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Measuring consistency of interval-valued preference relations: comments and comparison

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Abstract
The concepts of consistency definition and consistency index are usually used to measure the consistency of a preference relation. When interval numbers are used to express the preference information, the consistency of the derived interval-valued preference relations (IVPRs) is worth being investigated. In this study, a comment is provided for the ideas behind consistency definitions and consistency indexes of interval multiplicative reciprocal matrices (IMRMs) and interval additive reciprocal matrices (IARMs), respectively. A comparison is made by considering the two kinds of consistency definitions of IVPRs. It is found that the method of defining the consistency of IVPRs in terms of the imaginary intervals is equivalent to that of defining the approximate consistency. Numerical examples are reported to illustrate the differences of the two consistency definitions of IVPRs. The observations illustrate that the fundamental inconsistency of IVPRs is compatible with the underlying idea of fuzzy sets. It is revealed that a consistent preference relation is only a particular case with a fixed value of the defined consistency index. In general, the consistency index could be used to quantify the deviation degree from a consistent real-valued preference relation.

Keywords Interval-valued preference relation (IVPR) · Consistency definition · Consistency index · Deviation degree · Equivalence

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1 Introduction

In the 1970s, Saaty (1980, 1977) proposed the famous decision-making methodology of the Analytic Hierarchy Process (AHP). Based on the AHP method, one decomposes a complex decision-making problem into various factors, then integrates these factors into hierarchical structures through dominating relations, and determines the relative importance of various factors in the hierarchy through pairwise comparisons. According to pairwise comparisons of alternatives, the opinions given by decision makers are represented through preference relations. These preference relations are used to determine the weights of alternatives and finally an optimal alternative is chosen. Up to now, the AHP approach has been widely studied and used in group decision-making problems (Brunelli 2015; Golden et al. 1989; Vaidyaab and Kumar 2006) and decision support systems (Lu et al. 2011, 2007; Ma et al. 2010). In the typical AHP, the integers in-between 1 to 9 and their reciprocals are selected as the values of pairwise comparisons completed over a set of alternatives $X = \{x_1, x_2, \dots, x_n\}$. Then a multiplicative reciprocal matrix (MRM) is formed as $A = (a_{ij})_{n\times n}$, where a_{ij} denotes the preference intensity of the alternative x_i over the alternative x_j. In addition, following the idea of fuzzy sets (Zadeh 1965, 1996), the real number $b_{ij} \in [0, 1]$ is used to represent the preference degree of the alternative x_i over the alternative x_j. A binary relation on $X = \{x_1, x_2, \dots, x_n\}$ is defined and a preference relation $B = (b_{ij})_{n\times n}$ is given. The matrix $B = (b_{ij})_{n\times n}$ is called as fuzzy preference relation (FPR) (Kacprzyk 1986; Nurmi 1981; Orlovsky 1978; Tanino 1984) and it is further renamed as additive reciprocal matrix (ARM) by considering the reciprocity of $b_{ij} + b_{ji} = 1$ (Liu et al. 2014). From the definition of ARMs, it is seen that the values of b_{ij} are real numbers. However, owing to the complexity and uncertainty of real-world decision making problems, it is difficult to use precise values to express the preference information of decision makers. Saaty and Vargas (1987) applied interval numbers to model the uncertainty experienced by decision makers. Then an IMRM was defined by using the scale from 1/9 to 9. Various decision making models and their applications based on IMRMs have been studied (Herrera et al. 2005; Lin and Zhang 2017; Xu 2004; Zhou et al. 2016). Similarly, ARMs have been extended to IARMs to cope with the uncertainty experienced by decision makers in expressing their opinions (Herrera et al. 2005; Xu 2004; Zhou et al. 2014).

To avoid contradictory (conflicting) decisions, it is essential to define the transitivity and consistency of preference relations together with the judgements of decision makers. According to the typical AHP (Saaty 1980), if $a_{ij} = a_{ik}a_{kj}$ $(\forall i, j, k = 1, 2, \dots, n)$ is satisfied for $A = (a_{ij})_{n\times n}$, the comparison matrix is consistent. Moreover, there are two methods to define the consistency of ARMs (Tanino 1984) and they are generalized by using a function in Chiclana et al. (2009). One is the additive consistency of $B = (b_{ij})_{n\times n}$ satisfying the requirement $b_{ij} = b_{ik} - b_{jk} + 0.5$ or $b_{ij} + b_{jk} + b_{ki} = b_{ji} + b_{kj} + b_{ik}$ $(\forall i, j, k = 1, 2, \dots, n)$ (Herrera et al. 2005). The other is the multiplicative consistency of $B = (b_{ij})_{n\times n}$ satisfying the condition $b_{ij}b_{jk}b_{ki} = b_{ji}b_{kj}b_{ik} (\forall i, j, k = 1, 2, \dots, n)$. For the consistency of

