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ORIGINAL ARTICLE

Some Properties of Generalized Intuitionistic Fuzzy Nilpotent Matrices over Distributive Lattice

Amal Kumar Adak · Monoranjan Bhowmik · Madhumangal Pal

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Abstract In this paper, the concept of lattice over generalized intuitionistic fuzzy matrices (GIFMs) are introduced and have shown that the set of GIFMs forms a distributive lattice. Some algebraic properties of generalized intuitionistic fuzzy matrices (GIFMs) are presented over distributive lattice. Also, some characteristics of generalized intuitionistic fuzzy nilpotent matrices (GIFNMs) are discussed over distributive lattice. Finally, the reduction of GIFNMs over distributive lattice are given with some properties.

Keywords Intuitionistic fuzzy matrices · Generalized intuitionistic fuzzy matrices · Distributive lattice · Generalized intuitionistic fuzzy nilpotent matrices

1. Introduction

The theory of fuzzy sets is applied to many mathematical branches. Many researchers have done several works on fuzzy sets. Atanassov [5, 6] introduced the concept of intuitionistic fuzzy sets (IFSs). Also a lot of research works were done by several researchers on the field of IFS. Ragab and Emam [18] defined adjoint of a square fuzzy matrix. By the concept of IFSs, first time Pal [16] introduced intuitionistic fuzzy determinant. Later on Pal and Shyamal [20, 21] introduced intuitionistic fuzzy matrices and determined distance between intuitionistic fuzzy matrices. Bhowmik and Pal [7, 8] introduced some results on intuitionistic fuzzy matrices and intuitionistic circulant fuzzy matrices and generalized intuitionistic fuzzy matrices. Mondal and

Amal Kumar Adak (⊠) · Madhumangal Pal (⊠)
Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore-721102, India
email: amaladak17@gmail.com
mmpalvu@gmail.com
Monoranjan Bhowmik (⊠)
Department of Mathematics, V. T. T College, Midnapore, Paschim Medinipur-721101, India
email: mbvttc@gmail.com

Samanta [13] introduced another concept of IFSs called generalized IFSs. Bhowmik and Pal [9] defined generalized interval-valued intuitionistic fuzzy set (GIVIFS) and presented its various properties.

Algebraic structures play a prominent role in the mathematics with wide range of applications in many disciplines such as theoretical physics, computer science, control engineering, information sciences, coding theory, topological spaces etc. This provides sufficient motivation to the researchers to review various concepts and results from the area of abstract algebra in the broader framework of fuzzy setting. One of the structures which is most extensively used and discussed in the mathematics and its applications is lattice theory. As it is well known that lattice is considered as a relational, ordered structure and as an algebra.

Lattice matrices are useful tools in various domains like the theory of switching, automata theory and theory of finite graphs. The notions of nilpotent lattice matrices seem to appear first in the work of Give'on [11]. In [11], Give'on proved that an $n \times n$ lattice matrix is nilpotent if and only if $A^n = \mathbf{0}$. Since then, a number of researchers have studied the topic of the nilpotent lattice matrices.

Our aim is to introduce and study distributive lattice over GIFMs. The structure of this paper is organized as follows. In Section 2, the preliminaries and some definitions are given. In Section 3, some algebraic structures of GIFMs over distributive lattice are supplied and some results are given. In Section 4, we present some properties of generalized intuitionistic fuzzy determinant over distributive lattice (GIFD). In Section 5, the definition of generalized intuitionistic fuzzy nilpotent matrix (GIFNM) over distributive lattice is given. In Section 6, the reduction of generalized intuitionistic fuzzy nilpotent matrices over distributive lattice are given and some properties are studied. The conclusion is made in Section 7.

2. Preliminaries

Here some preliminaries, definitions of IFSs and GIFMs are recalled and some algebraic operations of GIFMs and different types of GIFMs are presented.

2.1. Fuzzy Set and Intuitionistic Fuzzy Set

Definition 2.1 (Fuzzy set) *A fuzzy set A in a universal set X is defined as* $A = \left\{\langle x, \mu_A(x)\rangle | x \in X\right\}$*, where* $\mu_A : X \to [0,1]$ *is a mapping called the membership function of the fuzzy set A.*

Definition 2.2 (Instuitionistic fuzzy set) *An instuitionistic fuzzy set (IFS) A over X is an object having the form* $A = \left\{x, \langle \mu_A(x), \nu_A(x)\rangle : x \in X\right\}$*, where* $\mu_A : X \to [0,1]$ *and* $\nu_A : X \to [0,1]$*,* $\mu_A(x)$ *and* $\nu_A(x)$ *are called the membership and non-membership values of x in A satisfying the condition* $0 \le \mu_A(x) + \nu_A(x) \le 1$.

Some operations on IFSs

In the following, we define some relational operations on IFSs. Let A and B be two IFSs on X, where

$$A = \left\{x, \langle \mu_A(x), \nu_A(x)\rangle : x \in X\right\}$$

and

$$B = \left\{x, \langle \mu_B(x), \nu_B(x)\rangle : x \in X\right\}.$$

Then,

(1) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ for all $x \in X$.

(2) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$.

(3) $\overline{A} = \left\{x, \langle \nu_A(x), \mu_A(x) \rangle : x \in X\right\}$.

(4) $A \cap B = \left\{\langle \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\} \rangle : x \in X\right\}$.

(5) $A \cup B = \left\{\langle \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\} \rangle : x \in X\right\}$.

2.2. Fuzzy Matrix and Intuitionistic Fuzzy Matrix

Definition 2.3 (Fuzzy matrix) *A fuzzy matrix of order $m \times n$ is defined as $A = \left[a_{ij\mu}\right]$, where $a_{ij\mu}$ is the membership value of the ij-th element in A.*

Definition 2.4 (Intuitionistic fuzzy matrix) *An intuitionistic fuzzy matrix of order $m \times n$ is defined as $A = \left[\langle a_{ij\mu}, a_{ij\nu} \rangle\right]$, where $a_{ij\mu}$ and $a_{ij\nu}$ are the membership and non-membership values of the ij-th element in A satisfying the condition $0 \leq a_{ij\mu} + a_{ij\nu} \leq 1$ for all i, j.*

Definition 2.5 (Generalized intuitionistic fuzzy matrix) *A generalized intuitionistic fuzzy matrix (GIFM) of order $m \times n$ is defined as $A = \left[\langle a_{ij\mu}, a_{ij\nu} \rangle\right]$, where $a_{ij\mu}$ and $a_{ij\nu}$ are the membership and non-membership values of the ij-th element in A satisfying the generalized intuitionistic fuzzy condition $0 \leq a_{ij\mu} \wedge a_{ij\nu} \leq 0.5$ for all i, j.*

Let $G_{m\times n}$ denotes the set of all GIFMs of order $m \times n$. In particular, G_n denotes the set of all GIFMs of order $n \times n$.

