ORIGINAL ARTICLE

Constraints in structural and rigid body mechanics: a frictional contact problem

S. R. Eugster · Ch. Glocker

Abstract In this work, we propose some basic approaches towards a unification of the theories for deformable and rigid bodies. This unification process is based on two fundamental mechanical concepts, which are the principle of virtual work and the principle of d'Alembert–Lagrange. The basic idea is to initially look upon structural elements as general continua, and to endow them later with specific properties like rigidity, imposed by perfect bilateral constraints. It is shown by the example of a simple flexible multibody system how this unifying and systematic approach has to be carried out. The system under consideration consists of a rigid disk and a nonlinear elastic string, which may come into contact with each other. The contact is modeled as a hard unilateral geometric constraint combined with a one-dimensional Coulomb friction element. The contact interactions are formulated as set-valued force laws and impact laws, and the system is consequently treated within the framework of nonsmooth dynamics. The model of the string allows for large deformations in time and for a nonlinear elastic material response. By constraining the kinematics of the string to finite dimensions, a nonlinear finite element formulation is achieved in a very natural way.

keywords Virtual work · Constraints · Contact · Friction · Nonsmooth dynamics · Elastic strings

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1 Introduction

The objective of this paper is to demonstrate the potential of constraints to serve as one of the key concepts in a unified theory for the mechanics of deformable and rigid bodies. Such a unification is especially desired for flexible multibody systems. In such systems, different types of solids as e.g. beams, shells, or rigid bodies interact with each other. A consistent theory has to rely on the very same fundamental principles for the whole mechanical system. The authors claim that these are the principle of virtual work, formulated for a set of material points, and the concept of constraints. The ideas of such a unified formulation are introduced in and carried out by a planar problem, where a rigid disk comes into contact with a nonlinear elastic string. The system under consideration therefore consists of two different structural elements, a rigid body and a deformable string.

In the spirit of analytical dynamics, the equations of motion for the entire mechanical system are derived from only and exclusively the principle of virtual work, but not from any balance laws as proposed by an increasing number of individuals in the recent past. The authors think that the principle of virtual work is the only acceptable and general approach to mechanics. This opinion is supported by many arguments, from which at least three should explicitly be mentioned here: In the mechanics of continua, the weak form of the balance laws is nowadays a fully accepted concept, used to perform existence and uniqueness proofs on the one hand, and to develop numerical schemes on the other hand. Precisely this concept has to be introduced artificially when starting off with the balance laws. In contrast, it is intrinsically anchored in the virtual work approach, with all of its variants. A second reason to favor the virtual work approach over the balance laws is

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that it provides the only possibility within classical mechanics to mathematically describe what perfect bilateral constraints are. The latter is done in form of a variational equality, known as the principle of d'Alembert–Lagrange, which puts the constraint forces into the annihilator space of the admissible virtual displacements. The concept of perfect constraints is omnipresent in each branch of mechanics, even in the numerical schemes as discussed later, and is strongly connected with all the results and statements obtained from the weak form of the balance equations. As a third reason, one may look at the mathematical definition of forces. Forces are linear functionals. They are 1-forms which have to be fed by primal entities to deliver a certain value as a result. The primal entities are the virtual displacements, and the value is the virtual work. Forces are abstract, and no man has ever seen them. The only way to access forces is to check them out by applying test displacements and to feel the result, which is again the virtual work.

Concerning the numerical schemes, one should be aware that the spatial discretization of continua by e.g. finite elements must not only be understood as a mathematical procedure to approximate the infinite-dimensional problem by a finite one. From the mechanical point of view, each and any reduction in the degrees of freedom is equivalent with the imposition of constraints. Finite elements have therefore to be regarded as (abstract) mechanisms, which restrict the system to move in only the directions permitted by them. The choice of these mechanisms together with all the constitutive laws belongs to the stage of modeling. As constraints are represented by constitutive laws, the selection of the particular finite element type and hence the numerical scheme has to be understood as part of the modeling process and therefore as part of the mechanical theory. Again, as the constitutive laws of constraints can only be expressed in variational form, the virtual work approach seems to be mandatory to capture them. A straightforward calculation of the latter will then naturally provide the well-known and well-suited numerical approaches. Furthermore, the virtual work approach is able to explicitly show the mechanical difference between a spatially discretized and a continuously formulated problem, if carefully applied. The transition from the infinite-dimensional to the finite-dimensional formulation goes along with the introduction of bilateral constraints together with their associated constraint forces, which have to be taken into account in the virtual work expression, at least initially. Only if these constraints are looked upon as perfect, the constraint forces will disappear from the equations by the projection performed when proceeding to admissible virtual displacements, as implicitly done in standard Galerkin finite element methods. Obviously, these forces are never mentioned or not even recognized when starting

from the strong form of the equations of motion, i.e. from the balance laws. From the above, one can see how much the numerical schemes are interwoven with the concept of perfect constraints and the issue of modeling.

According to the concepts described above, the following approach is chosen for the example of the rigid disk and the elastic string: Initially, both structural elements together are treated as one set of material points without any constraints. This is in particular noteworthy for the disk, on which the rigidity constraints are imposed later. The disk is formulated as a continuous body with infinitely many degrees of freedom and not overhasty considered as a finite-degree of freedom system. The principle of virtual work, formulated for the entire set of material points, serves then as the fundamental equation to describe the dynamics of the system. In this way, a variational formulation of the dynamics of the two structural elements together can be obtained from the very same fundamental principle. In a later step, perfect constraints are implicitly imposed on the system in the form of a constrained kinematics, which is described by a finite number of generalized coordinates. The kinematics for the material points forming the string results from two nested restrictions, which first turn the material points into a continuous string and then approximate it by a finite number of elements. Also the kinematics for the material points forming the disk follows from two restrictions and allows for general planar rigid body motion. In contrast to the string, the second restriction of the disk corresponds to the rigidity constraint and describes insofar the material law of a rigid body. The constraint forces occurring due to these restrictions are assumed to be perfect and consequently eliminated by the principle of d'Alembert–Lagrange. The contact between the disk and the string is modeled by unilateral frictional constraints. In contrast to the kinematic restrictions from above, these constraints are not perfect and can not be eliminated with the principle of d'Alembert–Lagrange. The contact forces are therefore explicitly considered in the equations of motion and equipped with their associated set-valued force laws. Hard frictional contacts may lead to discontinuities in the accelerations and velocities of the system, caused by the occurring slip-stick transitions and impacts. Impulsive motion is indeed observed in the discretized system, which is due to the condensed masses from the finite element approximation.

The mechanics of one-dimensional continua goes back to Jac. Bernoulli and Euler, see Antman [4] and Truesdell [41]. A vast amount of literature exists on the subsequent extensions, generalizations and new approaches. Short historical overviews are given in Antman [4] besides a detailed and extensive treatise of rod theories. Other treatments of one-dimensional structural elements can be found in Ballard and Millard [6], Rubin [38], Salençon

[39]. Internal (material) constraints play a crucial role within structural mechanics to derive desired rod theories. Internal constraints, as e.g. incompressibility constraints, restrict possible motions to those for which the constraint conditions are fulfilled. A treatment of internal mechanical constraints is given in Truesdell and Noll [42, Sect. 30]: Based on the assumption that the forces maintaining the constraints do no produce work, the "principle of determinism for simple materials subject to internal constraints" is formulated. The principle states that the stress can additively be split in a part determined by the deformation of the body, and a part which does not produce any work at any motion satisfying the constraints. The concept is successfully applied in Naghdi and Rubin [32] to derive several constrained theories of rods. According to Antman [3], Antman and Marlow [5], Rajagopal and Srinivasa [34], the treatment of internal constraints by Truesdell and Noll [42] is from the physical point of view not general enough. As an example, Rajagopal and Srinivasa [34] mentions the material law of an incompressible viscous fluid with pressure-dependent viscosity (cf. also Frémond et al. [18]). The stress state for such a material depends, in addition to the deformation of the body, on the pressure which is the constraint force guaranteeing the incompressibility condition. Exactly the same discussion arises in rigid body dynamics when frictional effects occur. In such Coulomb-type systems, constraint forces may not be workless as well. Following Glocker [19, Sect. 3.6], not the effective work but the virtual work due to virtual displacements is considered to treat constraints. The virtual displacements can be chosen compatible or incompatible to the constraints. The class of perfect bilateral constraint forces is then defined by the principle of d'Alembert–Lagrange as constraint forces which do not produce any virtual work at the whole system for compatible virtual displacements. In the current paper, we will introduce perfect bilateral internal constraints when spatially discretizing the system.

