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ORIGINAL PAPER

Evaluation of equations for the determination of the ultimate bearing capacity of shallow foundation in cohesionless soils

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Abstract

In most codes of practice for the building and construction, the conventional equations from Meyerhof, Hansen, Vesic, and Terzaghi are all acceptable for the determination of the ultimate bearing capacity of shallow foundation. It is difficult to evaluate which equation has the best performance against the experimental data on different types of soil. The stabilities of all the buildings, utilities, and other infrastructure in the urban area are much dependent on a reliable foundation. As a result, the methodology of the foundation design is important in the urban and rural development. However, in the codes of practice for the building and construction, it does not specify which method is most reliable for the determination of the ultimate bearing capacity of shallow foundation. It is good to understand the uncertainty associated with different equations in the determination of the ultimate bearing capacity. In this study, a total of 163 data sets from the experimental tests conducted on sandy soils were collected from different literatures. The ultimate bearing capacity values calculated using different equations were compared with the experimental data. It was observed that Vesic equation provides the best performance in calculating the ultimate bearing capacity of sandy soils. In addition, a new equation for the estimation of the bearing capacity factor, N_γ, is proposed in this study. The equation was developed using the regression analysis incorporating the collected database. The performance of the proposed equation was verified with independent data sets and presented in the Taylor diagram.

Keywords Ultimate bearing capacity · Empirical equation · Statistical analysis · Taylor's diagram

Introduction

In the process of urban and rural development, the design methodology in geotechnical engineering is constantly updated, and its innovation is commonly based on the original technology. The determination of the bearing capacity of foundations is a major focus of research in geotechnical engineering. Taghvamanesh and Moayed (2021) summarized the different calculation methods for different types of foundations. Those different methods can be categorized into five groups such as the limit equilibrium technique (Terzaghi (1943), Meyerhof (1963), Saran and Agarwal(1991), Zhu et al., (2003)); the method of characteristics (Booker (1969), Hansen (1970), Vesic (1973), Bolton and Lau (1993), Kumar (2003), Martin (2003), Smith (2005)); the limit analysis method (Chen (1990), Michalowski (1997), Kumar (2003), Hjiaj et al. (2005)); the empirical approach (Steenfelt (1977), Hettler and Gudehus (1988), Zadroga (1994)); and the statistical analysis method (Ingra and Baecher (1983)). Various researchers (Prandtl (1921), Meyerhof (1963), Hansen (1970), Vesic (1973), Chen (1990), Bolton and Lau (1993), Kumar (2003), Zhu et al., (2003), Martin (2003), and Smith (2005)) have proposed different semi-empirical equations for the determination of the ultimate bearing capacity. Bolton and Lau (1993), Kumar (2003), and Zhu et al. (2003) computed the bearing capacity

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of shallow foundation by assuming a non-plastic rigid wedge in the subgrade around foundation. Martin (2003) and Smith (2005) conducted the stress analysis and observed that failure surfaces in the subgrade soil around the foundation may not form the rigid wedges. Yang et.al (2021) used the upper limit analysis method and calculated the bearing capacity factors of shallow foundations near slopes isolated from the seismic action. Moayedi and Hayati (2018) presented the results of a study of analyzed ultimate bearing capacity of shallow foundation by using several nonlinear machine learning models such as feedforward neural networks (FFNN), radial basis neural networks (RBNN), general regression neural networks (GRNN), support vector machines (SVM), tree regression fitting models (TREE), and adaptive neuro-fuzzy inference systems (ANFIS).

Phoon and Tang (2019) stated that the characterization of model uncertainty was identified as one of the key elements in the geotechnical reliability design process. Moayedi et al. (2019) developed color intensity rating (CER) system to assess the capability of different methods in prediction of the ultimate bearing capacity of shallow foundations in double-layered soil conditions.

Padmini et al. (2008) calibrated new models for estimating the ultimate bearing capacity. The performance of the new models was also evaluated. The results show that the new models are credible, and all outperform the theoretical approach. Samui (2010) used two algorithms to determine the ultimate bearing capacity of shallow foundations on cohesionless soils and found that the new models predicted the ultimate bearing capacity with more confidence in the results. Khorrami and Derakhshani (2019) applied a new method of artificial intelligence (M5′-GP method) to predict the ultimate carrying capacity. The new model is also compared with different models, and the M5′-GP method is found to have a good advantage in predicting the ultimate bearing capacity. Ahmad et al. (2021) used the potential of Gaussian process regression (GPR) in predicting the ultimate bearing capacity of shallow foundations on cohesionless soils. The prediction results were evaluated against the theoretical approach, and the GPR results were found to be better than the theoretical approach. Prandtl (1921) assumed that the surface of the foundation base is smooth, and there is no friction between the foundation base and subgrade. Based on the failure surface shown in Fig. 1, Prandtl (1921) derived the ultimate bearing capacity for the strip footing without the embedment depth. Meyerhof (1963), Hansen (1970), and Vesic (1973) adopted this assumption from Prandtl (1921) and improved the equation by considering different scenarios. In general, the equation from Meyerhof (1963), Hansen (1970), or Vesic (1973) can be expressed in Eq. (1) as follows:

$$p_u = cN_c s_c d_c + qN_q s_q d_q + 0.5\gamma BN_\gamma s_\gamma d_\gamma \tag{1}$$

where p_u is the ultimate bearing capacity; N_c, N_q, and N_γ are the bearing capacity factors of load; s_c, s_q and s_γ are the shape factors; d_c, d_q, and d_γ are the depth influence factors; c is the cohesion of soil; q is the surcharge ($q = \gamma_m d$); γ_m is the unit weight of soil above the foundation base; d is the depth of foundation base from the ground surface; γ is the unit weight of soil below the foundation base; and B is the width of the foundation.

For the same assumption (i.e., there is no friction between foundation base and the subgrade), N_q and N_c in the three equations are consistent with each other and are illustrated in Eqs. (2) and (3), respectively:

$$N_q = e^{\pi \tan\phi}\tan^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \tag{2}$$

$$N_c = \left(N_q - 1\right)\cot\phi \tag{3}$$

where ϕ is the friction angle.

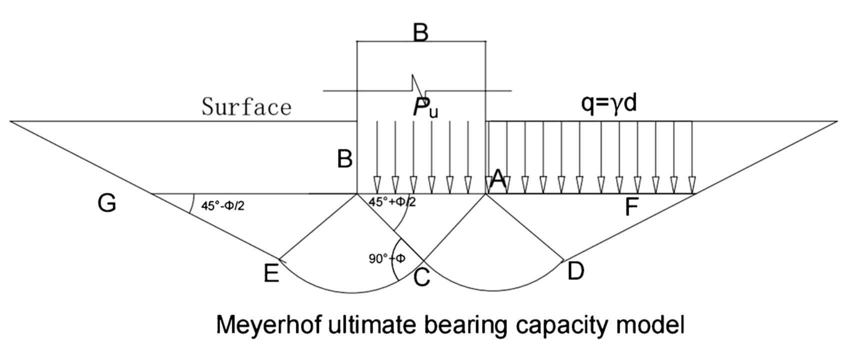
There is no theoretical derivation for N_γ, and only empirical equations are available for determination of N_γ. Meyerhof (1963), Hansen (1970), and Vesic (1973) defined N_γ as shown in Eqs. (4) to (6), respectively:

$$N_\gamma = \left(N_q - 1\right)\tan(1.4\phi) \tag{4}$$

$$N_\gamma = 1.5\left(N_q - 1\right)\tan(\phi) \tag{5}$$

$$N_\gamma = 2\left(N_q - 1\right)\tan(\phi) \tag{6}$$

Fig. 1 Failure surface adopted by Meyerhof (1963)



The shape factors and the depth influence factors from the three equations are different as illustrated in Table 1.

Terzaghi (1943) pointed out that the assumption of the smooth foundation base might not be the case in practical engineering. He assumed that the foundation base was rough and obtained a different failure surface as shown in Fig. 2. Chen (1990) indicated that the effect of roughness of the foundation base on the bearing capacity is insignificant if the width of foundation is small and is significant if the width of foundation is large. Terzaghi (1943) obtained the equation for the determination of the ultimate bearing capacity of shallow foundation as shown in Eq. (7):

$$p_u = cN_c\varsigma_c + qN_q\varsigma_q + 0.5\gamma BN_\gamma\varsigma_\gamma \tag{7}$$

where ζ_c, ζ_q, and ζ_γ are shape factors.

The failure surface from Terzaghi (1943), as shown in Fig. 2, was different from that of Prandtl (1921), as shown in Fig. 1; therefore, the definitions of the bearing capacity factors N_c, N_q, and N_γ in Eq. (7) are different from those in Eq. (1). Terzaghi (1943) indicated that N_c, N_q, and N_γ in Eq. (7) could be obtained from the following equations:

$$N_q = \frac{e^{\left(\frac{3\pi}{2}-\phi\right)\tan\phi}}{2\cos^2\left(\frac{\pi}{4}+\frac{\phi}{2}\right)} \tag{8}$$

$$N_c = \left(N_q - 1\right)\cot\phi \tag{9}$$

$$N_\gamma = \frac{1}{2}\left(\frac{K_{py}}{\cos^2\phi} - 1\right)\tan\phi \tag{10}$$

It should be noted that K_{py} is not the coefficient of passive earth pressure which could be computed from Rankine's theory or Coulomb's theory. Terzaghi (1943) indicated that K_{py} should be determined by means of the spiral or the friction circle method.

