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Intuitionistic Fuzzy Sublattices and Ideals

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Abstract We study the concept of intuitionistic fuzzy sublattices and intuitionistic fuzzy ideals of a lattice. Some characterization and properties of these intuitionistic fuzzy sublattices and ideals are established. Also we introduce the sum and product of two intuitionistic fuzzy ideals and prove that the sum and product of two Intuitionistic fuzzy ideals of a distributive lattice is again an intuitionistic fuzzy ideal. Moreover, we study the properties of intuitionistic fuzzy ideals under lattice homomorphism.

Keywords Lattice · Intuitionistic fuzzy sets · Fuzzy sublattice · Fuzzy ideal · Homomorphism · Epimorphism

1. Introduction

The concept of a fuzzy set was introduced by Zadeh [9] and it is now a rigorous area of research with applications in various fields. After that, many authors applied this concept to group and ring theory. In particular, N Ajmal and K.V Thomas [1, 2] applied the concept of fuzzy sets in lattice theory and systematically developed the theory of fuzzy sublattice.

With the work of fuzzy sets, 1986, Atanassov [4] introduced intuitionistic fuzzy sets which is very effective to ideal with vagueness. The concept of intuitionistic fuzzy sets is a generalization of fuzzy sets. Recently, many researchers applied the notion of intuitionistic fuzzy sets to Algebra and introduced intuitionistic fuzzy subgroups [5] and intuitionistic fuzzy subrings [6] etc.

In this paper, we study the concepts of intuitionistic fuzzy sublattices (IFL) and intuitionistic fuzzy ideals (IFI) of a lattice, and the properties of these IFLs and IFIs are established. Also we introduce the sum and product of two IFIs and prove that

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the sum and the product of two IFIs of a distributive lattice is again an IFI. Moreover, we study the properties of intuitionistic fuzzy ideals under lattice homomorphism.

2. Preliminaries

In this section, we recall the following definitions and results which are used in the sequel. Throughout this paper, L stands for a Lattice $(L, \vee, \wedge)$.

Definition 2.1 [1] *Let X be a non-empty set. An intuitionistic fuzzy set [IFS] A of X is an object of the following form* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$, *where* $\mu_A : X \to [0, 1]$ *and* $\nu_A : X \to [0, 1]$ *define the degree of membership and the degree of non membership of the element* $x \in X$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

The set of all IFSs on X is denoted by IFS (X).

Definition 2.2 [1] *If* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$ and B $= \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in X\}$ *are any two IFS of X, then*

(i) $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x), \forall x \in X$;
(ii) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x), \forall x \in X$;
(iii) $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle / x \in X\}$;
(iv) $[A] = \{\langle x, \mu_A(x), \mu^c{}_A(x)\rangle / x \in X\}$, where $\mu^c{}_A(x) = 1 - \mu_A(x)$;
(v) $\langle A\rangle = \{\langle x, \nu^c{}_A(x), \nu_A(x)\rangle / x \in X\}$, where $\nu^c{}_A(x) = 1 - \nu_A(x)$;
(vi) $A \cap B = \{\langle x, \mu_{A\cap B}(x), \nu_{A\cap B}(x)\rangle / x \in X\}$, where
$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$ and $\nu_{A\cap B}(x) = \max\{\nu_A(x), \nu_B(x)\}$;
(vii) $A \cup B = \{\langle x, \mu_{A\cup B}(x), \nu_{A\cup B}(x)\rangle / x \in X\}$, where
$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$ and $\nu_{A\cup B}(x) = \min\{\nu_A(x), \nu_B(x)\}$.

3. Intuitionistic Fuzzy Sublattices and Ideals

In this section, we introduce and study IFL and IFI and their characterizations.

Definition 3.1 *Let L be a lattice and* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in L\}$ *be an IFS of L. Then A is called an Intuitionistic fuzzy sublattice (IFL) of L if the following conditions are satisfied for all* $x, y \in L$:

(i) $\mu_A(x \vee y) \geq \min\{\mu_A(x), \mu_A(y)\}$; (ii) $\mu_A(x \wedge y) \geq \min\{\mu_A(x), \mu_A(y)\}$;
(iii) $\nu_A(x \vee y) \leq \max\{\nu_A(x), \nu_A(y)\}$; (iv) $\nu_A(x \wedge y) \leq \max\{\nu_A(x), \nu_A(y)\}$.

Example 1 Consider the lattice L of "divisors of 10". That is $L = \{1, 2, 5, 10\}$.

Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in L\}$ be given by

$\langle 1, 0.5, 0.1\rangle, \langle 2, 0.4, 0.5\rangle, \langle 5, 0.4, 0.3\rangle, \langle 10, 0.7, 0.3\rangle$.

Then A is an IFL of L.