Definition 2.6 (Comparable GIFMs) *Let A and B be two GIFMs such that $A = \left[\langle a_{ij\mu}, a_{ij\nu} \rangle\right]$ and $B = \left[\langle b_{ij\mu}, b_{ij\nu} \rangle\right] \in G_{m\times n}$. Then two matrices A and B are said to be comparable GIFMs if $a_{ij\mu} \leq b_{ij\mu}$ and $a_{ij\nu} \geq b_{ij\nu}$ for all i, j.*

Some algebraic operations of GIFMs

Let A and B be two GIFMs, such that $A = \left[\langle a_{ij\mu}, a_{ij\nu} \rangle\right]$ and $B = \left[\langle b_{ij\mu}, b_{ij\nu} \rangle\right] \in G_{m\times n}$.
(1) Matrix addition and subtraction are given by

$$A + B = \left[\langle \max\{a_{ij\mu}, b_{ij\mu}\}, \min\{a_{ij\nu}, b_{ij\nu}\} \rangle\right]$$

and

$$A - B = \left[\langle a_{ij\mu} - b_{ij\mu}, a_{ij\nu} - b_{ij\nu} \rangle\right],$$

where $a_{ij\mu} - b_{ij\mu} = \begin{cases} a_{ij\mu}, a_{ij\mu} \geq b_{ij\mu} \\ 0, \quad \text{elsewhere} \end{cases}$ and $a_{ij\nu} - b_{ij\nu} = \begin{cases} a_{ij\nu}, a_{ij\nu} < b_{ij\nu}, \\ 0, \quad \text{otherwise.} \end{cases}$
(2) Componentwise matrix multiplication is given by

$$A \odot B = \left[\langle \min\{a_{ij\mu}, b_{ij\mu}\}, \max\{a_{ij\nu}, b_{ij\nu}\} \rangle\right].$$

(3) Let A, B be two GIFMs of order $m \times n$ and $n \times p$. Then the matrix product AB is given by

$$AB = \left[\left\langle \sum_k \min\{a_{ik\mu}, b_{kj\mu}\}, \prod_k \max\{a_{ik\nu}, b_{kj\nu}\} \right\rangle\right] \in G_{m\times p}.$$

Different types of IFMs

(1) Intuitionistic fuzzy zero matrix is denoted by **O** and all entries of it are $\langle 0, 1\rangle$.

(2) Intuitionistic fuzzy identity matrix I_n is defined by $\left[\langle a_{ij\mu}, a_{ij\nu}\rangle\right]$ such that $a_{ij\mu} = 1, a_{ij\nu} = 0$ for $i = j$ and $a_{ij\mu} = 0, a_{ij\nu} = 1$ for all $i \neq j$.

(3) If all element of an IFM are $\langle 1, 0\rangle$, then it called intuitionistic fuzzy universal matrix and is denoted by J_n.

(4) An IFM A is reflexive if and only if $a_{ii} = \langle 1, 0\rangle$ for all i.

(5) If $a_{ii} = \langle 0, 1\rangle$ for all i of an IFM A, then it is called irreflexive.

2.3. Poset of Fuzzy Sets and GIFMs

Definition 2.7 *A binary relation '$\leq$' defined on a fuzzy set A is a partial order on the fuzzy set A if the following conditions hold identically in A:*
(i) $a \leq a$,
(ii) $a \leq b$ *and* $b \leq a$ *imply* $a = b$,
(iii) $a \leq b$ *and* $b \leq c$ *imply* $a \leq c$.
A nonempty fuzzy set A with a partial order on it is called a partially ordered set or briefly a poset and it is denoted by $(A, \leq)$.

Lemma 1 (Poset of GIFMs) *Let* G_n *be the set of all* $n \times n$ *GIFMs and '$\leq$' be comparable fuzzy matrix relation. Then* $(G_n, \leq)$ *is a poset.*

Proof Let A, B and $C \in G_n$. Then

(1) $A \leq A$ is true since $a_{ij\mu} \leq a_{ij\mu}$ and $a_{ij\nu} \geq a_{ij\nu}$. Hence the relation '$\leq$' is reflexive.

(2) $A \leq B$ and $B \leq A$ possible only when $A = B$, since $A \leq B$ when $a_{ij\mu} \leq b_{ij\mu}$ and $a_{ij\nu} \geq b_{ij\nu}$; $B \leq A$ when $b_{ij\mu} \leq a_{ij\mu}$ and $b_{ij\nu} \geq a_{ij\nu}$. Combining these two give $A = B$. Therefore the relation '$\leq$' is anti-symmetric.

(3) $A \leq B$ then $a_{ij\mu} \leq b_{ij\mu}$ and $a_{ij\nu} \geq b_{ij\nu}$; $B \leq C$ then $b_{ij\mu} \leq c_{ij\mu}$ and $b_{ij\nu} \geq c_{ij\nu}$. It is obvious that $A \leq C$ since $a_{ij\mu} \leq c_{ij\mu}$ and $a_{ij\nu} \geq c_{ij\nu}$. Hence the relation '$\leq$' is transitive.

Therefore a nonempty set of GIFMs G_n satisfies the partial order relation. Hence G_n is a partial order set i.e. poset.

Linearly ordered set of matrix.
If every pair of the elements of a poset $(G_n, \leq)$ are comparable, then G_n is said to be linearly ordered set of matrix.

Predecessor and successor
Let $(G_n, \leq)$ be a poset and A, $B \in G_n$. If $A \leq B$, then A is called predecessor and B is called successor.

Maximal and minimal elements

A matrix $A \in G_n$ is said to be maximal matrix if there exists no matrix B such that $A \leq B$.

Similarly, a matrix $A \in G_n$ is said to be minimal matrix if there exists no matrix B such that $B \leq A$.

Theorem 2.1 *Every finite nonempty poset* $(G_n, \leq)$ *has at least one maximal and one minimal elements.*

Proof Let $G_n = \{A_1, A_2, \cdots, A_n\}$ be a finite poset under $\leq$, containing n GIFMs. If A_1 is not a maximal GIFM, then by the definition there exists another GIFM $A_2 \in G_n$ such that $A_1 \leq A_2$. Again, if A_2 is not a maximal GIFM, then there exists another GIFM $A_3 \in G_n$ such that $A_2 \leq A_3$. Since G_n is finite, this process will terminate after a finite number of times. Hence, we obtain a finite sequence of GIFMs in G_n in the following ordered $A_1 \leq A_2 \leq A_3 \leq \cdots \leq A_n$. Therefore, there is no GIFM B such that $A_n \leq B$ for any $B \in G_n$. Hence A_n is a maximal GIFM of the poset $(G_n, \leq)$.

Similarly, it can be proved that poset $(G_n, \leq)$ has minimal element.

2.4. Lattice of Fuzzy Sets

Definition 2.8 (Lattice of fuzzy sets) *A lattice is a partial ordered set* $(L, \leq)$ *in which every two elements have a unique least upper bound and a greatest lower bound.*

For any two elements a and b in L, the least upper bound and greatest lower bound will be denoted by $a \vee b$ and $a \wedge b$. Lattice is also denoted by $(L, \leq, \wedge, \vee)$.

Definition 2.9 (Universal bounds) *An element a in the lattice L is called the universal upper bound if* $x \leq a$ *for all* $x \in L$ *and an element* $b \in L$ *is called universal lower bound if* $b \leq x$ *for all* $x \in L$.

The elements 0 and 1 are used to denote the universal lower and upper bounds respectively.

Definition 2.10 (Distributive lattice of FSs) *A lattice* $(L, \leq, \vee, \wedge)$ *is said to be distributive lattice if the operations* $\vee$ *and* $\wedge$ *are distributive with respect to each other, i.e.,*

(1) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,

(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, *where* a, b *and* $c \in L$.

