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Prime Fuzzy Ideals, Weakly Prime Fuzzy Ideals of Γ-semigroups

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Abstract Using fuzzy points the notions of prime fuzzy ideals and weakly prime fuzzy ideals of a Γ-semigroup have been introduced. Some important properties and characterizations of these ideals have been obtained. The concluding result shows that our work sharpens previous works on prime fuzzy ideals of Γ-semigroups.

Keywords Γ-semigroup · Prime fuzzy ideal · Weakly prime fuzzy ideal

1. Introduction

Uncertainty is an attribute of information and uncertain data are presented in various domains. The most appropriate theory for dealing with uncertainties was introduced by Zadeh [17] in 1965 by defining fuzzy set which has opened up keen insights and applications in vast range of scientific fields. Rosenfeld [9] pioneered the study of fuzzy algebraic structures by introducing the notions of fuzzy groups and showed that many results in groups can be extended in an elementary manner to develop algebraic concepts. After that Kuroki [7, 8] started the study of fuzzy ideal theory in semigroups. Xie [16] used the notion of fuzzy points to introduce prime fuzzy ideals in semigroups. The notion of Γ-semigroups was introduced by M.K. Sen [14] as a generalization of semigroups. T.K. Dutta and N.C. Adhikari [3] developed the theory of Γ-semigroups by introducing the notion of operator semigroups. Γ-semigroups have also been the object of study of many researchers like Chattopadhyay [1, 5], Chinram et al. [2]. The notion of Γ-semigroups have been extended to fuzzy setting by S.K. Sardar and S.K. Majumder [11-13]. They have studied fuzzy ideals, fuzzy prime ideals, fuzzy semiprime ideals and fuzzy ideal extensions in Γ-semigroups in usual way as well as via operator semigroups. Since Γ-semigroups generalize the

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notion of semigroups, in order to develop the theory of Γ-semigroups one takes the impetus from the results of semigroups and investigates their validity in the general setting. In this paper mainly focusing on this philosophy in the context of fuzzy setting, we investigate in Γ-semigroups the validity of various properties of fuzzy prime ideals of semigroups. We study here prime fuzzy ideals, weakly prime fuzzy ideals in Γ-semigroups by using the notion of fuzzy points. Results of Sardar and Majumder [12] are encompassed in our work (*cf.* Remark 6).

It is important to mention here as to why different types of prime ideals arise in fuzzy setting in contrast with the crisp setting of semigroups or Γ-semigroups. When we formulate some fuzzy notions, to check the correctness of the formulation, we always verify whether the level subset criterion and characteristic function criterion are satisfied. Some situations are very nice where translations of crisp notions to fuzzy setting become compatible with the level subset criterion and characteristic function criterion. But in case of prime fuzzy ideals, the situation is not so nice. Just by analogy with the definition of prime ideal in crisp algebra (*cf.* Definition 12), if we define prime fuzzy ideal (*cf.* Definition 14) in Γ-semigroups, then we see that (strong) level subset criterion does not hold (*cf.* Example 5). In order to make the notion compatible with the level subset and the strong level subset criteria (*cf.* Theorem 9), the notion of weakly prime fuzzy ideal (*cf.* Definition 15) is introduced. In this regard, we refer to [6].

We organize the paper as follows. In Section 2, we recall some preliminary notions of Γ-semigroups as well as of fuzzy subsets in Γ-semigroups. In Section 3, we define fuzzy points and their composition in a Γ-semigroup and subsequently characterize composition of two fuzzy points in Γ-semigroups (*cf.* Theorem 1). Also some related properties of fuzzy points are studied in this section. In Section 4, prime fuzzy ideals of Γ-semigroups are defined. We then obtain various properties of prime fuzzy ideals (*cf.* Propositions 5, 6, 7, Theorems 3, 5, Corollary 2). Some important characterizations of prime fuzzy ideals are also obtained (*cf.* Theorems 6, 7, 8). Weakly prime fuzzy ideals of Γ-semigroups are then defined and studied. It is shown that unlike prime fuzzy ideals they satisfy level subset criterion (*cf.* Theorem 9). Some other important properties of weakly prime fuzzy ideals are also obtained (*cf.* Theorem 10). To conclude it is shown that weakly prime fuzzy ideals in a commutative Γ-semigroup is nothing but the fuzzy prime ideals [12] of both sided Γ-semigroups which means, as mentioned earlier, that results of [12] are encompassed in our work.

2. Preliminaries

In this section, we discuss some elementary definitions that we will use later in this paper.

Definition 1 [14] *Let S and Γ be two non-empty sets. S is called a Γ-semigroup if there exist mappings from $S \times \Gamma \times S$ to S, written as $(a, \alpha, b) \longrightarrow a\alpha b$, and from $\Gamma \times S \times \Gamma$ to Γ, written as $(\alpha, a, \beta) \longrightarrow \alpha a\beta$ satisfying the following associative laws $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ and $\alpha(a\beta b)\gamma = (\alpha a\beta)b\gamma = \alpha a(\beta b\gamma)$ for all $a, b, c \in S$ and for all $\alpha, \beta, \gamma \in \Gamma$.*

Remark 1 Definition 1 is the definition of both sided Γ-semigroup. It may be noted here that in 1986, Sen and Saha [15] introduced the notion of one sided Γ-semigroups.

Definition 2 [15] *Let $S = \{x, y, z, \cdots\}$ and $\Gamma = \{\alpha, \beta, \gamma \cdots\}$ be two non-empty sets. Then S is called a Γ-semigroup (one sided) if there exists a mapping $S \times \Gamma \times S \to S$ (images to be denoted by $a\alpha b$) satisfying*
(1) $x\gamma y \in S$,
(2) $(x\beta y)\gamma z = x\beta(y\gamma z)$ *for all* $x, y, z \in S, \alpha, \beta, \gamma \in \Gamma$.

Throughout this paper unless otherwise mentioned S stands for a one sided Γ-semigroup.

Example 1 [14] Let S be the set of all 2×3 matrices over the set of positive integers and Γ be the set of all 3×2 matrices over same set. Then S is a both sided as well as a one sided Γ-semigroup with respect to the usual matrix multiplication.

Example 2 Let S be a set of all negative rational numbers. Obviously, S is not a semigroup under usual product of rational numbers. Let $\Gamma = \{-\frac{1}{p} \colon p \text{ is prime}\}$. Let $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Now if $a\alpha b$ is equal to the usual product of rational numbers a, α, b, then $a\alpha b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence S is a one sided Γ-semigroup. It is also clear that it is not a both sided Γ-semigroup.

