Fuzzy Inf. Eng. (2012) 4: 389-399
DOI 10.1007/s12543-012-0122-0

ORIGINAL ARTICLE

Fuzzy Generalized Bi-ideals of Γ-semigroups

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Received: 1 March 2012/ Revised: 10 July 2012/
Accepted: 28 October 2012/

Abstract In this paper, we introduce the concept of fuzzy generalized bi-ideal of a Γ-semigroup, which is an extension of the concept of a fuzzy bi-ideal of a Γ-semigroup and characterize regular Γ-semigroups in terms of fuzzy generalized bi-ideals.

Keywords Γ-semigroup · Regular Γ-semigroup · Fuzzy left (right) ideal · Fuzzy ideal · Fuzzy bi-ideal · Fuzzy generalized bi-ideal.

1. Introduction

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation [10]. The formal study of semigroups began in the early 20^{th} century. Semigroups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. The concept of fuzzy sets was introduced by L.A. Zadeh [27] in his classic paper in 1965. Azirel Rosenfeld [17] used the idea of fuzzy sets to introduce the notions of fuzzy subgroups. Nobuaki Kuroki [12-15] is the pioneer of fuzzy ideal theory of semigroups. The idea of fuzzy subsemigroup was also introduced by Kuroki [12, 14]. In [13], Kuroki characterized several classes of semigroups in terms of fuzzy left, fuzzy right and fuzzy bi-ideals. The notion of fuzzy generalized bi-ideals in semigroups was in introduced by Kuroki [15]. Others, who worked on fuzzy semigroup theory, such as X.Y. Xie [25, 26], Y.B. Jun [11] are mentioned in the bibliography. X.Y. Xie [25] introduced the idea of extensions of fuzzy ideals in semigroups.

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The notion of a Γ-semigroup was introduced by Sen and Saha [23] as a generalization of semigroups and ternary semigroup. Γ-semigroup have been analyzed by a lot of mathematicians, for instance by Chattopadhyay [1, 2], Dutta and Adhikari [4, 5], Hila [8, 9], Chinram [3], Saha [21], Sen et al [21-23], Seth [24]. S.K. Sardar and S.K. Majumder [6, 7, 18, 19] have introduced the notion of fuzzification of ideals, prime ideals, semiprime ideals and ideal extensions of Γ-semigroups and studied them via its operator semigroups. In [20], S.K. Sardar, S.K. Majumder and S. Kayal have introduced the concepts of fuzzy bi-ideals and fuzzy quasi-ideals in Γ-semigroups and obtained some important characterizations. The purpose of this paper is as stated in the abstract.

The structure of the paper is organized as follows. In Section 2, we recall some preliminaries of Γ-semigroups and fuzzy notions for their use in the sequel. In Section 3, we introduce the notion of generalized bi-ideal and fuzzy generalized bi-ideal of a Γ-semigroup. We use these concepts to characterize regular Γ-semigroup. Theorem 5 characterizes regular Γ-semigroup in terms of generalized bi-ideals and rest of the theorems characterize regular Γ-semigroup in terms of fuzzy generalized bi-ideals. The conclusion is drawn in Section 4.

2. Preliminaries

Definition 1 [22] *Let $S = \{x, y, z, \cdots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \cdots\}$ be two non-empty sets. Then S is called a Γ-semigroup if there exists a mapping $S \times \Gamma \times S \to S$ (images to be denoted by $a\alpha b$) satisfying*
(i) $x\gamma y \in S$.
(ii) $(x\beta y)\gamma z = x\beta(y\gamma z), \forall x, y, z \in S, \forall \gamma \in \Gamma$.

Example 1 Let $\Gamma = \{5, 7\}$. For any $x, y \in N$ and $\gamma \in \Gamma$, we define $x\gamma y = x.\gamma.y$ where "." is the usual multiplication on N. Then N is a Γ-semigroup.

