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ORIGINAL ARTICLE

Analysis of the non linear bending and membrane stresses associated to the fundamental non-linear mode shape of fully clamped skew plates at large vibration amplitudes

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Abstract Accurate prediction of the total geometrically non-linear dynamic stress, including both the membrane and bending stresses, is of a crucial importance in the engineering design. A semi-analytical model based on Hamilton's principle and spectral analysis has been developed recently to study the effects of large vibration amplitudes of fully clamped skew plates. The purpose of the present work was the extension of the model to the analysis of the stresses, including both the non-linear bending and membrane stresses associated to the fundamental non-linear mode shape. It was found that the non-linear frequency increases with increasing the amplitude of vibration, which corresponds to the hardening type effect due to the membrane forces induced by the large vibration amplitudes. The corresponding non-linear bending strains were obtained via the usual strains-displacement relationships, involving partial derivatives of only the transverse displacement function with respect to the space variables. To estimate the non-linear membrane stresses, without having to calculate the in-plane displacements, which would make the model much more laborious and time consuming, a simple practical engineering theory was presented here, which takes into account the contribution of the in-plane displacements u and v in an average sense, along lines parallel to the plate edges. The results show that, at large deflections, higher bending stresses occur near to the clamps, compared with those predicted by the linear theory. Numerical details are presented and comparison of the results obtained here with the ones previously ones treated in the literature shows a satisfactory agreement.

Keywords Skew plates · Non-linear bending stress · Non-linear membrane stress · Iterative method · Non-linear vibration

1 Introduction

When dealing with a multimodal approach to geometrically nonlinear transverse vibrations, two choices are possible corresponding to two different levels of descriptions:

(1) One can neglect the axial displacements u and v in the formulation and keep only the transverse displacement W. This assumption, which greatly simplify the formulation has been adopted and justified previously in many Benamar's works on non linear plate vibrations.
(2) A more complete description may be tempted by taking into account W, u and v. This choice leads to a big system of non-linear algebraic equations due to the number of basic functions which must be used in the series expansion for W, u and v. Such a system may be hand to solve and is in all case time consuming (the nonlinear tenser b_{ijkl} involves N^4 terms).

The method presented in this paper is an intermediate way between the two choices (1) and (2). On one hand, it benefits from the relative simplicity of the first method. On the other hand it allows easy estimation of the axial stresses induced by large vibration amplitudes which cannot be calculated by the first method.

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In addition to the point stated above, one can mention the following novelties of the present work, especially in comparison with Reference [12]:

(1) The extension of the method from the case of rectangular plates to that of skew plates which may be in some engineering applications of a high practical importance and for which many changes and adaptations have to be made such as the application of the necessary change of variables from the Cartesian coordinates to the Skew coordinates.
(2) The convergence study in the skew plate case has shown that because of the non symmetry of the skew plate it is necessary to use 18 basic functions when dealing with the non-linear fundamental mode shape instead of the 9 functions usually used for rectangular plates (consequently the nonlinear rigidity tensor in the present case involves now's $18^4 = 104{,}976$ terms instead of the $9^4 = 6561$ used in the rectangular case).
(3) To estimate the axial strains in the skew plate problem, it was necessary to consider oblique lines parallel to the plate oblique edges.

In contemporary structural design, skew plates may be encountered in diverse domains, such as the design of bridges, ship hulls, buildings floor systems, etc. The simulation of the nonlinear static and dynamic behaviour of skew plates at large displacement amplitudes is an interesting area of work for researchers because the problem has no exact analytical solution, even in the linear case. Approximate numerical methods, such as integral equations and series methods have been used in the literature to study the linear mode shapes and natural frequencies of fully clamped skew plates. A comprehensive treatment of the linear problem and references corresponding to the above-mentioned methods has been given in the monograph of Leissa [1]. for skew plates analysis, Nowinski [2] developed the governing equations for the nonlinear vibration of these plates and this was followed by several papers extending the study to the nonlinear vibration of orthotropic skew plates such as that of Alwar and Rao [3], who developed a non-linear analysis of orthotropic skew plates of constant thickness subjected to a uniform transverse load. Prathap and Varadan [4] studied the large amplitude free flexural vibrations of thin elastic anisotropic skew plates. Chia [5] considered the moderately large deflection elastic behaviour of homogenous isotropic and laminated anisotropic rectangular and skew plates by analytical methods. Sathyamoorthy [6] studied the nonlinear vibration of skew plates with attention to shear and rotary inertia. Nonlinear plate vibration has been studied by a large number of investigators. Benamar et al. [7] presented a theoretical formulation of the plate vibration problem at large displacement amplitudes. Han and Petyt [14] applied the hierarchical finite element method to study the variation of the natural frequencies and mode shapes with the change in the vibration amplitude of isotropic rectangular plates with clamped boundary conditions. El Kadiri et al. [8] and El Kadiri and Benamar [9, 10] presented a semi-analytical method, based on Hamilton's principle and spectral analysis, for determination of the geometrically non-linear free response of thin straight structures.

Many papers have made the assumption that only the transverse displacement W was considered in the plate theory, and the in-plane displacements u and v were neglected. This assumption introduces a simplification in the model and reduction in the computation time but, gives poor results concerning the membrane stresses. Haterbouch and Benamar [11] improved formulation for the geometrically non-linear free vibrations of clamped immovable circular plates by taking into account the in-plane displacement. It was found that the maximum membrane stress has a significant contribution (about 33%) to the maximum total stress, at vibration amplitudes equal to about twice the plate thickness. As a result, the membrane stress has to be taken into account in the stress analysis.

For the fully clamped rectangular plate, some papers has considered the transverse vibration and the average in-plane displacements u and v to examine the effect of large vibration amplitudes on the membrane and bending stress patterns associated to the fundamental non-linear mode shape.

As mentioned in Reference [12] and proved experimentally [13], for a homogeneous clamped clamped beam at large deflections, the axial strain has the same value along the beam. Therefore, the in-plane membrane strains have been assumed to be constant along lines parallel to the immovable edges of a rectangular plate.