Constraints with friction, or the above mentioned viscous material model which can be interpreted as a material with Coulomb friction, do not belong to the class of perfect bilateral constraints and are mostly ignored in classical analytical mechanics. A wider class of constraints, also including unilateral constraints, is treated in the field of nonsmooth mechanics and finds their applications amongst others in contact mechanics, Acary and Brogliato [1], Frémond [17], Glocker [19], Jean [24], Leine and Nijmeijer [26], Moreau [28, 30], and material formulation, Frémond [16], Romano and Sacco [37]. The treatment of unilateral frictional contacts arises in the posed problem when the disk interacts with the string via a hard contact with Coulomb friction. The contact interactions are formulated as set-valued force laws, which are subsequently written as normal cone inclusions by using the theory developed in

convex and nonsmooth analysis (Glocker [19], Moreau [31], Rockafellar [36]). Static frictional contact problems for deformable bodies with hyperelastic material laws have been treated in the context of nonsmooth mechanics in Alart and Curnier [2]. A general theory which discusses the dynamics of nonsmooth systems in finite degree of freedom mechanics has already been developed in Glocker [19], Jean [24], Moreau [28].

Various formulations of frictional contact problems for one-dimensional continua can be found in applications. There are self contact problems of rods, Chamekh et al. [11], Goyal et al. [23], descriptions of the pantograph-catenary interaction, Cho [12], Collina and Bruni [13], Rauter et al. [35], simulations of belt drive systems, Dufva et al. [14], Kerkkänen et al. [25], Yi et al. [43] or prediction of hair dynamics, Bertails et al. [7], Bertails-Descoubes et al. [8]. Except of Bertails-Descoubes et al. [8], all works treat the contact problems by compliant models, i.e. soft contacts. In the paper at hand, the unilateral constraints are introduced as hard frictional contacts, described by set-valued force laws. For a discussion about the advantages and disadvantages of the contact formulations by hard or soft contacts the reader is referred to Brogliato et al. [10]. In this review article, further literature on the topic of contacts is found. A force which is described by a set-valued force law, as e.g. Coulomb's friction law, has mostly an ambivalent character. The force acts either as an impressed force, where a functional dependence between the kinematics and the force exists, or as a constraint force which takes arbitrary values to fulfill the constraint. In the case of Coulomb friction, the friction force during slip is an impressed force, and the friction force during sticking is a constraint force.

The paper is organized as follows. In Sect. 2 the equations of motion of the discretized system are derived from the principle of virtual work and the principle of d'Alembert–Lagrange. General assumptions on force distributions are taken, and the discretization of the string is shown for a shape function being nonlinear in its generalized coordinates. In Sect. 3, the constitutive equations for the assumed force distributions are specified. For the bilateral constraints and the contact forces, set-valued force laws are introduced which require an impact law to fully describe the dynamics of the system. In Sect. 4 snapshots of two different simulations are shown and discussed.

2 Problem description and equations of motion

As an example, the dynamical system depicted in Fig. 1 is analyzed. It consists of a homogenous disk and a string in the vertical plane. The disk is considered as a rigid body and is characterized by its radius R and its density ρ_D. The

string which is suspended at A and B, has an undeformed length L, mass density ρ_0 in its reference configuration, stiffness k and is perfectly flexible, i.e. has no bending stiffness. The contact between the disk and the string is modeled as a hard unilateral constraint with Coulomb type friction. At a collision time-instant, a Newton-type impact law is imposed. The disk and the string are subjected to gravity with gravitational acceleration $\mathbf{g}$. The mechanical system is considered as a set of material points $\mathscr{S}$ of which each material point is placed in a three-dimensional Euclidean space by a corresponding time dependent position vector $\xi(\cdot, t)$. The dot $(\cdot)$ stands for a specific material point. The vector field $\xi(\cdot, t)$ defines the motion of the system $\mathscr{S}$. Differentiability with respect to time for almost all t allows for the definition of a velocity and an acceleration vector field as $\dot{\xi}(\cdot, t) = \partial \xi(\cdot, t)/\partial t$ and $\ddot{\xi}(\cdot, t) = \partial^2 \xi(\cdot, t)/\partial t^2$, respectively. The mass dm of a material point, placed at ξ, is subjected to internal and external forces. The principle of virtual work states that if the virtual work

$$\delta W = \int_{\mathscr{S}} \delta \xi^{\mathrm{T}}(\ddot{\xi} dm - d\mathbf{F}) = 0 \quad \forall \delta \xi, \forall t \tag{1}$$

vanishes for all variations $\delta \xi$, then the system $\mathscr{S}$ is in dynamic equilibrium.

The system consists of two subsystems: The string $\mathscr{S}_1$ and the disk $\mathscr{S}_2$. The problem is planar and we neglect the third vector component in the further derivation. Subsequently, since the integral is additive, derivations of the kinematics and force distributions are done separately for each body and indexed by $(\bullet)_1$ for the string and $(\bullet)_2$ for the disk.

2.1 Kinematics

The string is modeled as a one-dimensional deformable body which means that each material point may be addressed by a parameter $s \in \mathscr{S}_1 = [0, L]$. With foresight to the numerical evaluation, the kinematics of the string is constrained in such a way that the corresponding position vector

$$\xi_1(s, t) = \mathbf{r}_1(s, \mathbf{q}_1(t)) \tag{2}$$

can be expressed by introducing finitely many generalized coordinates $\mathbf{q}_1(t)$. Within the assumed constrained kinematics of Eq. (2) two different constraints are merged. The first constraint is that every material point of the string is addressed by a continuous function ξ_1. If the problem was formulated with this constraint only, the
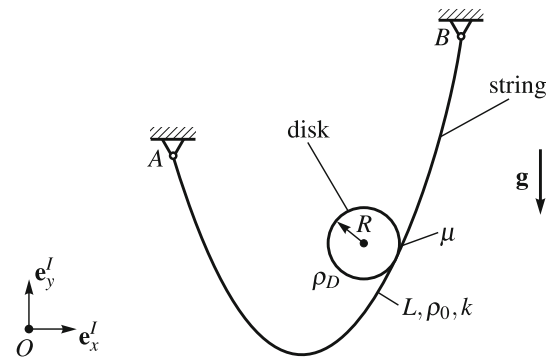
Fig. 1 In the dynamical system a rigid disk interacts with a suspended nonlinear elastic string by frictional contacts

dynamics of a continuous string would be described, as in classical continuum mechanics. Since a numerical evaluation of the system is desired, the second constraint corresponds to the spatial discretization of the string. The constraints enforce the position vectors of the string to remain on the curve described by the function $\mathbf{r}_1$. When the discretization of the string is understood as a constraint, it becomes obvious that a spatial discretization leads to additional forces in the string. Consequently the discretized formulation merely approximates the continuous formulation. The choice of generalized coordinates leads naturally together with the formulation of the virtual work of the system to the so-called finite element method. We speak of local finite elements if the string is divided in the sense of Biot [9]. Following Biot [9, p. 324], the system is divided into a certain number of cells, each of which is described by a small number of generalized coordinates, in such a way that interconnection constraints are satisfied. A cell, commonly called element and indexed by $(\bullet)^e$, is a region of the string $\Omega^e = [n^e, n^{e+1}] \subset [0, L] = \cup_{e=1}^{k_{el}} \Omega^e$. The material points at $s = n^e$ for $e = 1, \ldots, k_{el}$ are the nodes of the k_{el} numbers of elements. The kinematics of an element is described by the shape function $\mathbf{r}_1^e(s^e, \mathbf{q}_1^e(t))$. The parameter s^e (cf. Eq. (4)) is called element coordinate and takes values in the interval $[0,1]$. The connectivity matrix $\mathbf{C}_1^e$ of an element extracts the small number of generalized coordinates $\mathbf{q}_1^e = \mathbf{C}_1^e \mathbf{q}$ out of the generalized coordinates $\mathbf{q}(t)$ which describe the total system. With the choice of the shape functions $\mathbf{r}_1^e$ and the use of the characteristic function $\chi_{\Omega^e}(s)$, which equals unity inside and vanishes outside the region Ω^e, the motion of the string

$$\mathbf{r}_1(s, \mathbf{q}_1) = \sum_{e=1}^{k_{el}} \chi_{\Omega^e} \mathbf{r}_1^e(s^e, \mathbf{q}_1^e), \tag{3}$$