An important special case of a distributive lattice is the real unit interval $[0, 1]$ *with* 'max' *and* 'min' *is called fuzzy algebra.*

3. Distributive Lattice of GIFMs

In this section, we introduce the concept of distributive lattice of GIMFs and give some properties of GIFNMs over distributive lattice. We begin this section with some definitions:

3.1. Lattice of GIFMs

A nonempty poset $(G_n, \leq)$ with two binary operation $+$ and $\odot$ is called a lattice if the following axioms hold:

(1) Closure : $A, B \in G_n$ then $A + B \in G_n$ and $A \odot B \in G_n$.

(2) Commutative : $A, B \in G_n$ then $A + B = B + A$ and $A \odot B = B \odot A$.

(3) Associative : $A, B, C \in G_n$ then $(A + B) + C = A + (B + C)$ and $(A \odot B) \odot C = A \odot (B \odot C)$.

(4) Absorption : $A, B \in G_n$ then $A \odot (A + B) = A$ and $A + (A \odot B) = A$.

Therefore, the poset $(G_n, \leq)$ with two binary operation matrix addition and componentwise matrix multiplication of GIFMs form lattice.

It should be noted that the poset $(G_n, \leq)$ with two binary operation matrix addition and matrix product of GIFMs does not form lattice as matrix product is not commutative.

Idempotent law

Let A be an $n \times n$ GIFMs over distributive lattice $(G_n(L), \leq, +, \odot)$. Then A satisfies idempotent law, i.e., (i) $A + A = A$ and (ii) $A \odot A = A$.

Theorem 3.1 *Let A, B be two square GIFMs of n×n over distributive lattice* $(G_n(L), \leq, +, \odot)$. *Then* $A \odot B = A$ *if and only if* $A + B = B$.

Proof Let $A \odot B = A$, where $A, B \in G_n(L)$. Therefore, $\min\{a_{ij\mu}, b_{ij\mu}\} = a_{ij\mu}$ and $\max\{a_{ij\nu}, b_{ij\nu}\} = a_{ij\nu}$.

Hence, $\max\{a_{ij\mu}, b_{ij\mu}\} = b_{ij\mu}$ and $\min\{a_{ij\nu}, b_{ij\nu}\} = b_{ij\nu}$.
Now,

$$\begin{aligned} A + B &= \left[\left\langle \max\{a_{ij\mu}, b_{ij\mu}\}, \min\{a_{ij\nu}, b_{ij\nu}\}\right\rangle\right] \\ &= \left[\left\langle b_{ij\mu}, b_{ij\nu}\right\rangle\right] = B. \end{aligned}$$

The proof of converse part is similar.

Theorem 3.2 *Let* $(G_n(L), \leq, +, \odot)$ *be the lattice of GIFMs and A, B, C* $\in G_n$. *If* $A \leq B$ *and* $A \leq C$, *then* (1) $A \leq B + C$, (2) $A \leq B \odot C$.

Proof If $A \leq B$, then we have $a_{ij\mu} \leq b_{ij\mu}$ and $a_{ij\nu} \geq b_{ij\nu}$.

Again, $A \leq C$ we have $a_{ij\mu} \leq c_{ij\mu}$ and $a_{ij\nu} \geq c_{ij\nu}$.
Hence, $a_{ij\mu} \leq \max\{b_{ij\mu}, c_{ij\mu}\}$ and $a_{ij\nu} \geq \min\{b_{ij\nu}, c_{ij\nu}\}$.
Therefore,

$$\begin{aligned} A &= \left[\left\langle a_{ij\mu}, a_{ij\nu}\right\rangle\right] \\ &\leq \left[\left\langle \max\{b_{ij\mu}, c_{ij\mu}\}, \min\{b_{ij\nu}, c_{ij\nu}\}\right\rangle\right] \\ &= B + C. \end{aligned}$$

The proof of second part is similar.

Theorem 3.3 *Let* $(G_n(L), \leq, +, \odot)$ *be a lattice over GIFMs and A, B, C, D* $\in G_n$. *If* $A \leq B$ *and* $C \leq D$, *then* (1) $A + C \leq B + D$ *and* (2) $A \odot C \leq B \odot D$.

Proof If $A \leq B$, then we have $a_{ij\mu} \leq b_{ij\mu}$ and $a_{ij\nu} \geq b_{ij\nu}$.

Again, $C \leq D$, we have $c_{ij\mu} \leq d_{ij\mu}$ and $c_{ij\nu} \geq d_{ij\nu}$.
Hence, $\max\{a_{ij\mu}, c_{ij\mu}\} \leq \max\{b_{ij\mu}, d_{ij\mu}\}$ and $\min\{a_{ij\nu}, c_{ij\nu}\} \geq \min\{b_{ij\nu}, d_{ij\nu}\}$.
Therefore,

$$\begin{aligned} A + C &= \Big[\Big\langle \max\{a_{ij\mu}, c_{ij\mu}\}, \min\{a_{ij\nu}, c_{ij\nu}\}\Big\rangle\Big] \\ &\leq \Big[\Big\langle \max\{b_{ij\mu}, d_{ij\mu}\}, \min\{b_{ij\nu}, d_{ij\nu}\}\Big\rangle\Big] \\ &= B + D. \end{aligned}$$

Proof is similar for $A \odot C \leq B \odot D$.

3.2. Distributive Lattice of GIFMs

Let $A, B, C \in G_n$. Then the lattice of GIFMs $(G_n(L), \leq, +, \odot)$ is said to be distributive lattice of GIFMs if

(1) $A \odot (B + C) = (A \odot B) + (A \odot C)$.

(2) $A + (B \odot C) = (A + B) \odot (A + C)$.

Example 1 We shown by means of example of the distributive property of GIFMs. Let A, B, C be three 3×3 GIFMs, where

$$A = \begin{pmatrix} \langle 0.3, 0.8\rangle & \langle 0.4, 0.8\rangle & \langle 0.5, 0.9\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.5, 0.7\rangle & \langle 0.3, 0.8\rangle \\ \langle 0.3, 0.7\rangle & \langle 0.4, 0.5\rangle & \langle 0.3, 0.6\rangle \end{pmatrix}, B = \begin{pmatrix} \langle 0.4, 0.7\rangle & \langle 0.5, 0.7\rangle & \langle 0.5, 0.8\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.6, 0.4\rangle & \langle 0.4, 0.7\rangle \\ \langle 0.4, 0.7\rangle & \langle 0.5, 0.5\rangle & \langle 0.4, 0.6\rangle \end{pmatrix},$$

$$C = \begin{pmatrix} \langle 0.6, 0.5\rangle & \langle 0.6, 0.4\rangle & \langle 0.7, 0.5\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.7, 0.4\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.6\rangle & \langle 0.6, 0.5\rangle & \langle 0.5, 0.3\rangle \end{pmatrix}.$$