Definition 2 [20] *Let S be a Γ-semigroup. By a subsemigroup of S, we mean a non-empty subset A of S such that $A\Gamma A \subseteq A$.*

Definition 3 [20] *Let S be a Γ-semigroup. By a bi-ideal of S, we mean a subsemigroup A of S such that $A\Gamma S\Gamma A \subseteq A$.*

Definition 4 [4] *Let S be a Γ-semigroup. By a left (right) ideal of S, we mean a non-empty subset A of S such that $S\Gamma A \subseteq A (A\Gamma S \subseteq A)$. By a two sided ideal or simply an ideal, we mean a non-empty subset of S which is both a left ideal and right ideal of S.*

Definition 5 [27] *A fuzzy subset μ of a non-empty set X is a function $\mu : X \to [0, 1]$.*

Definition 6 [20] *A non-empty fuzzy subset μ of a Γ-semigroup S is called a fuzzy subsemigroup of S if $\mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\}\ \forall x, y \in S, \forall \gamma \in \Gamma$.*

Definition 7 [20] *A fuzzy subsemigroup μ of a Γ-semigroup S is called a fuzzy bi-ideal of S if $\mu(x\alpha y\beta z) \geq \min\{\mu(x), \mu(z)\}\ \forall x, y, z \in S, \forall \alpha, \beta \in \Gamma$.*

Definition 8 [18] *A non-empty fuzzy subset μ of a Γ-semigroup S is called a fuzzy left ideal of S if $\mu(x\gamma y) \geq \mu(y)$ $\forall x, y \in S$, $\forall \gamma \in \Gamma$.*

Definition 9 [18] *A non-empty fuzzy subset μ of a Γ-semigroup S is called a fuzzy right ideal of S if $\mu(x\gamma y) \geq \mu(x)$ $\forall x, y \in S$, $\forall \gamma \in \Gamma$.*

Definition 10 [18] *A non-empty fuzzy subset of a Γ-semigroup S is called a fuzzy ideal of S if it is both a fuzzy left ideal and a fuzzy right ideal of S.*

In the next section, we obtain some important properties of fuzzy generalized bi-ideal of a Γ-semigroup and characterizations of regular Γ-semigroup in terms of fuzzy generalized bi-ideal.

3. Fuzzy Generalized Bi-ideal

Definition 11 *Let S be a Γ-semigroup. A non-empty subset A of S is called a generalized bi-ideal of S if $A\Gamma S\Gamma A \subseteq A$.*

Definition 12 *A non-empty fuzzy subset μ of a Γ-semigroup S is called a fuzzy generalized bi-ideal of S if $\mu(x\alpha y\beta z) \geq \min\{\mu(x), \mu(z)\}$ $\forall x, y, z \in S$, $\forall \alpha, \beta \in \Gamma$.*

Remark 1 It is clear that fuzzy bi-ideal of S is a subset of fuzzy generalized bi-ideal of S. But in general, the converse inclusion does not hold which will be clear from the following example.

Example 2 Let $S = \{x, y, z, r\}$ and $\Gamma = \{\gamma\}$, where γ is defined on S with the following cayley table:

γ	x	y	z	r
x	x	x	x	x
y	x	x	x	x
z	x	x	y	x
r	x	x	y	y

Then S is a Γ-semigroup. We define a fuzzy subset $\mu : S \to [0, 1]$ as $\mu(x) = 0.5$, $\mu(y) = 0$, $\mu(z) = 0.2$, $\mu(r) = 0$. Then μ is a fuzzy generalized bi-ideal of S, but μ is not a fuzzy bi-ideal of S.

Theorem 1 *Let I be a non-empty set of a Γ-semigroup S and χ_I be the characteristic function of I. Then χ_I is a fuzzy generalized bi-ideal of S if and only if I is a generalized bi-ideal of S.*

Proof Let I be a generalized bi-ideal of a Γ-semigroup S and χ_I be the characteristic function of I. Let $x, y, z \in S$ and $\beta, \gamma \in \Gamma$. Then $x\beta y\gamma z \in I$ if $x, z \in I$. It follows that $\chi_I(x\beta y\gamma z) = 1 = \min\{\chi_I(x), \chi_I(z)\}$. Let either $x \notin I$ or $z \notin I$. Then

Case (i) If $x\beta y\gamma z \notin I$, then $\chi_I(x\beta y\gamma z) \geq 0 = \min\{\chi_I(x), \chi_I(z)\}$.

Case (ii) If $x\beta y\gamma z \in I$, then $\chi_I(x\beta y\gamma z) = 1 \geq 0 = \min\{\chi_I(x), \chi_I(z)\}$.

Hence χ_I is a fuzzy generalized bi-ideal of S.