IMRMs, some methods have been reviewed and the approximate consistency has been defined in Liu et al. (2017b). In addition, for the consistency of IARMs, some methods of defining the additive and multiplicative consistency have been investigated (Krejčí 2017, 2019; Liu et al. 2018b); and the additive approximation-consistency has been proposed in Liu et al. (2018b). Following the idea of quantifying inconsistency degrees of MRMs in Saaty (1980), the consistency indexes have attracted much attention to generally quantify the inconsistency degrees of IVPRs (Dong et al. 2015, 2016; Wan et al. 2018; Liu et al. 2018c, 2020). The methods of measuring the consistency of MRMs, ARMs, IMRMs, IARMs and hesitant reciprocal matrices have been further reviewed and investigated by Li et al. (2018, 2019). On the other hand, it is found that a new method of defining the consistency of IMRMs has been proposed in Meng and Tan (2017) by introducing the concept of quasi-positive intervals. Similarly, the multiplicative consistency and additive consistency of IARMs have been redefined (Meng et al. 2017a, c). It is noted that the idea behind the approximate consistency is based on the viewpoint that IVPRs are inconsistent as compared to the real numbers (Liu et al. 2017b, 2018b). The methods of defining a consistency of IVPRs in the series of works (Meng et al. 2017a; Meng and Tan 2017; Meng et al. 2017c) are the direct extensions of consistent preference relations with real-number entries. Hence, it is worth to compare the two different methods of defining the consistency of IVPRs, and clarify the concepts of consistency definition and consistency index. Motivated by the above discussions, here we comment on and compare the methods of defining the consistency of IVPRs (Liu et al. 2017b, 2018b; Meng et al. 2017a; Meng and Tan 2017; Meng et al. 2017c). The main objective is to reveal the ideas behind these consistency definitions of IVPRs. The main novelty is the determination that the two approaches to defining the consistency of IVPRs are equivalent. By the way, the consistency indexes of IVPRs are analyzed and compared with the consistency definitions.

This paper is structured as follows. In Sect. 2, we give the definitions of IMRMs and IARMs, then review the arithmetic operations of interval numbers. Section 3 focuses on the methods of defining the consistency of IMRMs. The equivalence of the two approaches to consistency definitions of IMRMs is proved. In Sect. 4, the methods of defining the consistency of IARMs are addressed. The equivalence of the two approaches to the multiplicative consistency and additive consistency of IARMs is investigated, respectively. The existing shortcoming is pointed out and a novel finding is offered. In Sect. 5, some further comparison and discussion are made by considering the meaning of imaginary intervals. It is found that the introduction of imaginary interval numbers is only used to satisfy the mathematical relations of consistent preference relations. Some comments are offered by comparing the concepts of consistency definition and consistency index. In Sect. 6, an algorithm for solving decision making problems with IVPRs is provided by considering all permutations of alternatives, and two numerical examples are reported. Some conclusions are covered in Sect. 7.

2 Preliminaries

In this section, we first recall the definitions of IMRMs and IARMs, respectively, then review the arithmetic operations of interval numbers.

2.1 The concepts of IMRMs and IARMs

Let $X = \{x_1, x_2, \ldots, x_n\}$ denote the set of alternatives in a complex decision making problem. The concept of IMRMs is introduced as follows (Saaty and Vargas 1987):

Definition 1 (Saaty and Vargas 1987) The IVPR $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ is multiplicatively reciprocal with

$$\bar{A} = \begin{bmatrix} [1,1] & [a_{12}^-, a_{12}^+] & \cdots & [a_{1n}^-, a_{1n}^+] \\ [a_{21}^-, a_{21}^+] & [1,1] & \cdots & [a_{2n}^-, a_{2n}^+] \\ \vdots & \vdots & \ddots & \vdots \\ [a_{n1}^-, a_{n1}^+] & [a_{n2}^-, a_{n2}^+] & \cdots & [1,1] \end{bmatrix}.$$

The interval number $\bar{a}_{ij} = [a_{ij}^-, a_{ij}^+]$ indicates that the alternative x_i is between a_{ij}^- and a_{ij}^+ times as important as the alternative x_j with the properties of $a_{ij}^\pm > 0$, $a_{ij}^- < a_{ij}^+$, $a_{ij}^- = 1/a_{ji}^+$, and $a_{ij}^+ = 1/a_{ji}^-$.

Similarly, the definition of IARMs is given as follows:

Definition 2 (Xu 2001, 2004) The IVPR $\bar{B} = (\bar{b}_{ij})_{n\times n}$ is additively reciprocal with

$$\bar{B} = \begin{bmatrix} [0.5,0.5] & [b_{12}^-, b_{12}^+] & \cdots & [b_{1n}^-, b_{1n}^+] \\ [b_{21}^-, b_{21}^+] & [0.5,0.5] & \cdots & [b_{2n}^-, b_{2n}^+] \\ \vdots & \vdots & \ddots & \vdots \\ [b_{n1}^-, b_{n1}^+] & [b_{n2}^-, b_{n2}^+] & \cdots & [0.5,0.5] \end{bmatrix}.$$

The interval number $\bar{b}_{ij} = [b_{ij}^-, b_{ij}^+]$ stands for the preference intensity of the alternative x_i over the alternative x_j with the properties of $0 \le b_{ij}^\pm \le 1$, $b_{ij}^- \le b_{ij}^+$, and $b_{ij}^- + b_{ji}^+ = b_{ij}^+ + b_{ji}^- = 1$.

One can see from Definitions 1 and 2 that the basic idea is to use interval numbers to express the opinions of decision makers. Thus some uncertainty experienced by decision makers can be captured. Moreover, when investigating the consistency of IVPRs $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ and $\bar{B} = (\bar{b}_{ij})_{n\times n}$, it seems inevitable to apply the arithmetic operations of interval numbers. In what follows, we review the arithmetic operations of interval numbers.

