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Study on critical conditions for rock failure by means of group renormalization*

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Abstract A study of the characteristics of the accumulative rock failure and its evolution by application of the group renormalization method were presented. In addition, the interaction and long-range correlated effects between the immediate neighboring units was studied. The concept of mechanical transference for model OFC, employed in the study of self-organized criticality, and the coefficient α were introduced into the calculation model for group renormalization. With the introduction, mechanisms for the drastic increase and decrease of failure intensity of rocks were investigated under similar macro-conditions.

Keywords group renormalization, rock failure, critical condition

Introduction

Under stress, rock materials are subject to failure and breakage, a phenomenon that is commonly witnessed. However, such a common phenomenon presents a most complicated and difficult problem for rock mechanical studies (Xia et al., 1995). The complication and difficulty can be attributed to the fact that the study involves the investigation of a dissipation system, including cracks of various sizes, the process of self-organization distant from equilibrium and various non-linear irreversible processes of inter-coupling (Li et al., 2002; Yin et al., 2002; Xu and Huang, 2005; Yin et al., 2006). At present, in applied mechanics, rock materials are treated as a continuous medium. In this traditional light, researches are carried out fruitfully but not without insurmountable difficulties. It can be safely said that in the study of failure and breakage, fundamental problems remain yet intact.

This paper will present an investigation into criticality in the rock failing process by turning to the theory of self-organized criticality, usually applied to the study of the extensive mechanical dissipation system, and by turning to the group renormalization method.

1 Researches on the critical conditions for rock destabilization

Owing to the differences in the properties of rock ingredients and the random distribution of the internal deficiency, the mechanical characteristics for rock materials are different, even in different places within a single rock. Therefore, in the description of rock deformation features, damage mechanics should be employed and the micro strength distribution should be studied in the light of statistical theory.

In 1982, D Krajcinovic et al. integrated the continuous failing theory with the theory of statistical strength and put forward a model for the statistical damage as shown in Fig.1 (Wang and Song, 2002). By abstracting the samples into an assembly consisting of N thin long beams, the basic relations can then be represented:

$$\sigma = E\varepsilon(1 - n/N), \tag{1}$$

where, E is the elastic modulus; ε is the strain; n is the

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number of broken beams; N is the total number of beams.

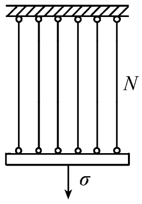

Fig.1 Thin and long beam assembly of samples

In Eq.(1), n/N is equal to the damage parameter D in damage mechanics. In other words:

$$D = n / N. \tag{2}$$

Under an external load, the linkage of cracks and the breakage are not only critical but also regardless of the scale. The damage can break a rock body as small as a grain or can develop into an extended crack of any size. In other words, although the damage in the self-organized criticality can be induced to various extents, the damage nature will remain the same. Therefore, it adapts to group renormalization method well. The group renormalization transform is based on the idea that first, the small-sized increases and decreases are equalized; then "the traces" left after equalization will be embodied in the effective interaction strengths for greater size increases and decreases. The set for such renormalization transforms will constitute the group renormalization (Huang and Xu, 1997). For this purpose, the idea (Tang et al., 2004) based on the model for statistical damage is introduced into the model for statistical failure induced by continuous accumulative damages done to a single stressed unit within a rock, as shown in Fig.2.

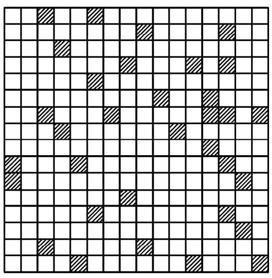

Fig.2 The damage and statistical model of rock failure

The present model is equal to the extension of model 1 horizontally. With the test sample under the vertical uni-axial compression, a section is obtained from the test sample along the strained crack. Then the section is divided horizontally and vertically into N rectangular units. Suppose the shadowed units are broken under stress while the shadowless units are not. It is further supposed that the area for a unit is S and the total area for all the units in the model is NS, and at a certain time, under a certain stress, the broken units are nS, the damage parameters can satisfy the demands of Eq.(2). Then from the viewpoint of statistics, the damage parameter D represents the unit breakage rate, which ranges from 0 to 1.

1.1 The establishment of a group renormalization model

In Fig.3, a kind of cell, consisting of four units, are shown and the kind of cell is on the lowest tier of the unit organization hierarchy. In Fig.3, the shadowed are broken units while others are unbroken units. When it comes to the kind of cell of four units, there are only five configurations and 16 different cells. Specifically, Fig.3(a) represents configuration 1, containing only 1 cell; Fig.3(b) represents configuration 2, consisting of 4 cells; Fig.3(c) represents configuration 3, composed of 6 cells; Fig.3(d) represents configuration 4, composed of 4 cells; and Fig.3(e) represents configuration 5, composed of 1 cell.