Conversely, let χ_I be a fuzzy generalized bi-ideal of S. Let $x, z \in I$. Then $\chi_I(x) = \chi_I(z) = 1$. Thus $\chi_I(x\beta y\gamma z) \geq \min\{\chi_I(x), \chi_I(z)\} = 1$ $\forall y \in S$, $\forall \beta, \gamma \in \Gamma$. Hence $x\beta y\gamma z \in I$ $\forall y \in S$, $\forall \beta, \gamma \in \Gamma$. Hence I is a generalized bi-ideal of S.

Definition 13 [4] *A Γ-semigroup S is called regular if for each element $x \in S$, there exist $y \in S$ and $\alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x$.*

Proposition 1 *Let S be a regular Γ-semigroup. Then every fuzzy generalized bi-ideal of S is a fuzzy bi-ideal S.*

Proof Let μ be a fuzzy generalized bi-ideal of S. Let $a, b \in S$. Since S is regular, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $b = b\alpha x\beta b$. Then for any $\gamma \in \Gamma$,

$$\mu(a\gamma b) = \mu(a\gamma(b\alpha x\beta b)) = \mu(a\gamma(b\alpha x)\beta b) \geq \min\{\mu(a), \mu(b)\}.$$

So μ is a fuzzy subsemigroup of S and consequently μ is a fuzzy bi-ideal of S. Hence the proof.

Remark 2 In view of above proposition, we can say that in a regular Γ-semigroup the concept of fuzzy generalized bi-ideal and fuzzy bi-ideal coincide.

Proposition 2 *Let μ and ν be two fuzzy generalized bi-ideals of a Γ-semigroup S. Then $\mu \cap \nu$ is a fuzzy generalized bi-ideal of S, provided $\mu \cap \nu$ is non-empty.*

Proof Let μ and ν be two fuzzy generalized bi-ideals of S and $x, y, z \in S$, $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned}(\mu \cap \nu)(x\alpha y\beta z) &= \min\{\mu(x\alpha y\beta z), \nu(x\alpha y\beta z)\}\\ &\geq \min\{\min\{\mu(x), \mu(z)\}, \min\{\nu(x), \nu(z)\}\}\\ &= \min\{\min\{\mu(x), \nu(x)\}, \min\{\mu(z), \nu(z)\}\}\\ &= \min\{(\mu \cap \nu)(x), (\mu \cap \nu)(z)\}.\end{aligned}$$

Hence $\mu \cap \nu$ is a fuzzy generalized bi-ideal of S.

Definition 14 *Let μ and σ be any two fuzzy subsets of a Γ-semigroups S. Then the product $\mu \circ \sigma$ is defined as*

$$(\mu \circ \sigma)(x) = \begin{cases} \sup\limits_{x=y\gamma z} [\min\{\mu(y), \sigma(z)\} : y, z \in S; \gamma \in \Gamma], \\ \qquad\qquad 0, \quad \textit{otherwise.} \end{cases}$$

Lemma 1 Let S be a Γ-semigroup and μ be a non-empty fuzzy subset of S. Then μ is a fuzzy generalized bi-ideal of S if and only if $\mu \circ \chi \circ \mu \subseteq \mu$, where χ is the characteristic function of S.

Proof Let μ be a fuzzy generalized bi-ideal of S. Suppose there exist x, y, p, $q \in S$ and β, $\gamma \in \Gamma$ such that $a = x\gamma y$ and $x = p\beta q$. Since μ is a fuzzy generalized bi-ideal of S, we obtain $\mu(p\beta q\gamma y) \geq \min\{\mu(p), \mu(y)\}$. Then

$$(\mu \circ \chi \circ \mu)(a) = \sup_{a=x\gamma y} [\min\{(\mu \circ \chi)(x), \mu(y)\}]$$

$$
\begin{aligned}
&= \sup_{a=x\gamma y}[\min\{\sup_{x=p\beta q}\{\min\{\mu(p),\chi(q)\}\},\mu(y)\}]\\
&= \sup_{a=x\gamma y}[\min\{\sup_{x=p\beta q}\{\min\{\mu(p),1\}\},\mu(y)\}]\\
&= \sup_{a=x\gamma y}[\min\{\mu(p),\mu(y)\}]\\
&\leq \mu(p\beta q\gamma y)=\mu(x\gamma y)=\mu(a).
\end{aligned}
$$

So we have $(\mu\circ\chi\circ\mu)\subseteq\mu$. Otherwise $(\mu\circ\chi\circ\mu)(a)=0\leq\mu(a)$. Thus $(\mu\circ\chi\circ\mu)\subseteq\mu$.