Definition 3 *A Γ-semigroup S is called a commutative Γ-semigroup if $a\alpha b = b\alpha a$ for all $a, b \in S$ and $\alpha \in \Gamma$.*

Definition 4 [17] *A fuzzy subset μ of a non-empty set X is a function $\mu : X \to [0, 1]$.*

Definition 5 *Let μ be a fuzzy subset of a non-empty set X. Then the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ for $t \in [0, 1]$, is called the level subset or t-level subset of μ and the set ${\mu_t}^{>} = \{x \in X : \mu(x) > t\}$ for $t \in [0, 1]$, is called the strong level subset or strong t-level subset of μ.*

Definition 6 [11] *Let S be a Γ-semigroup and μ_1, μ_2 be two fuzzy subsets of S. Then the composition of μ_1 and μ_2 is defined by*

$$(\mu_1 \circ \mu_2)(x) = \begin{cases} \bigvee\limits_{x=y\gamma z} \{\min\{\mu_1(y), \mu_2(z)\}\}, & \textit{if there exist } y, z \in S, \gamma \in \Gamma \textit{ with } x = y\gamma z, \\ 0, & \textit{otherwise.} \end{cases}$$

Definition 7 [3] *Let S be a Γ-semigroup. A non-empty subset I of S is said to be a right ideal (left ideal) of S if $I\Gamma S \subseteq I$ (resp. $S\Gamma I \subseteq I$). I is said to be an ideal of S if it is a right ideal as well as a left ideal of S.*

Definition 8 [11] *Let S be a Γ-semigroup. A non-empty fuzzy subset μ of S is called a fuzzy left ideal (fuzzy right ideal) of S if $\mu(x\alpha y) \geq \mu(y)$ (resp. $\mu(x\alpha y) \geq \mu(x)$) for all $x, y \in S$, and for all $\alpha \in \Gamma$. μ is said to be a fuzzy ideal of S if it is a fuzzy right ideal as well as a fuzzy left ideal of S. Equivalently, a non-empty fuzzy subset μ of S is called a fuzzy ideal if $S \circ \mu \circ S \subseteq \mu$, where S is the characteristic function of S.*

3. Fuzzy Points

In this section, we define fuzzy points and their composition in a Γ-semigroup. Many of their properties are also observed here for their use in the sequel.

Definition 9 *Let S be a Γ-semigroup of S. Let $a \in S$ and $t \in [0, 1]$. We now define a fuzzy subset a_t of S as follows*

$$a_t(x) = \begin{cases} t, & \text{if } x = a, \\ 0, & \text{otherwise} \end{cases}$$

for all $x \in S$. We call a_t a fuzzy point or fuzzy singleton of S.

Remark 2 For any fuzzy subset f of S, it is clear that $f = \bigcup_{a_t \subseteq f} a_t$.

Definition 10 *Let a_t and b_r be two fuzzy points of a Γ-semigroup S. Then the composition of these two fuzzy points is defined by*

$$a_t \circ b_r(x) = \begin{cases} \bigvee_{x=y\gamma z} \{\min\{a_t(y), b_r(z)\}\}, & \text{if there exist } y, z \in S, \gamma \in \Gamma \text{ with } x = y\gamma z, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1 *Let S be a Γ-semigroup, a_t and b_r be two fuzzy points of S. Then*

$$a_t \circ b_r = \bigcup_{\gamma \in \Gamma} (a\gamma b)_{t \wedge r}.$$

Proof Let $x \in S$. If there do not exist any $y, z \in S$ and $\gamma \in \Gamma$ such that $x = y\gamma z$, then $(a_t \circ b_r)(x) = 0 = (\bigcup_{\gamma \in \Gamma} (a\gamma b)_{t \wedge r})(x)$.

Again for $y, z \in S$ and $\gamma \in \Gamma$ if $x = y\gamma z$ implies either $y \neq a$ or $z \neq b$, then also $(a_t \circ b_r)(x) = 0 = (\bigcup_{\gamma \in \Gamma} (a\gamma b)_{t \wedge r})(x)$.

Now if $x = a\gamma' b$ for some $\gamma' \in \Gamma$, then

$$(a_t \circ b_r)(x) = a_t(a) \wedge b_r(b) = t \wedge r = (a\gamma' b)_{t \wedge r}(x) = (\bigcup_{\gamma \in \Gamma} (a\gamma b)_{t \wedge r})(x).$$

Hence $a_t \circ b_r = \bigcup_{\gamma \in \Gamma} (a\gamma b)_{t \wedge r}$.

Lemma 1 *Let S be a Γ-semigroup, f, g and h be fuzzy subsets of S. Then $f \circ (g \cup h) = (f \circ g) \cup (f \circ h)$.*

Proof Let $x \in S$. Then

$$\begin{aligned} (f \circ (g \cup h))(x) &= \sup_{\substack{x=y\gamma z \\ y,z \in S, \gamma \in \Gamma}} \{f(y) \wedge (g \cup h)(z)\} = \sup_{\substack{x=y\gamma z \\ y,z \in S, \gamma \in \Gamma}} \{f(y) \wedge \{g(z) \vee h(z)\}\} \\ &= \sup_{\substack{x=y\gamma z \\ y,z \in S, \gamma \in \Gamma}} \{\{f(y) \wedge g(z)\} \vee \{f(y) \wedge h(z)\}\} \\ &= \sup_{\substack{x=y\gamma z \\ y,z \in S, \gamma \in \Gamma}} \{f(y) \wedge g(z)\} \vee \sup_{\substack{x=y\gamma z \\ y,z \in S, \gamma \in \Gamma}} \{f(y) \wedge h(z)\} \\ &= ((f \circ g) \cup (f \circ h))(x). \end{aligned}$$

This completes the proof.

Definition 11 *Let S be a Γ-semigroup and a_t be a fuzzy point of S. Then the fuzzy ideal generated by a_t denoted by $< a_t >$, is the smallest fuzzy ideal containing a_t in S.*

Proposition 1 *Let S be a Γ-semigroup and a_t be a fuzzy point of S. Then the fuzzy ideal generated by a_t denoted by $< a_t >$, is*

$$< a_t > (x) = \begin{cases} t, & \text{if } x \in < a >, \\ 0, & \text{otherwise} \end{cases}$$

for any $x \in S$, where $< a >$ is an ideal of S generated by a.

Proof Let f be a fuzzy ideal of S such that $a_t \subseteq f$. Then $f(a) \geq a_t(a) = t$. Now let $z \in < a >= \{a\} \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S$. If $z = a$, then $f(z) = f(a) \geq t$. If $z = a\alpha x$ for some $x \in S$ and $\alpha \in \Gamma$, then $f(z) = f(a\alpha x) \geq f(a) \geq t$ (*cf.* Definition 8). Similarly, we can show that $f(z) \geq t$ if $z = y\beta a$ or $z = x\alpha a\beta y$ for some $x, y \in S$ and $\alpha, \beta \in \Gamma$. So $< a_t >\subseteq f$.