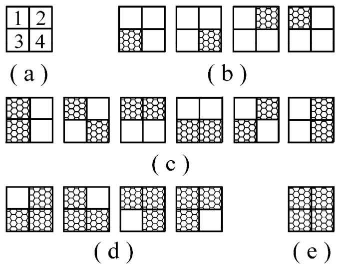

Fig.3 Possible configuration of a cell composed of four units

Suppose the failure probability for a unit is p and $(1-p)$ is the probability for unbroken units. It is further supposed that when at least two broken units are immediate neighbors, the cell in the next tier will be broken. If i and u are used to represent the broken and the unbroken unit respectively, then Fig.4 provides illustrations only for the failure mode of a cell represented by Fig.3(b), consisting of a single broken unit, and a cell represented by Fig.3(c), consisting of two broken units not arranged as immediate neighbors, when stress transfer is induced by breakage in the neighboring unit.

$$u_1u_2i_3u_4 \xrightarrow{\delta F} \begin{matrix} u_1u_2i_3i_4 \\ i_1u_2i_3u_4 \\ i_1u_2i_3i_4 \end{matrix} \qquad i_1u_2u_3i_4 \xrightarrow{2\delta F} \begin{matrix} i_1i_2u_3i_4 \\ i_1u_2i_3i_4 \\ i_1i_2i_3i_4 \end{matrix}$$

Fig.4 Failure modes for different units

Configuration 1 consists only of 1 cell, and it is clearly indicated that the cell is not subject to failure.

In configuration 2, a unit is broken, whose breaking force can be transferred to the immediate units, leading to the failure of the affected units and the failure of the cell. For example, if Unit 3 is broken, a force F will be transferred to Unit 4 or 1, leading to the failure of the unit and the cell. In that case, the failure probability for the cell will be $p(1-p)^3 p_{3.4}(1-p_{3.1})$ or $p(1-p)^3 p_{3.1}(1-p_{3.4})$. ($p_{m.\,n}$ is the conditional probability of the failure for Unit n in case Unit m is subject to failure, where m and n are the No. of units). In configuration 2 , the failure probability for the cell will be $8p(1-p)^3 p_{3.4}(1-p_{3.1})$.

In each cell of configuration 3, there are two damaged units, falling into two categories: ① in case that the two neighboring units are damaged, the failure probability will be $p^2(1-p)^2$, and in the configuration, 4 cells are in such a mode of failure; ② in a cell, two diagonally arranged units are damaged. For example, when Units 1 and 4 are damaged, the forces transferred to Unit 2 from the two units will be $2\delta F$. In other words, Unit 2 will have $2\delta F$ added. In this case, the failure probability will be $p^2(1-p)^2 p_{14.2}(1-p_{14.3})$. In case that the forces from Units 1 and 4 are transferred to Unit 3, the failure probability for the cell will be $p^2(1-p)^2 p_{14.3}(1-p_{14.2})$. If the force from Unit 1 is transferred to Unit 2 while the force from Unit 4 is transferred to Unit 3, then Units 2 and 3 will each have only δF added. If the force from Unit 1 is transferred to Unit 2 while the force from Unit 4 is transferred to Unit 3, or the force from Unit 1 is transferred to Unit 3 while the force from Unit 4 is transferred to Unit 2, then the two units, Units 2 and 3 will each have only δF added. In each of the two cases, the failure probability for the cell will be $2p^2(1-p)^2 p_{1.2}p_{4.3}$. And in configuration 3, there are two such cases. Therefore, the failure probability for configuration 3 will be $4p^2(1-p)^2[1+p_{14.2}(1-p_{14.3})+p_{1.2}p_{4.3}]$.

In configuration 4, 3 units are subject to failure and 1 unit is not. In that case, the failure probability of the cell will be $p^3(1-p)$. Also in configuration 4, there are four such cells. Therefore, the failure probability for the configuration will be $4p^3(1-p)$.

In configuration 5, the failure probability for each of all the units will be p^4, and configuration 5 consists of only one such cell. In other words, the cell is apparently completely subject to failure.

From the above analysis, we know that the probability for a single tier:

$$\begin{aligned} p^{(1)} = 8p^{(0)}[1-p^{(0)}]^3 p_{3.4}^{(0)}[1-p_{3.1}^{(0)}] + 4[p^{(0)}]^2 \times \\ [1-p^{(0)}]^2[1+p_{14.2}^{(0)}(1-p_{14.3}^{(0)}) + p_{1.2}p_{4.3}] + \\ 4[p^{(0)}]^3[1-p^{(0)}] + p^4, \end{aligned} \tag{3}$$

where, $p_{3.4}=p_{3.1}=p_{1.2}=p_{4.3}$; $p_{14.2}=p_{14.3}$.