Conversely, let us assume that $\mu\circ\chi\circ\mu\subseteq\mu$. Let $x, y, z\in S$ and $\beta,\gamma\in\Gamma$ and $a=x\beta y\gamma z$. Since $\mu\circ\chi\circ\mu\subseteq\mu$, we have

$$
\begin{aligned}
\mu(x\beta y\gamma z)=\mu(a) &\geq (\mu\circ\chi\circ\mu)(a)\\
&= \sup_{a=x\beta y\gamma z}[\min\{(\mu\circ\chi)(x\beta y),\mu(z)\}]\\
&\geq \min\{(\mu\circ\chi)(p),\mu(z)\}(\text{let } p=x\beta y)\\
&= \min[\sup_{p=x\beta y}\{\min\{\mu(x),\chi(y)\}\},\mu(z)]\\
&\geq \min[\min\{\mu(x),1\},\mu(z)]\\
&= \min\{\mu(x),\mu(z)\}.
\end{aligned}
$$

Hence μ is a fuzzy generalized bi-ideal of S.

In view of the above lemma, we have the following theorem.

Theorem 2 *The product of any two fuzzy generalized bi-ideals of a Γ-semigroup S is a fuzzy generalized bi-ideal of S.*

Proof Let μ and ν be two fuzzy generalized bi-ideals of S. Then

$$
\begin{aligned}
(\mu\circ\nu)\circ\chi\circ(\mu\circ\nu) &= \mu\circ\nu\circ(\chi\circ\mu)\circ\nu\\
&\subseteq \mu\circ(\nu\circ\chi\circ\nu)\\
&\subseteq \mu\circ\nu.
\end{aligned}
$$

Hence $\mu\circ\nu$ is a fuzzy generalized bi-ideal of S. Similarly, we can show that $\nu\circ\mu$ is also a fuzzy generalized bi-ideal of S.

Theorem 3 *Let S be a Γ-semigroup. Then following are equivalent:*
(1) *S is regular,*
(2) *For every fuzzy generalized bi-ideal μ of S, $\mu\circ\chi\circ\mu=\mu$ where χ is the characteristic function of S.*

Proof (1) $\Rightarrow$ (2) Let (1) hold, i.e., S is regular. Let μ be a fuzzy generalized bi-ideal of S and $a \in S$. Then there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Hence

$$\begin{aligned}(\mu \circ \chi \circ \mu)(a) &= \sup_{a=y\gamma z}[\min\{(\mu \circ \chi)(y), \mu(z)\}] \\ &\geq \sup_{a=(a\alpha x)\beta a}[\min\{(\mu \circ \chi)(a\alpha x), \mu(a)\}] \\ &\geq \min\{(\mu \circ \chi)(a\alpha x), \mu(a)\} \\ &= \min[\sup_{a\alpha x = p\gamma q}\{\min\{\mu(p), \chi(q)\}\}, \mu(a)] (\text{let } a\alpha x = p\gamma q) \\ &\geq \min\{\mu(a), \chi(x), \mu(a)\} \\ &= \min\{\mu(a), 1, \mu(a)\} = \mu(a).\end{aligned}$$

So $\mu \subseteq \mu \circ \chi \circ \mu$. By Lemma 1, $\mu \circ \chi \circ \mu \subseteq \mu$. Hence $\mu \circ \chi \circ \mu = \mu$.

(2) $\Rightarrow$ (1) Let us suppose that (2) holds. Let A be a generalized bi-ideal of S. Then by Theorem 1, χ_A is a fuzzy generalized bi-ideal of S, where χ_A is the characteristic function of A. Hence by hypothesis, $\chi_A \circ \chi \circ \chi_A = \chi_A$. Let $a \in A$. Then $\chi_A(a) = 1$. Thus

$$\begin{aligned}&(\chi_A \circ \chi \circ \chi_A)(a) = 1 \\ &\implies \sup_{a=b\gamma c}[\min\{(\chi_A \circ \chi)(b), \chi_A(c)\}] = 1 \\ &\implies \sup_{a=b\gamma c}[\min\{\sup_{b=p\delta q}\min\{\chi_A(p), \chi(q)\}, \chi_A(c)\}] = 1 \\ &\implies \sup_{a=b\gamma c}[\min\{\sup_{b=p\delta q}\min\{\chi_A(p), 1\}, \chi_A(c)\}] = 1 \\ &\implies \sup_{a=b\gamma c}[\min\{\sup_{b=p\delta q}\chi_A(p), \chi_A(c)\}] = 1.\end{aligned}$$