$$s^e = \frac{s - n^e}{n^{e+1} - n^e}, \qquad \mathbf{q}_1^e = \mathbf{C}_1^e \mathbf{q} \tag{4}$$

is discretized by k_{el} elements. The local shape functions have to be chosen in such a way that the motion $\mathbf{r}_1$ is at least continuous in s. This condition is asked for in standard, polynomial based, finite element analysis. Since the occurring kinks between the elements may corrupt the contact interaction, we strengthen this condition and ask for a twice differentiable C^2-function. A shape function satisfying this condition is e.g. a cubic Bézier spline of the form

$$
\begin{aligned}
\mathbf{r}_1^e(s^e, \mathbf{q}_1^e) &= \begin{pmatrix} x_e \\ y_e \end{pmatrix} + (s^e)^2 (3 - 2s^e) \begin{pmatrix} x_{e+1} - x_e \\ y_{e+1} - y_e \end{pmatrix} \\
&\quad + 3s^e(1 - s^e)^2 r_e \begin{pmatrix} \mathrm{c}(\varphi_e) \\ \mathrm{s}(\varphi_e) \end{pmatrix} \\
&\quad - 3(s^e)^2(1 - s^e) q_e \begin{pmatrix} \mathrm{c}(\varphi_{e+1}) \\ \mathrm{s}(\varphi_{e+1}) \end{pmatrix},
\end{aligned}
$$
$$\mathbf{q}_1^e(t) = (x_e\, y_e\, r_e\, \varphi_e\, x_{e+1}\, y_{e+1}\, q_e\, \varphi_{e+1})^{\mathrm{T}},$$

and is depicted in Fig. 2. Sine and cosine are abbreviated by $\mathrm{s}(\bullet)$ and $\mathrm{c}(\bullet)$, respectively. The element function is neither linear in its generalized coordinates $\mathbf{q}_1^e(t)$ nor restricted to nodal degrees of freedom as in standard finite element formulations.

The disk is modeled as a two-dimensional rigid body. The position of each material point $(r, \vartheta) \in \mathscr{S}_2 = [0, R] \times [0, 2\pi]$, as depicted in Fig. 5, is uniquely described by the position of the center of gravity $\mathbf{r}_{OM}(\mathbf{q}) = (x(t)\, y(t))^{\mathrm{T}}$ and the orientation $\varphi(t)$ of the body and can be formulated as

$$
\begin{aligned}
\xi_2((r, \vartheta), t) &= \mathbf{r}_2((r, \vartheta), \mathbf{q}_2(t)) = \begin{pmatrix} x \\ y \end{pmatrix} + r \begin{pmatrix} \mathrm{s}(\varphi + \vartheta) \\ -\mathrm{c}(\varphi + \vartheta) \end{pmatrix}, \\
\mathbf{q}_2 &= \mathbf{C}_2 \mathbf{q}.
\end{aligned}
\tag{5}
$$

Analogous to an element of the string, the generalized coordinates $\mathbf{q}_2(t) = (x\, y\, \varphi)^{\mathrm{T}}$ describing the disk are extracted by its connectivity matrix $\mathbf{C}_2$. In the same sense as for the string the kinematics merges two different
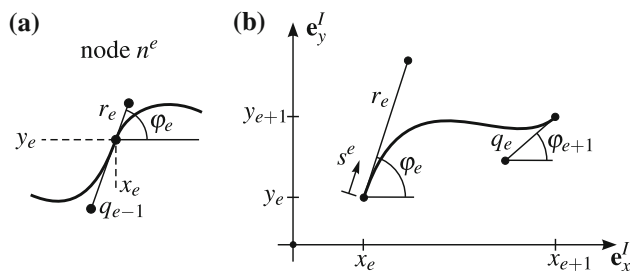
Fig. 2 **a** The position x_e, y_e and the angle φ_e belong to the node n^e. The weight factors r_e and q_{e-1} are degrees of freedom of the element. **b** The element on the domain $\Omega^e \subset [0, L]$ can be described by eight generalized coordinates $\mathbf{q}_1^e$. The element is depicted for $s^e \in [0, 1]$

constraints. The first constraint guarantees that the disk is a two-dimensional continuous body described by the function ξ_2. The second constraint, by introducing generalized coordinates, corresponds to the rigidity constraint. Since the dynamics of the rigid disk is of interest, the transition from the infinite dimensional to the finite dimensional description is necessary and does not lead to any approximation of the system.

2.2 Force distributions

Besides the kinematical parametrization of the system, the occurring forces have to be specified. In our consideration the string is modeled as perfectly flexible. Thus the stress $\mathbf{t}(s)$ in the string, at least once differentiable in s, is tangent to the current configuration of the string (cf. [4, p. 17])

$$\mathbf{t} \times \frac{\partial \mathbf{r}_1}{\partial s} = 0 \Leftrightarrow \mathbf{t} = T \frac{\partial \mathbf{r}_1 / \partial s}{\| \partial \mathbf{r}_1 / \partial s \|}, \tag{6}$$

with a scalar valued function T which contains the force law for a specific material. This assumption can be motivated by a reduction from a three-dimensional continuum model to a one-dimensional string model and coincide with the symmetry property of the Cauchy stress tensor [4, Sect. 16.6]. This symmetry property can be seen as a constitutive assumption of the three-dimensional continuum or as the consequence that we do not allow any distributed torques in classical continuum mechanics. Therefore the perfect flexibility of the string can be seen as an appropriate constitutive assumption. Using this assumption, we can draw the free body diagram as in Fig. 3 which gives us the force distributions on the string. The body forces $\mathbf{b}_1$ and the contact force distribution $-\mathbf{l}$ are defined per unit line segment. The contact force distribution $-\mathbf{l}$ models the contact interaction between the string and the disk. A perfect bilateral constraint force distribution $d\mathbf{z}_1$ guarantees that the string follows the kinematics, dictated by its discretization. The concept of atomic measures in space δ (cf. [19, 29]) allows us to include concentrated forces at the boundaries.

$$
\begin{aligned}
d\mathbf{F}_1 &= \frac{\partial \mathbf{t}}{\partial s} + (\mathbf{b}_1 - \mathbf{l})ds + d\mathbf{z}_1 && \text{for} \quad s \in (0, L) \\
d\mathbf{F}_1 &= (\mathbf{t} + \mathbf{F}_A)d\eta && \text{for} \quad s = 0 \\
d\mathbf{F}_1 &= (-\mathbf{t} + \mathbf{F}_B)d\eta && \text{for} \quad s = L.
\end{aligned}
\tag{7}
$$

The disk is subjected to the body force $\mathbf{b}_2$ defined per unit area segment $r\, dr\, d\vartheta$ and the reaction force $\mathbf{l}$ of the contact force distribution. To fulfill the rigidity conditions a perfect bilateral constraint force distribution $d\mathbf{z}_2$ is introduced on the interior of the disk and hence

$$
\begin{aligned}
d\mathbf{F}_2 &= \mathbf{b}_2 r\, dr\, d\vartheta + d\mathbf{z}_2 && \text{for} \quad (r, \vartheta) \in (0, R) \times (0, 2\pi) \\
d\mathbf{F}_2 &= \mathbf{l} R\, d\vartheta && \text{for} \quad (r, \vartheta) \in \{R\} \times (0, 2\pi).
\end{aligned}
\tag{8}
$$
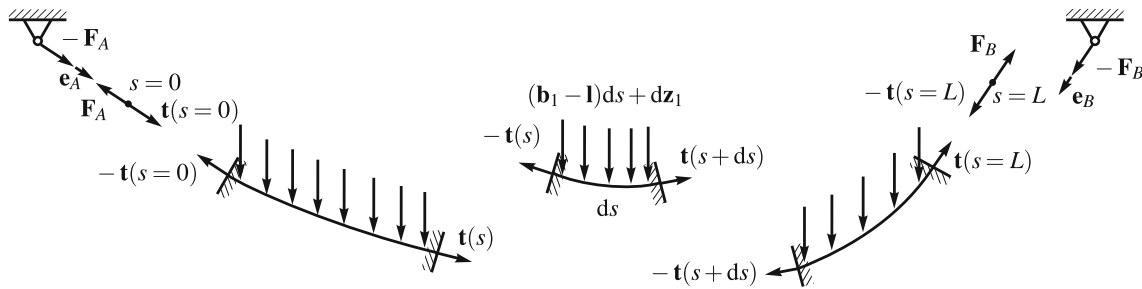
Fig. 3 Free body diagram of the string. The occurring forces in the domain of the string are the stress $\mathbf{t}$, the body forces $\mathbf{b}_1$, the contact force distribution $-\mathbf{l}$ and the perfect bilateral constraint forces $d\mathbf{z}_1$.