2.2 Arithmetic operations of interval numbers

Let $\bar{a} = [a_l, a_r]$ be an interval number with real bounds a_l and a_r. The interval $\bar{a} = [a_l, a_r]$ also means a set of real numbers denoted as $\{x | a_l \leq x \leq a_r\}$. Typically, the arithmetic operations of interval numbers have been defined in Moorse (1966). That is, if defining the negative and the reciprocal of $\bar{a} = [a_l, a_r]$ as

$$- \bar{a} = - [a_l, a_r] = [-a_r, -a_l] = \{-x | x \in \bar{a}\},$$

and

$$1/\bar{a} = \{1/x | x \in \bar{a}\},$$

one has the following arithmetic operations for two interval numbers $\bar{a} = [a_l, a_r]$ and $\bar{b} = [b_l, b_r]$ (Moorse 1966):

$$\begin{aligned} &\text{Sum:} \quad \bar{a} + \bar{b} = \{x + y | x \in \bar{a}, y \in \bar{b}\}, \\ &\text{Difference:} \quad \bar{a} - \bar{b} = \{x - y | x \in \bar{a}, y \in \bar{b}\}, \\ &\text{Product:} \quad \bar{a} \cdot \bar{b} = \{xy | x \in \bar{a}, y \in \bar{b}\}, \\ &\text{Quotient:} \quad \bar{a}/\bar{b} = \{x/y | x \in \bar{a}, y \in \bar{b}\}. \end{aligned}$$

In particular, when $a_{l,r} > 0$ and $b_{l,r} > 0$, it gives

$$\bar{a} + \bar{b} = [a_l + b_l, a_r + b_r], \tag{1}$$

$$\bar{a} - \bar{b} = [a_l - b_r, a_r - b_l], \tag{2}$$

$$\bar{a} \cdot \bar{b} = [a_l b_l, a_r b_r], \tag{3}$$

$$\bar{a}/\bar{b} = a_l/b_r, a_r/b_l]. \tag{4}$$

According to the operation law of difference, it is found that the substraction $(\bar{a} - \bar{a})$ does not equal to zero. For example, when $\bar{a} = [1, 2]$, it follows $\bar{a} - \bar{a} = [-1, 1]$. If the interval number $\bar{a}$ denotes the preference intensity of the alternative x_i over the alternative x_j, the substraction of two same preference intensities should be zero. In other words, the result of $\bar{a} - \bar{a} = [-1, 1]$ may be not reasonable (Hu and Wang 2006; Krejčí 2019). In general, for three interval numbers $\bar{a}, \bar{b}$ and $\bar{c}$, the relation $\bar{a} + \bar{b} = \bar{c}$ does not mean $\bar{a} = \bar{c} - \bar{b}$. Moreover, the relation of $\bar{a} \cdot \bar{b} = \bar{c}$ can not yield $\bar{a} = \bar{c}/\bar{b}$. The above phenomena reveal that the addition is not the inverse operation of subtraction and the multiplication is not the inverse operation of division according to the typical arithmetic operations of interval numbers (Su et al. 1997). Furthermore, we always suppose $a_r \geq a_l$ for an interval number $\bar{a} = [a_l, a_r]$. If one considers the case of $a_r < a_l$, the interval number $\bar{a} = [a_l, a_r]$ is defined as the "imaginary" interval (Su et al. 1997). Correspondingly, the interval number $\bar{a} = [a_l, a_r]$ with $a_r \geq a_l$ is called real. The real and imaginary intervals can be called uniformly the generalized interval numbers or intervals. In addition, the quasi-positive interval is defined as follows:

Definition 3 (Meng et al. 2017c; Meng and Tan 2017; Meng et al. 2016) If $\bar{a} = \left[a_l, a_r\right]$ satisfies $a_l > a_r$ and $a_l, a_r \in \mathfrak{R}_+$, the interval number $\bar{a}$ is said to be quasi-positive.

It is seen from Definition 3 that the so-called quasi-positive interval is also the imaginary interval with positive boundary values. On the other hand, the arithmetic operations of the generalized interval numbers have been discussed (Hu and Wang 2006; Su et al. 1997; Zhou et al. 1996). For example, when considering the generalized interval number $\bar{a} = \left[a_l, a_r\right]$ with $a_l, a_r \in \mathfrak{R}_+$, the negative interval is defined as $-\bar{a} = \left[-a_l, -a_r\right]$ and the reciprocal interval is $\bar{a}^{-1} = \left[1/a_l, 1/a_r\right]$. The arithmetic operations of the generalized interval numbers for $\bar{a} = [a_l, a_r]$ and $\bar{b} = [b_l, b_r]$ with $a_{l,r} > 0$ and $b_{l,r} > 0$ are given as follows:

$$\bar{a} + \bar{b} = [a_l + b_l, a_r + b_r], \tag{5}$$

$$\bar{a} - \bar{b} = [a_l - b_l, a_r - b_r], \tag{6}$$

$$\bar{a} \cdot \bar{b} = [a_l b_l, a_r b_r], \tag{7}$$

$$\bar{a}/\bar{b} = [a_l/b_l, a_r/b_r]. \tag{8}$$

As compared to (1)–(4), it is seen that the main differences are the substraction operation (6) and the division operation (8).

Based on the above discussion, one can see that the arithmetic operations of interval numbers are different from the arithmetic of real numbers. If the uncertainty experienced by decision makers is modelled by using interval numbers, the difference reflects the complexity and uncertainty of interval-valued comparison ratios. When defining consistency of IVPRs, it is found that the arithmetic operations of interval numbers may play an important role (Liu et al. 2017b, 2018b; Meng et al. 2017a; Meng and Tan 2017; Meng et al. 2017c).