Thus we get $p, c \in S$ such that $a = b\gamma c$ and $b = p\delta q$ with $\chi_A(p) = \chi_A(c) = 1$ whence $p, c \in A$. So $a = b\gamma c = p\delta q\gamma c \in A\Gamma S\Gamma A$. Consequently, $A \subseteq A\Gamma S\Gamma A$. Since A is a generalized bi-ideal of S, so $A\Gamma S\Gamma A \subseteq A$. Hence $A = A\Gamma S\Gamma A$ and so S is regular.

Theorem 4 *A Γ-semigroup S is regular if and only if for each fuzzy generalized bi-ideal μ of S and each fuzzy ideal ν of S, $\mu \cap \nu = \mu \circ \nu \circ \mu$.*

Proof Let S be regular. Let μ be a fuzzy generalized bi-ideal of S and ν be a fuzzy ideal of S. Then by Lemma 1, $\mu \circ \nu \circ \mu \subseteq \mu \circ \chi \circ \mu \subseteq \mu$. Again, by Theorem 4.3 [18], $\mu \circ \nu \circ \mu \subseteq \chi \circ \nu \circ \chi \subseteq \chi \circ \nu \subseteq \nu$. So $\mu \circ \nu \circ \mu \subseteq \mu \cap \nu$. Now let $a \in S$. Since S is regular, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a = a\alpha x\beta a\alpha x\beta a$. Then

$$\begin{aligned}(\mu \circ \nu \circ \mu)(a) &= \sup_{a=y\alpha z}\min\{\mu(y), (\nu \circ \mu)(z)\} \\ &\geq \min\{\mu(a), (\nu \circ \mu)(x\beta a\alpha x\beta a)\}\end{aligned}$$

$$= \min[\mu(a), \sup_{x\beta a\alpha x\beta a = p\rho q} \min\{\nu(p), \mu(q)\}]$$
$$\geq \min\{\mu(a), \nu(x\beta a\alpha x), \mu(a)\}$$
$$= \min\{\mu(a), \nu(x\beta a\alpha x)\}$$
$$\geq \min\{\mu(a), \nu(a)\} (\text{since } \nu \text{ is a fuzzy ideal of } S)$$
$$= (\mu \cap \nu)(a).$$

So $\mu \cap \nu \subseteq \mu \circ \nu \circ \mu$. Hence $\mu \circ \nu \circ \mu = \mu \cap \nu$.

Conversely, let us suppose that the necessary condition holds. Let μ be a fuzzy generalized bi-ideal of S. Then by hypothesis, $\mu = \mu \cap \chi = \mu \circ \chi \circ \mu$, where χ is the characteristic function of S. Hence by Theorem 3, S is regular.

Theorem 5 *Let S be a Γ-semigroup. Then the following are equivalent:*
(1) *S is regular,*
(2) *$A \cap L \subseteq A\Gamma L$ for each generalized bi-ideal A of S and each left ideal L of S,*
(3) *$R \cap A \cap L \subseteq R\Gamma A\Gamma L$ for each generalized bi-ideal A of S, each left ideal L of S and each right ideal R of S.*

Proof (1) $\Rightarrow$ (2) Let S be a regular Γ-semigroup and let $a \in A \cap L$. Then there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Then $a \in A$ and $a \in L$. Since L is a left ideal of S, so $x\beta a \in L$. This implies that $a = a\alpha x\beta a \in A\Gamma L$. Hence $A \cap L \subseteq A\Gamma L$.

(1) $\Rightarrow$ (3) Let S be a regular Γ-semigroup and $a \in R \cap A \cap L$. Then there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a = a\alpha x\beta a\alpha x\beta a$. Then $a \in R$ and $a \in A$ and $a \in L$. Since L is a left ideal and R is a right ideal of S, so $x\beta a \in L$ and $a\alpha x \in R$. This implies that $a = a\alpha x\beta a\alpha x\beta a \in R\Gamma A\Gamma L$. Hence $R \cap A \cap L \subseteq R\Gamma A\Gamma L$.