Definition 3.2 *IFS A of L is called an intutionistic fuzzy ideal (IFI) of L, if the following conditions are satisfied for all* $x, y \in L$:

(i) $\mu_A(x \vee y) \geq \min\{\mu_A(x), \mu_A(y)\}$; (ii) $\mu_A(x \wedge y) \geq \max\{\mu_A(x), \mu_A(y)\}$;
(iii) $\nu_A(x \vee y) \leq \max\{\nu_A(x), \nu_A(y)\}$; (iv) $\nu_A(x \wedge y) \leq \min\{\nu_A(x), \nu_A(y)\}$.

Example 2 Consider the lattice $L = \{1, 2, 3, 4, 6, 12\}$ of divisors of 12. We define $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in L\}$ by

$\langle 1, 0.7, 0.2\rangle, \langle 2, 0.5, 0.5\rangle, \langle 3, 0.6, 0.3\rangle, \langle 4, 0.4, 0.5\rangle, \langle 6, 0.5, 0.5\rangle, \langle 12, 0.4, 0.5\rangle$.

Then it can be easily verified that A is an IFI of L.

Theorem 3.1 *If A and B are two IFLs (IFIs) of a lattice L, then A∩B is an IFL (IFI) of L.*

Proof Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in X\}$ are any two IFS of L. Then $A \cap B = \{\langle x, \mu_{A\cap B}(x), \nu_{A\cap B}(x)\rangle / x \in X\}$, where $\mu_{A\cap B}(x)=\min\{\mu_A(x), \mu_B(x)\}$ and $\nu_{A\cap B}(x) = \max\{\nu_A(x), \nu_B(x)\}$, so that

$$\begin{aligned}\mu_{A\cap B}(x \vee y) &= \min\{\mu_A(x \vee y), \mu_B(x \vee y)\}\\ &\geq \min\{\min\{\mu_A(x), \mu_A(y)\}, \min\{\mu_B(x), \mu_B(y)\}\}\\ &= \min\{\min\{\mu_A(x), \mu_B(x)\}, \min\{\mu_A(y), \mu_B(y)\}\}\\ &= \min\{\mu_{A\cap B}(x), \mu_{A\cap B}(y)\}\end{aligned}$$

as A and B are IFLs of L,
i.e.,

$$\mu_{A\cap B}(x \vee y) \geq \min\{\mu_{A\cap B}(x), \mu_{A\cap B}(y)\}, \ \ \forall x, y \in L.$$

Similarly, we get

$$\mu_{A\cap B}(x \wedge y) \geq \min\{\mu_{A\cap B}(x), \mu_{A\cap B}(y)\}, \ \ \forall x, y \in L.$$

Also

$$\begin{aligned}\nu_{A\cap B}(x \vee y) &= \max\{\nu_A(x \vee y), \nu_B(x \vee y)\}\\ &\leq \max\{\max\{\nu_A(x), \nu_A(y)\}, \max\{\nu_B(x), \nu_B(y)\}\}\\ &= \max\{\max\{\nu_A(x), \nu_B(x)\}, \max\{\nu_A(y), \nu_B(y)\}\}\\ &= \max\{\nu_{A\cap B}(x), \nu_{A\cap B}(y)\}\end{aligned}$$

as A and B are IFLs of L,
i.e.,

$$\nu_{A\cap B}(x \vee y) \leq \max\{\nu_{A\cap B}(x), \nu_{A\cap B}(y)\}, \ \ \forall x, y \in L.$$

Similarly, we get

$$\nu_{A\cap B}(x \wedge y) \leq \max\{\nu_{A\cap B}(x), \nu_{A\cap B}(y)\}, \ \ \forall x, y \in L.$$

Hence $A \cap B$ IFL of L.

Proof for IFI is similar.

Proposition 3.1 *A is an IFL (IFI) of L if and only if [A] and ⟨A⟩ are IFLs (IFIs) of L.*

Proof We will prove the case of IFL. Proof for IFI is similar.

First assume that A is an IFL of L. We have $[A]=\{\langle x, \mu_A(x), \mu^c{}_A(x)\rangle / x \in L\}$, where $\mu^c{}_A(x) = 1 - \mu_A(x)$. Then $\forall x, y \in L$,

$$\mu_A(x \vee y) \geq \min\{\mu_A(x), \mu_A(y)\}$$

and

$$\mu_A(x \wedge y) \geq \min\{\mu_A(x), \mu_A(y)\}$$

as A IFL of L. Now

$$\begin{aligned}\mu^c{}_A(x \vee y) &= 1 - \mu_A(x \vee y)\\ &\leq 1 - \min\{\mu_A(x), \mu_A(y)\}\\ &= \max\{1-\mu_A(x), 1-\mu_A(y)\}\\ &= \max\{\mu^c{}_A(x), \mu^c{}_A(y)\}.\end{aligned}$$

Similarly, we get

$$\mu^c{}_A(x \wedge y) \leq \max\{\mu^c{}_A(x), \mu^c{}_A(y)\}.$$

Hence $[A]$ is an IFL of L.