3 The methods for defining consistency of IMRMs

One can see that the basic idea of consistent preference relations is to capture the cardinal transitivity of opinions provided by decision makers (Saaty 1980; Tanino 1984). In relative measurements, the consistency of the matrix $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ means that any entry $\bar{a}_{ij}$ can be obtained indirectly by using the product of $\bar{a}_{ik}$ and $\bar{a}_{kj}$ for $\forall k = 1, 2, \ldots, n$. As shown in Dubois (2011), according to the arithmetic operations of interval numbers, the consistency of $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ cannot be defined by directly extending the method of defining consistent MRMs in Saaty (1980). In order to define the consistency of an IMRM $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$, various approaches have been proposed and reviewed such as those in Liu et al. (2017b); Meng and Tan (2017). Here we only need to recall the method in Meng and Tan (2017) by using Definition 3 and that of defining the approximate consistency in Liu et al. (2017b).

3.1 Consistency definitions of IMRMs

Following the concept of quasi-positive interval, the definition of quasi IMRMs is given as follows:

Definition 4 (Meng and Tan 2017) Let $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ be an IMRM. $\bar{A}' = \left(\bar{a}'_{ij}\right)_{n\times n}$ is said to be a quasi IMRM with respect to A if defining

$$\begin{cases} \bar{a}'_{ij} = \bar{a}_{ij}, \\ \bar{a}'_{ji} = \bar{a}^{\circ}_{ji}, \end{cases} \quad \text{or} \quad \begin{cases} \bar{a}'_{ij} = \bar{a}^{\circ}_{ij}, \\ \bar{a}'_{ji} = \bar{a}_{ji}, \end{cases} \tag{9}$$

for all $i, j = 1, 2, \dots, n$, where $\bar{a}^{\circ}_{ij} = \left[a^-_{ij}, a^+_{ij}\right]^{\circ} = \left[a^+_{ij}, a^-_{ij}\right]$.

The condition (9) means that in order to produce $\bar{A}'$, half of all entries in $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ should be changed to the quasi-positive intervals $\bar{a}^{\circ}_{ij}$. For example, let us consider the following matrix (Meng and Tan 2017)

$$\bar{A}_1 = \begin{bmatrix} [1,1] & [2,3] & [1/2,1] \\ [1/3,1/2] & [1,1] & [1/6,1/2] \\ [1,2] & [2,6] & [1,1] \end{bmatrix}.$$

If changing the entries [2, 3], [1, 2] and [2, 6] as [3, 2], [2, 1] and [6, 2] respectively, it gives a quasi IMRM as

$$\bar{A}'_1 = \begin{bmatrix} [1,1] & [3,2] & [1/2,1] \\ [1/3,1/2] & [1,1] & [1/6,1/2] \\ [2,1] & [6,2] & [1,1] \end{bmatrix}.$$

It is found that the number of quasi IMRMs is $C^1_3 + C^2_3 + C^3_3 = 7$ by considering the reciprocal property of the matrix $\bar{A}_1$. In fact, it is sufficient to consider the case of $i > j$ or $i < j$. For instance, when $i > j$, there are $n(n-1)/2$ interval-valued entries in an IMRM $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$. If one entry is changed as a quasi-positive interval number, there are $C^1_{n(n-1)/2}$ cases. If two entries are changed to quasi-positive intervals, there are $C^2_{n(n-1)/2}$ cases, and so on. Generally, we can obtain the number of quasi IMRMs in terms of $A = \left(\bar{a}_{ij}\right)_{n\times n}$ as

$$C^1_{n(n-1)/2} + C^2_{n(n-1)/2} + \cdots + C^{n(n-1)/2}_{n(n-1)/2} = 2^{n(n-1)/2} - 1.$$

As shown in the above analysis, a quasi IVPR $\bar{A}' = \left(\bar{a}'_{ij}\right)_{n\times n}$ is constructed from $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ by changing one of two entries $\bar{a}_{ij}$ and $\bar{a}_{ji}$ to so called quasi-positive intervals. In addition, one has the following result:

Theorem 1 *Any quasi IMRM $\bar{A}' = \left(\bar{a}'_{ij}\right)_{n\times n}$ constructed from $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ can be changed to the matrix where all the entries above the diagonal are quasi-positive intervals and all the entries below the diagonal are standard intervals.*

Proof Without loss of generality, it is assumed that all the entries below the diagonal are quasi-positive intervals in $\bar{A}'$. That is, one has the following matrix:

$$\bar{A}' = (\bar{a}'_{ij})_{n\times n} = \left[\begin{array}{c|cccc} & x_1 & x_2 & \cdots & x_n \\ \hline x_1 & [1,1] & \left[a^-_{12}, a^+_{12}\right] & \cdots & \left[a^-_{1n}, a^+_{1n}\right] \\ x_2 & \left[a^+_{21}, a^-_{21}\right] & [1,1] & \cdots & \left[a^-_{2n}, a^+_{2n}\right] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & \left[a^+_{n1}, a^-_{n1}\right] & \left[a^+_{n2}, a^-_{n2}\right] & \cdots & [1,1] \end{array}\right].$$

In order to move the quasi-positive entries below the diagonal to the positions above the diagonal, one only needs to change the permutation of objectives as $(x_n, x_{n-1}, \ldots, x_1)$. Then one can obtain the changed matrix as

$$\bar{A}'_c = \left[\begin{array}{c|cccc} & x_n & x_{n-1} & \cdots & x_1 \\ \hline x_n & [1,1] & \left[a^+_{n(n-1)}, a^-_{n(n-1)}\right] & \cdots & \left[a^+_{n1}, a^-_{n1}\right] \\ x_{n-1} & \left[a^-_{(n-1)n}, a^+_{(n-1)n}\right] & [1,1] & \cdots & \left[a^+_{(n-1)1}, a^-_{(n-1)1}\right] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & \left[a^-_{1n}, a^+_{1n}\right] & \left[a^-_{1(n-1)}, a^+_{1(n-1)}\right] & \cdots & [1,1] \end{array}\right].$$