(3) $\Rightarrow$ (1) Let $R \cap A \cap L \subseteq R\Gamma A\Gamma L$, for each generalized bi-ideal A of S and for each left ideal L, each right ideal R of S. Let $a \in S$. Let $L = \{a\} \cup S\Gamma a$, $R = \{a\} \cup a\Gamma S$ and $A = \{a\} \cup a\Gamma a \cup a\Gamma S\Gamma a$. Then

$$\begin{aligned}
&R\Gamma A\Gamma L \\
&= (\{a\} \cup a\Gamma S)\Gamma(\{a\} \cup a\Gamma a \cup a\Gamma S\Gamma a)\Gamma(\{a\} \cup S\Gamma a) \\
&= (a\Gamma a \cup a\Gamma a\Gamma a \cup a\Gamma a\Gamma S\Gamma a \cup a\Gamma S\Gamma a \cup a\Gamma S\Gamma a\Gamma a \cup a\Gamma S\Gamma a\Gamma S\Gamma a)\Gamma(\{a\} \cup S\Gamma a) \\
&= a\Gamma a\Gamma a \cup a\Gamma a\Gamma S\Gamma a \cup a\Gamma a\Gamma a\Gamma a \cup a\Gamma a\Gamma a\Gamma S\Gamma a \cup a\Gamma a\Gamma S\Gamma a\Gamma a \cup a\Gamma a\Gamma S\Gamma a\Gamma S\Gamma a \\
&\cup a\Gamma S\Gamma a\Gamma a \cup a\Gamma S\Gamma a\Gamma S\Gamma a \cup a\Gamma S\Gamma a\Gamma a\Gamma a \cup a\Gamma S\Gamma a\Gamma a\Gamma S\Gamma a \cup a\Gamma S\Gamma a\Gamma S\Gamma a\Gamma a \\
&\cup a\Gamma S\Gamma a\Gamma S\Gamma a\Gamma S\Gamma a \subseteq a\Gamma S\Gamma a.
\end{aligned}$$

Since $a \in R \cap A \cap L$, then by hypothesis, $a \in R\Gamma A\Gamma L \subseteq a\Gamma S\Gamma a$. So there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Hence S is regular.

(2) $\Rightarrow$ (1) Let $A \cap L \subseteq A\Gamma L$ for each generalized bi-ideal A of S and for each left ideal L of S. Let $a \in S$. Let $L = \{a\} \cup S\Gamma a$ and $A = \{a\} \cup a\Gamma a \cup a\Gamma S\Gamma a$. Then

$$\begin{aligned}
A\Gamma L &= (\{a\} \cup a\Gamma a \cup a\Gamma S\Gamma a)\Gamma(\{a\} \cup S\Gamma a) \\
&= a\Gamma a \cup a\Gamma S\Gamma a \cup a\Gamma a\Gamma a \cup a\Gamma a\Gamma S\Gamma a \cup a\Gamma S\Gamma a\Gamma a \cup a\Gamma S\Gamma a\Gamma S\Gamma a \\
&\subseteq a\Gamma a \cup a\Gamma S\Gamma a.
\end{aligned}$$

Since $a \in A \cap L$, then by hypothesis, $a \in A\Gamma L \subseteq a\Gamma a \cup a\Gamma S\Gamma a$. If $a \in a\Gamma a$, then for some $\alpha \in \Gamma$, $a = a\alpha a = a\alpha a\alpha a \in a\Gamma S\Gamma a$. So there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Hence S is regular.

Theorem 6 *Let S be a Γ-semigroup. Then the following are equivalent:*
(1) *S is regular,*
(2) *$\mu \cap \nu \subseteq \mu \circ \nu$ for each fuzzy bi-ideal μ of S and for each fuzzy left ideal ν of S,*
(3) *$\mu \cap \nu \subseteq \mu \circ \nu$ for each fuzzy generalized bi-ideal μ of S and for each fuzzy left ideal ν of S,*
(4) *$\lambda \cap \mu \cap \nu \subseteq \lambda \circ \mu \circ \nu$ for each fuzzy bi-ideal μ of S, for each fuzzy left ideal ν of S and for each fuzzy right ideal λ of S,*
(5) *$\lambda \cap \mu \cap \nu \subseteq \lambda \circ \mu \circ \nu$ for each fuzzy generalized bi-ideal μ of S and for each fuzzy left ideal ν of S and for each fuzzy right ideal λ of S.*