This result has been used in the present work as a simple engineering tool to study fully clamped skew plates undergoing large vibration amplitudes. Following the approach used in [12], two averaging techniques were developed, the u–v–W average formulation and the W-average formulation respectively. In the first technique, the new expression for the fourth order non-linearity tensor b_{ijkl} was determined by substituting the averaged non-linear membrane strain in the associated energy integral. After integrating, the immovable in plane boundary conditions permitted naturally to eliminate the in plane displacements u and v. In the second technique, the membrane stresses were estimated by considering only the transverse deflection patterns, and the resulting averaged membrane strains, and the associated membrane stresses were determined by the averaging technique. The large amplitude free vibration problem has been modeled by a set of non-linear algebraic equations using Hamilton's principle and spectral analysis. The solution of the mathematical problem has been done by an iterative method. The main advantage of this approach is its simplicity and the large reduction it induces in the computation time.

The objective of the present work was the development of a theory taking into account the effect of the in plane displacements on the nonlinear vibrations and the membrane stresses of fully clamped thin elastic skew plates, by using an averaging method. The theoretical model using an iterative method and presented in Reference [7] was adapted here and applied to the analysis of the geometrically nonlinear free dynamic response of fully clamped isotropic skew plates. Backbone curves corresponding to the membrane stress and surface flexural stress are presented for various values of the skew angle and the plate aspect ratio in order to examine the effect of this parameter on the response of the plate at large vibration amplitudes. Comparison was made between the present results and those available in the literature and a good agreement was found.

2 General formulations and analysis

Consider the skew plate with a skew angle θ shown in Fig. 1. For the large vibration amplitudes formulation developed here, it is assumed that the material of the plate is elastic, isotropic and homogeneous. The thickness of the plate is considered to be sufficiently small so as to avoid the effects of shear deformation. The skew plate has the following characteristics: a, b, S: length, width and area of the plate; x–y: plate co-ordinates in the length and the width directions; ξ–η, H: Skew plate co-ordinates and plate thickness; E, ν: Young's modulus and Poisson's ratio; D, ρ: plate bending stiffness and mass per unit volume. By assuming harmonic motion and expanding the transverse displacement in the form of finite series of functions, the total strain energy V of a rectangular plate is given as the sum of the flexural strain energy V_f and the membrane strain energy V_m induced by large deflections. By discretization of the expressions for V_f, V_m and T, the tensors m_{ij}, k_{ij} and b_{ijkl} are defined. The frequency equation at large vibration amplitudes is then developed. Finally, the non dimensional formulation of the problem and details of the numerical solution are presented.

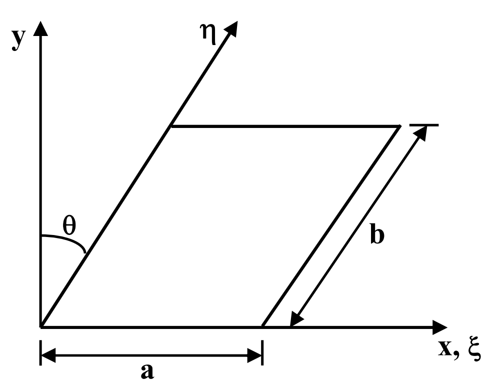


Fig. 1 Skew plate in x–y and ξ–η co-ordinate system

As presented in Reference [7], the flexural strain energy may be written in the x–y co-ordinate system as:

$$V_f = \frac{D}{2}\int_S \left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 2(1-\nu)\left(\frac{\partial^2 \nu W}{\partial x \partial y}\right)^2 + 2\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right] dS. \quad (1)$$

For a fully clamped rectangular plate, the above expression reduces to:

$$V_f = \frac{D}{2}\int_S \left[\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right]^2 dS. \quad (2)$$

The expression for the membrane strain energy, which is due to the stretching of the middle surface of the plate, is given by:

$$V_m = \frac{Eh}{2(1-\nu^2)}\int_S \Bigg\{ \left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial x}\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial v}{\partial y}\left(\frac{\partial W}{\partial y}\right)^2 + \frac{1}{4}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right]^2 + 2\nu\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{1}{2}\frac{\partial v}{\partial y}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{1}{2}\frac{\partial u}{\partial x}\left(\frac{\partial W}{\partial y}\right)^2 + \frac{1-\nu}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial x}\right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial W}{\partial x}\frac{\partial W}{\partial y} + 2\frac{\partial v}{\partial x}\frac{\partial W}{\partial x}\frac{\partial W}{\partial y}\right]\right]\Bigg\} dS. \quad (3)$$

The kinetic energy of the plate, T, is given in Reference [12] by:

$$T = \frac{\rho h}{2}\int_S \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial W}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2, \quad (4)$$

where rotatory inertia is neglected.

The problem formulated using the energy methods assuming in-plane displacements u, v, and transversal displacement W are rather difficult and computation time is more important.

The method is based on the statement proofed experimentally in Reference [13] and used in Reference [12] that the membrane strains are constant along lines parallel to the edges of the rectangular plate. This assumption can be applied for the skew plate. The in-plane boundary conditions allowed the simplification of the in-plane displacement; consequently, only transverse displacement W was used. Therefore, it can be possible to determine the normal membrane strains by considering their average value. Two techniques are presented below using the averaged in plane strains for the skew plate problem.

2.1 First technique

The main idea of the u–v–W averaged formulation is that the expression of the membrane strain energy $(V_m)_a$ are determined using the averaged normal non-linear membrane strains $(\varepsilon_x^m)_a$ and $(\varepsilon_y^m)_a$ at a point of coordinates (x_0, y_0) of the plate middle plane which the expression in Cartesian coordinate system are given as defined in Reference [12] by:

$$(\varepsilon_x^m)_a(y_0) = \frac{1}{a}\int_0^a \varepsilon_x^m dx. \tag{5}$$

And,

$$(\varepsilon_y^m)_a(x_0) = \frac{1}{b}\int_0^b \varepsilon_y^m dy. \tag{6}$$

Then,

$$(\varepsilon_x^m)_a(y_0) = \frac{1}{a}\left[\int_0^a \frac{\partial u}{\partial x}\bigg|_{y=y_0} dx + \frac{1}{2}\int_0^a \left(\frac{\partial W}{\partial x}\right)^2\bigg|_{y=y_0} dx\right]. \tag{7}$$

$$(\varepsilon_y^m)_a(x_0) = \frac{1}{b}\left[\int_0^b \frac{\partial v}{\partial y}\bigg|_{x=x_0} dy + \frac{1}{2}\int_0^b \left(\frac{\partial W}{\partial y}\right)^2\bigg|_{x=x_0} dy\right]. \tag{8}$$