Now,

$$B + C = \begin{pmatrix} \langle 0.6, 0.5\rangle & \langle 0.6, 0.4\rangle & \langle 0.7, 0.5\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.7, 0.4\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.6\rangle & \langle 0.6, 0.5\rangle & \langle 0.5, 0.3\rangle \end{pmatrix}, \text{ and}$$

$$A \odot (B + C) = \begin{pmatrix} \langle 0.3, 0.8\rangle & \langle 0.4, 0.8\rangle & \langle 0.5, 0.9\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.5, 0.7\rangle & \langle 0.3, 0.8\rangle \\ \langle 0.3, 0.7\rangle & \langle 0.4, 0.5\rangle & \langle 0.3, 0.6\rangle \end{pmatrix},$$

$$A \odot B = \begin{pmatrix} \langle 0.3, 0.8\rangle & \langle 0.4, 0.8\rangle & \langle 0.5, 0.9\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.5, 0.7\rangle & \langle 0.3, 0.8\rangle \\ \langle 0.3, 0.7\rangle & \langle 0.4, 0.5\rangle & \langle 0.3, 0.6\rangle \end{pmatrix}, A \odot C = \begin{pmatrix} \langle 0.3, 0.8\rangle & \langle 0.4, 0.8\rangle & \langle 0.5, 0.9\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.5, 0.7\rangle & \langle 0.3, 0.8\rangle \\ \langle 0.3, 0.7\rangle & \langle 0.4, 0.5\rangle & \langle 0.3, 0.6\rangle \end{pmatrix},$$

$$A \odot B + A \odot C = \begin{pmatrix} \langle 0.3, 0.8\rangle & \langle 0.4, 0.8\rangle & \langle 0.5, 0.9\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.5, 0.7\rangle & \langle 0.3, 0.8\rangle \\ \langle 0.3, 0.7\rangle & \langle 0.4, 0.5\rangle & \langle 0.3, 0.6\rangle \end{pmatrix}.$$

Therefore, $A \odot (B + C) = (A \odot B) + (A \odot C)$.

Similarly, it can be shown that $A + (B \odot C) = (A + B) \odot (A + C)$.

Theorem 3.4 *In a distributive lattice of GIFMs $(G_n(L), \leq, +, \odot)$ if $A, B, C \in G_n(L)$, $A + B = A + C$ and $A \odot B = A \odot C$, then $B = C$.*

Proof Since, $A, B, C \in (G_n(L), \leq, +, \odot)$, we have

$$
\begin{aligned}
B &= \Big[\min\{b_{ij\mu}, \max\{a_{ij\mu}, b_{ij\mu}\}\}, \max\{b_{ijv}, \min\{a_{ijv}, b_{ijv}\}\}\Big] \text{ [By absorption property]} \\
&= B \odot \Big[\max\{a_{ij\mu}, c_{ij\mu}\}, \min\{a_{ijv}, c_{ijv}\}\Big] \text{ [Since } A + B = A + C] \\
&= \Big[\min\{b_{ij\mu}, a_{ij\mu}\}, \max\{b_{ijv}, a_{ijv}\}\Big] + \Big[\max\{b_{ij\mu}, c_{ij\mu}\}, \min\{b_{ijv}, c_{ijv}\}\Big] \\
&\quad \text{[By distributive law]} \\
&= \Big[\max\{c_{ij\mu}, a_{ij\mu}\}, \min\{c_{ijv}, a_{ijv}\}\Big] + \Big[\min\{b_{ij\mu}, c_{ij\mu}\}, \max\{b_{ijv}, c_{ijv}\}\Big] \\
&\quad \text{[Since} B \odot A = C \odot A] \\
&= \Big[\min\{c_{ij\mu}, a_{ij\mu}\}, \max\{c_{ijv}, a_{ijv}\}\Big] + \Big[\min\{c_{ij\mu}, b_{ij\mu}\}, \max\{c_{ijv}, b_{ijv}\}\Big] \\
&\quad \text{[By commutative law]} \\
&= C \odot \Big[\max\{a_{ij\mu}, b_{ij\mu}\}, \min\{a_{ijv}, b_{ijv}\}\Big] \text{ [By distributive law]} \\
&= C \odot (A + C) = C \text{ [By absorption property].}
\end{aligned}
$$

Therefore, $B = C$.

4. Generalized Intuitionistic Fuzzy Determinant (GIFD) over Distributive Lattice

The generalized intuitionistic fuzzy determinant $|A|$ of an $n \times n$ GIFM A over a distributive lattice $(G_n(L), \leq, +, \odot)$ is defined as follows:

$$\det A = |A| = \sum_{\sigma \in S_n} \langle a_{1\sigma(1)\mu}, a_{1\sigma(1)v}\rangle \langle a_{2\sigma(2)\mu}, a_{2\sigma(2)v}\rangle \cdots \langle a_{n\sigma(n)\mu}, a_{n\sigma(n)v}\rangle,$$

where S_n denotes the symmetric group of all permutations of the indices $(1, 2, \cdots, n)$.

Proposition 4.1 *If a GIFM B over a distributive lattice* $(G_n(L), \leq, +, \odot)$*, is obtained from an* $n \times n$ *GIFM A by multiplying the i-th row of A (i-th column) by* $k = \langle k_1, k_2\rangle$ *such that* $0 \leq k_1 + k_2 \leq 1$*, then* $k|A| = |B|$.

Proof By definition of GIFD, we have

$$
\begin{aligned}
|B| &= \sum_{\sigma \in S_n} \langle b_{1\sigma(1)\mu}, b_{1\sigma(1)v}\rangle \langle b_{2\sigma(2)\mu}, b_{2\sigma(2)v}\rangle \cdots \langle b_{n\sigma(n)\mu}, b_{n\sigma(n)v}\rangle \\
&= \sum_{\sigma \in S_n} \langle a_{1\sigma(1)\mu}, a_{1\sigma(1)v}\rangle \langle a_{2\sigma(2)\mu}, a_{2\sigma(2)v}\rangle \cdots k\langle a_{i\sigma(i)\mu}, a_{i\sigma(i)v}\rangle \cdots \langle a_{n\sigma(n)\mu}, a_{n\sigma(n)v}\rangle \\
&= k \sum_{\sigma \in S_n} \langle a_{1\sigma(1)\mu}, a_{1\sigma(1)v}\rangle \langle a_{2\sigma(2)\mu}, a_{2\sigma(2)v}\rangle \cdots \langle a_{i\sigma(i)\mu}, a_{i\sigma(i)v}\rangle \cdots \langle a_{n\sigma(n)\mu}, a_{n\sigma(n)v}\rangle \\
&= k|A|.
\end{aligned}
$$

Proposition 4.2 *Let A be an* $n \times n$ *GIFM over a distributive lattice* $(G_n(L), \leq, +, \odot)$. *If all the elements of a row (column) are* $\langle 0, 1\rangle$*, then* $|A| = \langle 0, 1\rangle$.

Proof Since each term in $|A|$ contains a factor of each row (column) and hence contains a factor of $\langle 0, 1\rangle$ row (column), so that each term of $|A|$ is equal to $\langle 0, 1\rangle$ and consequently $|A| = \langle 0, 1\rangle$.

Proposition 4.3 *Let A be an $n \times n$ GIFM over a distributive lattice $(G_n(L), \leq, +, \odot)$. If A is triangular, then* $|A| = \prod_{i=1}^{n} \langle a_{ii\mu}, a_{ii\nu} \rangle$.