In case Unit 3 is subject to failure, the conditional probabilities of the failure for Units 1 and 4 are $p_{3.1}$ and $p_{3.4}$, respectively. In case Units 1 and 4 are subject to failure, the conditional probabilities of the failure for Unit 2 or 3 are $p_{14.2}$ and $p_{14.3}$, respectively. If the resultant force Units 1 and 2 bear is represented as F' and p' is equal to $p_{3.1}$ or $p_{14.2}$, then

$$p' = \frac{p_1(F') - p_1(F)}{1 - p_1(F)}. \tag{4}$$

1.2 On determination of F' by means of critical auto-organization theory

Under similar macro conditions, rock failure strength increases and decreases greatly, indicating that within the rock, there are randomly distributed cracks of various sizes. Under stress, the cracks will reorganize automatically, develop and couple in different ways, resulting in changes of macro mechanic properties of the rock. Numerous studies indicate that self-organization is present throughout the rock failure process (Gao et al., 2003). Consequently, if the intensity of the inter-unit action is not taken into account, the strength variation cannot be described. In the recent few decades, the theory of self-organized criticality is widely applied. Especially, the OFC model is applied, which is usually used to simulate movements in earthquakes by means of the cellular automata. The OFC model can be employed to describe the fact that when the sub-stability exceeds a certain critical value, there will be a phase space characterized by instability (Li, 2001). With the OFC model, a single parameter α is used to describe the intensity of the inter-unit action. When α=0, there will not be any interaction between neighboring units. When α=0.25, the interaction between neighboring units will reach the maximum. Therefore, the value for α serves as an indicator for the intensity of the interaction between neighboring units. The introduction of such a principle into the study of force transfer effects in the process of unit failure means that when the force F a unit bears exceeds the threshold value, the unit will release the force F acting on it. If the absorption coefficient for the neighboring unit is α, then the additional force the neighboring unit bears will increase by δF or $2\delta F$. Then F' can be represented:

$$F' = F + \delta F = F + \alpha F = (1+\alpha)F,$$

$$\text{or} \quad F' = F + 2\delta F = F + 2\alpha F = (1+2\alpha)F. \tag{5}$$

1.3 On determination of critical failure probability

Suppose the strengths of units obey the law of

Weibull distribution, then the failure probability for the unit under F can be expressed:

$$p(F)=1-\exp\left[-\left(\frac{F}{F_0}\right)^2\right], \quad (6)$$

here, F_0 is the parameter for Weibull distribution, an indicator for the mechanical properties of the rock material. From Eqs.(4)~(6), we can have:

$$\begin{aligned} p_{3.1} &= 1-[1-p(F)]^3, \\ p_{14.2} &= 1-[1-p(F)]^8. \end{aligned} \quad (7)$$

Utilizing Eqs. (3)~(7), the failure probability for the cells of the first tier can be expressed:

$$\begin{aligned} p^{(1)} = {} & 4p^{(0)}(1-p^{(0)})^3[1-(1-p^{(0)})^6]+ \\ & 2(p^{(0)})^2(1-p^{(0)})^2[1-(1-p^{(0)})^{(8\alpha+8\alpha^2)}]+ \\ & 4(p^{(0)})^2(1-p^{(0)})^2+4(p^{(0)})^3(1-p^{(0)})+(p^{(0)})^4. \end{aligned} \quad (8)$$

From Eq.(8), the failure probability for the cell of tier n+1 or the group renormalization formula can be expressed as:

$$\begin{aligned} p^{(n+1)} = {} & 4p^{(n)}(1-p^{(n)})^3[1-(1-p^{(n)})^{(4\alpha+2\alpha^2)}]+ \\ & 2(p^{(n)})^2(1-p^{(n)})^2[1-(1-p^{(n)})^{(8\alpha+8\alpha^2)}]+ \\ & 4(p^{(n)})^2(1-p^{(n)})^2+4(p^{(n)})^3(1-p^{(n)})+(p^{(n)})^4. \end{aligned}$$

From the theorem for the existence of unmovable points in the renormalized group, we will have:

$$\begin{aligned} f(p)=p= {} & 4p(1-p)^3[1-(1-p)^{(4\alpha+2\alpha^2)}]+ \\ & 2p^2(1-p)^2[1-(1-p)^{(8\alpha+8\alpha^2)}]+ \\ & 4p^2(1-p)^2+4p^3(1-p)+p^4. \end{aligned}$$

p_1=0, p_2=1, and p_c can be determined by:

$$\begin{aligned} & 4(1-p)^2[1-(1-p)^{(4\alpha+2\alpha^2)}]+2p(1-p)\times \\ & [1-(1-p)^{(8\alpha+8\alpha^2)}]-1+3p-p^2=0. \end{aligned}$$

Let $|\lambda|=\left|\frac{\partial f(p)}{\partial p}\right|$ serve as the condition for the determination of the properties of unmovable points, then the calculated values for p_c, λ and F/F_0 are listed in Table 1.