The boundaries of the string are cut free and belong to the mechanical system. On the boundaries the discrete bearing reactions $\mathbf{F}_A$ and $\mathbf{F}_B$ and the stresses at $s = 0$ and $s = L$ are acting

2.3 Virtual work for compatible virtual displacements

The virtual work of the system $\delta W = \delta W^{\mathrm{string}} + \delta W^{\mathrm{disk}}$ is the sum of the virtual work of its subsystems. Using Eq. (1) and Eq. (7) the virtual work for the string is

$$\delta W^{\mathrm{string}} = \int_{\mathscr{S}_1} \delta \xi_1^{\mathrm{T}} \ddot{\xi}_1 dm - \int_0^L \delta \xi_1^{\mathrm{T}} \left(\frac{\partial \mathbf{t}}{\partial s} + \mathbf{b}_1 - \mathbf{l} \right) ds$$
$$- \int_{\mathscr{S}_1} \delta \xi_1^{\mathrm{T}} d\mathbf{z}_1 - \delta \xi_1^{\mathrm{T}} (\mathbf{t} + \mathbf{F}_A)|_{s=0} - \delta \xi_1^{\mathrm{T}} (-\mathbf{t} + \mathbf{F}_B)|_{s=L}.$$

We want to mention that in contrast to the standard procedure, the discrete boundary forces which are relevant for the physical behavior, contribute to the virtual work of the system. Since the stress distribution $\mathbf{t}$ is at least once differentiable in s, the term $\int_0^L \delta \xi_1^{\mathrm{T}} \frac{\partial \mathbf{t}}{\partial s}$ can be integrated by parts whereupon the contribution of the stresses at the boundaries cancel out.

$$\delta W^{\mathrm{string}} = \int_{\mathscr{S}_1} \delta \xi_1^{\mathrm{T}} \ddot{\xi}_1 dm + \int_0^L \left(\frac{\partial \delta \xi_1}{\partial s} \right)^{\mathrm{T}} \mathbf{t} \, ds$$
$$- \int_0^L \delta \xi_1^{\mathrm{T}} (\mathbf{b}_1 - \mathbf{l}) ds - \int_{\mathscr{S}_1} \delta \xi_1^{\mathrm{T}} d\mathbf{z}_1 \qquad (9)$$
$$- \delta \xi_1^{\mathrm{T}} \mathbf{F}_A|_{s=0} - \delta \xi_1^{\mathrm{T}} \mathbf{F}_B|_{s=L}.$$

The virtual work for the disk is obtained using Eq. (1) together with Eq. (8) as follows

$$\delta W^{\mathrm{disk}} = \int_{\mathscr{S}_2} \delta \xi_2^{\mathrm{T}} \ddot{\xi}_2 dm - \int_0^{2\pi} \int_0^R \delta \xi_2^{\mathrm{T}} \mathbf{b}_2 r dr d\vartheta$$
$$- \int_{\mathscr{S}_2} \delta \xi_2^{\mathrm{T}} d\mathbf{z}_2 - \int_0^{2\pi} \delta \xi_2^{\mathrm{T}} \mathbf{l} R d\vartheta. \qquad (10)$$

As stated in Eq. (1) the virtual work vanishes for all virtual displacements $\delta \xi$. With the choice of the kinematics, i.e.

the discretization in Eq. (2) and (5) the problem is constrained in every material point of the string and the disk by bilateral constraints $g_1(s, t) = \xi_1 - \mathbf{r}_1 = 0$ and $g_2(s, t) = \xi_2 - \mathbf{r}_2 = 0$, respectively, from an infinite dimensional problem to a finite dimensional problem. The virtual displacements which are compatible to these constraints are of the form

$$\delta \xi_i = \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}} \delta \mathbf{q} \qquad \forall \delta \mathbf{q}, \qquad \text{for} \quad i = 1, 2. \qquad (11)$$

With respect to the principle of d'Alembert–Lagrange (cf. [19, p. 48]), the virtual work of the perfect bilateral constraints integrated over the total system $\mathscr{S}$ vanishes for all compatible virtual displacements. Using Eq. (1) together with Eqs. (9), (10) and the principle of d'Alembert–Lagrange, the virtual work of $\mathscr{S}$ for compatible virtual displacements (see Eq. (11)) is obtained as

$$\delta W = \delta \mathbf{q}^{\mathrm{T}} \left[\left\{ \int_{\mathscr{S}_1} \left(\frac{\partial \mathbf{r}_1}{\partial \mathbf{q}} \right)^{\mathrm{T}} \ddot{\mathbf{r}}_1 + \int_{\mathscr{S}_2} \left(\frac{\partial \mathbf{r}_2}{\partial \mathbf{q}} \right)^{\mathrm{T}} \ddot{\mathbf{r}}_2 dm \right\} \right.$$
$$+ \left\{ \int_0^L \left(\frac{\partial^2 \mathbf{r}_1}{\partial \mathbf{q} \partial s} \right)^{\mathrm{T}} \mathbf{t} \, ds \right\} - \left\{ \int_0^L \left(\frac{\partial \mathbf{r}_1}{\partial \mathbf{q}} \right)^{\mathrm{T}} \mathbf{b}_1 ds \right.$$
$$+ \left(\frac{\partial \mathbf{r}_1}{\partial \mathbf{q}} (s = 0) \right)^{\mathrm{T}} \mathbf{F}_A + \left(\frac{\partial \mathbf{r}_1}{\partial \mathbf{q}} (s = L) \right)^{\mathrm{T}} \mathbf{F}_B \qquad (12)$$
$$+ \int_0^{2\pi} \int_0^R \left(\frac{\partial \mathbf{r}_2}{\partial \mathbf{q}} \right)^{\mathrm{T}} \mathbf{b}_2 r dr d\vartheta \bigg\}$$
$$\left. - \left\{ \int_0^{2\pi} \left(\frac{\partial \mathbf{r}_2}{\partial \mathbf{q}} \right)^{\mathrm{T}} \mathbf{l} R d\vartheta - \int_0^L \left(\frac{\partial \mathbf{r}_1}{\partial \mathbf{q}} \right)^{\mathrm{T}} \mathbf{l} ds \right\} \right].$$

The expressions in the curly brackets are the generalized forces called inertia forces $\mathbf{f}^{\mathrm{inertia}}$, internal forces $\mathbf{f}^{\mathrm{int}}$, external forces $\mathbf{f}^{\mathrm{ext}}$ and contact forces $\mathbf{f}^{\mathrm{contact}}$. Hence the principle of virtual work can be written in short as

$$\delta W = \delta \mathbf{q}^{\mathrm{T}} \left[\mathbf{f}^{\mathrm{inertia}} + \mathbf{f}^{\mathrm{int}} - \mathbf{f}^{\mathrm{ext}} - \mathbf{f}^{\mathrm{contact}} \right] = 0 \qquad \forall \delta \mathbf{q}, \forall t. \qquad (13)$$

Since this equality holds for all virtual displacements $\delta\mathbf{q}$ and for all instants of time t, it relates directly to the classical partial differential equations.