Now$\langle A\rangle = \{\langle x, \nu^c{}_A(x), \nu_A(x)\rangle / x \in L\}$. Then $\forall x, y \in L$,

$$\nu_A(x \vee y) \leq \max\{\nu_A(x), \nu_A(y)\}$$

and

$$\nu_A(x \wedge y) \leq \max\{\nu_A(x), \nu_A(y)\}$$

as A IFL of L. Now

$$\begin{aligned} \nu^c{}_A(x \vee y) &= 1 - \nu_A(x \vee y) \\ &\geq 1 - \max\{\nu_A(x), \nu_A(y)\} \\ &= \min\{1- \nu_A(x), 1- \nu_A(y)\} \\ &= \min\{ \nu^c{}_A(x), \nu^c{}_A(y)\}. \end{aligned}$$

Similarly, we get

$$\nu^c{}_A(x \wedge y) \geq \min\{ \nu^c{}_A(x), \nu^c{}_A(y)\}.$$

Hence $\langle A\rangle$ is an IFL of L.

Conversly, assume that $[A]$ and $\langle A\rangle$ are IFLs of L. Then A IFL of L follow easily from definition.

Now let us prove certain simple results through counter examples.

Remark 1 The union of two IFLs need not be an IFL.

Consider the lattice given in Example 1. Define $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle/x \in L\}$ by

$$\langle 1, 0.7, 0.2\rangle, \langle 2, 0.4, 0.5\rangle, \langle 5, 0.1, 0.5\rangle, \langle 10, 0.2, 0.4\rangle$$

and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle/x \in L\}$ by

$$\langle 1, 0.6, 0.1\rangle, \langle 2, 0.1, 0.5\rangle, \langle 5, 0.3, 0.3\rangle, \langle 10, 0.2, 0.3\rangle.$$

Here note that A and B are IFLs of L. Now $A \cup B = \{\langle x, \mu_{A\cup B}(x), \nu_{A\cup B}(x)\rangle/x \in L\}$ is

$$\langle 1, 0.7, 0.1\rangle, \langle 2, 0.4, 0.5\rangle, \langle 5, 0.3, 0.3\rangle, \langle 10, 0.2, 0.3\rangle.$$

But $\mu_{A\cup B}(10) = \mu_{A\cup B}(5 \vee 2) = 0.2 \ngeq \min\{\mu_{A\cup B}(5), \mu_{A\cup B}(2)\} = 0.3$.

So $A \cup B$ is not an IFL.

Remark 2 Every IFI of L is an IFL. But the converse is not true.

Consider the lattice L given in Example 1. Define $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle/x \in L\}$ by

$$\langle 1, 0.5, 0.1\rangle, \langle 2, 0.4, 0.3\rangle, \langle 5, 0.4, 0.5\rangle, \langle 10, 0.7, 0.3\rangle.$$

Here A is an IFL of L but not an IFI, because

$$\mu_A(2) = \mu_A(2 \wedge 10) = 0.4 \ngeq \max\{\mu_A(2), \mu_A(10)\} = 0.7.$$

Remark 3 The union of two IFIs need not be an IFI.

Consider the lattice L given in Example 2. Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle/x \in L\}$ is defined by

$$\langle 1, 0.7, 0.2\rangle, \langle 2, 0.5, 0.5\rangle, \langle 3, 0.6, 0.3\rangle, \langle 4, 0.4, 0.5\rangle, \langle 6, 0.5, 0.5\rangle, \langle 12, 0.4, 0.5\rangle$$

and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle/x \in L\}$ is given by

$$\langle 1, 0.6, 0.2\rangle, \langle 2, 0.6, 0.4\rangle, \langle 3, 0.5, 0.5\rangle, \langle 4, 0.5, 0.4\rangle, \langle 6, 0.4, 0.5\rangle, \langle 12, 0.5, 0.5\rangle.$$

Here A and B are IFIs of L. Also $A \cup B = \{\langle x, \mu_{A\cup B}(x), \nu_{A\cup B}(x)\rangle/x \in L\}$ is

$$\langle 1, 0.7, 0.2\rangle, \langle 2, 0.6, 0.4\rangle, \langle 3, 0.6, 0.3\rangle, \langle 4, 0.5, 0.4\rangle, \langle 6, 0.5, 0.5\rangle, \langle 12, 0.5, 0.5\rangle.$$

Here $\nu_{A\cup B}(12) = \nu_{A\cup B}(3 \vee 4) = 0.5 \nleq \max\{\nu_{A\cup B}(3), \nu_{A\cup B}(4)\} = 0.4$.

Hence $A \cup B$ is not an IFI.

Remark 4 If A is an IFI and B an IFL of L, then $A \cap B$ is an IFL but not an IFI.