Proof Let A be a GIFM in triangular form below, i.e., $\langle a_{ij\mu}, a_{ij\nu} \rangle = \langle 0, 1 \rangle$ for $i < j$. Now consider a term b of $|A|$

$$b = \sum_{\sigma \in S_n} \langle a_{1\sigma(1)\mu}, a_{1\sigma(1)\nu} \rangle \langle a_{2\sigma(2)\mu}, a_{2\sigma(2)\nu} \rangle \cdots \langle a_{n\sigma(n)\mu}, a_{n\sigma(n)\nu} \rangle.$$

Let $\sigma(1) \neq 1$, so that $1 < \sigma(1)$ and therefore $\langle a_{1\sigma(1)\mu}, a_{1\sigma(1)\nu} \rangle = \langle 0, 1 \rangle$ and $b = \langle 0, 1 \rangle$. This means that each term is $\langle 0, 1 \rangle$ if $\sigma(1) \neq 1$. Now let $\sigma(1) = 1$ but $\sigma(2) \neq 2$. Then $2 < \sigma(2)$ and $\langle a_{2\sigma(2)\mu}, a_{2\sigma(2)\nu} \rangle = \langle 0, 1 \rangle$ and $b = \langle 0, 1 \rangle$. This means that each term is $\langle 0, 1 \rangle$ if $\sigma(1) \neq 1$ and $\sigma(2) \neq 2$. In the similar manner, we can prove that each term for which $\sigma(1) \neq 1$ or $\sigma(2) \neq 2 \cdots$ or $\sigma(n) \neq n$ must be $\langle 0, 1 \rangle$. Hence $|A| = \prod_{i=1}^{n} \langle a_{ii\mu}, a_{ii\nu} \rangle$.

4.1. Generalized Intuitionistic Fuzzy Principal Submatrix

Let $A \in (G_n(L), \leq, +, \odot)$ and $A(i_1, i_2, \cdots, j_t | j_1, j_2, \cdots, j_t)$ denote $(n - t) \times (n - t)$ submatrix obtained from A by eliminating rows $i_1, i_2, \cdots, i_t$ and columns $j_1, j_2, \cdots, j_t$ is called a principal submatrix of order $n - t$ of A.

The adjoint of an IFM over a distributive lattice is defined as below.

Definition 4.1 *Adjoint of an $n \times n$ GIFM A over a distributive lattice $(G_n(L), \leq, +, \odot)$, is denoted as* adj$A$ *and is defined as follows*

$$\text{adj } A = |A_{ji}|,$$

where $|A_{ji}|$ is the determinant of the $(n - 1) \times (n - 1)$ GIFM formed by deleting row j and column i from A.

Definition 4.2 *Let A be GIFM and $A \in (G_n(L), \leq, +, \odot)$. Then* $\det A = \sum_{i=1}^{n} a_{ij} A(i|j)$.

5. Generalized Intuitionistic Fuzzy Nilpotent Matrix (GIFNM) over a Distributive Lattice

If $A \in (G_n(L), \leq, +, \odot)$ and $A^m = \mathbf{0}$ for some $m \geq 1$, then A is called GIFNM over the distributive lattice $(G_n(L), \leq, +, \odot)$. The least positive integer m satisfying $A^m = \mathbf{0}$ is called the nilpotent index of A and is denoted by $h(A)$.

Definition 5.1 *Let A be a GIFM and $A \in G_n(L)$. Then A is said to be GIFNM if and only if every principal minor of A is $\langle 0, 1 \rangle$.*

Proposition 5.1 *Let A, B, C, A_1, A_2, $A_3, \cdots, A_n \in (G_n(L), \leq, +, \odot)$. Then*

(1) $A(B - C) \geq AB - AC$ *and* $(A - B)C \geq AC - BC$,

(2) $(A - B) - C \geq A - (B + C)$,

(3) *If* $A \leq B$, *then* $A - C \leq B - C$ *and* $C - A \geq C - B$,

(4) $(A_1 - A_2) + (A_2 - A_3) + \cdots + (A_{l-1} - A_l) + A_l = A_1 + A_2 + \cdots + A_l$.

Proof Since $A, B, C \in (G_n(L), \leq, +, \odot)$, we have

$$
\begin{aligned}
(A(B-C))_{ij} &= \Big[\Big\langle \sum_{k=1}^{n} a_{ik\mu} d_{kj\mu}, \prod_{k=1}^{n} a_{ik\nu} d_{kj\nu}\Big\rangle\Big] \\
&\quad [\text{where } d_{kj\mu} = b_{kj\mu} - c_{kj\mu} \textit{ and } d_{kj\nu} = b_{kj\nu} - c_{kj\nu}] \\
&= \Big[\Big\langle \sum_{k=1}^{n} a_{ik\mu}(b_{kj\mu} - c_{kj\mu}), \prod_{k=1}^{n} a_{ik\nu}(b_{kj\nu} - c_{kj\nu})\Big\rangle\Big] \\
&\geq \Big[\Big\langle \sum_{k=1}^{n}(a_{ik\mu} b_{kj\mu} - a_{ik\mu} c_{kj\mu}), \prod_{k=1}^{n}(a_{ik\nu} b_{kj\nu} - a_{ik\nu} c_{kj\nu})\Big\rangle\Big] \\
&\geq \Big[\Big\langle \sum_{k=1}^{n}(a_{ik\mu} b_{kj\mu}, \prod_{k=1}^{n} a_{ik\nu} b_{kj\nu})\Big\rangle\Big] - \Big[\Big\langle \sum_{k=1}^{n}(a_{ik\mu} c_{kj\mu}, \prod_{k=1}^{n} a_{ik\nu} c_{kj\nu})\Big\rangle\Big] \\
&= AB - AC.
\end{aligned}
$$

Hence, $A(B-C) \geq AB - AC$.

Similarly, it can prove the second part of this proposition.

Proofs of (2) and (3) are straight forward.

Proof (4): Let $A_1, A_2, A_3, \cdots, A_n \in (G_n(L), \leq, +, \odot)$. We have to prove
$(A_1 - A_2) + (A_2 - A_3) + \cdots + (A_{l-1} - A_l) + A_l = A_1 + A_2 + \cdots + A_l$.
We prove this proposition by means of induction on l. Now for $l = 2$, we have
$(A_1 - A_2) + A_2 = A_1 + A_2$.
Let us assume that the relation holds for $l-1$. Now,

$$
\begin{aligned}
&(A_1 - A_2) + (A_2 - A_3) + \cdots + (A_{l-1} - A_l) + A_l \\
&= (A_1 - A_2) + ((A_2 - A_3) + \cdots + (A_{l-1} - A_l) + A_l) \\
&= ((A_1 - A_2) + A_2) + A_3 + \cdots + A_l \text{ [By induction hypothesis]} \\
&= A_1 + A_2 + \cdots + A_l.
\end{aligned}
$$

Hence $(A_1 - A_2) + (A_2 - A_3) + \cdots + (A_{l-1} - A_l) + A_l = A_1 + A_2 + \cdots + A_l$.

Example 2 Let A, B, C be three GIFMs over distributive lattice $(G_n(L), \leq, +, \odot)$, where

$$
A = \begin{bmatrix} \langle 0.7, 0.4\rangle & \langle 0.7, 0.3\rangle & \langle 0.7, 0.5\rangle \\ \langle 0.6, 0.5\rangle & \langle 0.8, 0.5\rangle & \langle 0.7, 0.4\rangle \\ \langle 0.7, 0.5\rangle & \langle 0.8, 0.4\rangle & \langle 0.8, 0.3\rangle \end{bmatrix}, \quad B = \begin{bmatrix} \langle 0.5, 0.6\rangle & \langle 0.6, 0.4\rangle & \langle 0.5, 0.6\rangle \\ \langle 0.6, 0.5\rangle & \langle 0.6, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.6\rangle & \langle 0.5, 0.5\rangle & \langle 0.6, 0.4\rangle \end{bmatrix}
$$

and

$$
C = \begin{bmatrix} \langle 0.3, 0.8\rangle & \langle 0.4, 0.7\rangle & \langle 0.4, 0.8\rangle \\ \langle 0.4, 0.7\rangle & \langle 0.5, 0.7\rangle & \langle 0.5, 0.6\rangle \\ \langle 0.4, 0.8\rangle & \langle 0.3, 0.8\rangle & \langle 0.3, 0.9\rangle \end{bmatrix}.
$$