3 Constitutive laws

For a complete description of the system, the generalized forces have to be specified in detail by constitutive laws. For several force laws the mass distributions of the two bodies have to be known. The distribution on the boundary vanishes for the particular body. For the string one defines an arbitrary stress free state of the string as its reference state $\mathbf{r}_{1,ref}(s, \mathbf{q}_{1,ref})$. In this state the mass per unit line segment is ρ_0. For the homogenous disk the density ρ_D is defined per unit area segment. Hence the mass distributions of the two bodies are

$$dm = \rho_0 \|\partial\mathbf{r}_{1,ref}/\partial s\| ds \quad \text{for} \quad s \in (0,L)$$
$$dm = \rho_D r dr d\vartheta \qquad\qquad \text{for} \quad (r,\vartheta) \in (0,R) \times (0,2\pi).$$

The kinematics of the string is described by local shape functions. In many of the following derivations one has to integrate over the mass distribution of the string. To be short in those derivations, we perform the integration at the example of the total mass of the string. The notation $\sum_e$ stands for the summation defined in Eq. (3). When introducing the coordinates of the nodes $n^1 < \cdots < n^e < \cdots < n^{k_{el}+1}$ ascendingly for increasing index e, the partial derivative $\frac{\partial s^e}{\partial s} > 0$ is strictly positive due to Eq. (4).

$$m = \int_{\mathscr{S}_1} dm = \int_0^L \rho_0 \left\|\frac{\partial\mathbf{r}_{1,ref}}{\partial s}\right\| ds$$

$$= \sum_e \chi_{[n^e, n^{e+1}]} \int_0^L \rho_0 \left\|\frac{\partial\mathbf{r}_{1,ref}^e}{\partial s^e}\frac{\partial s^e}{\partial s}\right\| ds$$

$$= \sum_e \int_{n^e}^{n^{e+1}} \rho_0 \left\|\frac{\partial\mathbf{r}_{1,ref}^e}{\partial s^e}\right\|\frac{\partial s^e}{\partial s} ds = \sum_e \int_0^1 \rho_0 \left\|\frac{\partial\mathbf{r}_{1,ref}^e}{\partial s^e}\right\| ds^e.$$

3.1 Inertia forces

As stated in Papastavridis [33, p. 540] or Eugster [15] the following equivalence holds

$$\int_{\mathscr{S}} \left(\frac{\partial\mathbf{r}}{\partial\mathbf{q}}\right)^{\mathrm{T}} \ddot{\mathbf{r}} dm = \mathbf{M}\ddot{\mathbf{q}} + \Gamma : \dot{\mathbf{q}} \otimes \dot{\mathbf{q}},$$

$$\mathbf{M} = \int_{\mathscr{S}} dm \left(\frac{\partial\mathbf{r}}{\partial\mathbf{q}}\right)^{\mathrm{T}}\frac{\partial\mathbf{r}}{\partial\mathbf{q}}, \quad \Gamma = \int_{\mathscr{S}} dm \left(\frac{\partial\mathbf{r}}{\partial\mathbf{q}}\right)^{\mathrm{T}}\frac{\partial^2\mathbf{r}}{\partial\mathbf{q}\otimes\partial\mathbf{q}},$$

where $\mathbf{M} = \mathbf{M}(\mathbf{q})$ is the symmetric and positive definite mass matrix and $\Gamma = \Gamma(\mathbf{q},\dot{\mathbf{q}})$ are the Christoffel symbols of

the second kind. The colon denotes the double contraction between higher order tensors and $\otimes$ is the tensor product. During the derivation of this identity the derivative with respect to time is swapped with the derivative with respect to the generalized coordinates. Therefore we are restricted to holonomic constraints. Evaluation of the mass matrix and the Christoffel terms for the two bodies leads to

$$\mathbf{M}_1 = \sum_e (\mathbf{C}_1^e)^{\mathrm{T}} \left\{ \int_0^1 \rho_0 \left(\frac{\partial\mathbf{r}_1^e}{\partial\mathbf{q}_1^e}\right)^{\mathrm{T}} \left(\frac{\partial\mathbf{r}_1^e}{\partial\mathbf{q}_1^e}\right) \left\|\frac{\partial\mathbf{r}_{1,ref}^e}{\partial s^e}\right\| ds^e \right\} \mathbf{C}_1^e$$

$$\mathbf{M}_2 = (\mathbf{C}_2)^{\mathrm{T}} \left\{ \int_0^{2\pi}\int_0^R \rho_D \left(\frac{\partial\mathbf{r}_2}{\partial\mathbf{q}_2}\right)^{\mathrm{T}} \left(\frac{\partial\mathbf{r}_2}{\partial\mathbf{q}_2}\right) r dr d\vartheta \right\} \mathbf{C}_2$$

$$= (\mathbf{C}_2)^{\mathrm{T}} \left\{ \rho_D \begin{pmatrix} R^2\pi & 0 & 0 \\ 0 & R^2\pi & 0 \\ 0 & 0 & \frac{R^4}{2}\pi \end{pmatrix} \right\} \mathbf{C}_2$$

$$\Gamma_1 : \dot{\mathbf{q}} \otimes \dot{\mathbf{q}} = \sum_e (\mathbf{C}_1^e)^{\mathrm{T}} \left\{ \int_0^1 \rho_0 \left(\frac{\partial\mathbf{r}_1^e}{\partial\mathbf{q}_1^e}\right)^{\mathrm{T}} \frac{\partial^2\mathbf{r}_1^e}{\partial\mathbf{q}_1^e \otimes \partial\mathbf{q}_1^e} : \right.$$
$$\left. \dot{\mathbf{q}}_1^e \otimes \dot{\mathbf{q}}_1^e \left\|\frac{\partial\mathbf{r}_{1,ref}^e}{\partial s^e}\right\| ds^e \right\}$$

$$\Gamma_2 : \dot{\mathbf{q}} \otimes \dot{\mathbf{q}} = 0, \quad \text{where} \quad \mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2, \Gamma = \Gamma_1 + \Gamma_2.$$

3.2 Internal forces

Since the string is considered to be a deformable body, the reaction force against deformation, i.e. the stress, is not a constraint force as in a rigid body but contributes to the virtual work of the system for compatible virtual displacements. To evaluate the stress a constitutive equation is needed which combines kinematical quantities, e.g. the state of deformation, with force quantities. The measure of deformation compares an actual spatial configuration $\mathbf{r}_1$ with the reference configuration $\mathbf{r}_{1,ref}$, introduced above. An intuitive choice of the measure of deformation is the stretch ν^e, which is the quotient of the incremental lengthes of the actual curve and the curve in the reference configuration

$$\nu^e = \frac{dL}{dL_{ref}} = \frac{\|d\mathbf{r}_1^e\|}{\|d\mathbf{r}_{1,ref}^e\|} = \frac{\|\partial\mathbf{r}_1^e/\partial s^e\|}{\|\partial\mathbf{r}_{1,ref}^e/\partial s^e\|}.$$

The internal force vector from Eqs. (12) and (13) together with Eq. (6) is

$$\mathbf{f}^{\text{int}} = \sum_e (\mathbf{C}_1^e)^{\mathrm{T}} \int_0^1 \left(\frac{\partial^2\mathbf{r}_1^e}{\partial\mathbf{q}_1^e \partial s^e}\right)^{\mathrm{T}} \mathbf{t}^e ds^e,$$

where

$$\mathbf{t}^e = T^e(\nu^e) \frac{\partial\mathbf{r}_1^e/\partial s^e}{\|\partial\mathbf{r}_1^e/\partial s^e\|}.$$

As an example of a nonlinear material law, a neo-Hookean material is chosen which depends merely on the stretch at the specific material point, i.e.

$$T^e(v^e) = \frac{k}{3}\left(v^e - \frac{1}{(v^e)^2}\right).$$

The stiffness k is a quantity depending on the material property and the geometry of the physical string and is determined experimentally.

3.3 External forces

The external forces of the system are the body forces $\mathbf{b}_1 = \rho_0\|\partial\mathbf{r}_{1,ref}/\partial s\|\mathbf{g}$ and $\mathbf{b}_2 = \rho_D\,\mathbf{g}$ due to gravity $\mathbf{g} = (0\ -\ g)^{\mathrm{T}}$ and the bearing reactions $\mathbf{F}_i = \lambda_{Bi}\,\mathbf{e}_i$, for $i = \{A, B\}$. The reaction forces are modeled as perfect bilateral constraint forces. These constraint forces can be characterized by a set-valued force law as depicted in Fig. 4. For the most concise formulation of such a force law the concept of normal cone inclusions (cf. [19, 22]) is used. Hence the force law is written as follows

$$\mathbf{F}_i = \lambda_{Bi}\mathbf{e}_i, \qquad g_{Bi} \in \mathscr{N}_{\mathbb{R}}(-\lambda_{Bi}), \qquad \text{for } i = A, B$$

where g_{Bi} is the gap function between the actual position of the endpoints of the string and its desired suspension points $\mathbf{r}_{OA}$ and $\mathbf{r}_{OB}$, respectively. $\mathscr{N}_{\mathbb{R}}$ is the normal cone to the set of all reals $\mathbb{R}$, i.e. the classical constraint equation $g_{Bi} = 0 = \mathscr{N}_{\mathbb{R}}$. The vectors