Proof (1) $\Rightarrow$ (2) Let S be regular, μ be a fuzzy bi-ideal of S and ν be a fuzzy left ideal of S. Let $a \in S$. Then there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a = a\alpha x\beta a\alpha x\beta a$. Then

$$\begin{aligned}(\mu \circ \nu)(a) &= \sup_{a=y\rho z}[\min\{\mu(y), \nu(z)\}]\\ &\geq \min\{\mu(a\alpha x\beta a), \nu(x\beta a)\}(\text{since } a = a\alpha x\beta a = a\alpha x\beta a\alpha x\beta a)\\ &\geq \min\{\mu(a), \nu(a)\}(\text{since } \mu \text{ is a fuzzy bi-ideal of } S\\ &\quad \text{and } \nu \text{ is a fuzzy left ideal of } S)\\ &= (\mu \cap \nu)(a).\end{aligned}$$

Hence $\mu \circ \nu \supseteq \mu \cap \nu$.

Similarly we can prove that (1) implies (3).

(2) $\Rightarrow$ (1) Let (2) hold, i.e., $\mu \cap \nu \subseteq \mu \circ \nu$ for each fuzzy bi-ideal μ of S and for each fuzzy left ideal ν of S. Since every fuzzy right ideal of S is a fuzzy quasi ideal of S [20] and every fuzzy quasi ideal of S is a fuzzy bi-ideal of S (cf. Proposition 5.2 [20]), so $\mu \cap \nu \subseteq \mu \circ \nu$ for each fuzzy right ideal μ of S and for each fuzzy left ideal ν of S. Also $\mu \circ \nu \subseteq \mu \cap \nu$ always holds. Then $\mu \cap \nu = \mu \circ \nu$ and hence by Theorem 4.7 [18], S is regular.

(3) $\Rightarrow$ (1) Let us suppose that (3) holds. Let A be a generalized bi-ideal of S, L be a left ideal of S and $a \in A \cap L$. Then $a \in A$ and $a \in L$. Since A is a generalized bi-ideal of S, so by Theorem 1, χ_A is a fuzzy generalized bi-ideal of S, where χ_A is the characteristic function of A. By Theorem 3.1 [18], χ_L is a fuzzy left ideal of S, where χ_L is the characteristic function of L. Hence by hypothesis, $\chi_A \cap \chi_L \subseteq \chi_A \circ \chi_L$. Then $(\chi_A \circ \chi_L)(a) \geq (\chi_A \cap \chi_L)(a) = \min\{\chi_A(a), \chi_L(a)\} = 1$. Thus $\sup_{a=y\gamma z}[\min\{\chi_A(y), \chi_L(z)\}] = 1$.

So there exist $b, c \in S$ and $\delta \in \Gamma$ with $a = b\delta c$ such that $\chi_A(b) = \chi_L(c) = 1$. Consequently, $b \in A$ and $c \in L$. So $a = b\delta c \in A\Gamma L$. So $A \cap L \subseteq A\Gamma L$. Hence by Theorem 5, S is regular.

(1) $\Rightarrow$ (4) Let S be regular. Let μ be a fuzzy bi-ideal, ν be a fuzzy left ideal and λ be a fuzzy right ideal of S, respectively. Let $a \in S$. Then there exist $x \in S$ and

$\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a = a\alpha x\beta a\alpha x\beta a = a\alpha x\beta a\alpha x\beta a\alpha x\beta a$. Then

$$\begin{aligned}(\lambda \circ \mu \circ \nu)(a) &= \sup_{a=y\rho z}[\min\{\lambda(y), (\mu \circ \nu)(z)\}] \\ &\geq \min\{\lambda(a\alpha x), (\mu \circ \nu)(a\alpha x\beta a\alpha x\beta a)\} \\ &\geq \min\{\lambda(a), (\mu \circ \nu)(a\alpha x\beta a\alpha x\beta a)\}(\text{since } \lambda \text{ is a fuzzy right ideal of } S) \\ &= \min\{\lambda(a), \sup_{(a\alpha x\beta a)\alpha(x\beta a))=p\rho q} \min\{\mu(a\alpha x\beta a), \nu(x\beta a)\}\} \\ &\geq \min[\lambda(a), \min\{\mu(a\alpha x\beta a), \nu(x\beta a)\}] \\ &\geq \min[\lambda(a), \min\{\mu(a), \nu(a)\}](\text{since } \mu \text{ is a fuzzy bi-ideal of } S \\ &\text{and } \nu \text{ is a fuzzy left ideal of } S) \\ &\geq \min\{\lambda(a), \mu(a), \nu(a)\} \\ &= (\lambda \cap \mu \cap \nu)(a).\end{aligned}$$

Hence $\lambda \cap \mu \cap \nu \subseteq \lambda \circ \mu \circ \nu$.