Again let $x, y \in S$ and $\gamma \in \Gamma$. If $x, y \in < a >$, then $x\gamma y \in < a >$. So $< a_t > (x\gamma y) = t =< a_t > (x) =< a_t > (y)$. Again if $x, y \notin < a >$ but $x\gamma y \in < a >$, then $< a_t > (x\gamma y) = t \geq 0 =< a_t > (x) =< a_t > (y)$. If $x, y \notin < a >$ and $x\gamma y \notin < a >$, then $< a_t > (x\gamma y) = 0 =< a_t > (x) =< a_t > (y)$. Again if $x \in < a >$ and $y \notin < a >$, then $x\gamma y \in < a >$. So $< a_t > (x\gamma y) = t =< a_t > (x) \geq 0 =< a_t > (y)$. So $< a_t >$ is a fuzzy ideal. Since $< a_t > (x) \geq a_t(x)$ for all $x \in S$, $< a_t >$ contains a_t. Hence $< a_t >$ is the fuzzy ideal generated by a_t.

Proposition 2 *Let S be a Γ-semigroup and a_t be a fuzzy point of S. Then*

$$S \circ a_t \circ S(x) = \begin{cases} t, & \text{if } x \in S\Gamma a\Gamma S, \\ 0, & \text{otherwise} \end{cases}$$

for all $x \in S$.

Proof Let $x \in S$. If $x \neq w\alpha z\beta y$ for any $w, z, y \in S$ and $\alpha, \beta \in \Gamma$, then $x \notin S\Gamma a\Gamma S$ and $S \circ a_t \circ S(x) = 0$. Now let $x = w\alpha z\beta y$ for some $w, z, y \in S$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned} S \circ a_t \circ S(x) &= \bigvee_{x=p\gamma q} \{S \circ a_t(p) \wedge S(q)\} \\ &= \bigvee_{x=p\gamma q} \{S \circ a_t(p)\} \\ &= \bigvee_{x=s\delta r\gamma q} \{S(s) \wedge a_t(r)\} \\ &= \bigvee_{x=s\delta r\gamma q} a_t(r). \end{aligned}$$

If there exists one $r = a$, then $a_t(r) = 1$ and so $S \circ a_t \circ S(x) = 1$, i.e., if $x \in S\Gamma a\Gamma S$, then $S \circ a_t \circ S(x) = t$, otherwise $S \circ a_t \circ S(x) = 0$.

Proposition 3 *Let S be a Γ-semigroup and a_t be a fuzzy point of S. Then $< a_t >= a_t \cup a_t \circ S \cup S \circ a_t \cup S \circ a_t \circ S$.*

Proof By Proposition 1, we can say that for any $x \in S$,

$$< a_t > (x) = \begin{cases} t, & \text{if } x \in < a >, \\ 0, & \text{otherwise.} \end{cases}$$

Let $x \in S$. If $x \notin < a >= \{a\} \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S$, then $< a_t > (x) = 0$. Now $x \notin S\Gamma a\Gamma S$ implies that $S \circ a_t \circ S(x) = 0$, $x \notin S\Gamma a$ implies that $S \circ a_t(x) = 0$, $x \notin a\Gamma S$ implies that $a_t \circ S(x) = 0$ and $x \neq a$ implies that $a_t(x) = 0$. So $a_t \cup a_t \circ S \cup S \circ a_t \cup S \circ a_t \circ S(x) = 0$. Again if $x \in < a >= \{a\} \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S$, then $< a_t > (x) = t$. Now $x \in S\Gamma a\Gamma S$ implies that $S \circ a_t \circ S(x) = t$, $x \in S\Gamma a$ implies that $S \circ a_t(x) = t$, $x \in a\Gamma S$ implies that $a_t \circ S(x) = t$ and $x = a$ implies that $a_t(x) = t$. So $a_t \cup a_t \circ S \cup S \circ a_t \cup S \circ a_t \circ S(x) = t$. Hence $< a_t >= a_t \cup a_t \circ S \cup S \circ a_t \cup S \circ a_t \circ S$.

Corollary 1 *Let S be a Γ-semigroup and a_t be a fuzzy point of S. Then $< a_t >^3 \subseteq S \circ a_t \circ S$.*

Proof $< a_t >^2 = (a_t \cup a_t \circ S \cup S \circ a_t \cup S \circ a_t \circ S) \circ (a_t \cup a_t \circ S \cup S \circ a_t \cup S \circ a_t \circ S) \subseteq S \circ (a_t \cup a_t \circ S \cup S \circ a_t \cup S \circ a_t \circ S) = S \circ a_t \cup S \circ a_t \circ S$. So $< a_t >^3 = < a_t >^2 \circ < a_t > \subseteq (S \circ a_t \cup S \circ a_t \circ S) \circ (a_t \cup a_t \circ S \cup S \circ a_t \cup S \circ a_t \circ S) \subseteq (S \circ a_t \cup S \circ a_t \circ S) \circ S \subseteq S \circ a_t \circ S$.

Proposition 4 *Let S be a Γ-semigroup, A and B be subset of S and C_A be the characteristic function of A. Then for any $t, r \in (0, 1]$, the following statements hold.*

(i) $tC_A \circ rC_B = (t \wedge r)C_{A\Gamma B}$.

(ii) $tC_A \cap tC_B = tC_{A\cap B}$.

(iii) $tC_A = \bigcup\limits_{a\in A} a_t$.

(iv) $S \circ tC_A = tC_{S\Gamma A}$.

(v) *If A is an ideal (right ideal, left ideal) of S, then tC_A is a fuzzy ideal (fuzzy right ideal, fuzzy left ideal) of S.*

4. Prime Fuzzy Ideals and Weakly Prime Fuzzy Ideals

In this section, we deduce various properties and characterizations of prime fuzzy ideals and weakly prime fuzzy ideals of Γ-semigroups.

Definition 12 [5] *Let S be a Γ-semigroup. Then an ideal $I(\neq S)$ of S is called prime if for any two ideals A and B of S, $A\Gamma B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$.*

Definition 13 *Let S be a Γ-semigroup. Then an ideal $I(\neq S)$ of S is called completely prime if for any $a, b \in S$, $a\Gamma b \subseteq I$ implies $a \in I$ or $b \in I$.*

Definition 14 *Let S be a Γ-semigroup. Then a fuzzy ideal f of S is called prime fuzzy ideal if f is a nonconstant function and for any two fuzzy ideals g and h of S, $g \circ h \subseteq f$ implies $g \subseteq f$ or $h \subseteq f$.*

Example 3 Let $S = \mathbb{Z}_0^-$ and $\Gamma = \mathbb{Z}_0^-$, where $\mathbb{Z}_0^-$ denotes the set of all negative integers with 0. Then S is a Γ-semigroup. Let p be a prime number. Now we define a fuzzy subset f on S by

$$f(x) = \begin{cases} 1, & \text{for } x \in p\mathbb{Z}_0^-, \\ 0.6, & \text{otherwise.} \end{cases}$$

Then f is a prime fuzzy ideal of S.