In Terzaghi's (1943) method, the surface of the foundation base was not considered as smooth. Zone I is in an elastic state rather than the Rankine active state zone. Zone II

Table 1 The shape factors and depth influence factors for the three methods

Shape factor and depth influence factor	Meyerhof (1963)	Hansen (1970)	Vesic (1973)
s_c	$1 + 0.2K_pB/L$	$1 + N_qB/(N_cL)$	Same as Hansen (1970)
s_q	$\phi = 0°$, 1.0 $\phi > 10°$, $1 + 0.1\ K_pB/L$	$1 + B/L\ \tan\phi$	
s_γ	$\phi = 0°$, 1.0 $\phi > 10°$, $1 + 0.1\ K_pB/L$	$1 - 0.4B/L$	
d_c	$1 + 0.2\ K_p0.5\ d/B$	$d \le B$, $1 + 0.4d/B$ $d > B$, $1 + 0.4\tan^{-1}(d/B)$	
d_q	$\phi = 0°$, 1.0 $\phi > 10°$, $0.1\ K_p0.5(d/B)$	$d \le B$, $1 + 2\tan\phi(1\text{-}\sin\phi)^2\ (d/B)$ $d > B$ $1 + 2\tan\phi(1\text{-}\sin\phi)^2\ \tan^{-1}\ (d/B)$	
d_γ	$\phi = 0°$, 1.0 $\phi > 10°$, $0.1\ K_p^{0.5}(d/B)$	1.0	

$K_p = \tan^2(\pi/4 + \phi/2)$

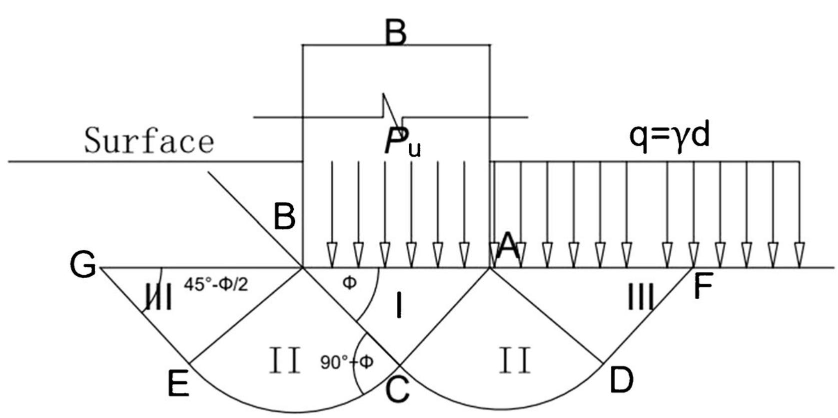


Fig. 2 Failure surface adopted by Terzaghi (1943)

is the transition zone, and Zone III is commonly considered as the passive zone.

Terzaghi (1943) recommended the values of ζ_c, ζ_q, and ζ_γ for different shapes of footing as shown in Table 2.

Each equation has its theoretical background but also includes the empirical term in the determination of the ultimate bearing capacity. It is difficult to judge which equation has the best performance in the determination of the ultimate bearing capacity.

As illustrated in Eqs. (1) to (10), the bearing capacity factors N_c and N_q can be obtained from the theoretical derivations. However, the theoretical backgrounds for the determination of N_γ, shape factors, and depth influence factors are weak. Table 3 illustrates the various equations for the determination of N_γ from different researchers. In this paper, the sandy soil ($c = 0$) is adopted, and the ultimate bearing capacity of foundation is mainly governed by the summation of the second and third terms in Eqs. (1) or (7). The calculated ultimate bearing capacities from those equations are compared with the experimental data from the literature. It is observed that the calculated bearing capacity using Meyerhof's (1963) equation provides the best agreement with the measured data as compared with the results from other equations. Subsequently, the collected data are divided into two groups; one group is used for the training of the data to obtain the correlation equation, and the other group is used as the verification of the developed relationship.

Table 2 Recommended values for the shape factors in Terzaghi's (1943) method

Shape factor	Strip footing	Square footing	Round footing
ζ_c	1.0	1.3	1.3
ζ_q	1.0	1.0	1.0
ζ_γ	1.0	0.8	0.63

Consequently, a new equation is proposed for the determination of the bearing capacity factor N_γ.

Soil database for the evaluation

To evaluate the calculated ultimate bearing capacities from those equations, the following parameters such as the width (B) and length (L) of the foundation, the ratio of L/B, the depth of foundation base (d), the surcharge (q), the unit weight of subgrade (γ), and the friction angle of the subgrade (ϕ) are compiled and summarized in Table 4. The measured ultimate bearing capacities ($p_{u,m}$) are also collected and illustrated in Table 4.

Michalowski (1997) indicated that N_γ was much related to the roundness of foundation base. Lyamin and Sloan (2002a, 2002b) and Hjiaj (2005) proposed different equations for the calculation of N_γ. Therefore, the N_γ in Eqs. (1) and (7) is an empirical factor because there is no unique solution for this factor.

Based on the information in Table 3, the ultimate bearing capacity can be easily calculated using Eqs. (1) and (7). On the other hand, the empirical bearing capacity factor N'_γ in Eqs. (1) and (7) can be recalculated from the measured $p_{u,m}$ as shown in Eq. (11) and (12), respectively:

$$N'_\gamma = \frac{p_{u,m} - qN_q s_q d_q}{0.5\gamma B s_\gamma d_\gamma} \tag{11}$$

$$N'_\gamma = \frac{p_{u,m} - qN_q \varsigma_q}{0.5\gamma B \varsigma_\gamma} \tag{12}$$

In this paper, there are two criteria adopted for the evaluation of the performance of the different methods in the determination of the ultimate bearing capacity. Both the average

Table 3 Summary of 10 equations for the determination of N_γ

No.	References	Equations for N_γ
1	Terzaghi (1943)	$N_\gamma = \frac{1}{2}\left(\frac{K_{p\gamma}}{\cos^2\phi} - 1\right)\tan\phi$
2	Meyerhof (1963)	$N_\gamma = (N_q - 1)\tan(1.4\phi)$
3	Booker (1969)	$N_\gamma = 0.145\exp(9.6\phi)$
4	,Hansen (1970)	$N_\gamma = 1.5(N_q - 1)\tan(\phi)$
5	Vesic (1973)	$N_\gamma = 2(N_q - 1)\tan(\phi)$
6	Steenfelt (1977)	$N_\gamma = \left[0.08705 + 0.3231\sin(2\phi) - 0.04836\sin^2(2\phi)\right]$ $\left[\tan^2(\pi/4 + (\phi)/2)\exp\left(1.5\pi\tan(\phi) - 1\right]\right.$
7	Ingra and Baecher (1983)	$N_\gamma = \exp(0.173(\phi) - 1.464)$($\phi$ in degrees)
8	Hettler and Gudehus (1988)	$N_\gamma = \exp[5.71(\tan(\phi)^{1.15}] - 1$
9	Saran and Agarwal (1991)	$N_\gamma = \exp\left[\frac{0.757}{\ln(\phi)} + 15.286(\phi) - 3.452\right]$
10	Michalowski (1997)	$N_\gamma = \exp(0.66 + 5.11\tan(\phi))\tan(\phi)$

Table 4 Measured ultimate bearing capacity for the foundation with different shapes and different physical properties of subgrade

S/N	Width B (m)	Depth d (m)	Length L (m)	L/B	Unit weight γ (kN/m^3)	Friction angle ϕ (°)	Surcharge q (kN/m^2)	Measured $p_{u,m}$ (kPa)	Reference
1	0.6	0.3	1.2	2	9.85	34.9	2.955	270.00	Muhs et al. (1969)
2	0.6	0	1.2	2	10.20	37.7	0	200.00	
3	0.6	0.3	1.2	2	10.20	37.7	3.06	570.00	
4	0.6	0	1.2	2	10.85	44.8	0	860.00	
5	0.6	0.3	1.2	2	10.85	44.8	3.255	1760.00	
6	0.5	0	0.5	1	10.20	37.7	0	154.00	Weiß (1970)
7	0.5	0	0.5	1	10.20	37.7	0	165.00	
8	0.5	0	1	2	10.20	37.7	0	203.00	
9	0.5	0	1	2	10.20	37.7	0	195.00	
10	0.5	0	1.5	3	10.20	37.7	0	214.00	
11	0.52	0	2.002	3.85	10.20	37.7	0	186.00	
12	0.5	0.3	0.5	1	10.20	37.7	3.06	681.00	
13	0.5	0.3	0.5	1	10.20	37.7	3.06	542.00	
14	0.5	0.3	1	2	10.20	37.7	3.06	530.00	
15	0.5	0.3	0.5	1	10.20	37.7	3.06	402.00	
16	0.52	0.3	2.08	4	10.20	37.7	3.06	413.00	
17	0.5	0	2	4	11.70	37	0	111.00	Muhs and Weiß (1971)
18	0.5	0	0.5	1	11.70	37	0	132.00	
19	0.5	0	0.5	1	11.70	37	0	143.00	
20	0.5	0.013	2	4	11.70	37	0.1521	137.00	
21	0.5	0.029	1	2	11.70	37	0.3393	109.00	
22	0.5	0.127	2	4	11.70	37	1.4859	187.00	
23	0.5	0.3	0.5	1	11.70	37	3.51	406.00	
24	0.5	0.3	0.5	1	11.70	37	3.51	446.00	
25	0.5	0.3	2	4	11.70	37	3.51	322.00	
26	0.5	0.5	1	2	11.70	37	5.85	565.00	
27	0.5	0.5	2	4	11.70	37	5.85	425.00	
28	0.5	0	0.5	1	12.41	44	0	782.00	
29	0.5	0	2	4	12.41	44	0	797.00	
30	0.5	0.3	0.5	1	12.41	44	3.723	1940.00	
31	0.5	0.5	1	2	12.41	44	6.205	2266.00	
32	0.5	0.5	1	2	12.41	44	6.205	2847.00	
33	0.5	0.5	2	4	12.41	44	6.205	2033.00	
34	0.5	0.49	2	4	12.27	42	6.0123	1492.00	
35	0.5	0	1	2	11.77	37	0	123.00	
36	0.5	0	1	2	11.77	37	0	134.00	
37	0.5	0.3	0.5	1	11.77	37	3.531	370.00	
38	0.5	0.5	1	2	11.77	37	5.885	464.00	
39	0.5	0	2	4	12.00	40	0	461.00	
40	0.5	0.5	2	4	12.00	40	6	1140.00	
41	1	0.2	3	3	11.97	39	2.394	710.00	Muhs and Weiß (1973)
42	1	0	3	3	11.93	40	0	630.00	
43	0.058	0.029	0.3451	5.95	15.70	34	0.4553	58.50	Gandhi (2003)
44	0.058	0.029	0.3451	5.95	16.10	37	0.4669	82.50	
45	0.058	0.058	0.3451	5.95	15.70	34	0.9106	70.91	
46	0.058	0.058	0.3451	5.95	16.10	37	0.9338	98.93	
47	0.058	0.029	0.3451	5.95	16.50	39.5	0.4785	121.50	
48	0.058	0.058	0.3451	5.95	16.50	39.5	0.957	142.90	
49	0.058	0.029	0.3451	5.95	16.80	41.5	0.4872	157.50	