Now we calculate the following

$$
A - B = \begin{bmatrix} \langle 0.7, 0.4\rangle & \langle 0.7, 0.3\rangle & \langle 0.7, 0.5\rangle \\ \langle 0.6, 0.0\rangle & \langle 0.8, 0.0\rangle & \langle 0.7, 0.4\rangle \\ \langle 0.7, 0.5\rangle & \langle 0.8, 0.4\rangle & \langle 0.8, 0.3\rangle \end{bmatrix}
$$

and

$$(A-B)-C=\begin{bmatrix}\langle 0.7,0.4\rangle & \langle 0.7,0.3\rangle & \langle 0.7,0.5\rangle\\ \langle 0.6,0.0\rangle & \langle 0.8,0.0\rangle & \langle 0.7,0.4\rangle\\ \langle 0.7,0.5\rangle & \langle 0.8,0.4\rangle & \langle 0.8,0.3\rangle\end{bmatrix}.$$

Again

$$B+C=\begin{bmatrix}\langle 0.5,0.6\rangle & \langle 0.6,0.4\rangle & \langle 0.5,0.6\rangle\\ \langle 0.6,0.5\rangle & \langle 0.6,0.5\rangle & \langle 0.5,0.5\rangle\\ \langle 0.5,0.6\rangle & \langle 0.5,0.5\rangle & \langle 0.6,0.4\rangle\end{bmatrix}$$

and

$$A-(B+C)=\begin{bmatrix}\langle 0.7,0.4\rangle & \langle 0.7,0.3\rangle & \langle 0.7,0.5\rangle\\ \langle 0.6,0.0\rangle & \langle 0.8,0.0\rangle & \langle 0.7,0.4\rangle\\ \langle 0.7,0.5\rangle & \langle 0.8,0.4\rangle & \langle 0.8,0.3\rangle\end{bmatrix}.$$

Therefore, $(A-B)-C \geq A-(B+C)$.

Example 3 Let A, B, C be three 3×3 GIFMs, where

$$A=\begin{pmatrix}\langle 0.3,0.8\rangle & \langle 0.7,0.5\rangle & \langle 0.4,0.5\rangle\\ \langle 0.5,0.6\rangle & \langle 0.5,0.6\rangle & \langle 0.6,0.4\rangle\\ \langle 0.6,0.4\rangle & \langle 0.4,0.7\rangle & \langle 0.5,0.8\rangle\end{pmatrix}, B=\begin{pmatrix}\langle 0.4,0.6\rangle & \langle 0.7,0.4\rangle & \langle 0.6,0.4\rangle\\ \langle 0.5,0.4\rangle & \langle 0.6,0.5\rangle & \langle 0.7,0.3\rangle\\ \langle 0.7,0.4\rangle & \langle 0.5,0.6\rangle & \langle 0.6,0.5\rangle\end{pmatrix}$$

and

$$C=\begin{pmatrix}\langle 0.4,0.5\rangle & \langle 0.8,0.4\rangle & \langle 0.6,0.5\rangle\\ \langle 0.6,0.4\rangle & \langle 0.6,0.3\rangle & \langle 0.7,0.3\rangle\\ \langle 0.8,0.4\rangle & \langle 0.6,0.5\rangle & \langle 0.7,0.4\rangle\end{pmatrix}.\text{ Here } A\leq B.$$

Now,

$$A-C=\begin{pmatrix}\langle 0.0,0.0\rangle & \langle 0.0,0.0\rangle & \langle 0.0,0.0\rangle\\ \langle 0.0,0.0\rangle & \langle 0.0,0.0\rangle & \langle 0.0,0.0\rangle\\ \langle 0.0,0.0\rangle & \langle 0.0,0.0\rangle & \langle 0.0,0.0\rangle\end{pmatrix}$$

and

$$B-C=\begin{pmatrix}\langle 0.4,0.0\rangle & \langle 0.4,0.0\rangle & \langle 0.6,0.4\rangle\\ \langle 0.0,0.0\rangle & \langle 0.6,0.0\rangle & \langle 0.7,0.0\rangle\\ \langle 0.0,0.0\rangle & \langle 0.0,0.0\rangle & \langle 0.0,0.0\rangle\end{pmatrix}.$$

Therefore, if $A\leq B$, then $A-C\leq B-C$.

Proposition 5.2 *If A is an GIFNM over a distributive lattice $(G_n(L),\leq,+,\odot)$, then* $\det A=\langle 0,1\rangle$ *but converse of the proposition is not true.*

Proof Let A be a GIFNM over a distributive lattice $(G_n(L),\leq,+,\odot)$. Then

$$\det A=\sum_{i=1}^{n} a_{ij}A(i|j),\ j=1,2,\cdots,n.$$

Again, by the definition of the GIFNM, a GIFM A will be a nilpotent intuitionistic fuzzy matrix if and only if every principal minor of A is $\langle 0,1\rangle$.

Therefore $\det A=\langle 0,1\rangle$.

Example 4 To show the converse part of the proposition, consider a GIFM over a distributive lattice $(G_n(L),\leq,+,\odot)$ as

$$A=\begin{bmatrix}\langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle\\ \langle 0.4,0.9\rangle & \langle 0.5,0.8\rangle & \langle 0.4,0.7\rangle\\ \langle 0.5,0.6\rangle & \langle 0.4,0.8\rangle & \langle 0.4,0.8\rangle\end{bmatrix}.$$

Therefore, by the Proposition 5.2, we have det $A = \langle 0, 1\rangle$ although it is not a GIFNM.

Proposition 5.3 *Let A be a GIFM and* $A \in (G_n(L), \le, +, \odot)$*. Then A is GIFNM if and only if* $a_{ii}^{(k)} = \langle 0, 1\rangle$*, where* $a_{ii}^{(k)}$ *is the diagonal elements of* A^k *for all k.*

Example 5 Let us consider a GIFM over a distributive lattice $(G_n(L), \le, +, \odot)$ as

$$A = \begin{pmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.4,0.7\rangle \\ \langle 0.5,0.6\rangle & \langle 0,1\rangle & \langle 0.3,0.8\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{pmatrix}.$$

Here $a_{ii}^{(1)} = \langle 0, 1\rangle$, $i = 1, 2, 3$. We obtain

$$A^2 = \begin{pmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.4,0.7\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{pmatrix}.$$

Here $a_{ii}^{(2)} = \langle 0, 1\rangle$, $i = 1, 2, 3$. and

$$A^3 = \begin{pmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{pmatrix},$$

also $a_{ii}^{(3)} = \langle 0, 1\rangle$, $i = 1, 2, 3$.

Thus GIFM A over a distributive lattice L is a GIFNM and index is $h(A) = 3$.

Proposition 5.4 *Let A be a GIFM over a distributive lattice L, i.e.,* $A \in (G_n(L), \le, +, \odot)$*. Then A is GIFNM if and only if A is irreflexive and transitive.*

Proof Let A be an IF irreflexive matrix, i.e., $a_{ii} = \langle 0, 1\rangle$ for all i. Since, A is IF transitive matrix, we have $A^2 \le A$ and so $A^k \le A$ for all k. Therefore, $a_{ii}^{(k)} \le a_{ii} = \langle 0, 1\rangle$ for all $i, k \in N$. So, A is a GIFNM.