Example 4 Let $S = \{a, b, c\}$. Let $\Gamma = \{\alpha, \beta\}$ be the non-empty set of binary operations on S with the following Cayley tables.

Table 1: Γ-semigroups of Example 4.

α	a	b	c	β	a	b	c
a	a	b	b	a	b	b	b
b	b	b	b	b	b	b	b
c	c	c	c	c	c	c	c

By a routine verification, we see that S is a Γ-semigroup. Now we define a fuzzy subset μ on S by $\mu(a) = 0.5$, $\mu(b) = 1 = \mu(c)$. It is easy to observe that μ is a prime fuzzy ideal of S.

Theorem 2 *Let S be a commutative Γ-semigroup and f be a fuzzy ideal of S. Then f is prime fuzzy ideal if and only if for any fuzzy subsets g and h of S, $g \circ h \subseteq f$ implies $g \subseteq f$ or $h \subseteq f$.*

Proof Let f be a prime fuzzy ideal and g, h be two fuzzy subsets such that $g \circ h \subseteq f$ and $g \nsubseteq f$. Then there exists $x_t \subseteq g$ such that $x_t \nsubseteq f$. Let $y_r \subseteq h$. Then $x_t \circ y_r \subseteq g \circ h$ whence $x_t \circ y_r \subseteq f$. By Proposition 3, $< x_t > \circ < y_r >= (x_t \cup x_t \circ S \cup S \circ x_t \cup S \circ x_t \circ S) \circ (y_r \cup y_r \circ S \cup S \circ y_r \cup S \circ y_r \circ S)$. Since S is commutative, $< x_t > \circ < y_r >\subseteq (x_t \circ y_r) \cup (x_t \circ y_r \circ S) \cup (S \circ x_t \circ y_r) \cup (S \circ x_t \circ y_r \circ S)$. So $< x_t > \circ < y_r >\subseteq (f \circ S) \cup (S \circ f) \cup (S \circ f \circ S)$. Hence $< x_t > \circ < y_r >\subseteq f$ (*cf.* Definition 8). Since f is a prime fuzzy ideal, $< x_t >\subseteq f$ or $< y_r >\subseteq f$. As $x_t \nsubseteq f$, $< x_t >\nsubseteq f$. So $< y_r >\subseteq f$. Hence $y_r \subseteq f$. Consequently, $h \subseteq f$.

Converse part is obvious.

Theorem 3 *Let S be a Γ-semigroup and I be an ideal of S. Then I is a prime ideal of S if and only if C_I is a prime fuzzy ideal of S where C_I is the characteristic function of I.*

Proof Let I be a prime ideal of S. Then C_I is a fuzzy ideal of S (*cf.* Proposition 4). Now let f and g be two fuzzy ideals of S such that $f \circ g \subseteq C_I$ and $f \nsubseteq C_I$. Then there exists a fuzzy point $x_t \subseteq f$ $(t > 0)$ such that $x_t \nsubseteq C_I$. Let $y_r \subseteq g$ $(r > 0)$. Then $< x_t > \circ < y_r >\subseteq f \circ g \subseteq C_I$. Again for all $z \in S$, in view of Proposition 1 and Proposition 4, we obtain

$$< x_t > \circ < y_r > (z) = \begin{cases} t \wedge r, & \text{if } z \in < x > \Gamma < y >, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $< x > \Gamma < y > \subseteq I$. This together with the hypothesis, implies that $< x > \subseteq I$ or $< y > \subseteq I$ (*cf.* Definition 12). Since $x_t \nsubseteq C_I$, $t = x_t(x) > C_I(x)$. So $C_I(x) = 0$ whence $x \notin I$. Hence $< x > \nsubseteq I$. Consequently, $< y > \subseteq I$. Then $y_r \subseteq C_I$ and so $g \subseteq C_I$. Hence C_I is a prime fuzzy ideal of S.

Conversely, suppose C_I is a prime fuzzy ideal of S. Let A and B be two fuzzy ideals of S such that $A\Gamma B \subseteq I$. Then by Proposition 4, C_A and C_B are fuzzy ideals of S and $C_A \circ C_B = C_{A\Gamma B} \subseteq C_I$. So by hypothesis, $C_A \subseteq C_I$ or $C_B \subseteq C_I$. Hence $A \subseteq I$ or $B \subseteq I$. Consequently, I is a prime ideal of S.

Proposition 5 *Let S be a Γ-semigroup and f be a prime fuzzy ideal of S. Then* $|Imf| = 2$.

Proof By Definition 14, f is a nonconstant fuzzy ideal. So $|Imf| \geq 2$. Suppose $|Imf| > 2$. Then there exist $x, y, z \in S$ such that $f(x), f(y), f(z)$ are distinct. Let us assume, without loss of generality, $f(x) < f(y) < f(z)$. Then there exist $r, t \in (0, 1)$ such that $f(x) < r < f(y) < t < f(z) \cdots (1)$. Then for all $u \in S$,

$$< x_r > \circ < y_t > (u) = \begin{cases} r \wedge t, & \text{if } u \in < x > \Gamma < y >, \\ 0, & \text{otherwise.} \end{cases}$$

Let $u \in < x > \Gamma < y >$. Then $f(u) \geq f(x) \vee f(y) > r \wedge t$. Therefore $< x_r > \circ < y_t > \subseteq f$ which, by Definition 14, implies that $< x_r > \subseteq f$ or $< y_t > \subseteq f$. Suppose $< x_r > \subseteq f$. Then $f(x) \geq < x_r > (x) = r$ which contradicts (1). Similarly, $< y_t > \subseteq f$ contradicts (1). Hence $|Imf| = 2$.

Theorem 4 *Let S be a Γ-semigroup and f be a prime fuzzy ideal of S. Then there exists an $x_0 \in S$ such that $f(x_0) = 1$.*

Proof By Proposition 5, we have $|Imf| = 2$. Suppose $Imf = \{t, s\}$ such that $t < s$. Let if possible $f(x) < 1$ for all $x \in S$. Then $t < s < 1$. Let $f(x) = t$ and $f(y) = s$ for some $x, y \in S$ such that $f(x) = t < s = f(y) < 1$. Now we choose $t_1, t_2 \in (0, 1)$ such that $f(x) < t_1 < s < t_2 < 1$. Then by the similar argument as applied in the proof of Proposition 5, we obtain $< x_{t_1} > \circ < y_{t_2} > \subseteq f$. Since f is a prime fuzzy ideal of S, $< x_{t_1} > \subseteq f$ or $< y_{t_2} > \subseteq f$ whence $f(x) \geq t_1$ or $f(y) \geq t_2$. This contradicts the choices of t_1 and t_2. Hence there exists an $x_0 \in S$ such that $f(x_0) = 1$.