$$\mathbf{e}_A = \frac{\mathbf{r}_1^1(s^e = 0) - \mathbf{r}_{OA}}{\|\mathbf{r}_1^1(s^e = 0) - \mathbf{r}_{OA}\|}, \quad \mathbf{e}_B = \frac{\mathbf{r}_1^{k_{el}}(s^e = 1) - \mathbf{r}_{OB}}{\|\mathbf{r}_1^{k_{el}}(s^e = 1) - \mathbf{r}_{OB}\|}$$

are the normalized direction vectors between these points. Due to Eqs. (12) and (13) the external forces can be written as

$$\begin{aligned}
\mathbf{f}^{ext} = \sum_e (\mathbf{C}_1^e)^{\mathrm{T}} &\left\{\int_0^1 \rho_0 \left(\frac{\partial\mathbf{r}_1^e}{\partial\mathbf{q}_1^e}\right)^{\mathrm{T}} \mathbf{g}\left\|\frac{\partial\mathbf{r}_{1,ref}^e}{\partial s^e}\right\| ds^e\right\} \\
&+ (\mathbf{C}_1^1)^{\mathrm{T}}\left\{\left(\frac{\partial\mathbf{r}_1^1(s^e = 0)}{\partial\mathbf{q}_1^1}\right)^{\mathrm{T}} \mathbf{e}_A\lambda_{BA}\right\} \\
&+ (\mathbf{C}_1^{k_{el}})^{\mathrm{T}}\left\{\left(\frac{\partial\mathbf{r}_1^{k_{el}}(s^e = 1)}{\partial\mathbf{q}_1^{k_{el}}}\right)^{\mathrm{T}} \mathbf{e}_B\lambda_{BB}\right\} \\
&+ (\mathbf{C}_2)^{\mathrm{T}}\left\{(0\ -\ \rho_D R^2\pi g\ \ 0)^{\mathrm{T}}\right\}.
\end{aligned}$$

3.4 Contact forces

During the motion of the system, the string constrains the disk by surrounding its contour. The force distribution $\mathbf{l}$ as the contact interaction between the two bodies seems to be an appropriate choice for a continuous formulation. But in
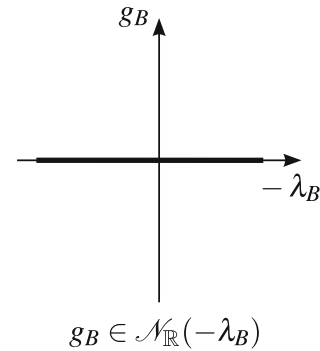
Fig. 4 Set-valued force law for perfect bilateral constraint on displacement level. The force $-\lambda_B$ can take arbitrary values to fulfill the constraint $g_B = 0$
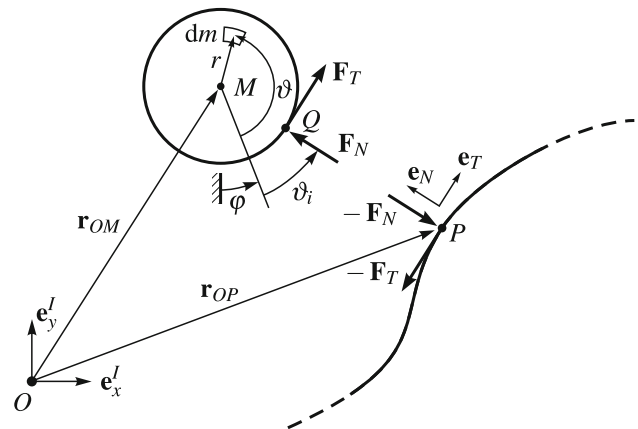
Fig. 5 Parametrization of the disk's material points and contact kinematics

the kinematically discretized model the contact force distribution will typically degenerate into discrete forces at a limited number of points, the positions of which are a priori unknown. A convenient approach is to introduce a dense grid of k_{cp} possible contact points at arbitrarily given material points of the string and to approximate the force distribution $\mathbf{l}$ by introducing discrete contact forces in normal and tangential direction in the sense that (cf. Eq. (12) and (13))

$$\begin{aligned}
\mathbf{f}^{contact} &= \int_0^{2\pi}\left(\frac{\partial\mathbf{r}_2}{\partial\mathbf{q}}\right)^{\mathrm{T}}\mathbf{l}R d\vartheta - \int_0^L\left(\frac{\partial\mathbf{r}_1}{\partial\mathbf{q}}\right)^{\mathrm{T}}\mathbf{l}ds \\
&\approx \sum_{i=1}^{k_{cp}}\left[\left(\frac{\partial\mathbf{r}_2}{\partial\mathbf{q}} - \frac{\partial\mathbf{r}_1}{\partial\mathbf{q}}\right)^{\mathrm{T}}(\mathbf{F}_N + \mathbf{F}_T)\right]_i \qquad (14) \\
&= \sum_{i=1}^{k_{cp}}[\mathbf{w}_N\lambda_N + \mathbf{w}_T\lambda_T]_i.
\end{aligned}$$

In the following the contact model is developed for an arbitrary point P on an element e with element coordinate s_i^e placed at $\mathbf{r}_{OP} = \mathbf{r}_1^e(s_i^e, \mathbf{q}_1^e)$. The corresponding contact

point on the disk at (R, ϑ_i) is placed at $\mathbf{r}_{OQ} = \mathbf{r}_2((R, \vartheta_i), \mathbf{q}_2)$. With respect to Fig. 5 the discrete contact forces can be written as $\mathbf{F}_N = \lambda_N \mathbf{e}_N$ and $\mathbf{F}_T = \lambda_T \mathbf{e}_T$ as normal and tangential contact forces, respectively. The normal direction $\mathbf{e}_N$ for the contact point is the normalized connection line between the contact point $\mathbf{r}_{OP}$ and the position of the disk's center of gravity $\mathbf{r}_{OM}$. The tangential direction $\mathbf{e}_T$ is orthogonal to the normal direction and is introduced as depicted in Fig. (5)

$$\mathbf{e}_N = \frac{\mathbf{r}_{OM} - \mathbf{r}_{OP}}{\|\mathbf{r}_{OM} - \mathbf{r}_{OP}\|}, \qquad \mathbf{e}_T = (\mathbf{e}_N \cdot \mathbf{e}_y^I)\mathbf{e}_x^I - (\mathbf{e}_N \cdot \mathbf{e}_x^I)\mathbf{e}_y^I.$$

The contact point Q on the boundary of the disk is at (R, ϑ_i). Due to Fig. (5) the normal and tangential unit vectors can be expressed as well in terms of sine and cosine of the angle $\varphi_0 = \varphi + \vartheta_i$. Using Eq. (5) it can be seen that the angle ϑ_i does not have to be known explicitly, since it holds that

$$\left(\frac{\partial \mathbf{r}_{OQ}}{\partial \mathbf{q}_2}\right)^{\mathrm{T}} \mathbf{e}_N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ Rc(\varphi_0) & Rs(\varphi_0) \end{pmatrix} \begin{pmatrix} -s(\varphi_0) \\ -c(\varphi_0) \end{pmatrix} = \begin{pmatrix} \mathbf{e}_N \\ 0 \end{pmatrix}$$

$$\left(\frac{\partial \mathbf{r}_{OQ}}{\partial \mathbf{q}_2}\right)^{\mathrm{T}} \mathbf{e}_T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ Rc(\varphi_0) & Rs(\varphi_0) \end{pmatrix} \begin{pmatrix} c(\varphi_0) \\ s(\varphi_0) \end{pmatrix} = \begin{pmatrix} \mathbf{e}_T \\ R \end{pmatrix}.$$

Together with Eq. (14) the generalized force directions for an arbitrary contact point on an element e can be written as

$$\mathbf{w}_N = (\mathbf{C}_2)^{\mathrm{T}} \begin{pmatrix} \mathbf{e}_N \\ 0 \end{pmatrix} - (\mathbf{C}_1^e)^{\mathrm{T}} \left(\frac{\partial \mathbf{r}_1^e}{\partial \mathbf{q}_1^e}\right)^{\mathrm{T}} \mathbf{e}_N,$$

$$\mathbf{w}_T = (\mathbf{C}_2)^{\mathrm{T}} \begin{pmatrix} \mathbf{e}_T \\ R \end{pmatrix} - (\mathbf{C}_1^e)^{\mathrm{T}} \left(\frac{\partial \mathbf{r}_1^e}{\partial \mathbf{q}_1^e}\right)^{\mathrm{T}} \mathbf{e}_T.$$