Consider the lattice L given in Example 2. Define $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle/x \in L\}$ by

$$\langle 1, 0.7, 0.2\rangle, \langle 2, 0.5, 0.5\rangle, \langle 3, 0.6, 0.3\rangle, \langle 4, 0.4, 0.5\rangle, \langle 6, 0.5, 0.5\rangle, \langle 12, 0.4, 0.5\rangle$$

and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle/x \in L\}$ by

$$\langle 1, 0.2, 0.7\rangle, \langle 2, 0.4, 0.4\rangle, \langle 3, 0.2, 0.5\rangle, \langle 4, 0.3, 0.6\rangle, \langle 6, 0.5, 0.5\rangle, \langle 12, 0.6, 0.3\rangle.$$

Here A an IFI of L and B is an IFL of L. Then $A \cap B = \{\langle x, \mu_{A\cap B}(x), \nu_{A\cap B}(x)\rangle/x \in X\}$

is

$$\langle 1, 0.2, 0.7\rangle, \langle 2, 0.4, 0.5\rangle, \langle 3, 0.2, 0.5\rangle, \langle 4, 0.3, 0.6\rangle, \langle 6, 0.5, 0.5\rangle, \langle 12, 0.4, 0.5\rangle.$$

Clearly, $A \cap B$ is an IFL of L, but not an IFI because

$$\mu_{A\cap B}(1) = \mu_{A\cap B}(2 \wedge 3) = 0.2 \ngeq \max\{\mu_{A\cap B}(2), \mu_{A\cap B}(3)\} = 0.4.$$

4. Sum and Product of Two IFIs

In this section, we introduce two operations $A + B$ and $A \otimes B$ on IFS of L and prove that in a distributive lattice these are IFIs of L if both A and B are IFIs of L.

Definition 4.1 *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle/x \in L\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle/x \in L\}$ are two IFSs in L. Then their sum A+B is defined as*

$$A{+}B{=}\{\langle x, \mu_{A+B}(x), \nu_{A+B}(x)\rangle/x \in L\},$$

where

$$\mu_{A+B}(x) = \underset{x=a\vee b}{Sup}\{\min\{\mu_A(a), \mu_B(b)\}\}$$

and

$$\nu_{A+B}(x) = \underset{x=a\vee b}{Inf}\{\max\{\nu_A(a), \nu_B(b)\}\}.$$

Theorem 4.1 *The sum of two IFIs in a distributive lattice L is again an IFI of L.*

Proof Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle/x \in L\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle/x \in L\}$ are two IFIs in L. Then $A+B{=}\{\langle x, \mu_{A+B}(x), \nu_{A+B}(x)\rangle/x \in L\}$. Let $x, y \in L$ and let $\min\{\mu_{A+B}(x), \mu_{A+B}(y)\} = k$. Then for any $\varepsilon > 0$,

$$k - \varepsilon < \mu_{A+B}(x) = \underset{x=a\vee b}{Sup}\{\min\{\mu_A(a), \mu_B(b)\}\}$$

and

$$k - \varepsilon < \mu_{A+B}(y) = \underset{y=c\vee d}{Sup}\{\min\{\mu_A(c), \mu_B(d)\}\}.$$

So their exist representations $x = a \vee b$ and $y = c \vee d$ such that,

$$k - \varepsilon < \min\{\mu_A(a), \mu_B(b)\}$$

and

$$k - \varepsilon < \min\{\mu_A(c), \mu_B(d)\}.$$

Then

$$k - \varepsilon < \mu_A(a), k - \varepsilon < \mu_B(b), k - \varepsilon < \mu_A(c), k - \varepsilon < \mu_B(d).$$

Therefore

$$k - \varepsilon < \min\{\mu_A(a), \mu_A(c)\} \leq \mu_A(a \vee c),$$

since A is IFI of L .

Also

$$k - \varepsilon < \min\{\mu_B(b), \mu_B(d)\} \leq \mu_B(b \vee d),$$

since B is IFI of L.

Therefore

$$k - \varepsilon < \min\{\mu_A(a \vee c), \mu_B(b \vee d)\}.$$

Note that $x \vee y = (a \vee b) \vee (c \vee d) = (a \vee c) \vee (b \vee d)$. So

$$\begin{aligned} \mu_{A+B}(x \vee y) &= \underset{x\vee y=u\vee v}{Sup}\{\min\{\mu_A(u), \mu_B(w)\}\} \\ &\geq \min\{\mu_A(a \vee c), \mu_B(b \vee d)\} \\ &> k - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\mu_{A+B}(x \vee y) \geq k = \min\{\mu_{A+B}(x), \mu_{A+B}(y)\}. \quad (1)$$

Now let p=max$\{\mu_{A+B}(x), \mu_{A+B}(y)\} = \mu_{A+B}(x)$(say). Then for any $\varepsilon > 0$,

$$p - \varepsilon < \mu_{A+B}(x) = \underset{x=a\vee b}{Sup}\{\min\{\mu_A(a), \mu_B(b)\}\}.$$