It is seen from $\bar{A}'_c$ that all the entries above the diagonal are quasi-positive entries. This completes the proof. □

In fact, for any quasi IMRM, there is a permutation of objectives such that all the quasi-positive entries are moved to the positions above the diagonal of the corresponding quasi IVPR. Then the definition of consistent IMRMs is given as follows:

Definition 5 (Meng and Tan 2017) Let $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ be an IMRM. $\bar{A}$ is said to be consistent if there is an associated consistent quasi IMRM $\bar{A}' = \left(\bar{a}'_{ij}\right)_{n\times n}$ satisfying $\bar{a}'_{ij} = \bar{a}'_{ik} \cdot \bar{a}'_{kj}$ for all $i, k, j = 1, 2, \ldots, n$.

It is seen from Definition 5 that the method of defining the consistency of $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ is dependent on the multiplicative transitivity of consistent MRMs (Saaty 1980) and the arithmetic operations of interval numbers. However, in Meng and Tan (2017), the authors did not discuss how to compute the product of two quasi-positive intervals. In fact, the multiplication operation of generalized interval numbers (7) has been used after some analysis. For example, it is easy to verify that the entries in A'_1 satisfy the relation of $\bar{a}'_{ij} = \bar{a}'_{ik} \cdot \bar{a}'_{kj}$ $(i, k, j = 1, 2, \ldots, n)$ according to the multiplication operation (7). In other words, the objective of introducing the quasi-positive intervals is only to satisfy the mathematical relation of multiplicative transitivity.

On the other hand, we recall the method of defining the approximate consistency of IMRMs in Liu et al. (2017b). Let $\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ be a bijective mapping. For convenience, it is also assumed that σ denotes a permutation of $\{1, 2, \dots, n\}$. The application of σ to $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ leads to an IMRM with permutations $\bar{A}^\sigma = \left(\bar{a}^\sigma_{ij}\right)_{n\times n}$ where $\bar{a}^\sigma_{ij} = [a^-_{\sigma(i)\sigma(j)}, a^+_{\sigma(i)\sigma(j)}]$. We further define two MRMs $C^\sigma = \left(c^\sigma_{ij}\right)_{n\times n}$ and $D^\sigma = \left(d^\sigma_{ij}\right)_{n\times n}$ where

$$c^\sigma_{ij} = \begin{cases} a^-_{\sigma(i)\sigma(j)}, & i < j, \\ 1, & i = j, \\ a^+_{\sigma(i)\sigma(j)}, & i > j, \end{cases} \quad d^\sigma_{ij} = \begin{cases} a^+_{\sigma(i)\sigma(j)}, & i < j, \\ 1, & i = j, \\ a^-_{\sigma(i)\sigma(j)}, & i > j. \end{cases} \tag{10}$$

Then the approximate consistency of IMRMs is defined as follows:

Definition 6 (Liu et al. 2017b) An IMRM $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ exhibits approximate consistency, if there is a permutation σ such that $C^\sigma = \left(c^\sigma_{ij}\right)_{n\times n}$ and $D^\sigma = \left(d^\sigma_{ij}\right)_{n\times n}$ are all consistent.

One can see from Definition 6 that the approximate consistency of IMRMs is based on the consistency of two boundary matrices $C^\sigma = \left(c^\sigma_{ij}\right)_{n\times n}$ and $D^\sigma = \left(d^\sigma_{ij}\right)_{n\times n}$. The basic idea behind the approximate consistency is to stress that IMRMs are inconsistent in nature, which is in agreement with the idea of fuzzy sets (Zadeh 1965). The consistency property of IMRMs is characterized by using the consistency property of the two boundary matrices offered by the decision maker. The method of defining the approximate consistency is related to the process of constructing IMRMs. As shown in Dong et al. (2016) and Li et al. (2019), Definition 6 can be called the boundary consistency of IMRMs under permutations of objectives. Moreover, the reciprocal property of $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ has been used in Definition 6, and it is independent of the arithmetic operations of interval numbers.

3.2 The equivalence of the two approaches

We further investigate the relation of consistent IMRMs according to Definitions 5 and 6. The following result is obtained:

Theorem 2 *Let $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ be an IMRM. The method of defining approximate consistency of $\bar{A}$ in Definition 6 is equivalent to that of defining the consistency of $\bar{A}$ in Definition 5.*

Proof Let $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ be an IMRM with approximate consistency according to Definition 6. There exists a permutation σ of $\{1, 2, \dots, n\}$ such that the two multiplicative reciprocal relations C^σ and D^σ are all consistent. For convenience, we write C^σ and D^σ as follows:

$$C^\sigma = \begin{bmatrix} 1 & a^-_{\sigma(1)\sigma(2)} & \cdots & a^-_{\sigma(1)\sigma(n)} \\ a^+_{\sigma(2)\sigma(1)} & 1 & \cdots & a^-_{\sigma(2)\sigma(n)} \\ \vdots & \vdots & \ddots & \vdots \\ a^+_{\sigma(n)\sigma(1)} & a^+_{\sigma(n)\sigma(2)} & \cdots & 1 \end{bmatrix},$$

$$D^\sigma = \begin{bmatrix} 1 & a^+_{\sigma(1)\sigma(2)} & \cdots & a^+_{\sigma(1)\sigma(n)} \\ a^-_{\sigma(2)\sigma(1)} & 1 & \cdots & a^+_{\sigma(2)\sigma(n)} \\ \vdots & \vdots & \ddots & \vdots \\ a^-_{\sigma(n)\sigma(1)} & a^-_{\sigma(n)\sigma(2)} & \cdots & 1 \end{bmatrix}.$$