Being concerned here only with immovable edge conditions, the first terms appearing in the above averaged membrane expressions vanish

$$\int_0^a \frac{\partial u}{\partial x}dx = \int_0^b \frac{\partial v}{\partial y}dy = 0. \tag{9}$$

$$\gamma_{xy} \approx \frac{\partial W}{\partial x}\frac{\partial W}{\partial y}. \tag{10}$$

Thus, the expression for the averaged membrane strain energy $(V_m)_a$ is given in Reference [12] by:

$$\begin{aligned}(V_m)_a = \frac{Eh}{2(1-\nu^2)}\Bigg\{ & \frac{1}{4a}\int_0^b\left[\int_0^a \left(\frac{\partial W}{\partial x}\right)^2\bigg|_{y=y_0} dx\right]^2 dy \\ & + \frac{1}{4b}\int_0^a\left[\int_0^b \left(\frac{\partial W}{\partial y}\right)^2\bigg|_{x=x_0} dy\right]^2 dx \\ & + \frac{2\nu}{4ab}\int_0^b\left[\int_0^a \left(\frac{\partial W}{\partial x}\right)^2\bigg|_{y=y_0} dx\right]dy \\ & \int_0^a\left[\int_0^b \left(\frac{\partial W}{\partial y}\right)^2\bigg|_{x=x_0} dy\right]dx \\ & + \frac{1-\nu}{2}\int_0^a\int_0^b \left(\frac{\partial W}{\partial x}\right)^2\left(\frac{\partial W}{\partial y}\right)^2 dxdy\Bigg\}.\end{aligned} \tag{11}$$

The skew co-ordinates are related to the rectangular co-ordinate (ξ, η) by: $\xi = x - y\ tan\ \theta;\ \eta = y/cos\theta$. So, the strain energy due to bending V_f, the averaged membrane strain energy $(V_m)_a$ and the kinetic energy of the plate, T are given in the ξ–η co-ordinate system by:

$$\begin{aligned}V_f = \frac{D}{2}\int_A \Bigg[& \frac{1}{cos^4\theta}\left(\frac{\partial^2 w}{\partial^2} + \frac{\partial^2 \xi w}{\partial \eta^2} - 2sin\theta\frac{\partial^2 w}{\partial\xi\,\partial\eta}\right)^2 \\ & + \frac{2(1-\nu)}{cos^2\theta}\left(\left(\frac{\partial^2 w}{\partial\,\xi\partial\eta}\right)^2 - \left(\frac{\partial^2 w}{\partial\xi^2}\frac{\partial^2 w}{\partial\eta^2}\right)\right)\Bigg] cos\theta\ d\xi d\eta,\end{aligned} \tag{12}$$

$$(V_m)_a = \frac{Eh}{2(1-\nu^2)}\left[V_{1m} + V_{2m} + V_{3m} + V_{4m}\right], \tag{13}$$

where:

$$V_{1m} = \frac{1}{4a}\int_0^b\left[\int_0^a \left(\frac{\partial W}{\partial \xi}\right)^2\bigg|_{\eta=\eta_0} d\xi\right]^2 cos\theta d\eta, \tag{14}$$

$$V_{2m} = \frac{1}{4b}\int_0^a\left[\int_0^b \left(-tan\theta\frac{\partial W}{\partial \xi} + \frac{1}{cos\theta}\frac{\partial W}{\partial \eta}\right)^2\bigg|_{\xi=\xi_0} cos\theta d\eta\right]^2 d\xi, \tag{15}$$

$$\begin{aligned}V_{3m} = \frac{2\nu}{4ab}& \int_0^b\left[\int_0^a \left(\frac{\partial W}{\partial \xi}\right)^2\bigg|_{\eta=\eta_0} d\xi\right]cos\theta d\eta \\ & \int_0^a\left[\int_0^b \left(-tan\theta\frac{\partial W}{\partial \xi} + \frac{1}{cos\theta}\frac{\partial W}{\partial \eta}\right)^2\bigg|_{\xi=\xi_0} cos\theta d\eta\right]d\xi,\end{aligned} \tag{16}$$

$$V_{4m} = \frac{1-\nu}{2}\int_0^a\int_0^b \left(\frac{\partial W}{\partial \xi}\right)^2 \left(-tan\theta\frac{\partial W}{\partial \xi} + \frac{1}{cos\theta}\frac{\partial W}{\partial \eta}\right)^2 d\xi cos\theta d\eta. \quad (17)$$

On the other hand, if the rotatory inertia as well as the in plane inertia terms are neglected, the expression for the kinetic energy, T, of the skew plate reduces to:

$$T = \frac{\rho h}{2}\int_S \left(\frac{\partial W}{\partial t}\right)^2 d\xi cos\theta d\eta, \quad (18)$$

where D is the bending stiffness, given by: $D = EH^3/12(1-\nu^2)$, and W (ξ, η, t) is the transverse displacement function, ρ is the mass per unit volume of the plate material. This assumption is valid for slander beams and thin plates [7]. The analysis performed in the present work was made using plate functions obtained as products of clamped–clamped beam functions in the x and y directions in the manner detailed in Reference [7] for rectangular plates. If the time and space functions are supposed to be separated and harmonic motion is assumed, the transverse displacement can be written as:

$$W(\xi,\eta,t) = W(\xi,\eta)sin\omega t, \quad (19)$$

where (ξ, η) is the vector co-ordinate of the current point of the plate midplane. After expansion of W(ξ, η) in the form of a finite series:

$$W(\xi,\eta) = a_i W_i(\xi,\eta). \quad (20)$$

In which the usual summation convention is used, the discretization of the strain energy expression, V_b, can be carried out. Using matrix notation, Eq. 19 can be written as:

$$W(\xi,\eta,t) = \{A\}^T\{W\}\sin(\omega t). \quad (21)$$

where $\mathbf{\{W\}^T=[W_1\ (\xi,\eta)\ W_2\ (\xi,\eta)...W_n\ (\xi,\eta)]}$ is the basic spatial function vector depending on the current midplane point co-ordinate (ξ, η) and $\mathbf{\{A\}^T = [a_1\ a_2 \cdot\cdot a_n]}$ is the vector of contribution coefficients. The discretization of the total strain and kinetic energies are respectively given in Reference [7] by:

$$V_b = \frac{1}{2}sin^2(\omega t)a_i a_j k_{ij}, \quad (22)$$

$$V_a = \frac{1}{2}sin^4(\omega t)a_i a_j a_k a_l b_{ijkl}, \quad (23)$$

$$T = \frac{1}{2}\omega^2 cos^2(\omega t)a_i a_j m_{ij}. \quad (24)$$

In which $\mathbf{k_{ij}}$ is the general term of the rigidity tensor, $\mathbf{b_{ijkl}}$ is the fourth order tensor and $\mathbf{m_{ij}}$ the mass matrix.