So $\exists$ representation x=$a \vee b$ such that

$$p - \varepsilon < \min\{\mu_A(a), \mu_B(b)\} \Rightarrow p - \varepsilon < \mu_A(a), p - \varepsilon < \mu_B(b).$$

So for y=$c \vee d$, we have

$$p - \varepsilon < \max\{\mu_A(a), \mu_A(c \vee d)\} \leq \mu_A(a \wedge (c \vee d)),$$

since A is an IFI of L.
Also

$$p - \varepsilon < \max\{\mu_B(b), \mu_B(c \vee d)\} \leq \mu_B(b \wedge (c \vee d)),$$

since B is an IFI of L.
Therefore

$$p - \varepsilon < \min\{\mu_A(a \wedge (c \vee d)), \mu_B(b \wedge (c \vee d))\}.$$

Note that

$$x \wedge y = (a \vee b) \wedge (c \vee d) = (a \wedge (c \vee d)) \vee (b \wedge (c \vee d)).$$

So

$$\begin{aligned}\mu_{A+B}(x \wedge y) &= \underset{x\wedge y=u\vee v}{Sup}\{\min\{\mu_A(u), \mu_B(w)\}\}\\ &\geq \min\{\mu_A(a \wedge (c \vee d)), \mu_B(b \wedge (c \vee d))\}\\ &> p - \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\mu_{A+B}(x \wedge y) \geq p = \max\{\mu_{A+B}(x), \mu_{A+B}(y)\}. \quad (2)$$

Next, let $\max\{\nu_{A+B}(x), \nu_{A+B}(y)\} = q$. Then for any $\varepsilon > 0$,

$$q + \varepsilon > \nu_{A+B}(x) = \underset{x=a\vee b}{Inf}\{\max\{\nu_A(a), \nu_B(b)\}\}$$

and

$$q + \varepsilon > \nu_{A+B}(y) = \underset{y=c\vee d}{Inf}\{\max\{\nu_A(c), \nu_B(d)\}\}.$$

So their exist representations $x = a \vee b$ and $y = c \vee d$ such that,

$$q + \varepsilon > \max\{\nu_A(a), \nu_B(b)\}$$

and

$$q + \varepsilon > \max\{\nu_A(c), \nu_B(d)\}.$$

Then

$$q + \varepsilon > \nu_A(a), q + \varepsilon > \nu_B(b), q + \varepsilon > \nu_A(c), \text{ and } q + \varepsilon > \nu_B(d).$$

Therefore

$$q + \varepsilon > \max\{\nu_A(a), \nu_A(c)\} \geq \nu_A(a \vee c),$$

since A is IFI of L
and

$$q + \varepsilon > \max\{\nu_B(b), \nu_B(d)\} \geq \nu_B(b \vee d),$$

since B is IFI of L.
Therefore

$$q + \varepsilon > \max\{\nu_A(a \vee c), \nu_B(b \vee d)\}.$$

Note that

$$x \vee y = (a \vee b) \vee (c \vee d) = (a \vee c) \vee (b \vee d).$$

So

$$\begin{aligned}\nu_{A+B}(x \vee y) &= \underset{x\vee y=u\vee v}{Inf}\{\max\{\nu_A(u), \nu_B(w)\}\} \\ &\leq \max\{\nu_A(a \vee c), \nu_B(b \vee d)\} \\ &< q + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\nu_{A+B}(x \vee y) \leq q = \max\{\nu_{A+B}(x), \nu_{A+B}(y)\}. \tag{3}$$

Finally, let $h = \min\{\nu_{A+B}(x), \nu_{A+B}(y)\} = \nu_{A+B}(x)$ (say). Then for any $\varepsilon > 0$,

$$h + \varepsilon > \nu_{A+B}(x) = \underset{x=a\vee b}{Inf}\{\max\{\nu_A(a), \nu_B(b)\}\}.$$

So $\exists$ representation $x = a \vee b$ such that

$$h + \varepsilon > \max\{\nu_A(a), \nu_B(b)\} \Rightarrow h + \varepsilon > \nu_A(a), h + \varepsilon > \nu_B(b).$$

So for $y = c \vee d$, we have

$$h + \varepsilon > \min\{\nu_A(a), \nu_A(c \vee d)\} \geq \nu_A(a \wedge (c \vee d)),$$

since A is an IFI of L and

$$h + \varepsilon > \min\{\nu_B(b), \nu_B(c \vee d)\} \geq \nu_B(b \wedge (c \vee d)),$$

since B is an IFI of L.