In what follows, the entries in C^σ and D^σ are used to form a quasi IVPR $\bar{A}'^\sigma = (\bar{a}^\sigma_{ij})_{n\times n}$ or $\tilde{A}'^\sigma = (\tilde{a}^\sigma_{ij})_{n\times n}$ as follows:

$$\bar{A}'^\sigma = \begin{bmatrix} [1,1] & \left[a^-_{\sigma(1)\sigma(2)}, a^+_{\sigma(1)\sigma(2)}\right] & \cdots & \left[a^-_{\sigma(1)\sigma(n)}, a^+_{\sigma(1)\sigma(n)}\right] \\ \left[a^+_{\sigma(2)\sigma(1)}, a^-_{\sigma(2)\sigma(1)}\right] & [1,1] & \cdots & \left[a^-_{\sigma(2)\sigma(n)}, a^+_{\sigma(2)\sigma(n)}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \left[a^+_{\sigma(n)\sigma(1)}, a^-_{\sigma(n)\sigma(1)}\right] & \left[a^+_{\sigma(n)\sigma(2)}, a^-_{\sigma(n)\sigma(2)}\right] & \cdots & [1,1] \end{bmatrix},$$

or

$$\tilde{A}'^\sigma = \begin{bmatrix} [1,1] & \left[a^+_{\sigma(1)\sigma(2)}, a^-_{\sigma(1)\sigma(2)}\right] & \cdots & \left[a^+_{\sigma(1)\sigma(n)}, a^-_{\sigma(1)\sigma(n)}\right] \\ \left[a^-_{\sigma(2)\sigma(1)}, a^+_{\sigma(2)\sigma(1)}\right] & [1,1] & \cdots & \left[a^+_{\sigma(2)\sigma(n)}, a^-_{\sigma(2)\sigma(n)}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \left[a^-_{\sigma(n)\sigma(1)}, a^+_{\sigma(n)\sigma(1)}\right] & \left[a^-_{\sigma(n)\sigma(2)}, a^+_{\sigma(n)\sigma(2)}\right] & \cdots & [1,1] \end{bmatrix}.$$

By considering the consistency of C^σ and D^σ, it is verified that the consistent relations $\bar{a}^\sigma_{ij} = \bar{a}^\sigma_{ik} \cdot \bar{a}^\sigma_{kj}$ and $\tilde{a}^\sigma_{ij} = \tilde{a}^\sigma_{ik} \cdot \tilde{a}^\sigma_{kj}$ are satisfied for $\tilde{A}'^\sigma$ and $\bar{A}'^\sigma$ respectively. As shown in Meng and Tan (2017), the consistency in Definition 5 is invariant with respect to the permutations σ. It is concluded that $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ is consistent according to Definition 5.

On the contrary, suppose that $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ is an IMRM with consistency following Definition 5. There exists an associated quasi IMRM $\bar{A}' = \left(\bar{a}'_{ij}\right)_{n\times n}$ satisfying $\bar{a}'_{ij} = \bar{a}'_{ik} \cdot \bar{a}'_{kj}$ for all $i, k, j = 1, 2, \ldots, n$. Without loss of generality, it is assumed that

$$\bar{A}' = \begin{bmatrix} [1,1] & \left[a^+_{12}, a^-_{12}\right] & \cdots & \left[a^+_{1n}, a^-_{1n}\right] \\ \left[a^-_{21}, a^+_{21}\right] & [1,1] & \cdots & \left[a^+_{2n}, a^-_{2n}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \left[a^-_{n1}, a^+_{n1}\right] & \left[a^-_{n2}, a^+_{n2}\right] & \cdots & [1,1] \end{bmatrix}.$$

In fact, if another quasi IMRM $\tilde{A}'$ satisfies the condition $\bar{a}'_{ij} = \bar{a}'_{ik} \cdot \bar{a}'_{kj}$, we can perform a permutation σ of $\tilde{A}'$ such that $\bar{A}'$ is determined from $\tilde{A}'$ according to Theorem 1. Applying the condition $\bar{a}'_{ij} = \bar{a}'_{ik} \cdot \bar{a}'_{kj}$, we have the following two consistent MRMs:

$$C = \begin{bmatrix} 1 & a_{12}^{-} & \cdots & a_{1n}^{-} \\ a_{21}^{+} & 1 & \cdots & a_{2n}^{-} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^{+} & a_{n2}^{+} & \cdots & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & a_{12}^{+} & \cdots & a_{1n}^{+} \\ a_{21}^{-} & 1 & \cdots & a_{2n}^{+} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^{-} & a_{n2}^{-} & \cdots & 1 \end{bmatrix}.$$

Based on Definition 6, $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ is of approximate consistency. The proof of the theorem is completed. □