Theorem 5 *Let S be a Γ-semigroup and f be a prime fuzzy ideal of S. Then*

(i) *each level subset f_t ($\neq S$), $t \in (0, 1]$, if non-empty, is a prime ideal of S, and*

(ii) *each strong level subset $f_t^{>}$ ($\neq S$), $t \in [0, 1]$, if non-empty, is a prime ideal of S.*

Proof (i) Since f is a fuzzy ideal, each level subset f_t, $t \in (0, 1]$, if non-empty, is an ideal of S (*cf.* Theorem 3.3 [11]). Let $t \in (0, 1]$ be such that f_t ($\neq S$) is non-empty. Now let I, J be two ideals of S such that $I\Gamma J \subseteq f_t$. Then $tC_{I\Gamma J} \subseteq f$. By Proposition 4(v), $g = tC_I$ and $h = tC_J$ are fuzzy ideals of S. Since $g \circ h = tC_I \circ tC_J = tC_{I\Gamma J}$ (*cf.* Proposition 4(i)), $g \circ h \subseteq f$. Since f is a prime fuzzy ideal, $g \subseteq f$ or $h \subseteq f$. Hence either $tC_I \subseteq f$ or $tC_J \subseteq f$ whence we obtain $I \subseteq f_t$ or $J \subseteq f_t$. Hence f_t is a prime ideal of S.

(ii) Let $t \in [0, 1]$ be such that $f_t^{>}$ ($\neq S$) is non-empty. Then $f_t^{>}$ is an ideal of S and $t < \sup_{x \in S} f(x)$. Now let $t \in Imf$. Then there exists $x \in S$ such that $f(x) = t$. Since Imf is finite (*cf.* Proposition 5), there exists $t' \in Imf$ satisfying $t < t'$ and there is no element in Imf lying between t and t'. Hence $f_t^{>} = f_{t'}$. By the first part of this theorem, $f_{t'}$ is a prime ideal of S. Hence $f_t^{>}$ is a prime ideal of S. Again let $t \notin Imf$. Since $f_t^{>}$ is non-empty and $f_t^{>} \subseteq f_t$, f_t is also non-empty. If $x \in f_t$, $f(x) \geq t$. But $t \notin Imf$. Hence $f(x) > t$, i.e., $x \in f_t^{>}$. Thus $f_t \subseteq f_t^{>}$ whence $f_t = f_t^{>}$. In view of the first part of this theorem, f_t is a prime ideal of S. Consequently, $f_t^{>}$ is a prime ideal of S.

As a consequence of Theorems 4 and 5 (i), we obtain the following result.

Corollary 2 *If f is a prime fuzzy ideal of a Γ-semigroup S, then f_1 is a prime ideal of S.*

Remark 3 The converse of Theorem 5 is not true which is illustrated in the following example.

Example 5 Let S be a Γ-semigroup and A be a prime ideal of S. Let

$$f(x) = \begin{cases} t, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then f is a fuzzy ideal of S. Here $f_{t_1} = A = f_{t_2}{}^{>}$, where $0 < t_1 \leq t$ and $0 < t_2 < t$. Hence each of non-empty level subsets and strong level subsets of f is a prime ideal of S. But if $0 < t < 1$, then f is not a prime fuzzy ideal of S (*cf.* Theorem 4).

Lemma 2 *Let S be a Γ-semigroup. Then a fuzzy subset f of S satisfying (i) and (ii)*

(i) $|Imf| = 2$.

(ii) *$f_1 \neq \emptyset$ and f_1 is a prime ideal of S,*

is a fuzzy ideal of S.

The following result characterizes a prime fuzzy ideal of a Γ-semigroup.

Theorem 6 *Let S be a Γ-semigroup. Then a fuzzy subset f of S is a prime fuzzy ideal of S if and only if f satisfies the following conditions:*

(i) $|Imf| = 2$.

(ii) *$f_1 \neq \emptyset$ and f_1 is a prime ideal of S.*

Proof The direct implication follows easily from Proposition 5, Theorem 4 and Corollary 2.

To prove the converse, we first observe that f is a fuzzy ideal of S (*cf.* Lemma 2). Then let g and h be two fuzzy ideals of S such that $g \circ h \subseteq f$. If $g \nsubseteq f$ and $h \nsubseteq f$, then there exist $x, y \in S$ such that $g(x) > f(x)$ and $h(y) > f(y)$. Thus $x, y \notin f_1$, which implies $x\Gamma S\Gamma y \not\subseteq f_1$ (*cf.* [5]), then there exist $s \in S$, $\alpha, \beta \in \Gamma$ such that $x\alpha s\beta y \notin f_1$ which means $f(x\alpha s\beta y) < 1$. So $f(x\alpha s\beta y) = t = f(x) = f(y)$, where $Imf = \{t, 1\}$. But by using Definition 6 and Definition 8, we obtain $(g \circ h)(x\alpha s\beta y) \geq g(x) \wedge h(s\beta y) \geq g(x) \wedge h(y) > f(x) \wedge f(y) = t$. Hence $g \circ h \not\subseteq f$ which is a contradiction. Hence f is a prime fuzzy ideal of S.

Corollary 3 *Let S be a Γ-semigroup and f be a prime fuzzy ideal of S. Then there exist a prime fuzzy ideal g of S such that f is properly contained in g.*

Proof By Theorem 6, there exists $x_0 \in S$ such that $f(x_0) = 1$ and $Im(f) = \{t, 1\}$ for some $t \in [0, 1)$. Let g be a fuzzy subset of S defined by $g(x) = 1$, if $x \in f_1$ and $g(x) = r$, if $x \notin f_1$, where $t < r < 1$. Then by Theorem 6, g is a prime fuzzy ideal and $f \subsetneq g$.

Proposition 6 *Let S be a Γ-semigroup, f be a prime fuzzy ideal of S and g, h be fuzzy subsets of S such that $g \circ S \circ h \subseteq f$. Then $g \subseteq f$ or $h \subseteq f$.*

Proof Let $a_t \subseteq g$, $b_r \subseteq h$, where $t, r \in (0, 1]$. Then $a_t \circ S \circ b_r \subseteq g \circ S \circ h \subseteq f$ and so $(S \circ a_t \circ S) \circ (S \circ b_r \circ S) \subseteq S \circ (a_t \circ S \circ b_r) \circ S \subseteq S \circ f \circ S$. Since f is a fuzzy ideal of S, $S \circ f \circ S \subseteq f$. Hence f being a prime fuzzy ideal and $S \circ a_t \circ S$ and $S \circ b_r \circ S$ are fuzzy ideals of S, we deduce that $S \circ a_t \circ S \subseteq f$ or $S \circ b_r \circ S \subseteq f$. Suppose $S \circ a_t \circ S \subseteq f$. Then $< a_t >^3 \subseteq S \circ a_t \circ S \subseteq f$ (*cf.* Corollary 1) which together with primeness of f implies $a_t \in < a_t > \subseteq f$. Hence $g \subseteq f$. Similarly, $S \circ b_r \circ S \subseteq f$ implies $h \subseteq f$.