Within the range of $0\le p\le 1$, there exist three unmovable points. When p=0 and p=1, $|\lambda|\le 1$ and the unmovable points are stable. When $p=p_c$, $|\lambda|>1$ and the unmovable points are unstable. In other words, p_c is a critical point. When $0<p<p_c$, the iteration approaches the limit for the nonoccurrence of failure p_∞=0. When $p_c<p_0<1$, the iteration approaches the limit for the occurrence of failure p_∞=1. In other words, when the failure probability for a unit is more than p_c, all the other units in a cell will possibly be subject to failure due to the force transference. From Table 1, we know when F/F_0<1, the stress transfer will be lower

Table 1 Calculated values for p_c, λ and F/F_0

α	p_c	λ	F/F_0
0	0.382	1.528	0.694
0.01	0.362	1.545	0.670
0.02	0.344	1.562	0.649
0.03	0.327	1.576	0.629
0.04	0.312	1.590	0.612
0.06	0.284	1.611	0.578
0.07	0.273	1.622	0.565
0.08	0.263	1.633	0.552
0.09	0.251	1.640	0.538
0.10	0.242	1.650	0.527
0.11	0.234	1.660	0.516
0.12	0.224	1.664	0.504
0.13	0.218	1.673	0.500
0.14	0.210	1.678	0.485
0.15	0.202	1.682	0.475
0.16	0.200	1.690	0.468
0.17	0.190	1.695	0.459
0.18	0.185	1.703	0.452
0.19	0.179	1.707	0.444
0.20	0.174	1.711	0.437
0.21	0.171	1.729	0.433
0.22	0.164	1.718	0.423
0.23	0.159	1.721	0.416
0.24	0.155	1.725	0.410
0.25	0.150	1.728	0.404

than when a single unit is taken into consideration. However, with the increase of the self-organization coefficient α, the resistance of the rock to failure will decrease. And with the decrease of critical probability, the probability for failure occurrence will increase. Therefore, p_c is the critical point at which a failure may break out suddenly in a unit. Also it serves to mark that the system is in the critical state of self-organization. This means that a tiny disturbance of part of the rock can lead to the evolution of the cracks in the direction of instability, just like a "domino effect". In that case, the failure will be accumulative and extensible to the whole system, triggering a "snow-slide" behavior.

In application of pressure onto the rock, when the pressure-induced rock failure probability is smaller than the critical probability, the unit with rather low strength distributed at random will be the first to be subject to failure. In addition, the failure is only triggered by the action from a neighboring unit without coupling within any long range. Also the failure occurs at random. When the applied pressure exceeds the critical failure probability, the rock system and its local components are characterized by desirable self-organization capacity. The unit failure can be attributed to the impacts of the immediate neighboring units and the coupling occurs within a long range. From macro viewpoint, random and disordered cracks are in the transition to well-ordered cracks. Finally, cracks will develop through the maximal failure plane, which will serve as a kinetic attractor of self-organized criticality properties.

2 Conclusions

(1) Under the impacts of an external force, rock failure presents a state distant from equilibrium or a process of self-organization of cracks, composed of a number of nonlinear irreversible coupling processes. If the external load cannot enable the rock unit failure probability to reach the critical value, the breaking behaviors for various units are independent and coupling takes place only within a short range. In this case, failure occurs at random and disorderly. On the contrary, if the external load is so strong as to exceed the critical probability for the unit failure, the unit failure will be in the form of interactions between neighboring units and crack coupling will be effective distantly. Then the random and disorderly failure accumulates in an attraction zone, forming a breakthrough breakage plane. In this case, the breaking plane will be a kinetic attractor of self-organized criticality properties.

(2) Taking into consideration that a force is transferred from a failing unit to a neighboring unit while the remaining force is lower than the force for a single unit, then with the increase of the self-organization coefficient α, the failure resistance capacity for the rock will decrease. With the decrease of the critical probability, the failure will occur increasingly. With α, the increasing and decreasing characteristics of failure strength of rock materials can be adequately described under similar macro conditions.

(3) By turning to the group renormalization method, the values obtained through calculation for rock failure critical probabilities are similar to those obtained through experiments. However, taking into consideration the non-equilibrium effects for rock failure, further researches should be conducted along this line. Nevertheless, the group normalization and self-organization theories have provided an adequate approach for the study of failure criticality.

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