In what follows, we use an example to illustrate Theorem 2. Let us consider the following IVPR:

$$\bar{A}_2 = \left[\begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline x_1 & [1,1] & [1,1.3] & [1.2,1.3] & [1.2,1.6] \\ x_2 & \left[10/13,1\right] & [1,1] & [1,1.2] & \left[1.2,16/13\right] \\ x_3 & \left[10/13,10/12\right] & \left[10/12,1\right] & [1,1] & \left[1,16/13\right] \\ x_4 & \left[10/16,10/12\right] & \left[13/16,10/12\right] & \left[13/16,1\right] & [1,1] \end{array}\right].$$

By considering the permutation $\sigma = (1, 3, 2, 4)$, it gives

$$C^{\sigma} = \begin{bmatrix} 1 & 1.2 & 1 & 1.2 \\ 10/12 & 1 & 10/12 & 1 \\ 1 & 1.2 & 1 & 1.2 \\ 10/12 & 1 & 10/12 & 1 \end{bmatrix},$$

$$D^{\sigma} = \begin{bmatrix} 1 & 1.3 & 1.3 & 1.6 \\ 10/13 & 1 & 1 & 16/13 \\ 10/13 & 1 & 1 & 16/13 \\ 10/16 & 13/16 & 13/16 & 1 \end{bmatrix}.$$

One can determine that C^{σ} and D^{σ} are all consistent. This means that the matrix $\bar{A}_2$ is of approximate consistency in virtue of Definition 6. Moreover, based on the concept of quasi-positive intervals, we construct a quasi IMRM as follows:

$$\bar{A}'_2 = \left[\begin{array}{c|cccc} & x_1 & x_3 & x_2 & x_4 \\ \hline x_1 & [1,1] & [1.2,1.3] & [1,1.3] & [1.2,1.6] \\ x_3 & [10/12,10/13] & [1,1] & [10/12,1] & [1,16/13] \\ x_2 & [1,10/13] & [1.2,1] & [1,1] & [1.2,16/13] \\ x_4 & [10/12,10/16] & [1,13/16] & [10/12,13/16] & [1,1] \end{array}\right].$$

It is computed that the entries in $\bar{A}'_2$ satisfy the consistent relation $\bar{a}'_{ij} = \bar{a}'_{ik} \cdot \bar{a}'_{kj}$ in terms of the multiplication operation (7). This means that the matrix $\bar{A}_2$ is consistent in terms of Definition 5.

4 The methods for defining consistency of IARMs

Let us recall the methods of defining the consistency of IARMs. There are two kinds of consistency definitions for IARMs, which is following the methods of defining the consistency of ARMs (Tanino 1984). One is multiplicative consistency similar to that of IMRMs and the other is additive consistency.

4.1 Consistency definitions of IARMs

It is convenient to recall the definition of quasi IARMs as follows:

Definition 7 (Meng et al. 2017c) Assume that $\bar{B} = \left(\bar{b}_{ij}\right)_{n\times n}$ is an IARM. $\bar{B}' = \left(\bar{b}'_{ij}\right)_{n\times n}$ is a quasi IARM with respect to $\bar{B}$ if

$$\begin{cases} \bar{b}'_{ij} = \bar{b}_{ij}, \\ \bar{b}'_{ji} = \bar{b}^{\circ}_{ji}, \end{cases} \quad \text{or} \quad \begin{cases} \bar{b}'_{ij} = \bar{b}^{\circ}_{ij}, \\ \bar{b}'_{ji} = \bar{b}_{ji}, \end{cases} \tag{11}$$

for all $i, j = 1, 2, \dots, n$, where $\bar{b}^{\circ}_{ij} = \left[b^-_{ij}, b^+_{ij}\right]^{\circ} = \left[b^+_{ij}, b^-_{ij}\right]$.

Obviously, Definition 7 is similar to Definition 4 and one can construct $2^{n(n-1)/2} - 1$ quasi IARMs with respect to $\bar{B}$. As shown in Theorem 1, there is a permutation such that all the entries above the diagonal of the quasi IARM are quasi-positive intervals. Following the idea in Definition 5, the multiplicative consistency of IARMs is defined as follows:

Definition 8 (Meng et al. 2017c) Let $\bar{B} = \left(\bar{b}_{ij}\right)_{n\times n}$ be an IARM and $\bar{B}' = \left(\bar{b}'_{ij}\right)_{n\times n}$ be a quasi one with respect to $\bar{B}$. $\bar{B} = \left(\bar{b}_{ij}\right)_{n\times n}$ is of multiplicative consistency, if there is a quasi-matrix $\bar{B}' = \left(\bar{b}'_{ij}\right)_{n\times n}$ satisfying

$$\bar{b}'_{ij} \cdot \bar{b}'_{jk} \cdot \bar{b}'_{ki} = \bar{b}'_{ji} \cdot \bar{b}'_{ik} \cdot \bar{b}'_{kj}, \tag{12}$$

for all $i, k, j = 1, 2, \dots, n$.

It is noted from Definition 8 that the multiplicative consistency of IARMs is dependent on the method of defining multiplicative consistency of ARMs in Tanino (1984) and the multiplication operation of interval numbers. The objective of introducing quasi-positive intervals is only to satisfy the relation (12). In addition, based on the addition and substraction operations of interval numbers together with the quasi-positive intervals, the additive consistency of IARMs can be defined as follows:

Definition 9 (Meng et al. 2017a) It is assumed that $\bar{B}' = \left(\bar{b}'_{ij}\right)_{n\times n}$ is a quasi IARM constructed from $\bar{B} = \left(\bar{b}_{ij}\right)_{n\times n}$. If there is a $\bar{B}' = \left(\bar{b}'_{ij}\right)_{n\times n}$ such that one of the following three relations is satisfied:

$$\bar{b}'_{ij} = \bar{b}'_{ik} - \bar{b}'_{jk} + 0.5, \tag{13}$$

$$\bar{b}'_{ij} + 0.5 = \bar{b}'_{ik} + \bar{b}'_{kj}, \tag{14}$$

$$\bar{b}'_{ij} + \bar{b}'_{jk} + \bar{b}'_{ki} = \bar{b}'_{ji} + \bar{b}'_{ik} + \bar{b}'_{kj}, \tag{15}$$

for all $i, k, j = 1, 2, \dots, n$, $\bar{B} = \left(\bar{b}_{ij}\right)_{n\times n}$ is of additive consistency.