Table 4 (continued)

S/N	Width B (m)	Depth d (m)	Length L (m)	L/B	Unit weight γ (kN/m^3)	Friction angle ϕ (°)	Surcharge q (kN/m^2)	Measured $p_{u,m}$ (kPa)	Reference
50	0.058	0.058	0.3451	5.95	16.80	41.5	0.9744	184.90	
51	0.058	0.029	0.3451	5.95	17.10	42.5	0.4959	180.50	
52	0.058	0.058	0.3451	5.95	17.10	42.5	0.9918	211.00	
53	0.094	0.047	0.564	6	15.70	34	0.7379	74.40	
54	0.094	0.094	0.564	6	16.10	34	1.5134	91.50	
55	0.094	0.047	0.564	6	15.70	37	0.7379	104.80	
56	0.094	0.094	0.564	6	16.10	37	1.5134	127.50	
57	0.094	0.047	0.564	6	16.50	39.5	0.7755	155.80	
58	0.094	0.094	0.564	6	16.50	39.5	1.551	185.60	
59	0.094	0.047	0.564	6	16.80	41.5	0.7896	206.80	
60	0.094	0.094	0.564	6	16.80	41.5	1.5792	244.60	
61	0.094	0.047	0.564	6	17.10	42.5	0.8037	235.60	
62	0.094	0.094	0.564	6	17.10	42.5	1.6074	279.60	
63	0.152	0.075	0.9044	5.95	15.70	34	1.1775	98.20	
64	0.152	0.15	0.9044	5.95	16.10	34	2.415	122.30	
65	0.152	0.075	0.9044	5.95	15.70	37	1.1775	143.30	
66	0.152	0.15	0.9044	5.95	16.10	37	2.415	176.40	
67	0.152	0.075	0.9044	5.95	16.50	39.5	1.2375	211.20	
68	0.152	0.15	0.9044	5.95	16.50	39.5	2.475	254.50	
69	0.152	0.075	0.9044	5.95	16.80	41.5	1.26	285.30	
70	0.152	0.15	0.9044	5.95	16.80	41.5	2.52	342.50	
71	0.152	0.075	0.9044	5.95	17.10	42.5	1.2825	335.30	
72	0.152	0.15	0.9044	5.95	17.10	42.5	2.565	400.60	
73	0.094	0.047	0.094	1	15.70	34	0.7379	67.70	
74	0.094	0.094	0.094	1	16.10	34	1.5134	90.50	
75	0.094	0.047	0.094	1	15.70	37	0.7379	98.80	
76	0.094	0.094	0.094	1	16.10	37	1.5134	131.50	
77	0.094	0.047	0.094	1	16.50	39.5	0.7755	147.80	
78	0.094	0.094	0.094	1	16.50	39.5	1.551	191.60	
79	0.094	0.047	0.094	1	16.80	41.5	0.7896	196.80	
80	0.094	0.094	0.094	1	16.80	41.5	1.5792	253.60	
81	0.094	0.047	0.094	1	17.10	42.5	0.8037	228.80	
82	0.094	0.094	0.094	1	17.10	42.5	1.6074	295.60	
83	0.152	0.075	0.152	1	15.70	34	1.1775	91.20	
84	0.152	0.15	0.152	1	16.10	34	2.415	124.40	
85	0.152	0.075	0.152	1	15.70	37	1.1775	135.20	
86	0.152	0.15	0.152	1	16.10	37	2.415	182.40	
87	0.152	0.075	0.152	1	16.50	39.5	1.2375	201.20	
88	0.152	0.15	0.152	1	16.50	39.5	2.475	264.50	
89	0.152	0.075	0.152	1	16.80	41.5	1.26	276.30	
90	0.152	0.15	0.152	1	16.80	41.5	2.52	361.50	
91	0.152	0.075	0.152	1	17.10	42.5	1.2825	325.30	
92	0.152	0.15	0.152	1	17.10	42.5	2.565	423.60	
93	0.08	0	0.08	1	17.20	42.8	0	133.00	Golder (1941)
94	0.15	0	0.15	1	17.20	42.8	0	246.00	
95	0.05	0	0.05	1	17.20	42.8	0	109.00	Eastwood (1951)
96	0.08	0	0.08	1	17.10	42.8	0	130.00	
97	0.1	0	0.1	1	17.10	42.8	0	152.00	
98	0.15	0	0.15	1	17.10	42.8	0	214.00	

Table 4 (continued)

S/N	Width B (m)	Depth d (m)	Length L (m)	L/B	Unit weight γ (kN/m^3)	Friction angle ϕ (°)	Surcharge q (kN/m^2)	Measured $p_{u,m}$ (kPa)	Reference
99	0.2	0	0.2	1	17.10	42.8	0	266.00	
100	0.25	0	0.25	1	17.10	42.8	0	333.00	
101	0.3	0	0.3	1	17.10	42.8	0	404.00	
102	0.03	0	0.03	1	15.89	42	0	52.00	Subrahmanyam (1967)
103	0.04	0	0.04	1	15.89	42	0	92.00	
104	0.05	0	0.05	1	15.89	42	0	95.00	
105	0.06	0	0.06	1	13.20	32	0	14.00	Cerato (2007)
106	0.06	0	0.06	1	14.80	42	0	72.00	
107	0.06	0	0.06	1	15.40	42	0	106.00	
108	1	0	1	1	19.50	38.75	0	377.00	Akbas and Kulhawy (2009)
109	1	0	1	1	19.50	38.75	0	368.00	
110	1	0	1	1	19.50	38.75	0	335.00	
111	1	0	1	1	19.50	38.75	0	305.00	
112	1	0	1	1	19.50	38.75	0	400.00	
113	1	0	1	1	19.50	38.75	0	296.00	
114	1	0	1	1	19.50	38.75	0	390.00	
115	1	0	1	1	19.50	38.75	0	435.00	
116	0.71	0	0.71	1	19.50	38.75	0	773.00	
117	1	0	1	1	16.80	40.55	0	685.00	
118	1	0	1	1	16.80	40.55	0	560.00	
119	1	0	1	1	16.80	40.55	0	500.00	
120	1	0	1	1	16.80	40.55	0	598.00	
121	1	0	1	1	16.80	40.55	0	584.00	
122	1	0	1	1	16.80	40.55	0	716.00	
123	1	0	1	1	16.80	40.55	0	825.00	
124	1	0	1	1	16.80	40.55	0	922.00	
125	1	0	1	1	16.80	40.55	0	659.00	
126	1	0	1	1	16.80	40.55	0	640.00	
127	1	0	1	1	16.80	40.55	0	626.00	
128	1	0	1	1	16.80	40.55	0	726.00	
129	1	0	1	1	16.80	40.55	0	927.00	
130	0.7	0	0.7	1	16.80	40.55	0	612.25	
131	0.75	0	0.75	1	20.80	44.95	0	856.90	
132	0.75	0	0.75	1	20.80	44.95	0	1020.44	
133	0.45	0	0.45	1	20.80	44.95	0	953.09	
134	0.45	0	0.45	1	20.80	44.95	0	454.32	
135	0.3	0	0.3	1	20.80	45.7	0	422.23	
136	0.3	0	0.3	1	20.80	45.7	0	600.00	
137	0.3	0	0.3	1	20.80	45.7	0	900.00	
138	0.3	0	0.3	1	20.80	45.7	0	1688.89	
139	0.91	0	0.91	1	14.60	31.95	0	324.84	
140	0.61	0	0.61	1	14.60	31.95	0	94.06	
141	0.61	0	0.61	1	14.60	31.95	0	196.18	
142	0.61	0	0.61	1	14.60	31.95	0	322.49	
143	0.46	0	0.46	1	14.60	31.95	0	259.92	
144	0.61	0	0.61	1	19.00	37	0	257.99	
145	0.8	0	0.8	1	17.10	39.75	0	348.44	
146	0.63	0	0.63	1	17.10	39.75	0	365.33	
147	0.46	0	0.46	1	17.10	39.75	0	103.97	