Similarly, we can prove that (1) implies (5).

(4) $\Rightarrow$ (1) Let (4) hold. Let λ and ν be any fuzzy right ideal and fuzzy left ideal of S, respectively. Since χ_S is itself a fuzzy bi-ideal of S, where χ_S is the characteristic function of S, by assumption, we have $\lambda \cap \nu = \lambda \cap \chi_S \cap \nu \subseteq \lambda \circ \chi_S \circ \nu \subseteq \lambda \circ \nu$. Also $\lambda \circ \nu \subseteq \lambda \cap \nu$. Therefore $\lambda \circ \nu = \lambda \cap \nu$. Hence by Theorem 4.7 [18], S is regular.

(5) $\Rightarrow$ (1) Let us suppose that (5) holds. Let A be a generalized bi-ideal of S, L be a left ideal of S, R be a right ideal of S and $a \in R \cap A \cap L$. Then $a \in R$, $a \in A$ and $a \in L$. Since A is a generalized bi-ideal of S, so by Theorem 1, χ_A is a fuzzy generalized bi-ideal of S, where χ_A is the characteristic function of A. By Theorem 3.1 [18], χ_L is a fuzzy left ideal of S and χ_R is a fuzzy right ideal of S, where χ_L and χ_R are the characteristic functions of L and R respectively. Hence by hypothesis, $\chi_R \cap \chi_A \cap \chi_L \subseteq \chi_R \circ \chi_A \circ \chi_L$. Then $(\chi_R \circ \chi_A \circ \chi_L)(a) \geq (\chi_R \cap \chi_A \cap \chi_L)(a) = \min\{\chi_R(a), \chi_A(a), \chi_L(a)\} = 1$. Thus $\sup_{a=y\gamma z}[\min\{(\chi_R \circ \chi_A)(y), \chi_L(z)\}] = 1$.

So there exist $b, c \in S$ and $\delta \in \Gamma$ with $a = b\delta c$ such that $(\chi_R \circ \chi_A)(b) = \chi_L(c) = 1$. Then $c \in L$ and $\sup_{b=p\rho q}[\min\{\chi_R(p), \chi_A(q)\}] = 1$. Then $b = d\theta e$ for some $d, e \in S$ and $\theta \in \Gamma$ with $\chi_R(d) = \chi_A(e) = 1$. Consequently, $d \in R$ and $e \in A$. So $a = b\delta c = d\theta e\delta c \in R\Gamma A\Gamma L$. So $R \cap A \cap L \subseteq R\Gamma A\Gamma L$. Hence by Theorem 5, S is regular.

4. Conclusion

Definition 1 is the definition of one sided Γ-semigroup introduced by M.K. Sen. It may be noted here that in 1981 M.K. Sen [22] introduced the notion of both sided Γ-semigroups, later T.K. Dutta and N.C. Adhikari [4] introduced the notion of both sided Γ-semigroup and also introduced the notions of operator semigroups of a both sided Γ-semigroup. Throughout this paper, S serves the role of one sided Γ-semigroup. In this paper, the concept of fuzzy generalized bi-ideal of a Γ-semigroup has been introduced and a regular Γ-semigroup has been characterized in terms of fuzzy generalized bi-ideal. Theorems 3-6 illustrate this fact. It is also worthwhile noting that the corresponding fuzzy generalized bi-ideals of the operator semigroups of a Γ-semigroup may play important roles in furthering the study of the properties of fuzzy generalized bi-ideals of a Γ-semigroup.

Acknowledgments

The authors are very grateful to Dr. Sujit Kumar Sardar, Associate Professor, Department of Mathematics, Jadavpur University, Kolkata, India for his inspiration to write this paper. We are also thankful to the learned referees for their valuable comments and suggestions for improving the paper.

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