neq \phi$ for $i \in$

$\{1, 2, 3, \cdots, n\}$. Then $AX = \phi$ implies that $\sum_{k=1}^{n} a_{jk}x_k = \phi$ for each $j \in \{1, 2, 3, \cdots, n\}$. This implies $a_{jk}x_k = \phi$ for each j and k. Since $x_i \neq \phi$, $a_{ij} = \phi$ for each j, therefore the ith column of A is zero.

Definition 5.3 *Let $\sigma(A)$ be the set of all eigenvalues of A. Then $\delta(A) = \sup\{\lambda \mid \lambda \in \sigma(A)\}$ is called the spectral radius of A.*

Theorem 5.7 *Let $A \in F_n$. Then $\delta(A)$ is either ϕ or I.*

Proof If $\sigma(A) = \{\phi\}$, then $\delta(A) = \phi$, otherwise, if there exist $\lambda \in \sigma(A)$ $(\lambda \neq \phi)$, then there is a non-zero eigenvector $X \in V^n$ (set of column vectors of A of order n) such that $AX = \lambda X$. Also we know that for any β with $\lambda \leq \beta \leq I$, $\beta \cdot \lambda = \lambda$ and $\lambda \cdot \lambda = \lambda$.

Therefore,

$\lambda X = (\beta \cdot \lambda)X = \beta(\lambda X)$

$\Rightarrow A(\lambda X) = \lambda(AX) = \lambda(\lambda X) = (\lambda \cdot \lambda)X = \lambda X = \beta(\lambda X)$.

Hence, $\beta \in \sigma(A)$.

Since β is arbitrary, $I \in \sigma(A)$. Therefore $\delta(A) = I$.

Theorem 5.8 *For any $A, B \in F_n$, if $A \leq B$, then $\delta(A) \leq \delta(B)$.*

Proof From Theorem 5.7, $\delta(A)$ is either ϕ or I.

If $\delta(A) = \phi$, then $\delta(A) \leq \delta(B)$ holds trivially.

If $\delta(A) = I$, we have to prove that $\delta(B) = I$.

Since $\delta(A) = I$, then by definition $I \in \sigma(A)$ and $AX = IX = X$ for some non-zero column vector X. We consider $e = [I, I, I, \cdots, I]^T$, then $X \leq e$.

Also $A^nX = A^{n-1}AX = A^{n-1}X = A^{n-2}X = \cdots = A^2X = AX = X$,

i.e., $X = A^nX \leq A^ne \leq B^ne$. (Since $X \leq e$ and $A \leq B$.)

Since X is non-zero, hence B^ne is non-zero.

Now, if $Y = B^ne$, then $BY = B^{n+1}e = B^ne = Y = IY$. Hence $I \in \sigma(B)$.

Thus, $\delta(B) = I$.

5. Conclusion

We study the properties of similarity relations, invertibility conditions and eigenvalues of IFMs. A very few work are available to find the eigenvalues of a fuzzy matrix. In this paper, first time we investigate the eigenvalues and eigenvectors of an IFMs and illustrated with suitable examples. Also an outline has been given to find the eigenvectors and eigenvalues of an IFMs. An attempt has been made to find eigenvectors and eigenvalues for some particular types of IFMs. This is the first attempt to find the eigenvectors and eigenvalues of IFMs. More investigations are required for find them for all other IFMs.

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