Table 4 (continued)

S/N	Width B (m)	Depth d (m)	Length L (m)	L/B	Unit weight γ (kN/m^3)	Friction angle ϕ (°)	Surcharge q (kN/m^2)	Measured $p_{u,m}$ (kPa)	Reference
148	0.31	0	0.31	1	15.80	37.9	0	478.67	
149	1.2	0	1.2	1	20.40	41	0	1129.86	
150	1.2	0	1.2	1	20.40	41	0	978.47	
151	0.3	0	0.3	1	20.40	41	0	1277.78	
152	0.3	0	0.3	1	20.40	41	0	522.22	
153	0.3	0	0.3	1	20.40	41	0	811.11	
154	0.3	0	0.3	1	20.40	41	0	333.33	
155	0.3	0	0.3	1	20.40	41	0	233.33	
156	0.76	0	0.76	1	16.20	40.8	0	744.46	
157	0.31	0	0.31	1	16.20	40.8	0	260.15	
158	0.31	0	0.31	1	16.20	40.8	0	468.26	
159	1	0.71	1	1	15.50	35.3	11.005	1550.00	Briaud and Gibbens (1997)
160	1.5	0.76	1.5	1	15.50	35.3	11.78	1355.56	
161	2.5	0.76	2.5	1	15.50	35.3	11.78	1152.00	
162	3	0.76	3	1	15.50	35.3	11.78	1144.44	
163	3	0.89	3	1	15.50	35.3	13.795	1011.11	

relative error (ARE), which is defined in Eq. (13), and the normalized sum of squared error (SSE_{norm}), which is defined in Eq. (14), are adopted. Less values of ARE and SSE_{norm} indicate better performance of the equation in the determination of the ultimate bearing capacity:

$$ARE = \frac{1}{N}\sum_{i=1}^{N}\left|\frac{\left(y_i - \hat{y}_i\right)}{y_i}\right| \tag{13}$$

$$SSE_{norm} = \sqrt{\sum_{i=1}^{N}\left(\frac{y_i - \hat{y}_i}{y_i}\right)^2} \tag{14}$$

where y_i is the measured value of i_{th} data, $\hat{y}_i$ is the calculated value of i_{th} data, and N is the total number of collected datasets.

In addition, Taylor's diagram from Taylor (2001) is adopted to evaluate the performance of the different methods in the determination of the ultimate bearing capacity. In Taylor's diagram, the correlation coefficient (R), as defined in Eq. (15); standard deviation (σ_x), as defined in Eq. (16); and root mean square error ($RMSE$), as defined in Eq. (17), are used for the evaluation. In general, the scatter points in the Taylor diagram represent the calculated results from the different methods, the radial lines represent the R, the horizontal and vertical axes represent the σ_x, and the dashed line represents the centered pattern $RMSE$. The advantage of using Taylor diagram to evaluate the performance of a model is that it provides an intuitive and convenient representation of the performance metrics of R, σ_x, and $RMSE$:

$$R = \frac{\frac{1}{N}\sum_{n=1}^{N}\left(f_n - \bar{f}\right)\left(r_n - \bar{r}\right)}{\sigma_f \sigma_r} \tag{15}$$

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)}{n}} \tag{16}$$

$$RMSE = \sqrt{\frac{1}{N}\sum_{n=1}^{N}\left[\left(f_n - \bar{f}\right) - \left(r_n - \bar{r}\right)\right]^2} \tag{17}$$

where f_n and r_n are the variables of nth data, $\bar{f}$ and $\bar{r}$ are the mean values, and σf and σr are the standard deviations of f and r, respectively. N is the total number of collected datasets.

Results and discussions

Comparison between the calculated the ultimate bearing capacities and the measured data

The calculated ultimate bearing capacities from Meyerhof (1963), Hansen (1970), Vesic (1973), and Terzaghi (1943)

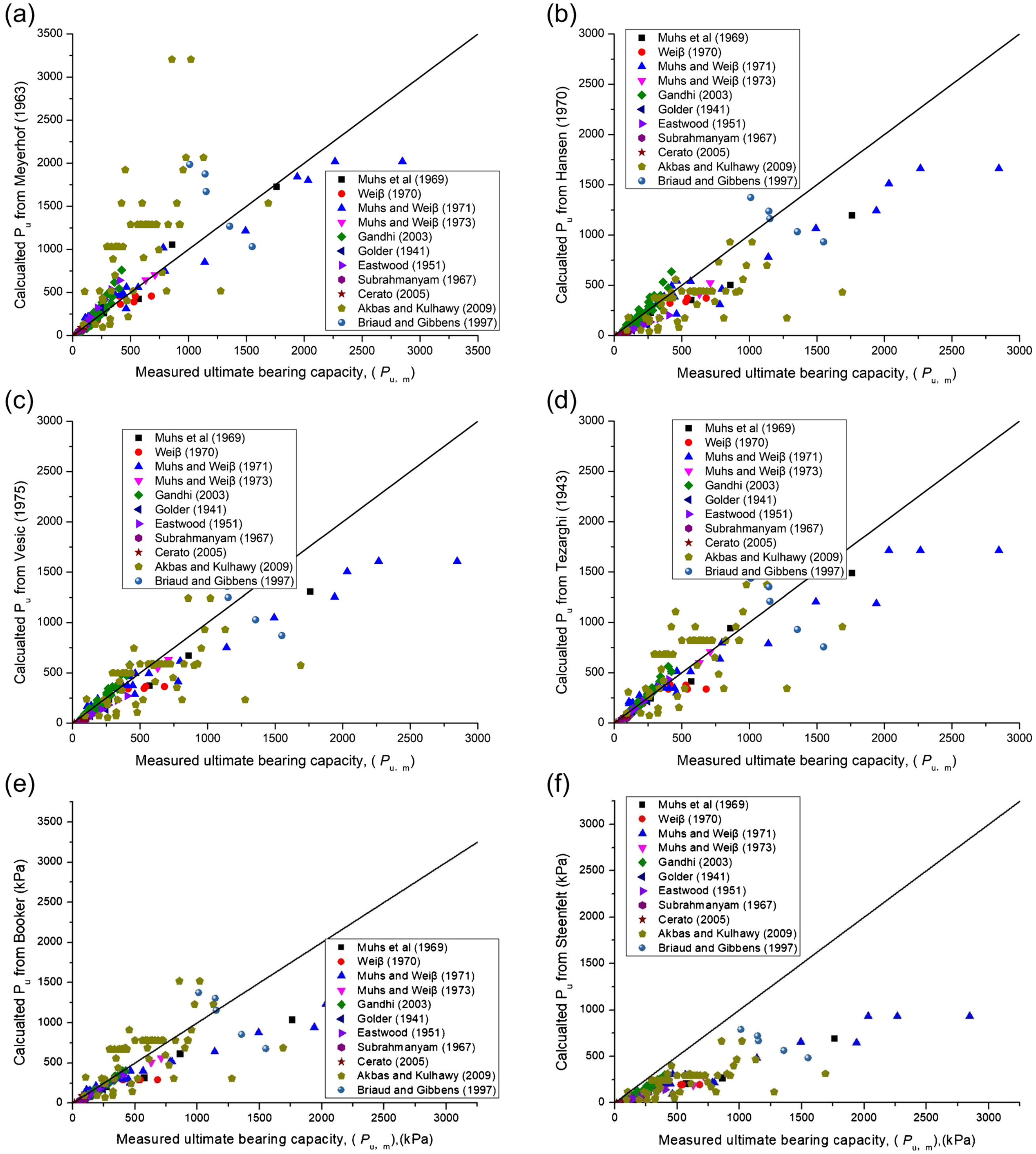


Fig. 3 Illustration of the comparison of calculated ultimate bearing capacities from different equations and measured data. **a** From Meyerhof (1963), **b** from Hansen (1970), **c** from Vesic (1973), **d** from Terzaghi (1943), **e** from Booker (1969), **f** from Steenfelt (1977), **g** from Ingra and Baecher(1983), **h** from Hettler and Gudehus (1988), **i** from Saran and Agarwal (1991), and **j** from Michalowski(1997)