Conversely, suppose that A is a GIFNM. If A is not a GIFNM, then $A \ne \mathbf{0}$. If $A^2 = A$, then $A = A^2 = \cdots = A^n$ and so $A^n = A \ne \mathbf{0}$. Thus a contradiction to the assumption that A is GIFNM.

Again, if $A^2 \ge A$, then $A \le A^2 \le \cdots \le A^n$ and therefore $A^n \le A \ne \mathbf{0}$, which is also a contradiction.

Hence, A must be a generalized intuitionistic fuzzy transitive matrix.

Now suppose that A is not generalized intuitionistic fuzzy irreflexive matrix. Then $a_{ii} \ne \langle 0, 1\rangle$ for some $i \in N$. Therefore A is not a GIFNM, which is also a contradiction.

Hence A must be a generalized intuitionistic fuzzy irreflexive and transitive matrix.

Proposition 5.5 *Let A be a GIFM and* $A \in (G_n(L), \le, +, \odot)$*. If A generalized intuitionistic fuzzy nilpotent matrix, then*

(1) A (adj A) = $\mathbf{0}$ *and* (adj A) $A = \mathbf{0}$.

(2) (adj $A)^2 = \mathbf{0}$.

Proof Let $B = A(\text{adj } A)$. Then for any $i, j \in N$ (set of natural numbers) with $i \ne j$. We have

$$\langle b_{ij\mu}, b_{ij\nu}\rangle = \sum_{\sigma \in S_n} \langle a_{1\sigma(1)\mu}, a_{1\sigma(1)\mu}\rangle \cdots \langle a_{i\sigma(i)\mu}, a_{i\sigma(i)\nu}\rangle \cdots \langle a_{n\sigma(n)}, a_{i\sigma(i)\nu}\rangle.$$

Let $\sigma \in S_n$ be an arbitrary.

Case (i) $\sigma^l(i) \neq j$ for $l \geq 1$. Then there exists d such that $1 \leq d \leq n$, $\sigma^l(i) = i$ and i, $\sigma^l(i) \cdots \sigma^{d-1}(i)$ are mutually different and belong to N. Then

$$\begin{aligned}
&\langle a_{1\sigma(1)\mu}, a_{1\sigma(1)\nu}\rangle \cdots \langle a_{i\sigma(i)\mu}, a_{i\sigma(i)\nu}\rangle \cdots \langle a_{n\sigma(n)\mu}, a_{n\sigma(n)\nu}\rangle \\
&\quad \leq \langle a_{i\sigma(i)\mu}, a_{i\sigma(i)\nu}\rangle \cdots \langle a_{\sigma(i)\sigma^2(i)\mu}, a_{\sigma(i)\sigma^2(i)\nu}\rangle \cdots \langle a_{\sigma^{d-1}(i)i\mu}, a_{\sigma^{d-1}(i)i\nu}\rangle \\
&\quad \leq (A^d)_{ii}[\text{since for GIFNM } a_{ii}^k = \langle 0, 1\rangle \text{ for all } i, k \in N \text{ where } A^k = \left[a_{ii}^k\right]] \\
&\quad = \left[\langle 0, 1\rangle\right].
\end{aligned}$$

Case (ii*)* There exists l such that $\sigma^l(i) = j$. Then there exists d such that $1 \leq d \leq n$, $\sigma^l(i) = j$ and i, $\sigma^l(j) \cdots \sigma^{d-1}(j)$ are mutually different and belong to N. Then

$$\begin{aligned}
&\langle a_{1\sigma(1)\mu}, a_{1\sigma(1)\nu}\rangle \cdots \langle a_{i\sigma(i)\mu}, a_{i\sigma(i)\nu}\rangle \cdots \langle a_{i\sigma(j)\mu}, a_{i\sigma(j)\nu}\rangle \cdots \langle a_{n\sigma(n)\mu}, a_{n\sigma(n)\nu}\rangle \\
&\quad \leq \langle a_{i\sigma(j)\mu}, a_{i\sigma(j)\nu}\rangle \cdots \langle a_{\sigma(j)\sigma^2(j)\mu}, a_{\sigma(j)\sigma^2(j)\nu}\rangle \cdots \langle a_{\sigma^{d-1}(j)i\mu}, a_{\sigma^{d-1}(j)i\nu}\rangle \\
&\quad \leq (A^d)_{ii}[\text{since for GIFNM } a_{ii}^k = \langle 0, 1\rangle \text{ for all } i, k \in N \text{ where } A^k = \left[a_{ii}^k\right]] \\
&\quad = \left[\langle 0, 1\rangle\right].
\end{aligned}$$

Therefore, for any $i, j \in N$ with $i \neq j$, we have

$$\langle b_{ij\mu}, b_{ij\nu}\rangle = \sum_{\sigma \in s_n} \langle a_{1\sigma(1)\mu}, a_{1\sigma(1)\mu}\rangle \cdots \langle a_{i\sigma(i)\mu}, a_{i\sigma(i)\nu}\rangle \cdots \langle a_{n\sigma(n)}, a_{i\sigma(i)\nu}\rangle = \langle 0, 1\rangle.$$

If $i = j$, it is clear that $b_{ii} = \langle 0, 1\rangle$.

Thus $B = A(\text{adj } A) = \mathbf{0}$.

Example 6 Let $A = \begin{pmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.4,0.7\rangle \\ \langle 0.3,0.6\rangle & \langle 0,1\rangle & \langle 0.3,0.8\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{pmatrix}$ be a GIFNM over distributive lattice $(G_n(L), \leq, +, \odot)$. Now, the adjoint matrix of GIFM A is

$$\text{adj } A = \begin{pmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.3,0.7\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{pmatrix}.$$

Also

$$\begin{aligned}
(\text{adj } A)^2 &= \begin{pmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,0\rangle & \langle 0,1\rangle & \langle 0.3,0.7\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{pmatrix} \begin{pmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.3,0.7\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{pmatrix} \\
&= \begin{pmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,0\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{pmatrix} = \mathbf{0}.
\end{aligned}$$

Proposition 5.6 *Let A be a GIFNM over a distributive lattice L, i.e., $A \in G_n(L)$. Then $h(A) = 3$ if and only if $AA^T = \mathbf{0}$ and for some $i, j \in N$, $R_i \wedge R_j^T \neq \langle 0, 1\rangle$, where R_i and R_j are the i-th and j-th rows of the GIFNM A.*

Proof Let the index of a GIFM A be 3, i.e., $h(A) = 3$. Since $A^2 \neq \mathbf{0}$, then there must exist some rows say s-th and t-th such that $R_i \wedge R_j^T \neq \langle 0, 1\rangle$. Now we show that $AA^T = \mathbf{0}$. Suppose $AA^T \neq \mathbf{0}$. Then we can find p, q such that $\langle a_{pq\mu}, a_{pq\nu}\rangle\langle a_{qp\mu}, a_{qp\nu}\rangle > \langle 0, 1\rangle$. Therefore $\langle a_{pq\mu}, a_{pq\nu}\rangle\langle a_{qp\mu}, a_{qp\nu}\rangle\langle a_{pq\mu}, a_{pq\nu}\rangle > \langle 0, 1\rangle$, which is a term of (p, q)-th entry of the matrix A^3. So

$$\sum_{1\leq i_1, i_2\leq n} \langle a_{qi_1\mu}, a_{qi_1\nu}\rangle\langle a_{i_1i_2\mu}, a_{i_1i_2\nu}\rangle\langle a_{i_2p\mu}, a_{i_2p\nu}\rangle > \langle a_{qp\mu}, a_{qp\nu}\rangle\langle a_{pq\mu}, a_{pq\nu}\rangle\langle a_{qp\mu}, a_{qp\nu}\rangle > \langle 0, 1\rangle,$$

which leads to a contradiction, since $A^3 = \mathbf{0}$.