The contact forces contribute to the equations of motion as stated in Eq. (14). For the normal and tangential force law, the contact kinematics in normal and tangential direction are needed. The gap function g_N and its corresponding constraint velocity γ_N are given by

$$g_N = \mathbf{e}_N^{\mathrm{T}}(\mathbf{r}_{OM} - \mathbf{r}_{OP}) - R,$$

$$\gamma_N = \mathbf{e}_N^{\mathrm{T}}(\dot{\mathbf{r}}_{OM} - \dot{\mathbf{r}}_{OP}) = \mathbf{e}_N^{\mathrm{T}}\left(\frac{\partial \mathbf{r}_{OM}}{\partial \mathbf{q}} - \frac{\partial \mathbf{r}_{OP}}{\partial \mathbf{q}}\right)\dot{\mathbf{q}} = \mathbf{w}_N^{\mathrm{T}}\dot{\mathbf{q}}.$$

For $g_N > 0$, the bodies are separated, for a vanishing contact distance $g_N = 0$, the contact is closed and if $g_N < 0$, then the two bodies penetrate. The constraint velocity in tangential direction is given as

$$\gamma_T = \mathbf{e}_T^{\mathrm{T}}(\dot{\mathbf{r}}_{OQ} - \dot{\mathbf{r}}_{OP}) = \mathbf{e}_T^{\mathrm{T}}(\dot{\mathbf{r}}_{OM} + R\dot{\varphi}\mathbf{e}_T - \dot{\mathbf{r}}_{OP})$$

$$= \mathbf{e}_T^{\mathrm{T}}\left(R\dot{\varphi}\mathbf{e}_T + \left(\frac{\partial \mathbf{r}_{OM}}{\partial \mathbf{q}} - \frac{\partial \mathbf{r}_{OP}}{\partial \mathbf{q}}\right)\dot{\mathbf{q}}\right) = \mathbf{w}_T^{\mathrm{T}}\dot{\mathbf{q}}.$$

The contact laws, sketched in Fig. 6a and b, are formulated as set-valued force laws which guarantee the

impenetrability condition of a contact. The force law in normal direction is the law for a unilateral constraint on velocity level

$$g_N > 0 \Rightarrow -\lambda_N = 0$$
$$g_N = 0 \Rightarrow -\gamma_N \in \mathcal{N}_{\mathbb{R}_0^-}(-\lambda_N),$$

where $\mathbb{R}_0^-$ are all negative reals including zero. In tangential direction we introduce a force law for plane Coulomb friction which depends on the normal contact force λ_N, i.e.

$$\gamma_T \in \mathcal{N}_{[-\mu\lambda_N, \mu\lambda_N]}(-\lambda_T).$$

By the results obtained so far, the virtual work of Eq. (13) can be written in the form

$$\mathbf{M}\dot{\mathbf{u}} - \mathbf{h} - \sum_{i \in \mathcal{H}} \mathbf{w}_i \lambda_i = 0, \quad \mathbf{u} = \dot{\mathbf{q}},$$
$$\gamma_i = \mathbf{w}_i^{\mathrm{T}}\mathbf{u} + \chi_i, \quad \gamma_i \in \mathcal{N}_{\mathcal{C}_i}(-\lambda_i),$$

$$(15)$$

which constitutes the equations of motion of the system. The index set $\mathcal{H}(\mathbf{q}, t)$ consists of all closed and (numerically) penetrated contacts. The contact forces λ_i act in their appropriate generalized force direction $\mathbf{w}_i(\mathbf{q}, t)$. The remaining forces in the equations of motion are collected in $\mathbf{h}(\mathbf{u}, \mathbf{q}, t)$. The contact force laws are formulated as normal cone inclusions on velocity level. The constraint velocities $\gamma_i(\mathbf{q}, t)$ are split into a term linear in the generalized velocities $\mathbf{u}$ and a remaining term $\chi_i(\mathbf{q}, t)$.

3.5 Impact laws

The formulation in Eq. (15) describes the dynamics of the spatially discretized system as long as closed contacts remain closed or will detach, open contacts remain open, and stick-slip transition occur. This is briefly called the non-impulsive dynamics of the system. Because of the introduction of set-valued force laws which may e.g. fulfill the impenetrability condition exactly, discontinuities in velocities may occur in addition. For these discontinuities
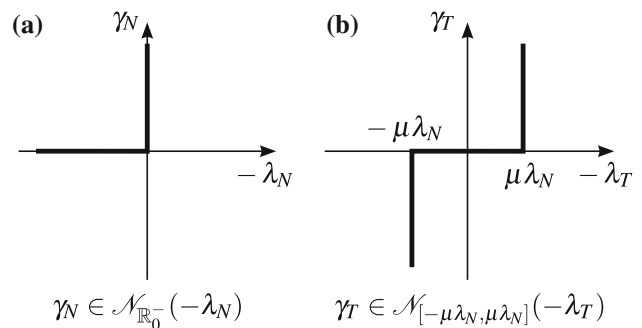
Fig. 6 Set-valued force laws for a unilateral frictional contact with Coulomb friction. a Unilateral constraint in normal direction on velocity level. b Coulomb friction law in tangential direction

Eq. (15) is not valid anymore or has to be understood in a measure sense, see Glocker [19], Moreau [30]. Hence, an impact equation and a corresponding impact law (cf. [21, 22]) is needed for the specific instant of time where the solution jumps. In the considered problem the following impact equation with an impact law of Newton type is used

$$
\mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) - \sum_{i \in \mathscr{H}} \mathbf{w}_i \Lambda_i = 0
$$

$$
\gamma_i^\pm = \mathbf{w}_i^\mathrm{T} \mathbf{u}^\pm + \chi_i, \quad \gamma_i^+ + \varepsilon_i \gamma^- \in \mathscr{N}_{\mathscr{D}_i}(-\Lambda_i).
$$

(16)

Superscript $(\bullet)^-$ and $(\bullet)^+$ denote the pre- and post impact state, Λ_i is the impulsive contact force of a particular contact and the convex set $\mathscr{D}_i$ corresponds to the reservoir of possible impulsive forces. The convex sets for the normal and tangential contacts are $\mathscr{D}_N = \mathbb{R}_0^-$ and $\mathscr{D}_T = [-\mu\Lambda_N, \mu\Lambda_N]$, respectively. Since we deal with elastic structures and due to the argumentation in Glocker [20] the restitution coefficient ε_i is chosen to be zero for all contact laws.

4 Examples

Two different examples are simulated to show the possibilities of our formulation. The chosen parameters are listed in Table 1. The transient dynamic behavior of the system is evaluated with Moreau's timestepping algorithm which is outlined in the following. For a detailed treatment of the algorithm, especially how to solve the inclusion problem, we refer to Glocker [22], Leine and Nijmeijer [26], Möller [27], Studer [40]. The spatial integrations are evaluated numerically by a rectangle integration scheme with equidistant points.

Moreau's midpoint rule uses one difference scheme to approximate both, for impact-free and impulsive motion (15), (16), together with their associated inclusions. Starting from a known state $\mathbf{u}^\mathrm{B} = \mathbf{u}(t^\mathrm{B})$ and $\mathbf{q}^\mathrm{B} = \mathbf{q}(t^\mathrm{B})$ at the time t^B of a time step Δt, the coordinates $\mathbf{q}^\mathrm{M} = \mathbf{q}(t^\mathrm{M})$ at the midpoint are calculated as

$$
\mathbf{q}^\mathrm{M} = \mathbf{q}^\mathrm{B} + \frac{\Delta t}{2} \mathbf{u}^\mathrm{B}, \quad t^\mathrm{M} = t^\mathrm{B} + \frac{\Delta t}{2}.
$$

(17)

In a second step, by using the midpoint coordinates $\mathbf{q}^\mathrm{M}$ and the velocities at the beginning of the time step $\mathbf{u}^\mathrm{B}$, the following inclusion is solved with respect to the velocities $\mathbf{u}^\mathrm{E}$ and the impulsive contact forces Λ_i.

$$
\mathbf{M}(\mathbf{u}^\mathrm{E} - \mathbf{u}^\mathrm{B}) - \mathbf{h}\Delta t - \sum_{i \in \mathscr{H}} \mathbf{w}_i \Lambda_i = 0,
$$

$$
\gamma_i^\mathrm{B} = \mathbf{w}_i^\mathrm{T} \mathbf{u}^\mathrm{B} + \chi_i, \quad \gamma_i^\mathrm{E} = \mathbf{w}_i^\mathrm{T} \mathbf{u}^\mathrm{E} + \chi_i,
$$

$$
\gamma_i^\mathrm{E} + \varepsilon_i \gamma_i^\mathrm{B} \in \mathscr{N}_{\mathscr{D}_i}(-\Lambda_i).
$$