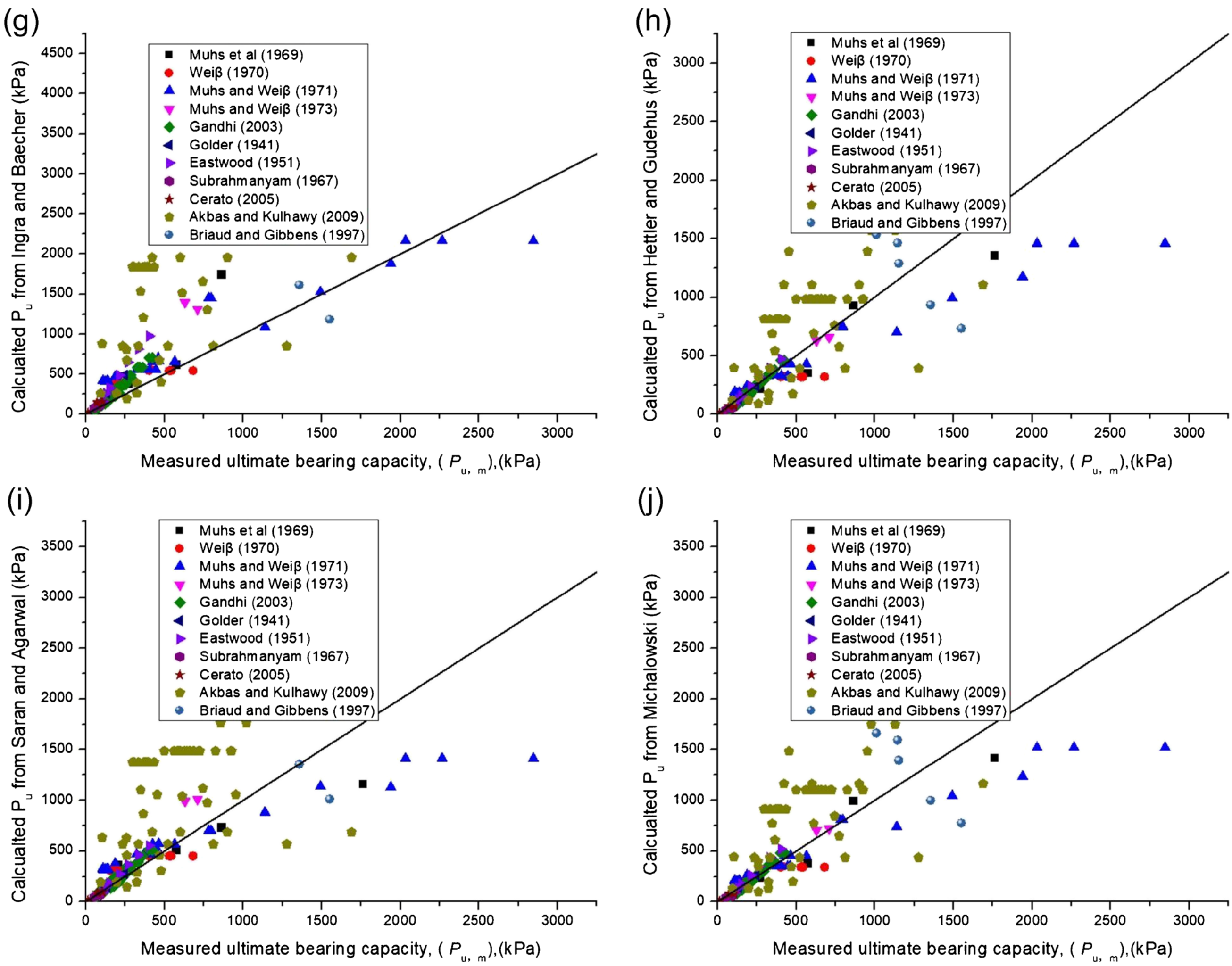


Fig. 3 (continued)

are compared with the measured ultimate bearing capacity, $p_{u,m}$. The comparison results are illustrated in Fig. 3.

The values of *ARE* and SSE_{norm} for the calculated ultimate bearing capacity, p_u, from the different equations are illustrated in Table 5.

As can be seen in Table 5, the four groups of semi-empirical formulae generally fit the field data well, while the fit results for the empirical formulae and the statistical fit method models are more discrete.

It is suspected that the eccentrical load may be applied on the foundation. Therefore, the experimental data with *B*/*L* equaling to 1 are adopted for the re-comparison. The compared results between the calculated ultimate bearing capacity (p_u) and the measured ultimate bearing capacity

Table 5 Illustration of ARE and SSE_{norm} of calculated p_u from different equations

	Meyerhof (1963)	Hansen (1970)	Vesic (1973)	Terzaghi (1943)
ARE	0.536	0.325	0.279	0.295
SSE_{norm}	113.76	24.73	19.80	29.77
	Booker(1969)	Steenfelt (1977)	Ingra and Baecher(1983)	Hettler and Gudehus(1988)
ARE	0.327	0.545	1.180	0.364
SSE_{norm}	28.87	53.13	478.84	49.93
	Saran and Agarwal (1991)	Michalowski(1997)		
ARE	0.628	0.410		
SSE_{norm}	166.95	67.23		

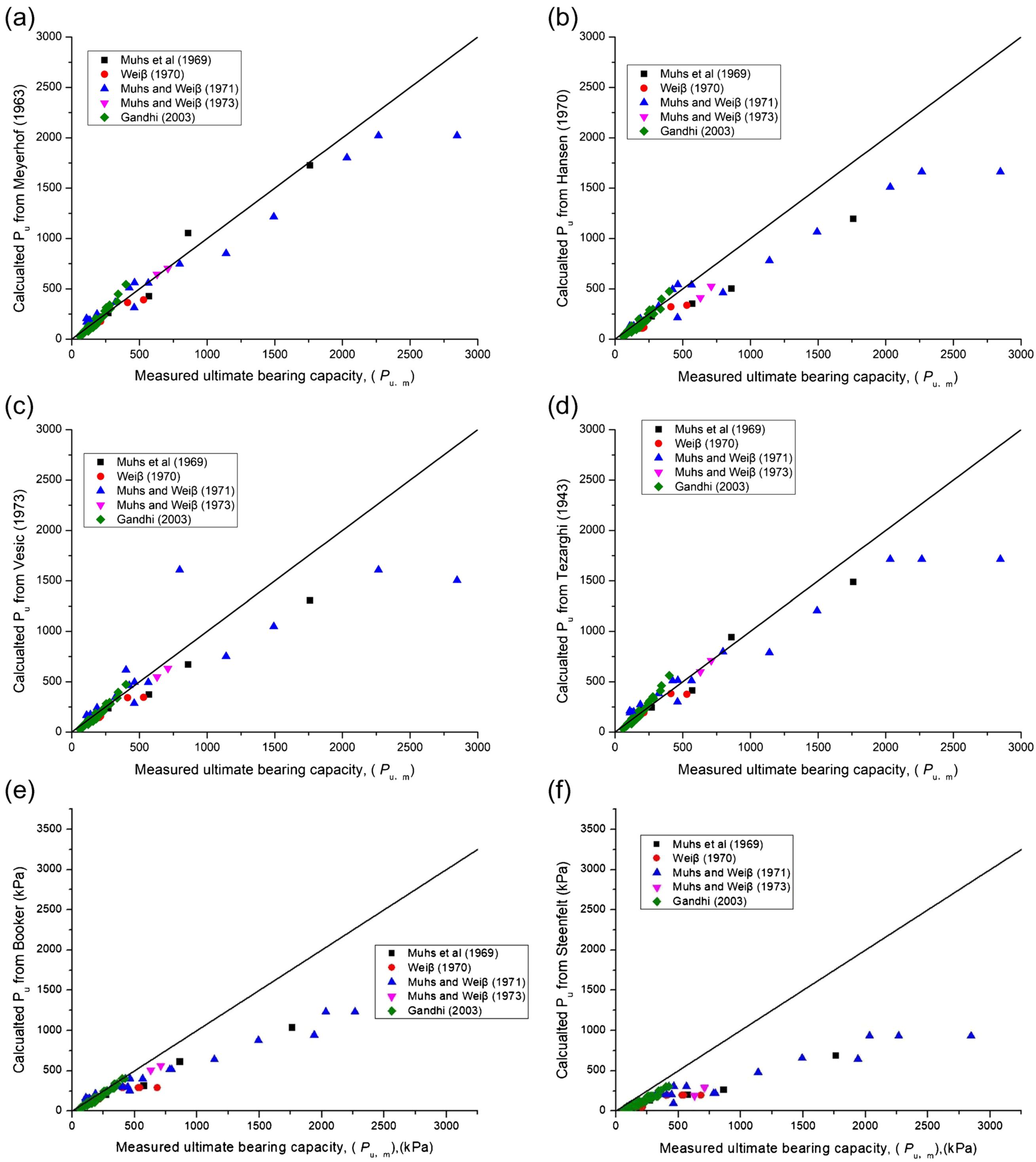


Fig. 4 Comparisons of the calculated ultimate bearing capacity p_u from different equations and measured data, $p_{u,m}$ for the case of $B/L = 1$. **a** From Meyerhof (1963), **b** from Hansen (1970), **c** from Vesic (1973), **d** from Terzaghi (1943), **e** from Booker (1969), **f** from Steenfelt (1977), **g** from Ingra and Baecher(1983), **h** from Hettler and Gudehus (1988), **i** from Saran and Agarwal (1991) and **j** from Michalowski (1997)

($p_{u,m}$) are illustrated in Fig. 4. Both the *ARE* and SSE_{norm} is recalculated and illustrated in Table 6.

The values of *ARE* and SSE_{norm} for the calculated ultimate bearing capacity, p_u, from different equations for the case of $B/L = 1$ (a total of 60 sets of experimental data) are recalculated and illustrated in Table 6.