The non-dimensional formulation of the non-linear vibration problem has been carried out as follows:

$$W_i(\xi,\eta) = HW_i^*\left(\frac{\xi}{a},\frac{\eta}{b}\right) = HW_i^*(\xi^*,\eta^*). \quad (25)$$

where ξ^* and η^* are non-dimensional co-ordinates $\xi^* = \xi/a$ and $\eta^* = \eta/b$. One then obtains:

$$k_{ij} = \frac{D\cdot H^2\cdot a}{cos^3\theta\cdot b^3}k_{ij}^*, \quad (26)$$

$$b_{ijkl} = \frac{D\cdot H^2\cdot a}{cos^3\theta\cdot b^3}b_{ijkl}^*, \quad (27)$$

$$m_{ij} = \rho H^3\cdot a\cdot b\cdot cos\theta\cdot m_{ij}^*. \quad (28)$$

The dynamic behavior of the isotropic skew plate is governed by Hamilton's principle which can be symbolically written as:

$$\partial\int_0^{2\pi/\omega}(V-T) = 0. \quad (29)$$

This leads, after substituting Eqs. (22, 23, 24) into Eq. 29 and integrating the time functions over a period of vibration, to the following set of non linear algebraic equations:

$$2a_i k_{ir} + 3a_i a_j a_k b_{ijkr} - 2\omega^2 a_i m_{ir} = 0, \quad r = 1\ldots n, \quad (30)$$

which may also be written, using non-dimensional parameters, as:

$$2a_i k_{ir}^* + 3a_i a_j a_k b_{ijkr}^* - 2\omega^{*2} a_i m_{ir}^* = 0, \quad r = 1\ldots n, \quad (31)$$

where: ω^{*2} is the non-dimensional frequency parameter, which can be obtained by pre-multiplying Eq. 27 by $\mathbf{\{A\}^T}$ from the left-hand side, which leads to the following equation:

$$\omega^{*2} = \frac{a_i a_j k_{ij}^* + \frac{3}{2}a_i a_j a_k a_l b_{ijkl}^*}{a_i a_j m_{ij}^*}. \quad (32)$$

Substituting Eqs. (26, 27, 28) into Eq. (30), one obtains:

$$\omega^2 = \frac{D}{\rho\cdot b^4\cdot cos^4\theta}\omega^{*2}. \quad (33)$$

2.2 Second technique

In this part, the W-formulation developed in Reference [12] and consisting of using the transverse displacement W is used here to determine the averaged membrane strains, and then the associated membrane stresses.

Following the assumption used in Reference [12] specifically the classical thin plate assumption of plane stress state and Hooke's law, the expressions of the surface normal flexural stresses, and the averaged normal membrane stresses presented in Reference [12] were given by:

$$\sigma_{xb}^{f} = \frac{EH}{2(1-\nu^2)}\left(\frac{\partial^2 W}{\partial x^2} + \nu\left(\frac{\partial^2 W}{\partial y^2}\right)\right), \tag{34}$$

$$\sigma_{yb}^{f} = \frac{EH}{2(1-\nu^2)}\left(\frac{\partial^2 W}{\partial y^2} + \nu\left(\frac{\partial^2 W}{\partial x^2}\right)\right), \tag{35}$$

$$\sigma_{xa}^{m}(x_0, y_0) = \frac{E}{2(1-\nu^2)}\left[\frac{1}{a}\int_0^a \left(\frac{\partial W}{\partial x}\right)^2\Big|_{y=y_0} dx + \frac{\nu}{b}\int_0^b \left(\frac{\partial W}{\partial y}\right)^2\Big|_{x=x_0} dy\right], \tag{36}$$

$$\sigma_{ya}^{m}(x_0, y_0) = \frac{E}{2(1-\nu^2)}\left[\frac{1}{b}\int_0^b \left(\frac{\partial W}{\partial y}\right)^2\Big|_{x=x_0} dy + \frac{\nu}{a}\int_0^a \left(\frac{\partial W}{\partial x}\right)^2\Big|_{y=y_0} dx\right]. \tag{37}$$

2.3 Bending stress analysis

The bending stress is considered here, which allows a qualitative understanding of the non-linearity effects. The maximum bending strains $\varepsilon_{\xi b}$ and $\varepsilon_{\eta b}$ obtained are given by:

$$\varepsilon_{\xi b} = \frac{H}{2}\left(\frac{\partial^2 W}{\partial \xi^2}\right), \tag{38}$$

$$\varepsilon_{\xi b} = \frac{H}{2}\left(\frac{\partial^2 W}{\partial \eta^2}\frac{1}{cos\theta} - \frac{2tan\theta}{cos\theta}\frac{\partial^2 W}{\partial \xi \partial \eta} + \frac{\partial^2 W}{\partial \xi^2}tan^2\theta\right). \tag{39}$$

For the skew plate, the stresses $\sigma_{\xi b}$ and $\sigma_{\eta b}$ can be obtained by:

$$\sigma_{\xi b} = \frac{EH}{2(1-\nu^2)}\left(\frac{\partial^2 W}{\partial \xi^2} + \nu\left(\frac{\partial^2 W}{\partial \eta^2}\frac{1}{cos\theta} - \frac{2tan\theta}{cos\theta}\frac{\partial^2 W}{\partial \xi \partial \eta} + \frac{\partial^2 W}{\partial \xi^2}tan^2\theta\right)\right). \tag{40}$$

$$\sigma_{\eta b} = \frac{EH}{2(1-\nu^2)}\left(\frac{\partial^2 W}{\partial \eta^2}\frac{1}{cos\theta} - \frac{2tan\theta}{cos\theta}\frac{\partial^2 W}{\partial \xi \partial \eta} + \frac{\partial^2 W}{\partial \xi^2}tan^2\theta + \nu\frac{\partial^2 W}{\partial \xi^2}\right). \tag{41}$$