Proposition 7 *Let S be a Γ-semigroup, f be a fuzzy ideal of S satisfying that for all fuzzy subsets g, h of S, $g \circ S \circ h \subseteq f$ implies $g \subseteq f$ or $h \subseteq f$. Then*

(i) *For each fuzzy right ideal g and each fuzzy subset h of S, $g \circ h \subseteq f$ implies $g \subseteq f$ or $h \subseteq f$.*

(ii) *For each fuzzy subset g and each fuzzy left ideal h of S, $g \circ h \subseteq f$ implies $g \subseteq f$ or $h \subseteq f$.*

Proof (i) Let g be a fuzzy right ideal and h be a fuzzy subset of S such that $g \circ h \subseteq f$. Let $a_t \subseteq g$, $b_r \subseteq h$, where $t, r \in (0, 1]$. Then $a_t \circ S \circ b_r \subseteq g \circ S \circ h$. Since g is a fuzzy right ideal of S, $g \circ S \circ h \subseteq f$. Hence by hypothesis, $a_t \subseteq f$ or $b_r \subseteq f$. Consequently, $g \subseteq f$ or $h \subseteq f$.

Proof of (ii) is similar to that of (i).

Combination of Propositions 6 and 7 gives rise to the following characterization of a prime fuzzy ideal of a Γ-semigroup.

Theorem 7 *Let S be a Γ-semigroup and f be a fuzzy ideal of S. Then f is prime fuzzy ideal if and only if for all fuzzy subsets g, h of S, $g \circ S \circ h \subseteq f$ implies $g \subseteq f$ or $h \subseteq f$.*

Proof The necessity of the condition follows from Proposition 6.

In order to prove the sufficiency, let g and h be two fuzzy ideals of S such that $g \circ h \subseteq f$. Then g is a fuzzy right ideal and h is a fuzzy subset of S as well. So by Proposition 7, $g \subseteq f$ or $h \subseteq f$.

Lemma 3 *Let S be a Γ-semigroup. Then the fuzzy ideals of S are idempotent if and only if for all fuzzy ideals f, g of S, we have $f \cap g = f \circ g$.*

Proof Let f and g be two fuzzy ideals of S. Then $f \cap g$ is also a fuzzy ideal of S. Let $a_t \subseteq f \cap g$, where $t \in (0, 1]$. Then $a_t \subseteq f$ and $a_t \subseteq g$ whence by Definition 11, $< a_t >\subseteq f$ and $< a_t >\subseteq g$. By hypothesis, $< a_t >=< a_t >^2$ and so $< a_t >\subseteq f \circ g$. Hence $a_t \in f \circ g$. Consequently, $f \cap g \subseteq f \circ g$. Again clearly, $f \circ g \subseteq f \cap g$. Hence $f \cap g = f \circ g$.

Converse part is obvious.

The following result shows when every fuzzy ideal of a Γ-semigroup is a prime fuzzy ideal.

Theorem 8 *Let S be a Γ-semigroup. Then the set $\mathbb{F}$ of all fuzzy ideals of S and the set $\mathbb{P}$ of all prime fuzzy ideals of S coincide if and only if each element of $\mathbb{F}$ is idempotent and $\mathbb{F}$ forms a chain under fuzzy set inclusion.*

Proof Let each element of $\mathbb{F}$ be prime and $f, g \in \mathbb{F}$. Then $f \cap g \in \mathbb{F}$. Let $a_t \subseteq f \cap g$, where $t \in (0, 1]$. Then $a_t \subseteq f$ and $a_t \subseteq g$ whence $< a_t >\subseteq f$ and $< a_t >\subseteq g$. So $< a_t > \circ < a_t >\subseteq f \circ g$. Since $f \circ g$ is also a fuzzy ideal, it is prime by hypothesis. Hence the above inclusion implies $< a_t >\subseteq f \circ g$ and so $a_t \subseteq f \circ g$. Hence $f \cap g \subseteq f \circ g$ and $f \cap g = f \circ g$. Consequently, by Lemma 3, each element of $\mathbb{F}$ is idempotent.

Now for $f, g \in \mathbb{F}$, let $a_t \subseteq f$ and $b_r \subseteq g$, where $t, r \in (0, 1]$. Then $< a_t >\subseteq f$ and $< b_r >\subseteq g$. Hence $< a_t > \circ < b_r >\subseteq f \circ g$. Since $f \circ g \in \mathbb{F}$, by hypothesis, $f \circ g \in \mathbb{P}$. Hence $< a_t >\subseteq f \circ g$ or $< b_r >\subseteq f \circ g$. Again $f \circ g \subseteq g \cap f$. Hence $a_t \subseteq g$ or $b_r \subseteq f$ whence $f \subseteq g$ or $g \subseteq f$. Consequently, $\mathbb{F}$ forms a chain.

Conversely, let the elements of $\mathbb{F}$ be idempotent and form a chain. Let $f, g, h \in \mathbb{F}$ such that $g \circ h \subseteq f$. Let $a_t \subseteq g$ and $b_r \subseteq h$, where $t, r \in (0, 1]$. Then $< a_t >\subseteq g$ and $< b_r >\subseteq h$. Since $< a_t >, < b_r >\in \mathbb{F}$, by hypothesis $< a_t >\subseteq< b_r >$ or $< b_r >\subseteq< a_t >$. Suppose $< a_t >\subseteq< b_r >$. Then $< a_t >=< a_t > \cap < b_r >\subseteq g \cap h = g \circ h$ (using hypothesis and Lemma 3) whence $a_t \subseteq f$. Hence $g \subseteq f$. Similarly, $< b_r >\subseteq< a_t >$ implies that $h \subseteq f$. Consequently, $f \in \mathbb{P}$.

In Theorem 5 we have shown that every non-empty level subset of a prime fuzzy ideal is a prime ideal. But Example 5 shows that the converse need not be true. In order to make the level subset criterion to hold, a new type of fuzzy primeness in ideals of a Γ-semigroup can be defined what is called weakly prime fuzzy ideal.

Definition 15 *Let S be a Γ-semigroup. A nonconstant fuzzy ideal f of S is called a weakly prime fuzzy ideal of S if for all ideals A and B of S and for all $t \in (0, 1]$, $tC_A \circ tC_B \subseteq f$ implies $tC_A \subseteq f$ or $tC_B \subseteq f$.*

Theorem 9 *Let S be a Γ-semigroup and f be a fuzzy ideal of S.*

(i) *Then* f *is a weakly prime fuzzy ideal of* S *if and only if each level subset* f_t $(\neq S)$, $t \in (0,1]$, *is a prime ideal of* S *for* $f_t \neq \emptyset$;

(ii) *If* Imf *is finite, then* f *is a weakly prime fuzzy ideal of* S *if and only if each strong level subset* $f_t^>$ $(\neq S)$, $t \in [0,1]$, *is a prime ideal of* S *for* $f_t^> \neq \emptyset$.