As can be seen in Table 6, the four sets of semi-empirical formulations fit significantly better than before with the

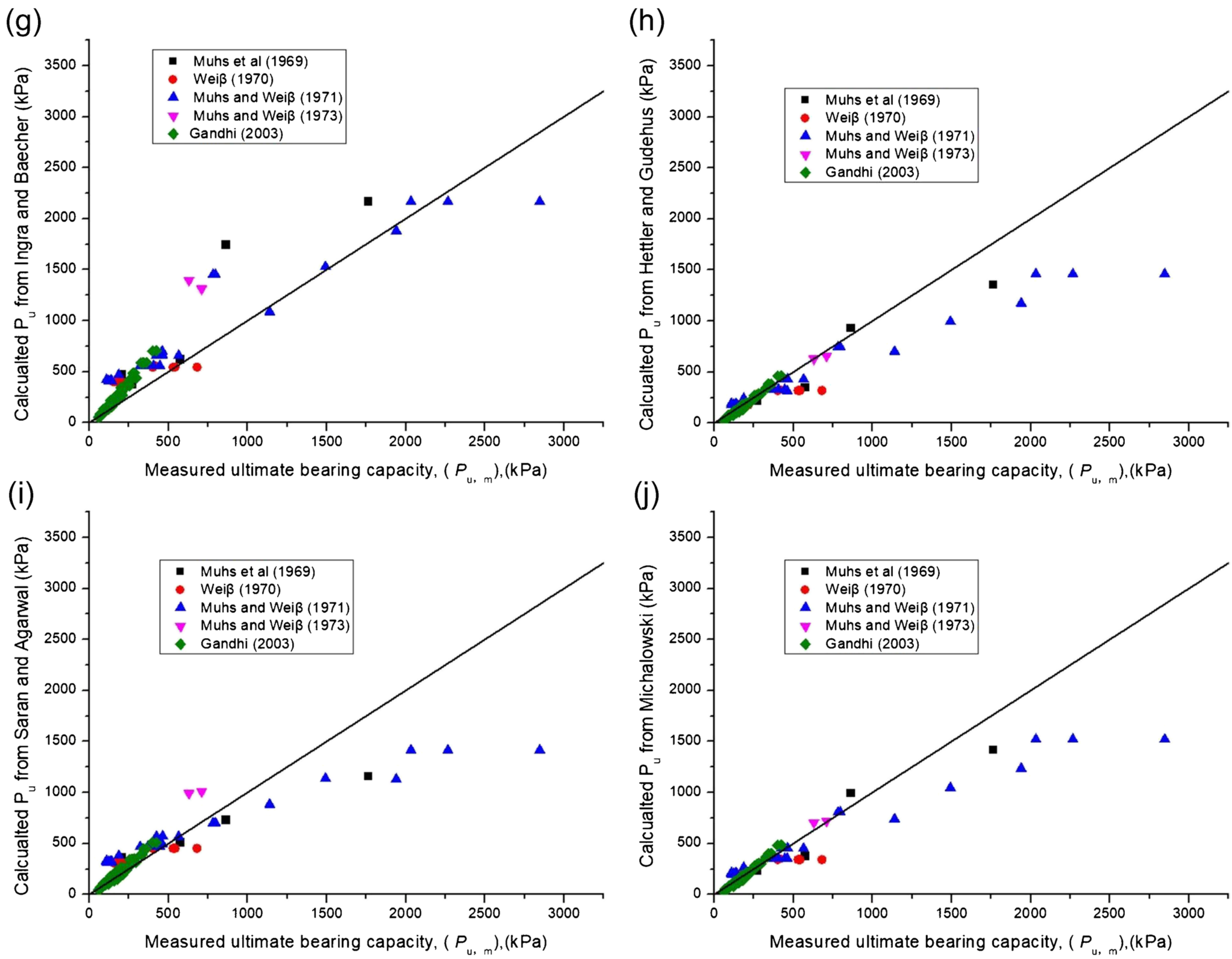


Fig. 4 (continued)

effect of eccentric loading removed, while the dispersion of the fit results between the empirical formulations and the statistical fitting method models increases further, indicating that the semi-empirical formulations have a higher degree of accuracy.

Both Figs. 3 and 4 indicate that when the p_u is less than 750 kPa, the calculated result from either equation (i.e., Meyerhof (1963), or Hansen (1970), or Vesic (1973), or Terzaghi (1943)) provides a good estimation as compared with the experimental data. On the other hand, when the

Table 6 Illustration of *ARE* and SSE_{norm} of calculated p_u from different equations for the case of $B/L < 1$

	Meyerhof (1963)	Hansen (1970)	Vesic (1973)	Terzaghi (1943)
ARE	0.194	0.240	0.209	0.208
SSE_{norm}	3.81	4.59	3.60	4.56
	Booker(1969)	Steenfelt (1977)	Ingra and Baecher(1983)	Hettler and Gudehus(1988)
ARE	0.359	0.556	1.497	0.449
SSE_{norm}	23.27	35.42	429.59	45.51
	Saran and Agarwal (1991)	Michalowski(1997)		
ARE	0.778	0.523		
SSE_{norm}	147.02	62.20		

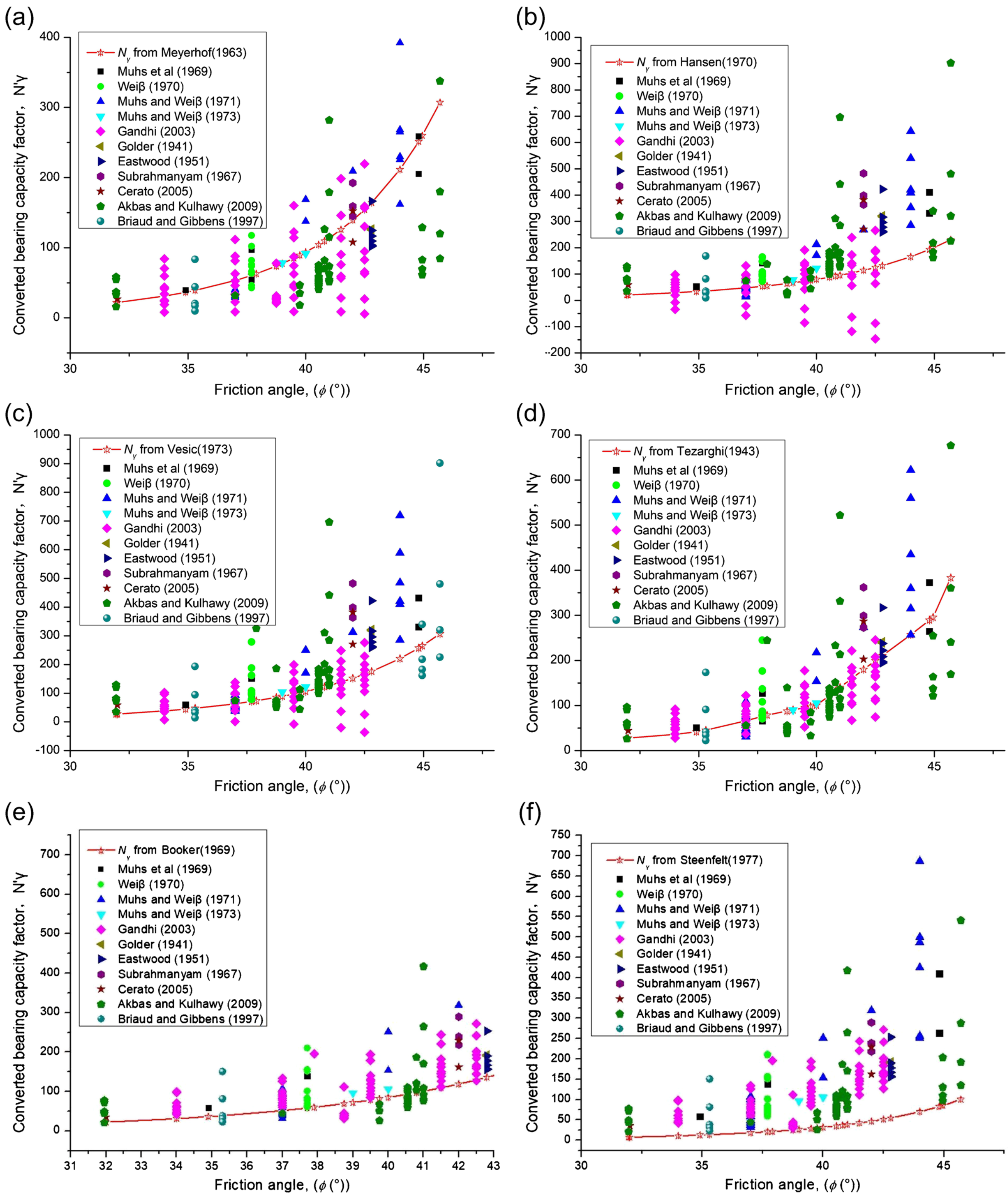


Fig. 5 Illustration of the comparison of N_γ from different equations and N'_γ from the measured data. **a** From Meyerhof (1963), **b** from Hansen (1970), **c** from Vesic (1973), **d** from Terzaghi (1943), **e** from Booker (1969), **f** from Steenfelt (1977), **g** from Ingra and Baecher(1983), **h** from Hettler and Gudehus (1988), **i** from Saran and Agarwal (1991) and **j** from Michalowski(1997)