Therefore

$$h + \varepsilon > \max\{\nu_A(a \wedge (c \vee d)), \nu_B(b \wedge (c \vee d))\}.$$

Note that

$$x \wedge y = (a \vee b) \wedge (c \vee d) = (a \wedge (c \vee d)) \vee (b \wedge (c \vee d)).$$

So

$$\begin{aligned}\nu_{A+B}(x \wedge y) &= \underset{x\wedge y=u\vee v}{Inf}\{\max\{\nu_A(u), \nu_B(w)\}\} \\ &\leq \max\{\nu_A(a \wedge (c \vee d)), \nu_B(b \wedge (c \vee d))\} \\ &< h + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\nu_{A+B}(x \wedge y) \leq h = \min\{\nu_{A+B}(x), \nu_{A+B}(y)\}. \tag{4}$$

From (1), (2), (3) and (4) $A + B$ is an IFI of L.

Definition 4.2 *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in L\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in L\}$ are two IFSs in L. Then their product $A \otimes B$ is defined as $A\otimes B=\{\langle x, \mu_{A\otimes B}(x), \nu_{A\otimes B}(x)\rangle / x \in L\}$, where $\mu_{A\otimes B}(x) = \underset{x\leq a\wedge b}{Sup}\{\min\{\mu_A(a), \mu_B(b)\}\}$ and $\nu_{A\otimes B}(x) = \underset{x\leq a\wedge b}{Inf}\{\max\{\nu_A(a), \nu_B(b)\}\}$.*

Theorem 4.2 *The product of two IFIs in a distributive lattice L is again an IFI of L.*

Proof Proof for this theorem is same in sprit of Theorem 4.1 and hence omitted.

5. Intuitionistic Fuzzy Ideals and Homomorphism

In this section, the properties of intuitionistic fuzzy ideals under lattice homomorphism are studied.

Definition 5.1 *Let $f : L \to L'$ be a mapping from a lattice L to another lattice L' and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in L\}$ be an IFS of L. Then the image f(A) is defined by $f(A) = \{\langle y, f(\mu_A)(y), f(\nu_A)(y)\rangle / y \in L'\}$, where*

$$f(\mu_A)(y) = \begin{cases}\sup\{\mu_A(x)/x \in f^{-1}(y), & \text{if } f^{-1}(y) \neq \phi, \\ o, & \text{if } f^{-1}(y) = \phi\end{cases}$$

and

$$f(\nu_A)(y) = \begin{cases} \inf\{\nu_A(x)/x \in f^{-1}(y), & \text{if } f^{-1}(y) \neq \phi, \\ 1, & \text{if } f^{-1}(y) = \phi. \end{cases}$$

Similarly, if $A' = \{\langle y, \mu_{A'}(y), \nu_{A'}(y)\rangle / y \in L'\}$ *be an IFS of* L'*, then*

$$f^{-1}(A') = \{\langle x, f^{-1}(\mu_{A'})(x), f^{-1}(\nu_{A'})(x)\rangle / x \in L\},$$

where $f^{-1}(\mu_{A'}(x)) = \mu_{A'}(f(x))$ *and* $f^{-1}(\nu_{A'}(x)) = \nu_{A'}(f(x))$.

Theorem 5.1 *If f : L* $\to$ *L′ is a lattice epimorphism and A is an IFI of L, then f(A) is an IFI of L′.*

Proof Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in L\}$ be an IFI of L. Then

$$f(A) = \{\langle y, f(\mu_A)(y), f(\nu_A)(y)\rangle / y \in L'\}.$$

Let $y, z \in L'$. Then

$$\begin{aligned} f(\mu_A)(y \vee z) &= \sup\{\mu_A(x)/x \in f^{-1}(y \vee z)\} \\ &\geq \sup\{\mu_A(u \vee v)/u \in f^{-1}(y) \& v \in f^{-1}(z)\} \\ &\geq \sup\{\min\{\mu_A(u), \mu_A(v)\}/u \in f^{-1}(y) \& v \in f^{-1}(z)\} \\ &= \min\{\sup \mu_A(u)/u \in f^{-1}(y), \sup \mu_A(v)/v \in f^{-1}(z)\} \\ &= \min\{f(\mu_A)(y), f(\mu_A)(z)\}, \end{aligned}$$

since A is an IFI of L.

Also

$$\begin{aligned} f(\mu_A)(y \wedge z) &= \sup\{\mu_A(x)/x \in f^{-1}(y \wedge z)\} \\ &\geq \sup\{\mu_A(u \wedge v)/u \in f^{-1}(y) \& v \in f^{-1}(z)\} \\ &\geq \sup\{\max\{\mu_A(u), \mu_A(v)\}/u \in f^{-1}(y) \& v \in f^{-1}(z)\} \\ &= \max\{\sup \mu_A(u)/u \in f^{-1}(y), \sup \mu_A(v)/v \in f^{-1}(z)\} \\ &= \max\{f(\mu_A)(y), f(\mu_A)(z)\}, \end{aligned}$$

since A is an IFI of L.