Conversely, $R_i \wedge R_j^T \neq \langle 0, 1\rangle$ for some rows say i, j, then $A^2 \neq \mathbf{0}$. Therefore, from $AA^T = \mathbf{0}$, we get $A^3 = \mathbf{0}$. Hence A is an IFNM of index 3.

Note: Let A be a GIFNM over a distributive lattice L, i.e., $A \in G_n(L)$, and $AA^T = \mathbf{0}$ and for all $i, j \in N$, $R_i \wedge R_j^T = \langle 0, 1\rangle$. Then A is neither GIFNM nor converge to a GIFM.

Example 7 Let $A = \begin{pmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.5,0.7\rangle \\ \langle 0.3,0.8\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0.4,0.6\rangle & \langle 0,1\rangle \end{pmatrix}$ be a GIFM over a distributive lattice $(G_n(L), \leq, +, \odot)$ of order 3×3 and $AA^T = \mathbf{0}$, and for all $i, j \in N$, $R_i \wedge R_j^T = \langle 0, 1\rangle$. Now,

$$A^2 = \begin{pmatrix} \langle 0,1\rangle & \langle 0.4,0.7\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.3,0.8\rangle \\ \langle 0.3,0.8\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{pmatrix}, A^3 = \begin{pmatrix} \langle 0.3,0.8\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0.3,0.8\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.3,0.8\rangle \end{pmatrix},$$

$$A^4 = \begin{pmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.3,0.8\rangle \\ \langle 0.3,0.8\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0.3,0.8\rangle & \langle 0,1\rangle \end{pmatrix}, A^5 = \begin{pmatrix} \langle 0,1\rangle & \langle 0.3,0.8\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.3,0.8\rangle \\ \langle 0.3,0.8\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{pmatrix},$$

$$A^6 = \begin{pmatrix} \langle 0.3,0.8\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0.3,0.8\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.3,0.8\rangle \end{pmatrix}$$

and if continue this process, we get $A_3^k = A_3^{k+3}$ where $k \in N$, set of natural numbers and $k \geq 3$.

Here A is neither GIFNM nor converge to any GIFM.

6. Reduction of GIFNMs over a Distributive Lattice

Let L be a distributive lattice and $A \in (G_n(L), \leq, +, \odot)$. Then GIFM $A/A = A - A^2$ is called a reduction of GIFM A. It is clear that $A/A \leq A$ for all GIFM A in $(G_n(L), \leq, +, \odot)$. Hashimoto [12] discussed the reduction of irreflexive and transitive fuzzy matrices and obtained some properties and applied these properties to nilpotent fuzzy matrices.

In this section, we shall consider the reduction of GIFNMs over distributive lattice. The results obtained in this section generalize the previous results on nilpotent matrices by Hasimoto [12].

Theorem 6.1 *Let A be a GIFNM and $A \in (G_n(L), \leq, +, \odot)$. Then $(A/A)^+ = A^+$, where $A/A = A - A^2$ and $A^+ = A + A^2 + A^3 + \cdots + A^n$.*

Let $S = A/A$. We note that A and S are GIFNM. It is clear that $(A/A)^T \leq A^T$ since $A/A = A - A^2 \leq A$.

In the following, we shall prove that $A^T \leq (A/A)^T$. To do this, we shall prove that $S^l \leq A^l - A^{l-1}$ for all l.

We shall prove $S^l \leq A^l - A^{l-1}$ by induction on l. It is clear that it holds for $l = 1$, since $S = A - A^2$, and we may assume that it holds for $l - 1$. Then $S^l = SS^{l-1} \geq S(A^{l-1} - A^l) \geq SA^{l-1} - SA^l$.

Since $A - A^2 = S \leq A$. We have

$$\begin{aligned} S^l &\geq (A - A^2)A^{l-1} - A^{l+1} \\ &\geq (A^l - A^{l+1}) - A^{l+1} \\ &\geq A^l - (A^{l+1} + A^{l+1}) \\ &= A^l - A^{l+1}. \end{aligned}$$

Now,

$$\begin{aligned} (A/A)^+ &= S^+ \\ &= S + S^2 + S^3 + \cdots + S^n \\ &\geq (A - A^2) + (A^2 - A^3) + \cdots + (A^n - A^{n+1}) \\ &= (A - A^2) + (A^2 - A^3) + \cdots + (A^{n-1} - A^n) + A^n (since A^{n+1} = \mathbf{0}) \\ &= A + A^2 + A^3 + \cdots + A^n (\text{ by Proposition 5.1}) \\ &= A^+. \end{aligned}$$

Therefore, $(A/A)^+ = A^+$.

Theorem 6.2 *Let A be an $n \times n$ irreflexive and transitive matrix over $(G_n(L), \leq, +, \odot)$. Then $(A/A)^+ = A$.*

Proof Since A is IF irreflexive and transitive matrix, we have A is IF nilpotent matrix and $A = A^+$, and so $(A/A)^+ = A$.

Theorem 6.3 *Let A be an $n \times n$ irreflexive and transitive matrix over $(G_n(L), \leq, +, \odot)$. Then the following conditions are equivalent*
(1) $A/A \leq S \leq A$, (2) $S^+ = A$.

Proof Suppose that $A/A \leq S \leq A$, clearly by Theorem 6.2, $S^+ = A$. Thus we have that (1) implies (2).

Now suppose that $S^+ = A$. Then we have that $S \leq A$ and $A^2 = (S^+)^2 = S^2 + S^3 + \cdots + S^n + \cdots + S^{2n}$.

Since A is IF irreflexive and transitive, A is GIFNM and so $S^l = \mathbf{0}$ for $l \geq n$ (because $S \leq A$).
Therefore, $A^2 = S^2 + S^3 + \cdots + S^{n-1}$ and so $S + A^2 = S + S^2 + S^3 + \cdots + S^{n-1} = S^+ = A$. Thus, $\langle s_{ij\mu}, s_{ij\nu} \rangle + \langle a^{(2)}_{ij\mu}, a^{(2)}_{ij\nu} \rangle = \langle a_{ij\mu}, a_{ij\nu} \rangle$ for all i and j, i.e., $\langle s_{ij\mu}, s_{ij\nu} \rangle = \langle a_{ij\mu}, a_{ij\nu} \rangle - \langle a^{(2)}_{ij\mu}, a^{(2)}_{ij\nu} \rangle = (A/A)_{ij}$ for all i and j.
Hence we have that $A/A \leq S \leq A$ and that (2) implies (1).

7. Conclusion

In this paper, we have proved that GIFMs form a lattice and also shown that this lattice is distributive. But we have not studied whether this lattice is modular or not.

Some properties of GIFMs are investigated, including the nilpotency of it. It can be seen that all GIFMs are not nilpotent, but under certain condition some GIFMs are nilpotent. we expect some other conditions may exist for nilpotency of GIFMs. Convergence of GIFMs over lattice is a very important property for any type of fuzzy matrices. At present, we are investigating this property for GIFMs.

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