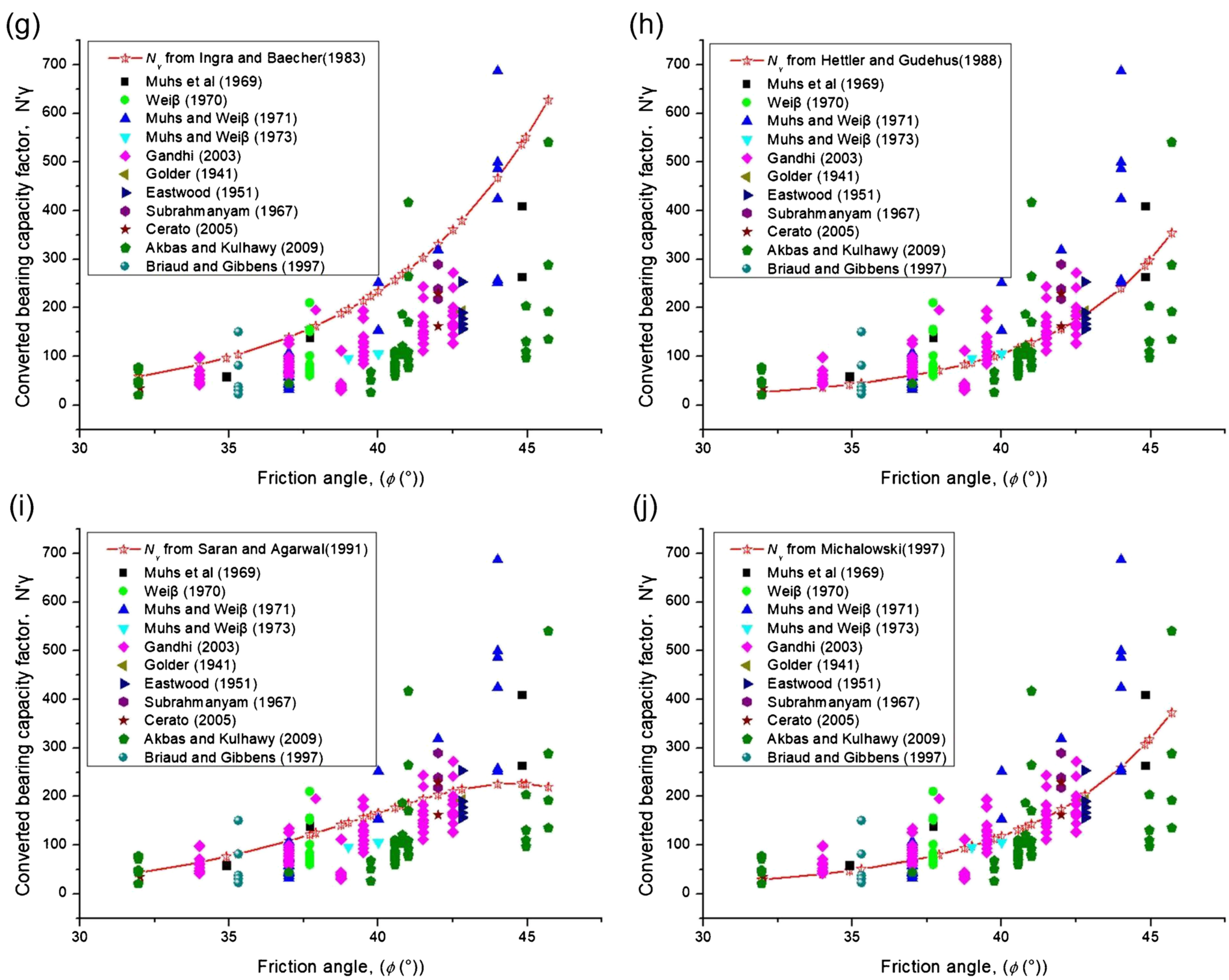


Fig. 5 (continued)

p_u is higher than 750 kPa, the estimated results from those four equations overestimate the p_u as compared with the experimental data. However, for most of the shallow foundation design, the p_u is commonly less than 750 kPa. Therefore, it is quite reasonable for code of practice of building and construction to accept those four equations in determination of the p_u. Booker's (1969) equation can provide also accurate estimation within the low bearing capacity range and more conservative when the suction is higher than 750 kPa. It seems that the predicted results from Steenfelt (1977) are too conservative while the predicted results from Ingra and Baecher (1983) are relatively accurate when the bearing capacity is less than 500 kPa. It is also observed that the predicted results from Hettler and Gudehus(1988), Saran and Agarwal (1991), and Michalowski(1997) are less accurate than those from other equations.

Evaluation of the bearing capacity factors

It is noted the calculated results from those four equations are mainly governed by the bearing capacity factors. Therefore, the bearing capacity factors N'_γ are back-calculated using Eqs. (11) and (12) from the measured data. The calculated values of N'_γ are compared with the results of N_γ which are computed from the different methods (i.e., Eqs. (4) to (6) and Eq. (10)), and illustrated in Fig. 5.

For the similar reason, the experimental data with B/L less than 1 are adopted for the re-comparison and illustrated in Fig. 6.

As observed in Eqs. (4) to (6) and Table 1, the bearing capacity factor N_γ, the shape factor s_γ, and the depth influence factor, d_γ, can be expressed in a general term as illustrated in Eqs. (18) to (20), respectively:

$$N_\gamma = a_f\left(N_q - 1\right)\tan\left(b_f\phi\right) \tag{18}$$

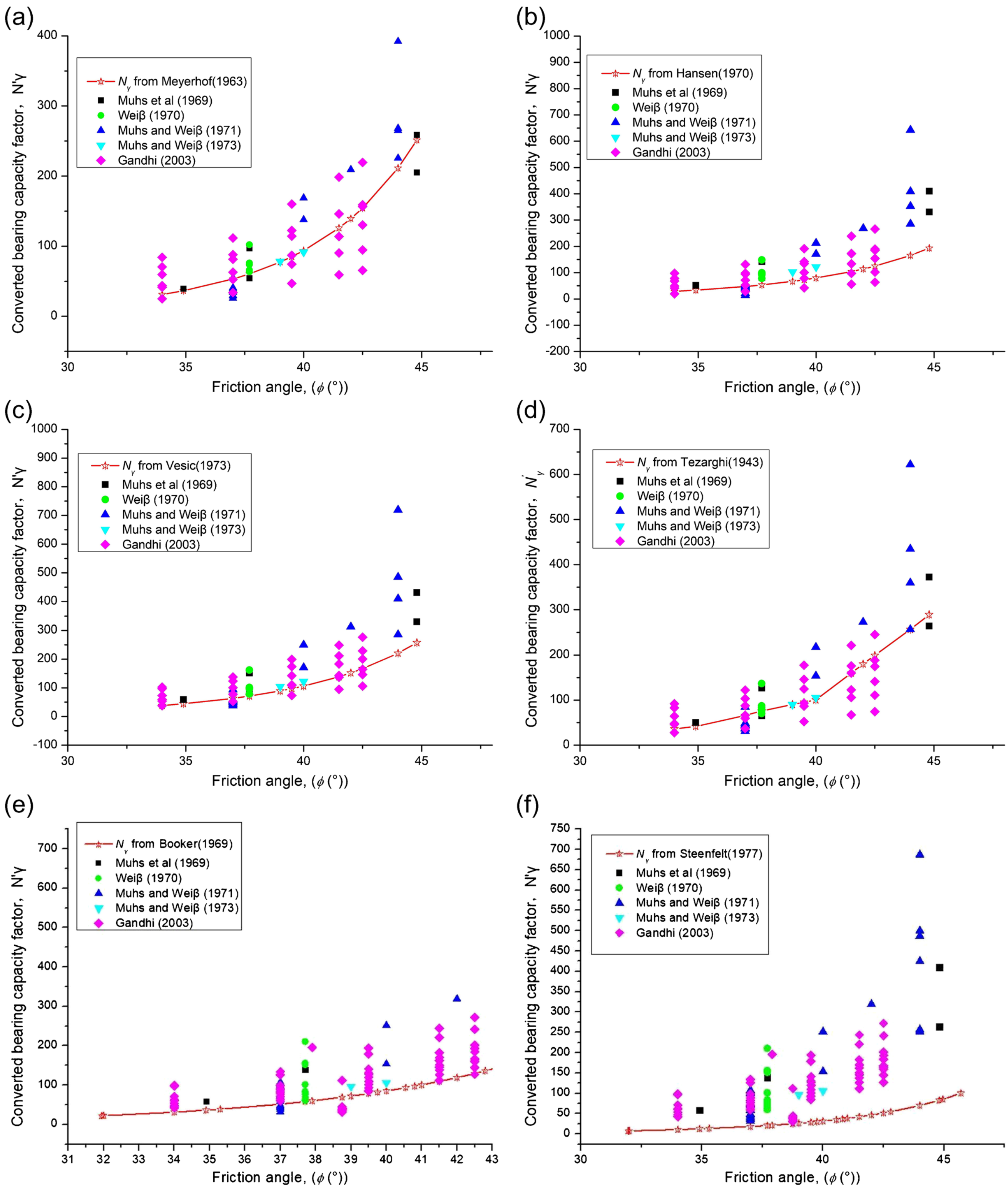


Fig. 6 Comparison of $N\gamma$ from different equations and $N'\gamma$ from the measured data for the case of $B/L = 1$. **a** From Meyerhof (1963), **b** from Hansen (1970), **c** from Vesic (1973), **d** from Terzaghi (1943), **e** from Booker (1969), **f** from Steenfelt (1977), **g** from Ingra and Baecher(1983), **h** from Hettler and Gudehus (1988), **i** from Saran and Agarwal (1991) and **j** from Michalowski (1997). Figures 5 and 6 indicate that the results of the converted N'_γ from four equations do not agree with their definitions