Similarly,

$$\begin{aligned} f(\nu_A)(y \vee z) &= \inf\{\nu_A(x)/x \in f^{-1}(y \vee z)\} \\ &\leq \inf\{\nu_A(u \vee v)/u \in f^{-1}(y) \& v \in f^{-1}(z)\} \\ &\leq \inf\{\max\{\nu_A(u), \nu_A(v)\}/u \in f^{-1}(y) \& v \in f^{-1}(z)\} \\ &= \max\{\inf \nu_A(u)/u \in f^{-1}(y), \inf \nu_A(v)/v \in f^{-1}(z)\} \\ &= \max\{f(\nu_A)(y), f(\nu_A)(z)\}, \end{aligned}$$

since A is an IFI of L,

and

$$\begin{aligned} f(\nu_A)(y \wedge z) &= \inf\{\nu_A(x)/x \in f^{-1}(y \wedge z)\} \\ &\leq \inf\{\nu_A(u \wedge v)/u \in f^{-1}(y) \& v \in f^{-1}(z)\} \\ &\leq \inf\{\min\{\nu_A(u), \nu_A(v)\}/u \in f^{-1}(y) \& v \in f^{-1}(z)\} \\ &= \min\{\inf \nu_A(u)/u \in f^{-1}(y), \inf \nu_A(v)/v \in f^{-1}(z)\} \\ &= \min\{f(\nu_A)(y), f(\nu_A)(z)\}, \end{aligned}$$

since A is an IFI of L. Hence $f(A)$ is an IFI of L'.

Theorem 5.2 *Let f: L* $\to$ *L′ be a lattice homomorphism and A′ is an IFI of L′. Then* $f^{-1}(A')$ *is an IFI of L′.*

Proof Proof follows easily from Definition 5.1 and the fact that A' IFI of L'. Hence omitted.

Theorem 5.3 *Let f: $L \to L'$ be an onto mapping and A and A' are IFSs of the lattices L and L', respectively. Then*

(i) $f[f^{-1}(A')] = A'$; (ii) $A \subseteq f^{-1}[f(A)]$.

Proof (i) We have

$$\begin{aligned} f(f^{-1}(\mu_{A'})(y)) &= \sup\{f^{-1}(\mu_{A'})(x)/x \in f^{-1}(y)\} \\ &= \sup\{\mu_{A'}\, f(x)/x \in L, f(x) = y\} \\ &= \mu_{A'}(y). \end{aligned}$$

Similarly, $f(f^{-1}(\nu_{A'})(y)) = \nu_{A'}(y)$. Hence $f[f^{-1}(A')] = A'$.

(ii) We have

$$\begin{aligned} f^{-1}(f(\mu_A))(x) &= f(\mu_A)(f(x)) \\ &= \sup\{\mu_A(x)/x \in f^{-1}(f(x))\} \\ &\geq \mu_A(x) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(f(\nu_A))(x) &= f(\nu_A)(f(x)) \\ &= \inf\{\nu_A(x)/x \in f^{-1}(f(x))\} \\ &\leq \nu_A(x). \end{aligned}$$

Hence $A \subseteq f^{-1}[f(A)]$.

Definition 5.2 *Let f: $L \to L'$ be a function from a lattice L to another lattice L' and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in L\}$ be an IFS of L. Then A is said to be f-invariant if, f(x) = f(y)$\Rightarrow \mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$.*

Proposition 5.1 *If an IFS A is f-invariant, then $f^{-1}[f(A)] = A$.*

Proof Follow from Theorem 5.3 and Definition 5.2.

Theorem 5.4 *Let f: $L \to L'$ be a function from a lattice L to another lattice L' and A ,B are two IFSs of L and A',B' are IFSs of L'. Then*

(i) $A \subseteq B \Rightarrow f(A) \subseteq f(B)$;

(ii) $A' \subseteq B' \Rightarrow f^{-1}(A') \subseteq f^{-1}(B')$.

Proof Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in L\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in L\}$ be two IFS in L. Then

$$A \subseteq B \Rightarrow \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x).$$

Then

$$f(A) = \{\langle y, f(\mu_A)(y), f(\nu_A)(y)\rangle / y \in L'\}$$

and

$$f(B) = \{\langle y, f(\mu_B)(y), f(\nu_B)(y)\rangle / y \in L'\}.$$

Now

$$\begin{aligned} f(\mu_A)(y) &= \sup\{\mu_A(x)/x \in f^{-1}(y)\} \\ &\leq \sup\{\mu_B(x)/x \in f^{-1}(y)\} \\ &= f(\mu_B(y), \text{ as } \mu_A(x) \leq \mu_B)(x). \end{aligned}$$

Also

$$\begin{aligned} f(\nu_A)(y) &= \inf\{\nu_A(x)/x \in f^{-1}(y)\} \\ &\geq \inf\{\nu_B(x)/x \in f^{-1}(y)\} \\ &= f(\nu_B)(y), \text{ as } \nu_A(x) \geq \nu_B)(x). \end{aligned}$$

Hence $f(A) \subseteq f(B)$.

Similarly, we can prove (ii).