$$\sigma_{\xi a}^{m}(\xi_0, \eta_0) = \frac{E}{2(1-\nu^2)}\left[\frac{1}{a}\int_0^a \left(\frac{\partial^2 W}{\partial \xi^2}\right)^2\Big|_{\eta=\eta_0} d\xi + \frac{\nu}{b}\int_0^b \left(\frac{\partial^2 W}{\partial \eta^2}\frac{1}{cos\theta} - \frac{2tan\theta}{cos\theta}\frac{\partial^2 W}{\partial \xi \partial \eta} + \frac{\partial^2 W}{\partial \xi^2}tan^2\theta\right)^2\Big|_{\xi=\xi_0} cos\theta d\eta\right]. \tag{42}$$

$$\sigma_{\eta a}^{m}(\xi_0, \eta_0) = \frac{E}{2(1-\nu^2)}\left[\frac{1}{b}\int_0^b \left(\frac{\partial^2 W}{\partial \eta^2}\frac{1}{cos\theta} - \frac{2tan\theta}{cos\theta}\frac{\partial^2 W}{\partial \xi \partial \eta} + \frac{\partial^2 W}{\partial \xi^2}tan^2\theta\right)^2\Big|_{\xi=\xi_0} cos\theta d\eta + \frac{\nu}{a}\int_0^a \left(\frac{\partial^2 W}{\partial \xi^2}\right)^2\Big|_{\eta=\eta_0} d\xi\right]. \tag{43}$$

In terms of the non-dimensional parameters defined in previous section, non- dimensional stresses $\sigma_{\xi b}^{*}$ and $\sigma_{\eta b}^{*}$ can be defined by:

$$\sigma_{\xi b}^{*} = \left(^2\frac{\partial^2 W^*}{\partial \xi^{*2}} + \nu\left(\frac{\partial^2 W^*}{\partial \eta^{*2}}\frac{1}{cos\theta} - \alpha\frac{2tan\theta}{cos\theta}\frac{\partial^2 W^*}{\partial \xi^* \partial \eta^*} + \alpha^2\frac{\partial^2 W^*}{\partial \xi^{*2}}tan^2\theta\right)\right). \tag{44}$$

$$\sigma_{\eta b}^{*} = \left(\frac{\partial^2 W^*}{\partial \xi^{*2}}\frac{1}{cos\theta} - \alpha\frac{2tan\theta}{cos\theta}\frac{\partial^2 W^*}{\partial \xi^* \partial \eta^*} + \alpha^2\frac{\partial^2 W^*}{\partial \xi^{*2}}tan^2\theta + \nu\alpha^2\frac{\partial^2 W^*}{\partial \xi^{*2}}\right). \tag{45}$$

And the non- dimensional membrane stresses $\sigma_{\xi a}^{*m}$ and $\sigma_{\eta b}^{*m}$ can be given by:

$$\sigma_{\xi a}^{*m}(\xi_0^*, \eta_0^*) = \left[\alpha^4\int_0^1 \left(\frac{\partial^2 W^*}{\partial \xi^{*2}}\right)^2\Big|_{\eta^*=\eta_0^*} d\xi^* + \nu\int_0^1 \left(\frac{\partial^2 W^*}{\partial \eta^{*2}}\frac{1}{\cos\theta} - \frac{2\alpha\tan\theta}{\cos\theta}\frac{\partial^2 W^*}{\partial \xi^* \partial \eta^*} + \alpha^2\frac{\partial^2 W^*}{\partial \xi^{*2}}\tan^2\theta\right)^2\Big|_{\xi^*=\xi_0^*} \cos\theta d\eta^*\right]. \tag{46}$$

$$\sigma_{\eta a}^{*m}(\xi_0, \eta_0) = \left[\int_0^1 \left(\frac{\partial^2 W^*}{\partial \eta^{*2}}\frac{1}{\cos\theta} - \frac{2\alpha\tan\theta}{\cos\theta}\frac{\partial^2 W^*}{\partial \xi^* \partial \eta^*} + \alpha^2\frac{\partial^2 W^*}{\partial \xi^{*2}}\tan^2\theta\right)^2\Big|_{\xi^*={\xi^*}_0} \times \cos\theta d\eta^* + \nu\alpha^4\int_0^1 \left(\frac{\partial^2 W^*}{\partial \xi^{*2}}\right)^2\Big|_{\eta^*={\eta^*}_0} d\xi^*\right]. \tag{47}$$

The relationships between the dimensional and non dimensional stresses are:

$$\sigma = \frac{EH^2}{2(1-\nu^2)b^2}\sigma^*. \quad (48)$$

$$\sigma^m = \frac{EH^2}{2(1-\nu^2)b^4}\sigma^{*m}. \quad (49)$$

3 Numerical results and discussion

In this section, the results and curves obtained by applying the two averaged techniques mentioned above to a fully clamped isotropic skew plate are presented and discussed for various values of the skew angle and aspect ratio. Material properties considered for the isotropic skew plate are: Young's modulus of elasticity $E = 200$ GPa, Poisson's ratio $\nu = 0.33$, and material density $\rho = 7850$ Kg/m^3.

Non displacement and slope are considered along the clamped edges and product of beam functions are used as an appropriate mode shape. A system of non-linear vibrations equations of skew plate obtained using Hamilton's principle and integration of the time functions over the range [0, $2\pi/\omega$]. These equations are a set of non-linear algebraic equations, involving the parameters $\mathbf{m}^*_{\mathbf{ij}}$, $\mathbf{k}^*_{\mathbf{ij}}$ and $\mathbf{b}^*_{\mathbf{ijkl}}$ which have been computed numerically by a routine called PREP [7]. In order to obtain the numerical solution for the nonlinear problem in the neighborhood of a given mode shape, the contribution of the dominant function participating in this mode was chosen and those of the other functions were calculated numerically.

Non linear frequency parameters for fully clamped square plate and for various vibration amplitudes are presented in Table 1 using the u–v–W-average formulation. Comparison was made with the W-average formulation and previous results present in the literature taken from [12, 14]. Good agreement can be seen between those results.