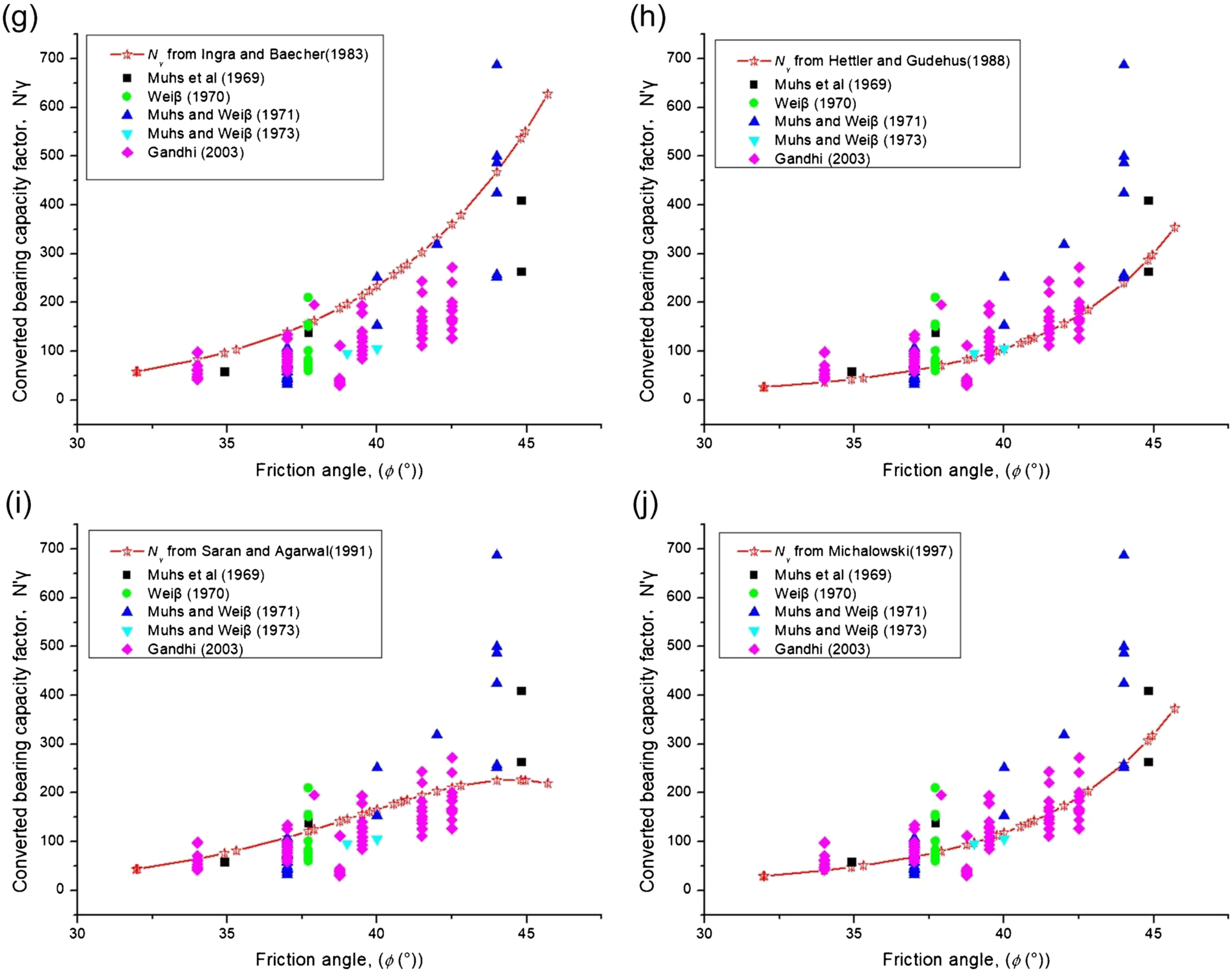


Fig. 6 (continued)

Table 7 Illustration of a_f, b_f, c_f, and d_f from the different equations

Methods	a_f	b_f	c_f	d_f
Meyerhof (1963)	1.039	1.543	0.087	0
Hansen (1970)	1.037	1.544	0.087	0.018
Vesic (1973)	1.053	1.536	0.087	0
Terzaghi (1943)	0.973	1.320	0.010	− 0.184

$$s_\gamma = 1 - c_f K_p B/L \tag{19}$$

$$d_\gamma = 1 - d_f \sqrt{K_p} d/B \tag{20}$$

where a_f, b_f, c_f, and d_f are the fitting parameters.

Equations (18) to (20) are used to best fit with the two-thirds of 60 sets of data as shown in Fig. 5 and the fitting parameters a_f, b_f, c_f, and d_f are obtained from the different methods as illustrated in Table 7. The remaining one-third

Table 8 Illustration of *ARE* and *SSE* of calculated p_u using Eqs. (18) to (20) from the regression analysis (total 40 sets of experimental data)

	Meyerhof (1963)	Hansen (1970)	Vesic (1973)	Terzaghi (1943)
ARE	0.180	0.174	0.181	0.181
SSE_{norm}	2.17	2.20	2.21	2.33

Table 9 Illustration of *ARE* and SSE_{norm} of calculated p_u using Eqs. (18) to (20) for the verification (total 20 sets of experimental data)

	Meyerhof (1963)	Hansen (1970)	Vesic (1973)	Terzaghi (1943)
ARE	0.214	0.222	0.206	0.226
SSE_{norm}	1.25	1.23	1.20	1.29

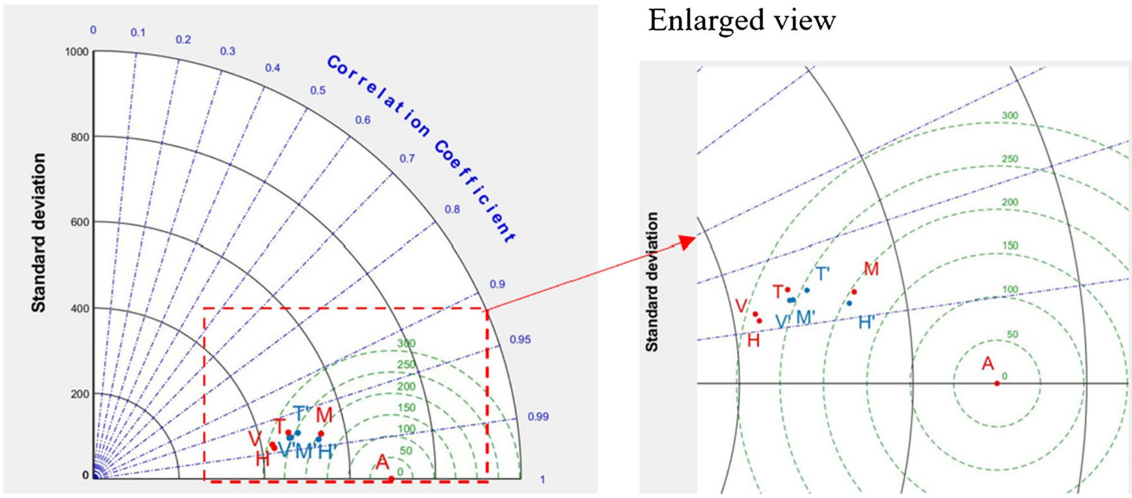


Fig. 7 Taylor diagram of determined p_u from the different equations and measured $p_{u,m}$ for the case of $B/L = 1$

of 60 sets of data are used for the verification of Eqs. (18) to (20).

Both the ARE and SSE_{norm} for the regression analysis (40 sets of data) and verification process (independent 20 sets of data) for the different methods are illustrated in Tables 8 and 9, respectively.

To have a better comparison, the Taylor diagram for the calculated ultimate bearing capacity, p_u, from the different methods using the original equations (Eq. (1) to (10)) and using the equations proposed in this paper (Eq. (18) to (20)) are illustrated in Fig. 7. *A* denotes measured data, *M* denotes obtained results from the original Meyerhof's (1963) method, and *M′* denotes obtained results from Meyerhof (1963) using N_γ proposed in this paper. Similarly, *H*, *V*, and *T* denote obtained results from the original Hansen's (1970), Vesic's (1973), and Terzaghi's (1943) method, respectively, while *H′*, *V′*, and *T′* denote obtained results from Hansen's (1970), Vesic's (1973), and Terzaghi's (1943) method using N_γ proposed in this paper, respectively.

It is observed that both values for *ARE* and SSE_{norm} are relatively high in Table 4. In addition, it is also observed that there is high variability in the measured ultimate bearing capacity $p_{u,m}$ for the case of $B/L = 1$ as shown in Table 3.

As shown in Table 5, both values of *ARE* and SSE_{norm} can be significantly reduced by removing the experimental data for the case of $B/L = 1$. In addition, it is observed that the results from Meyerhof (1963) and Terzaghi (1943) have better agreements with the experimental data than those from the other two methods. As shown in Tables 5 and 8, the values of ARE in Table 8 are not much different as those in Table 5. However, the values of SSE_{norm} in Table 8 are significantly reduced.

Figure 7 indicates that only the point *M′* is located further to the point *A* than the distance between *A* and *M*. The points of *H′*, *V′*, and *T′* are located closer to point *A* as compared with the points of *H*, *V*, and *T*, respectively. It indicates that the equation for N_γ proposed in this paper can improve the performance of Hansen's (1970), Vesic's (1973), and Terzaghi's (1943) equations in the determination of the ultimate bearing capacity.

Conclusions

It is observed that Meyerhof (1963), Hansen (1970), Vesic (1973), and Terzaghi (1943) equations are commonly acceptable for determination of the ultimate bearing capacity of shallow foundation in most codes of practices. The performances of those four equations in the determination of the ultimate bearing capacity are evaluated by using 163 sets of experimental data from literature. It is observed that Meyerhof (1963) and Terzaghi (1943) equations have better performance than Hansen (1970) and Vesic (1973) equations when compared with the experimental data adopted in this paper. A new equation for the determination of the bearing capacity factor N_γ is proposed in this paper. The parameters in the new equations are determined using the regression analysis. The proposed equations are verified with independent experimental data, and they are proven to improve the performance of the conventional equations in the determination of the ultimate bearing capacity.

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Declarations

Competing interests The authors declare that they have no competing interests.

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