(18)

In a last step, the end state is updated as

$$
\mathbf{q}^\mathrm{E} = \mathbf{q}^\mathrm{M} + \frac{\Delta t}{2} \mathbf{u}^\mathrm{E}, \quad t^\mathrm{E} = t^\mathrm{M} + \frac{\Delta t}{2}.
$$

(19)

In our first application problem (see Fig. 7), a heavy disk falls into the suspended string whereupon large deformation of the string occurs. Multiple contacts close and the string wraps around the disk. The string is stretched until the motion reverses, the string relaxes and the disk is ejected whereupon contact with the string is lost. This sequence repeats until enough energy is dissipated such that the contacts between the disk and the string remain closed and the disk does not slide on the string, i.e. every contact point is in a sticking phase. From this instant of time the friction force behaves like a perfect constraint force, no more energy is dissipating and the string swings together with the disk in a motion up and down. In a regularized friction model, where the ambivalent character of the force is not considered, the disk will always slide on the string, the energy of the system is dissipated completely and the dynamical prediction differs from our formulation. Due to the coarse mesh of only two finite elements some artifacts occur at the position of the clamping. The string should be in a straight line between the clamping and the first contact point. This behavior can be eliminated easily by increasing the number of elements.

In Fig. 8 one can see how a light disk rolls on the string. The decoupling of the number of contact points from the number of elements allows to introduce a dense grid of 60 contact points. Even though the contacts between the disk and the string open and close permanently, a rolling and sliding motion can be performed. To show the high performance of shape functions which are nonlinear in their generalized coordinates, both problems have been simulated with merely two finite elements. Neither convergence nor stability problems occurred.

Table 1 Simulation parameters for example 1 and 2

Parameter	Value
Gravity	$g = 9.81$ kg ms^{-2}
Distance suspension points	$\overline{AB} = 0.8$ m
Friction coefficient	$\mu = 0.2$
Radius disk ex.1/ex.2	$R = 0.3/0.1$ m
Density disk ex.1/ex.2	$\rho_D = 150/0.1$ kg m^{-2}
Length string (undeformed)	$L = 1$ m
Density string (undeformed)	$\rho_0 = 0.3142$ kg m^{-1}
Stiffness string	$k = 853.75$ N
Number of elements	$k_{el} = 2$
Number of contact points ex.1/ex.2	$k_{cp} = 20/60$
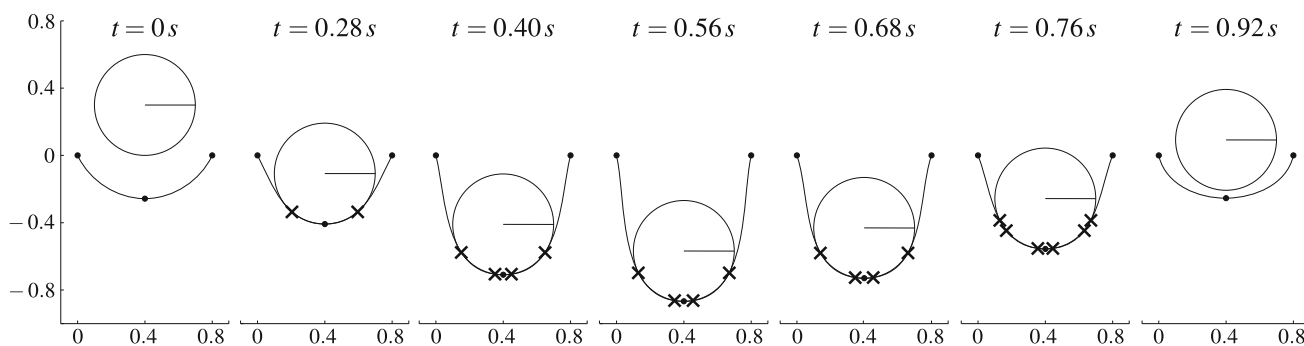
Fig. 7 Due to large deformation of the string it is possible that multiple contact points are closed at the same time. The closed contact points are marked by *crosses*. The string is discretized by two nonlinear finite elements. The nodal points are depicted by *bullets*
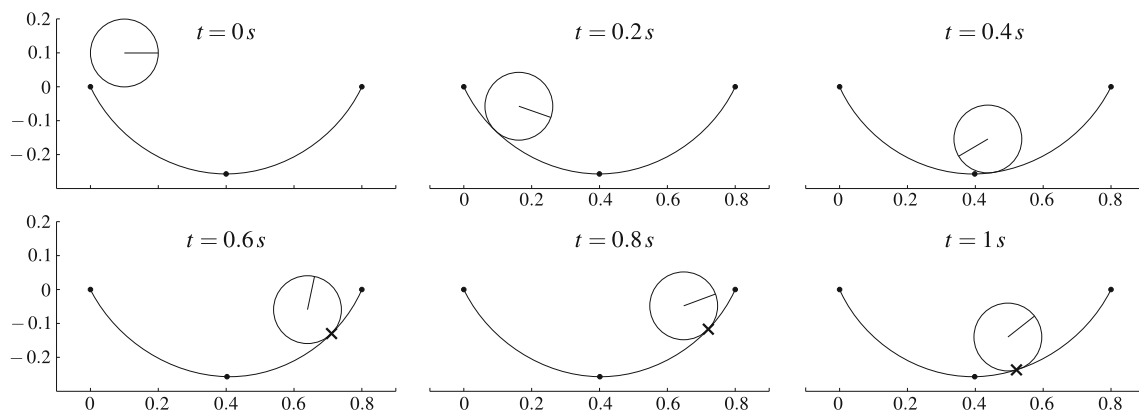
Fig. 8 Rolling of the disk is possible due to a dense grid of contact points and frictional contact laws in *each contact point*. The first three snapshots are taken at an instant of time, where the disk and the string are not in contact

5 Conclusions

In this paper we have shown a systematic approach for the formulation of the dynamics of deformable and rigid bodies. Both the deformable and the rigid body are introduced as continuous bodies. The discretization of the two bodies reduces the problem from an infinite dimensional to a finite dimensional problem. The choice of the kinematics, i.e. the choice of shape functions and generalized coordinates, goes along with the introduction of perfect bilateral constraints. These constraints are enforced by the constraint force distributions $d\mathbf{z}_1$ and $d\mathbf{z}_2$ introduced in Eq. (7) and (8), respectively. By treating the kinematic assumptions as perfect bilateral constraints, the mechanical theory includes the numerical approximation by finite elements. The principle of virtual work contains the projection of the local equations of motion in arbitrary directions, i.e. the virtual work vanishes for all virtual displacements. Together with the principle of d'Alembert–Lagrange and the projection onto the direction compatible to the bilateral constraints of the kinematics, the constraint forces can be eliminated. This formulation proposed here, allows it as well to introduce additional constraints e.g. inextensibility of the string. For the inextensibility constraint the

introduced kinematics of this paper would not lead to compatible virtual displacements. This means, that the constraint forces would not vanish for the virtual displacements chosen in Eq. (11) and would contribute to the equations of motion in Eq. (13) by a Lagrange multiplier. Since constraints are defined in a variational form, also the dynamics of the bodies need to be formulated variationally. Within such a formulation, one is completely free to choose the level, i.e. numerically or analytically, on which one wants to fulfill the constraints. For example the rigid disk could also be introduced as a deformable body discretized by finite elements satisfying additional rigidity constraints. A consequent implementation of these ideas may lead to a unification of the concepts used for deformable and rigid bodies.

Generally, discrete forces are not included in the concept of classical continuum mechanics. With the introduction of an atomic measure in space, discrete forces are applied to the deformable body in the continuous formulation. Even though the boundary of the string is zero-dimensional the boundary forces and so the boundary condition appear explicitly in the virtual work of the system. The contact force distribution, being the interaction between the two bodies, has been approximated by finitely

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many contact points of which each holds a set-valued force law. Eventually, the nonlinear finite element formulation has been obtained from the virtual work and the discretization of the deformable body by finitely many generalized coordinates. This natural formulation restricts an element shape function neither to be linear in its generalized coordinates nor to be described only by nodal degrees of freedom. A non-standard choice of shape function has been shown using Bézier splines.

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