The membrane stress distribution associated to a fully clamped skew plate has been determined and plotted using the u–v–W-average formulation. Also the surface flexural stress distribution was calculated with this technique. Comparisons with previous results for the square plate case are made in order to prove the convergence of the model and a good agreement were found. In Fig. 2, the membrane stresses along the section $\xi^* = 0.5$ of a fully clamped square plate for a non-dimensional maximum amplitude $W^*_{max} = 2$ is shown and the present results and the results from Reference [14] are depicted. It can be seen clearly that in the center of the plate the membrane stress values obtained previously and those obtained here are very close while the difference

Table 1 Non linear frequency parameters for a fully clamped square plate and for various vibrations amplitudes, comparison between the results obtained here and those previously obtained in the literature

References	Averaged [7]	Benamar et al. [7]	Han and Petyt [14] with u and v (P0 = 5. π = 6)	Present work	Error %
$W^*_{(0.5,0.5)}$					
0.2	1.0083	1.006	1.0072	1.007	0.02
0.4	1.0283	1.027	1.0285	1.027	0.09
0.6	1.061	1.060	1.0631	1.061	0.03
0.8	1.106	1.104	1.1095	1.105	0.14
1.0	1.1591	1.159	1.1663	1.160	0.15
1.5	1.3423	1.335	1.3442	1.323	1.32

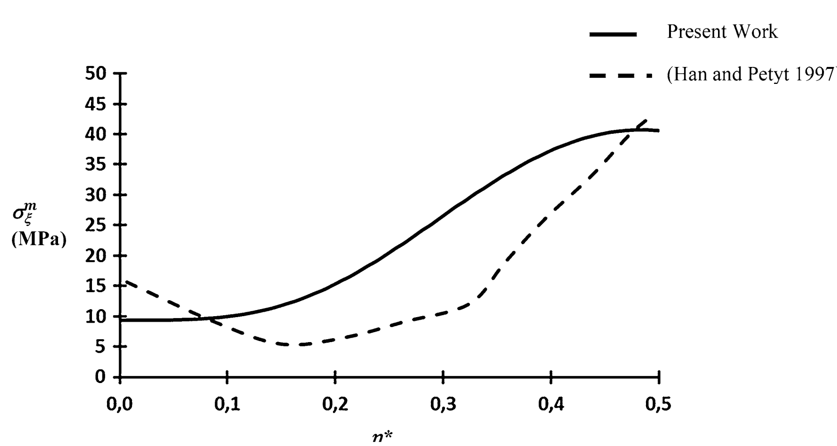


Fig. 2 Comparison of the membrane stresse σ^m_ξ; σ^m_η along the section $\xi^* = 0.5$ of a fully clamped skew plate for $\theta = 0°$, $\alpha = 1$ and $W^*_{max} = 2$

Table 2 Convergence study of fundamental frequency parameter for a fully clamped skew plate

Aspect ratio α	References	Fundamental parameter for different values of skew angle θ (°)					
		15 (°)	20 (°)	30 (°)	35 (°)	45 (°)	60 (°)
1	[15]	35.636	35.376	34.624	34.172	–	–
1	[16]	35.625	–	34.788	–	32.795	30.323
1	Present work	35.635	35.364	34.607	34.123	32.964	30.844
0.5	[15]	24.484	24.388	24.196	24.096	–	–
0.5	Present work	24.471	24.388	24.165	24.027	23.709	23.186

Table 3 Non linear frequency parameters for fully clamped skew plates for various values of the vibration amplitude, for an aspect ratio $\alpha=1$ and for different values of skew angle θ

$\theta°$	0°	15°	30°	45°
$W^*_{(0.5,0.5)}$				
0.2	1.007	1.008	1.013	1.025
0.4	1.027	1.032	1.048	1.100
0.6	1.061	1.069	1.107	1.222
0.8	1.105	1.119	1.182	1.380
1.0	1.160	1.179	1.278	1.590
1.5	1.323	1.359	1.532	2.236

Table 4 Non linear frequency parameters for fully clamped skew plate for various vibrations amplitudes, for an aspect ratio $\alpha=0.6$ and for different values of skew angle θ

$\theta°$	0°	15°	30°
$W^*_{(0.5,0.5)}$			
0.2	1.007	1.008	1.013
0.4	1.029	1.033	1.051
0.6	1.065	1.074	1.115
0.8	1.113	1.129	1.198
1.0	1.181	1.205	1.312
1.5	1.354	1.397	1.580

seen near to the clamps may be attributed to the averaging assumption.

In order to verify the accuracy of the results obtained in the present work, comparisons have been made with some available results obtained in References [15, 16]. Table 2 shows the comparison of the fundamental parameter $\omega b^2\sqrt{\frac{\rho}{D}}\cos^2\theta$ for a fully clamped skew plate for different values of skew angle θ.

Tables 3 and 4 present respectively non linear frequency parameters for fully clamped skew plate for various vibration amplitudes and different skew angle for aspect ratio $\alpha=1$ and 0.6. The non linear frequency increase with skew angle and aspect ratio.

Figures 2 and 3 show the membrane stresses σ^m_ξ; σ^m_η along the section $\xi^*=0.5$ of a fully clamped skew plate for $\theta=0°$, $\alpha=1$ and $W^*_{max}=2$ calculated by using the *u*–*v*–*W*-average formulation and compared with those given Reference [14] based on the finite element method. It can be seen that this technique gives a satisfactory results.

Figures 4 and 5 show the non dimensional membrane stresses σ^m_ξ; σ^m_η along the section $\xi^*=0.5$ of a fully clamped skew plate for different skew angles: $\theta=0°$; 15°; 30° and 45°, aspect ratio $\alpha=1$ and non dimensional maximal amplitude $W^*_{max}=2$ calculated by using the *u*–*v*–*W*-average formulation. The increasing in the values of the skew angle induces an increasing in the values of the membrane stresses.