Theorem 5.5 *If f: $L \to L'$ is an epimorphism, then there is one to one order preserving correspondence between the the IFIs of L' and those of L which are f-invariant.*

Proof Let I(L') denote the set of all IFIs of L' and I (L) denote the set of all IFIs of L which are f-invariant. Define ϕ : I (L) → I(L') and ψ : I(L')→ I(L) such that $\phi(A)$ = f(A) and $\psi(A') = f^{-1}(A')$. By Theorem 5.1 and Theorem 5.2, ϕ and ψ are well defined. By Theorem 5.3 and Proposition 5.1, ϕ and ψ inverse to each other which gives the one to one correspondence. Also by Theorem 5.4, we have $A \subseteq B \Rightarrow f(A) \subseteq f(B)$. Thus the correspondence is order preserving.

Theorem 5.6 *If $f: L \to L'$ is an epimorphism and A and B are IFIs of L, then $f(A \cap B) \subseteq f(A) \cap f(B)$ and equality holds if at least one of A or B is f-invariant.*

Proof Since $A \cap B \subseteq A, A \cap B \subseteq B$, by Theorem 5.4 $f(A \cap B) \subseteq f(A) \cap f(B)$. Next suppose that B is f-invariant. We prove that $f(A) \cap f(B) \subseteq f(A \cap B)$.

Let $\alpha = [f(\mu_A) \wedge f(\mu_B)](y)$ and $\beta = f(\mu_{A\cap B})(y)$. Then

$$\alpha = \min\{f(\mu_A)(y), f(\mu_B)(y)\} = \min\{\sup \mu_A(x)/x \in f^{-1}(y), f(\mu_B)(y)\}.$$

Thus $\alpha \leq \{\sup \mu_A(x)/x \in f^{-1}(y)\}$ and $\alpha \leq f(\mu_B)(y)$. Therefore, for any $\epsilon > 0$, $\exists x \in f^{-1}(y)$ such that $\alpha - \epsilon < \mu_A(x)$ and $\alpha - \epsilon < f(\mu_B(y))$.

Now

$$\alpha - \epsilon < f(\mu_B(y)) \Rightarrow \alpha - \epsilon < f(\mu_B(f(x))) = f^{-1}(f(\mu_B(x))) = \mu_B(x),$$

since B is f-invariant $f^{-1}(f(\mu_B)) = \mu_B$.

This implies

$$\alpha - \epsilon < \min\{\mu_A(x), \mu_B(x)\} = \mu_{A\cap B}(x).$$

Hence $\alpha - \epsilon < \sup\{\mu_{A\cap B}(x)/x \in f^{-1}(y)\}$. That is $\alpha - \epsilon < f(\mu_{A\cap B})(y) = \beta$.

Since $\epsilon > 0$ is arbitrary,

$$\alpha \leq \beta. \tag{5}$$

Now let $\chi = [f(\nu_A) \vee f(\nu_B)](y)$ and $\delta = f(\nu_{A\cap B})(y)$. Then

$$\begin{aligned}\chi &= \max\{f(\nu_A)(y), f(\nu_B)(y)\}\\ &= \max\{\inf \nu_A(x)/x \in f^{-1}(y), f(\nu_B)(y)\}.\end{aligned}$$

Thus $\chi \geq \{\inf \nu_A(x)/x \in f^{-1}(y)\}$ and $\chi \geq f(\nu_B)(y)$. Therefore, for any $\epsilon > 0$, $\exists x \in f^{-1}(y)$ such that $\chi + \epsilon > \nu_A(x)$ and $\chi + \epsilon > f(\nu_B(y))$.

Now

$$\chi + \epsilon > f(\nu_B(y)) \Rightarrow \chi + \epsilon > f(\nu_B(f(x))) = f^{-1}(f(\nu_B(x))) = \nu_B(x),$$

since B is f-invariant $f^{-1}(f(\nu_B) = \nu_B$.

This implies

$$\chi + \epsilon > \min\{\nu_A(x), \nu_B(x)\} = \nu_{A\cap B}(x).$$

Hence $\chi + \epsilon > \inf\{\nu_{A\cap B}(x)/x \in f^{-1}(y)\}$. That is $\chi + \epsilon > f(\nu_{A\cap B})(y) = \delta$.

Since $\epsilon > 0$ is arbitrary,

$$\chi \geq \delta. \tag{6}$$

From (5) and (6) $f(A) \cap f(B) \subseteq f(A \cap B)$.

This completes the proof.

6. Conclusion

In this paper, we study about intuitionistic fuzzy sublattices (IFL) and ideals (IFI), and established their properties. We also present certain counter examples to prove some properties of them. Here we defined the sum and product of two IFIs and prove that in a distributive lattice they are again IFIs. Finally, we study about homomorphic images and preimages of IFIs and define f-invariant class of IFIs. A correspondence theorem between IFIs of a lattice which are f-invariant and IFIs of it's homomorphic image are established.

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