Proof (i) Let f be a weakly prime fuzzy ideal of S, A and B be ideals of S and $t \in (0,1]$ such that $f_t \neq \emptyset$ and $f_t \neq S$. Let $A\Gamma B \subseteq f_t$. Then $tC_{A\Gamma B} \subseteq f$ which means $tC_A \circ tC_B \subseteq f$ (*cf.* Proposition 4). Hence by hypothesis, $tC_A \subseteq f$ or $tC_B \subseteq f$ (*cf.* Definition 15). Hence either $A \subseteq f_t$ or $B \subseteq f_t$. Consequently, f_t is a prime ideal of S.

Conversely, suppose each f_t $(\neq S)$, $t \in (0,1]$, is a prime ideal of S for $f_t \neq \emptyset$. Let A and B be ideals of S such that $tC_A \circ tC_B \subseteq f$ where $t \in (0,1]$. Then $tC_{A\Gamma B} \subseteq f$ (*cf.* Proposition 4) whence $A\Gamma B \subseteq f_t$. Hence either $A \subseteq f_t$ or $B \subseteq f_t$ whence $tC_A \subseteq f$ or $tC_B \subseteq f$. Hence f is a weakly prime fuzzy ideal of S.

(ii) Let f be a weakly prime fuzzy ideal of S. Then by applying similar argument as applied in the proof of Theorem 5 (ii), we deduce from the first part, that each $f_t^>$ $(\neq S)$, $t \in [0,1]$, is a prime ideal of S for $f_t^> \neq \emptyset$.

Conversely, suppose each strong level subset $f_t^>$ $(\neq S)$, $t \in [0,1]$, is a prime ideal of S for $f_t^> \neq \emptyset$. Let $t \in [0,1]$ with $f_t \neq \emptyset$ and $f_t \neq S$. Now let $t \in Imf$. Then three cases may arise.

Case 1 $t = \inf_{y \in S} f(y)$. Then $f_t = S$. So t cannot be $\inf_{y \in S} f(y)$.

Case 2 $t = \sup_{y \in S} f(y)$. Then since Imf is finite and f is nonconstant by Definition 15, there exists $t' \in Imf$ satisfying $t' < t$ and there is no element in Imf lying between t' and t. So $f_{t'}^> = f_t$.

Case 3 $\inf_{y \in S} f(y) < t < \sup_{y \in S} f(y)$. Then since Imf is finite, there exists $t' \in Imf$ satisfying $t' < t$ and there is no element in Imf lying between t' and t. So $f_{t'}^> = f_t$.

Now for $t \notin Imf$, clearly $f_t = f_t^>$. By hypothesis, $f_t^>$ is a prime ideal of S and hence so is f_t. Consequently, by the first part we conclude that f is a weakly prime fuzzy ideal of S.

As an easy consequence of Theorems 5 and 9, we obtain the following corollary.

Corollary 4 *In a* Γ*-semigroup* S, *every prime fuzzy ideal is a weakly prime fuzzy ideal.*

The following example shows that the converse of the above corollary is not always true.

Example 6 Let $S = \{0, a, b, c\}$. Let $\Gamma = \{\alpha, \beta, \gamma\}$ be the non-empty set of binary operations on S with the following Cayley tables.

Table 2: Γ-semigroups of Example 6.

α	0	a	b	c	β	0	a	b	c	γ	0	a	b	c
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	0	0	b	a	0	0	0	b	a	0	0	0	0
b	0	0	0	b	b	0	0	0	b	b	0	0	0	0
c	b	b	b	c	c	b	b	b	b	c	0	0	0	0

By a routine but tedious verification, we see that S is a Γ-semigroup. Now we define a fuzzy subset f on S by $f(0) = f(a) = 0.8$, $f(b) = 0.3$ and $f(c) = 0$. It can be checked that f is a weakly prime fuzzy ideal of S. But f is not a prime fuzzy ideal of S (*cf.* Theorem 6).

Remark 4 The above corollary and the example together show that the notion of weakly prime fuzzy ideal generalizes the notion of prime fuzzy ideal. For similar notion in semigroups, we refer to [8].

The following theorem characterizes weakly prime fuzzy ideals of Γ-semigroups.

Theorem 10 *Let S be a Γ-semigroup and f be a fuzzy ideal of S. Then the followings are equivalent.*

(i) *f is a weakly prime fuzzy ideal of S.*

(ii) *For any $x, y \in S$ and $r \in (0, 1]$, if $x_r \circ S \circ y_r \subseteq f$, then $x_r \subseteq f$ or $y_r \subseteq f$.*

(iii) *For any $x, y \in S$ and $r \in (0, 1]$, if $< x_r > \circ < y_r > \subseteq f$, then $x_r \subseteq f$ or $y_r \subseteq f$.*

(iv) *If A and B are right ideals of S such that $tC_A \circ tC_B \subseteq f$, then $tC_A \subseteq f$ or $tC_B \subseteq f$.*

(v) *If A and B are left ideals of S such that $tC_A \circ tC_B \subseteq f$, then $tC_A \subseteq f$ or $tC_B \subseteq f$.*

(vi) *If A is a right ideal of S and B is a left ideal of S such that $tC_A \circ tC_B \subseteq f$, then $tC_A \subseteq f$ or $tC_B \subseteq f$.*

Proof (i) ⇒ (ii).

Let f be a weakly prime fuzzy ideal of S. Let $x, y \in S$ and $r \in (0, 1]$ be such that $x_r \circ S \circ y_r \subseteq f$. Then by Proposition 2, $rC_{S\Gamma x\Gamma S} \circ rC_{S\Gamma y\Gamma S} = (S \circ x_r \circ S) \circ (S \circ y_r \circ S) \subseteq S \circ (x_r \circ S \circ y_r) \circ S \subseteq S \circ f \circ S \subseteq f$. Hence by hypothesis, $rC_{S\Gamma x\Gamma S} \subseteq f$ or $rC_{S\Gamma y\Gamma S} \subseteq f$ whence $S \circ x_r \circ S \subseteq f$ or $S \circ y_r \circ S \subseteq f$. Let $S \circ x_r \circ S \subseteq f$. Then $< x_r >^3 \subseteq f$ (*cf.* Corollary 1). Hence $(rC_{<x>})^3 \subseteq f$. Since f is weakly prime fuzzy ideal, this implies that $< x_r > \subseteq f$ whence $x_r \subseteq f$.