Figures 6 and 7 shows the surface flexural stresses σ^f_ξ; σ^f_η along the section $\xi^*=0.5$ of a fully clamped skew plate for $\theta=0°$, $\alpha=1$ and $W^*_{max}=2$ calculated by using the *u*–*v*–*W*-average formulation and compared with Reference [14] based

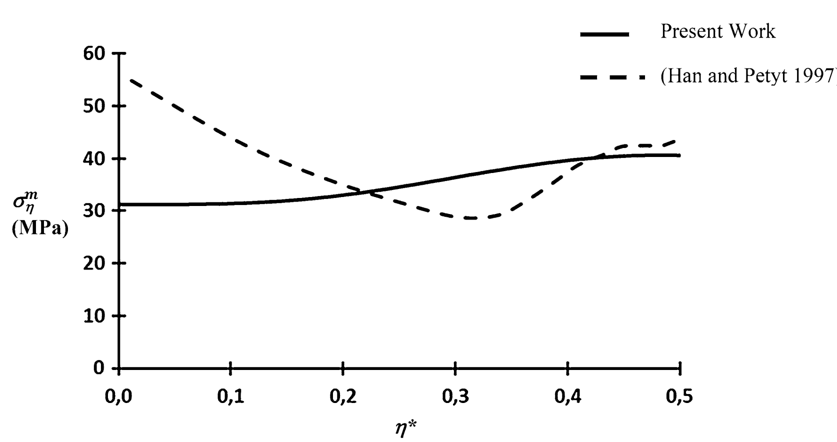


Fig. 3 Comparison of the membrane stress σ^m_ξ along the section $\xi^*=0.5$ of a fully clamped skew plate for $\theta=0°$, $\alpha=1$ and $W^*_{max}=2$

Fig. 4 Comparison of the membrane stress σ_{ξ}^{m*} along the section $\xi^* = 0.5$ of a fully clamped skew plate for $\theta = 0°$; 15°; 30° and 45°, $\alpha = 1$ and $W^*_{max} = 2$

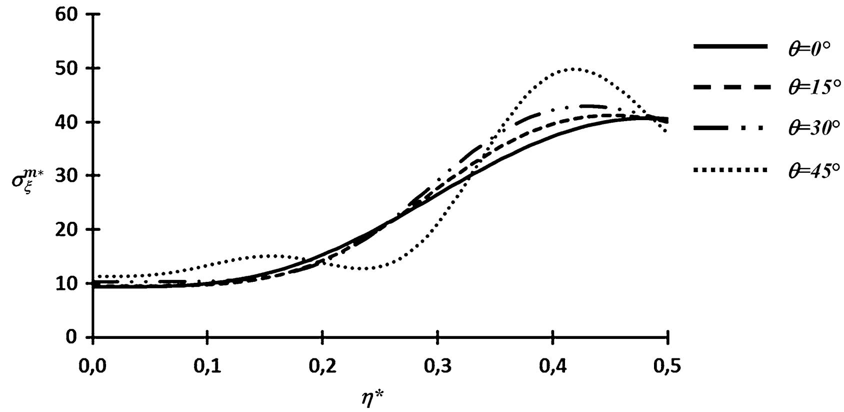


Fig. 5 Comparison of the membrane stress σ_{η}^{m*} along the section $\xi^* = 0.5$ of a fully clamped skew plate for $\theta = 0°$; 15°; 30° and 45°, $\alpha = 1$ and $W^*_{max} = 2$

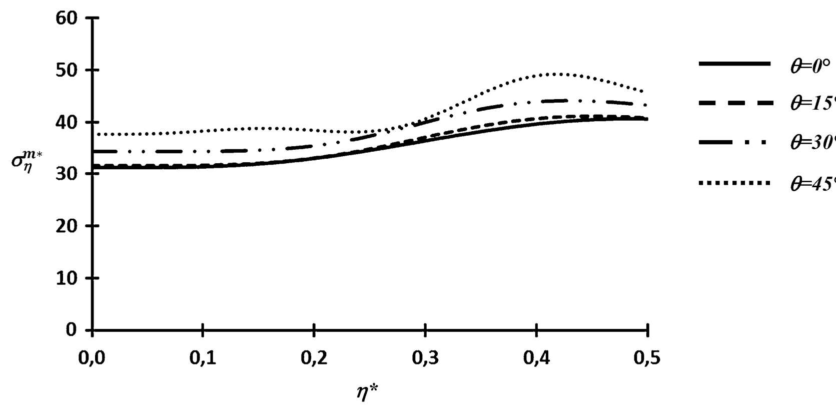


Fig. 6 Comparison of the surface flexural stress σ_{ξ}^{m} along the section $\xi^* = 0.5$ of a fully clamped skew plate for $\theta = 0°$, $\alpha = 1$ and $W^*_{max} = 2$

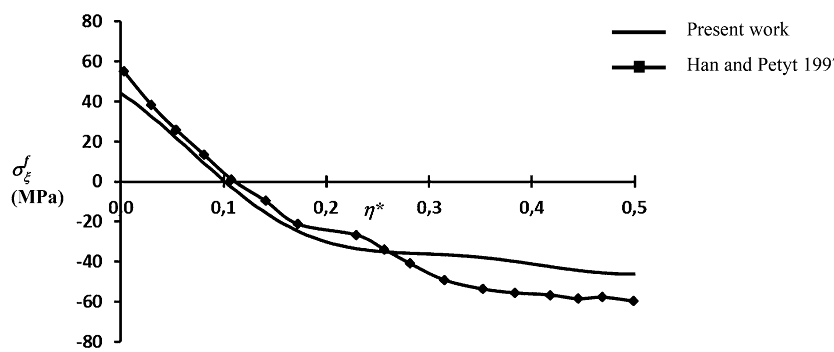
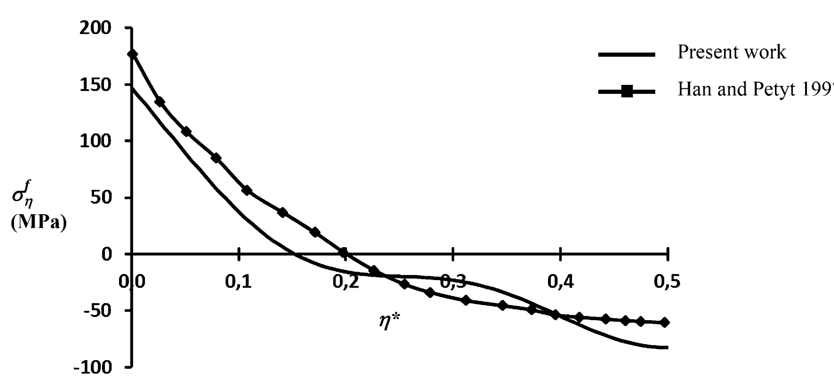


Fig. 7 Comparison of the surface flexural stress σ_{η}^{m} along the section $\xi^* = 0.5$ of a fully clamped skew plate for $\theta = 0°$, $\alpha = 1$ and $W^*_{max} = 2$



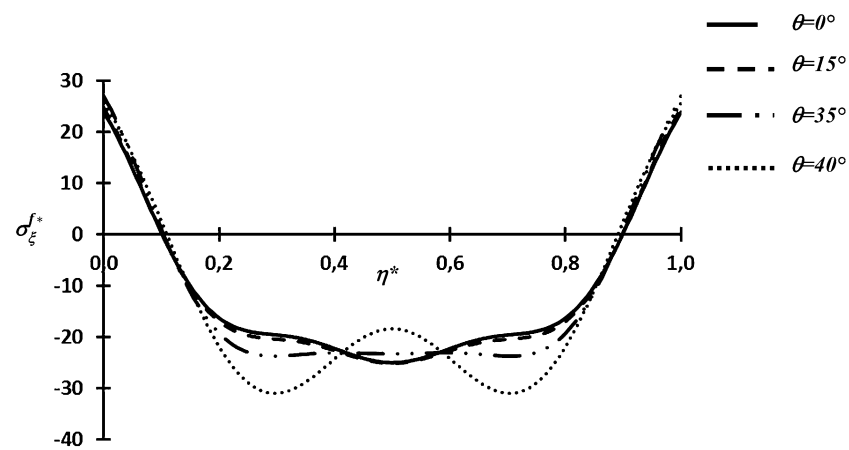


Fig. 8 Comparison of the surface flexural stress σ_{ξ}^{f*} along the section $\xi^* = 0.5$ of a fully clamped skew plate for $\theta = 0°$; 15°; 35° and 40°, $\alpha = 1$ and $W_{max}^* = 2$

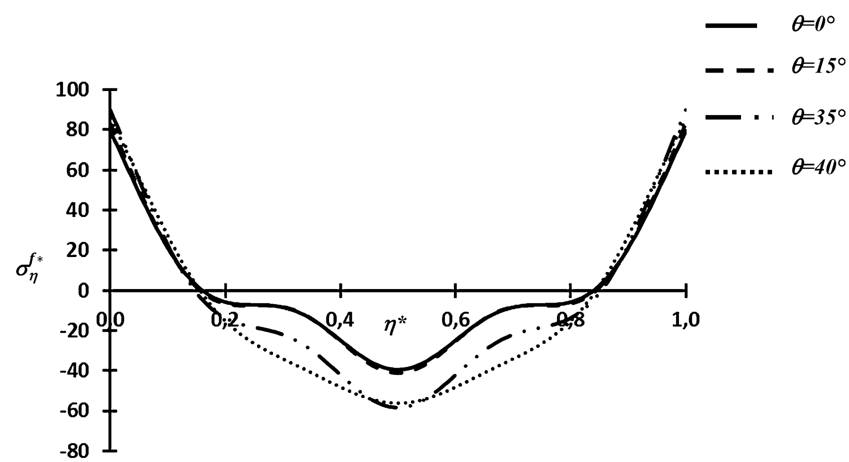


Fig. 9 Comparison of the surface flexural stress σ_{η}^{f*} along the section $\xi^* = 0.5$ of a fully clamped skew plate for $\theta = 0°$; 15°; 35° and 40°, α = 1 and $W_{max}^* = 2$

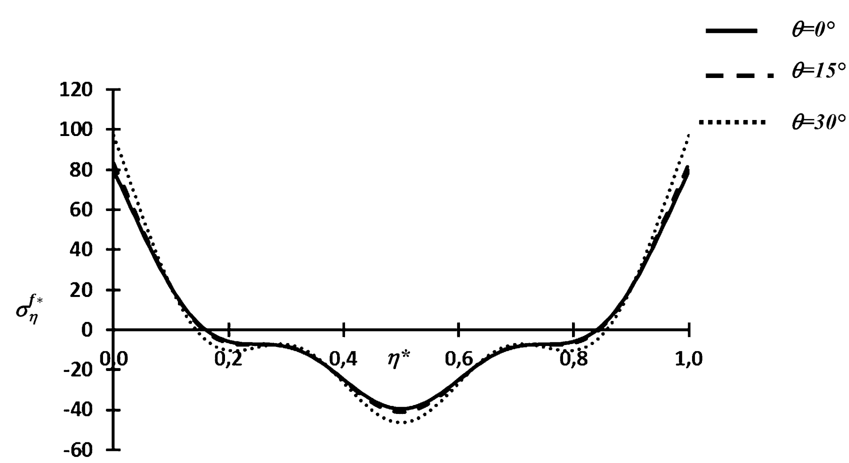


Fig. 10 Comparison of the surface flexural stress σ_{η}^{f*} along the section $\xi^* = 0.5$ of a fully clamped skew plate for $\theta = 0°$; 15° and 30°, $\alpha = 0.6$ and $W_{max}^* = 2$

on the finite element method. Good agreement between the results was achieved.

Figures 8 and 9 show the non dimensional surface flexural stresses σ_{ξ}^{f*}; σ_{η}^{f*} along the section $\xi^* = 0.5$ of a fully clamped skew plate for different skew angles : $\theta = 0°$; 15°; 35° and 40°, aspect ratio $\alpha = 1$ and non dimensional maximal amplitude $W_{max}^* = 2$ calculated by using the u–v–W-average formulation. An increasing in the values of the surface flexural stresses can be noticed with increasing the value of the skew angle θ. Beyond the value of 40°, a kind of instability is observed in the variation of the flexural stress distribution.

Figure 10 present the non dimensional surface flexural stresses σ_{η}^{f*} along the section $\xi^{*}=0.5$ of a fully clamped skew plate for different skew angles: $\theta=0°$; 15° and 30°, aspect ratio $\alpha=0.6$ and non dimensional maximal amplitude $W_{max}^{*}=2$ calculated by using the *u*–*v*–*W*-average formulation. It can be seen that the aspect ratio has a limited influence on the values of the surface flexural stresses.

4 Conclusion

An averaging method was proposed for analyzing the non-linear vibration of fully clamped skew plate. The membrane stress distribution and the flexural stress distribution have been determined using the assumption, according to which the in plane membrane strains are constant over lines parallel to the plate with immovable edges at large deformation amplitudes. Results obtained for different values of the plate skew angle and aspect ratio showed an increasing in the membrane and surface flexural stresses values with increasing the skew angle between 0° and 45°. It can be seen also that increasing the skew angle induces a more accentuated hardening effect. Comparison in the rectangular plate case with the previous results available shows a satisfactory agreement.

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