(ii) ⇒ (iii).

Let $x, y \in S$ and $r \in (0, 1]$ be such that $< x_r > \circ < y_r > \subseteq f$. Then since $x_r \circ S \subseteq < x_r >$ and $S \circ y_r \subseteq < y_r >$, $x_r \circ S \circ y_r \subseteq f$. Hence by (ii), $x_r \subseteq f$ or $y_r \subseteq f$.

(iii) ⇒ (iv).

Let A, B be two right ideals of S such that $tC_A \circ tC_B \subseteq f$ and $tC_A \nsubseteq f$. Then there exists $a \in A$ such that $a_t \nsubseteq f$. Now for any $b \in B$, $< a_t > \circ < b_t > = tC_{<a>} \circ tC_{<b>}$

$= tC_{<a>\Gamma<b>} \subseteq tC_{S\Gamma A\Gamma B} = S \circ tC_{A\Gamma B} = S \circ tC_A \circ tC_B$. Again $S \circ tC_A \circ tC_B \subseteq S \circ f \subseteq f$ whence $< a_t > \circ < b_t > \subseteq f$. Hence by (iii), $b_t \subseteq f$. Consequently, $tC_B \subseteq f$.

(iii) $\Rightarrow$ (vi).

Let A be a right ideal and B be a left ideal of S such that $tC_A \circ tC_B \subseteq f$ and $tC_A \nsubseteq f$. Then there exists $a \in A$ such that $a_t \nsubseteq f$. Now for any $b \in B$, $< a_t > \circ < b_t > = tC_{<a>} \circ tC_{<b>} = tC_{<a>\Gamma<b>} \subseteq tC_{S\Gamma A\Gamma B\Gamma S} = S \circ tC_A \circ tC_B \circ S$. Again $S \circ tC_A \circ tC_B \circ S \subseteq S \circ f \circ S \subseteq f$ whence $< a_t > \circ < b_t > \subseteq f$. Hence by (iii), $b_t \subseteq f$. Consequently, $tC_B \subseteq f$.

(iv) $\Rightarrow$ i, (v) $\Rightarrow$ (i), (vi) $\Rightarrow$ (i) are obvious and (iii) $\Rightarrow$ (v) is similar to (iii) $\Rightarrow$ (iv).

Remark 5 The above result is analogous to Theorem 3.4 [4]. This theorem of [4] plays an important role in radical theory of Γ-semigroups. So the above theorem will help study radical theory in Γ-semigroups via fuzzy subsets.

Theorem 11 *Let S be a commutative Γ-semigroup and f be a fuzzy ideal of S. Then f is weakly prime fuzzy ideal if and only if* $\inf_{\gamma\in\Gamma} f(x\gamma y) = \max\{f(x), f(y)\}$ *for all* $x, y \in S$.

Proof Let f be a weakly prime fuzzy ideal and $x, y \in S$. Then f being a fuzzy ideal, $f(x\gamma y) \geq \max\{f(x), f(y)\}$ for all $\gamma \in \Gamma$. So $\inf_{\gamma\in\Gamma} f(x\gamma y) \geq \max\{f(x), f(y)\}$. Now let $\inf_{\gamma\in\Gamma} f(x\gamma y) = t$ where $t \in [0, 1]$. If $t = 0$, then $\inf_{\gamma\in\Gamma} f(x\gamma y) \leq \max\{f(x), f(y)\}$. Otherwise, $f(x\gamma y) \geq t$ for all $\gamma \in \Gamma$, *i.e.*, $(x\gamma y)_t \subseteq f$ for all $\gamma \in \Gamma$. So $\bigcup_{\gamma\in\Gamma} (x\gamma y)_t \subseteq f$ which means $x_t \circ y_t \subseteq f$ (*cf.* Theorem 1). By Proposition 3, $< x_t > \circ < y_t > = (x_t \cup x_t \circ S \cup S \circ x_t \cup S \circ x_t \circ S) \circ (y_t \cup y_t \circ S \cup S \circ y_t \cup S \circ y_t \circ S)$. Since S is commutative, $< x_t > \circ < y_t > \subseteq (x_t \circ y_t) \cup (x_t \circ y_t \circ S) \cup (S \circ x_t \circ y_t) \cup (S \circ x_t \circ y_t \circ S)$. This implies $< x_t > \circ < y_t > \subseteq (f \circ S) \cup (S \circ f) \cup (S \circ f \circ S)$. Hence $< x_t > \circ < y_t > \subseteq f$ (*cf.* Definition 8). Since f is a weakly prime fuzzy ideal, in view of Theorem 10 (iii), we deduce that $x_t \subseteq f$ or $y_t \subseteq f$. Hence $f(x) \geq t$ or $f(y) \geq t$, whence $\max\{f(x), f(y)\} \geq t = \inf_{\gamma\in\Gamma} f(x\gamma y)$. Consequently, $\inf_{\gamma\in\Gamma} f(x\gamma y) = \max\{f(x), f(y)\}$.

Conversely, suppose the condition holds. Let x_t and y_t be two fuzzy points of S such that $< x_t > \circ < y_t > \subseteq f$, where $t \in (0, 1]$. Then $x_t \circ y_t \subseteq f$ whence $\bigcup_{\gamma\in\Gamma} (x\gamma y)_t \subseteq f$ (*cf.* Theorem 1). So $(x\gamma y)_t \subseteq f$ for all $\gamma \in \Gamma$, i.e., $f(x\gamma y) \geq t$ for all $\gamma \in \Gamma$. Hence $\inf_{\gamma\in\Gamma} f(x\gamma y) \geq t$. So by the hypothesis, $\max\{f(x), f(y)\} \geq t$. Then $f(x) \geq t$ or $f(y) \geq t$ whence $x_t \subseteq f$ or $y_t \subseteq f$. Hence in view of Theorem 10 (iii), f is a weakly prime fuzzy ideal of S.

Remark 6 The above characterization of weakly prime fuzzy ideal coincides with the definition of fuzzy prime ideal of both sided Γ-semigroups [12]. This indicates that all the results obtained in [12] for fuzzy prime ideals in both sided Γ-semigroups are valid for weakly prime fuzzy ideals in commutative one sided Γ-semigroups.

5. Concluding Remark

Semigroup theory has a wide range of applications in many significant disciplines like computer sciences, information sciences and coding theory etc. Γ-semigroup, being a generalization of semigroup, carries forward those applications. On the other hand, fuzzy sets generalize the theories of crisp sets. Prime ideal plays an important role in semigroup theory. So fuzzy notion of prime ideal in Γ-semigroups also characterizes Γ-semigroups. We, taking into account the fuzzy points in describing prime fuzzy ideals in Γ-semigroups, generalize semigroups in a significant way. Using this work in future we can characterize fuzzy m-system and fuzzy radicals in Γ-semigroups (*cf.* Remark 5).

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