In order to illustrate the above consistency definitions, we investigate the following matrix:

$$\bar{B}_2 = \begin{bmatrix} [0.5, 0.5] & [0.3, 0.4] & [0.25, 0.55] \\ [0.6, 0.7] & [0.5, 0.5] & [0.45, 0.65] \\ [0.45, 0.75] & [0.35, 0.55] & [0.5, 0.5] \end{bmatrix}.$$

When the entries [0.3, 0.4], [0.25, 0.55] and [0.45, 0.65] are replaced by [0.4, 0.3], [0.55, 0.25] and [0.65, 0.45], respectively, one has

$$\bar{B}'_2 = \begin{bmatrix} [0.5, 0.5] & [0.4, 0.3] & [0.55, 0.25] \\ [0.6, 0.7] & [0.5, 0.5] & [0.65, 0.45] \\ [0.45, 0.75] & [0.35, 0.55] & [0.5, 0.5] \end{bmatrix}.$$

It can be easily verified that the entries in $\bar{B}'_2$ satisfy the relation (15), meaning that $\bar{B}_2$ is of additive consistency in terms of Definition 9.

Furthermore, we recall the method of defining the additive approximation-consistency of IARMs in Liu et al. (2018b). Applying σ to $\bar{B} = \left(\bar{b}_{ij}\right)_{n\times n}$, we obtain $\bar{B}^{\sigma} = \left(\bar{b}^{\sigma}_{ij}\right)_{n\times n}$ where $\bar{b}^{\sigma}_{ij} = [b^{-}_{\sigma(i)\sigma(j)}, b^{+}_{\sigma(i)\sigma(j)}]$. Defining two ARMs as $P^{\sigma} = (p^{\sigma}_{ij})_{n\times n}$ and $Q^{\sigma} = (q^{\sigma}_{ij})_{n\times n}$ where

$$p^{\sigma}_{ij} = \begin{cases} b^{-}_{\sigma(i)\sigma(j)}, & i < j, \\ 0.5, & i = j, \\ b^{+}_{\sigma(i)\sigma(j)}, & i > j, \end{cases} \qquad q^{\sigma}_{ij} = \begin{cases} b^{+}_{\sigma(i)\sigma(j)}, & i < j, \\ 0.5, & i = j, \\ b^{-}_{\sigma(i)\sigma(j)}, & i > j, \end{cases} \tag{16}$$

the definition of additive approximation-consistency is given as follows:

Definition 10 (Liu et al. 2018b) Suppose that $\bar{B}^{\sigma}$ is an IARM with permutations σ. If there exists a permutation σ such that P^{σ} and Q^{σ} are all of additive consistency, $\bar{B}$ is of additive approximation-consistency.

Similarly, the multiplicative approximation-consistency of IARMs can be also defined. That is, one has the following definition:

Definition 11 Assume that $\bar{B}^{\sigma}$ is an IARM with permutations σ. If there exists a permutation σ such that P^{σ} and Q^{σ} are all of multiplicative consistency, $\bar{B}$ is of multiplicative approximation-consistency.

Different to Definitions 8 and 9, the additive and multiplicative approximation-consistency of IARMs in Definitions 10 and 11 are based on two boundary matrices $P^{\sigma} = (p^{\sigma}_{ij})_{n\times n}$ and $Q^{\sigma} = (q^{\sigma}_{ij})_{n\times n}$, and they can be called the boundary consistency of IARMs under permutations of alternatives following the existing concept (Dong et al. 2016; Li et al. 2019).

4.2 The equivalence of the two approaches

The relations of consistent IARMs according to Definitions 8–11 are considered, respectively. In terms of Definitions 8 and 11, we have the following result:

Theorem 3 *Let $\bar{B} = \left(\bar{b}_{ij}\right)_{n\times n}$ be an IARM. The method of defining multiplicative consistency of $\bar{B} = \left(\bar{b}_{ij}\right)_{n\times n}$ according to Definition 8 is equivalent to that of multiplicative approximation-consistency in terms of Definition 11.*

Proof It is assumed that $\bar{B} = \left(\bar{b}_{ij}\right)_{n\times n}$ is of multiplicative approximation-consistency in virtue of Definition 11. There is a permutation σ such that the two matrices P^{σ} and Q^{σ} defined by using (16) are of multiplicative consistency. It is convenient to express P^{σ} and Q^{σ} as

$$P^{\sigma} = \begin{bmatrix} 0.5 & b^{-}_{\sigma(1)\sigma(2)} & \cdots & b^{-}_{\sigma(1)\sigma(n)} \\ b^{+}_{\sigma(2)\sigma(1)} & 0.5 & \cdots & b^{-}_{\sigma(2)\sigma(n)} \\ \vdots & \vdots & \ddots & \vdots \\ b^{+}_{\sigma(n)\sigma(1)} & b^{+}_{\sigma(n)\sigma(2)} & \cdots & 0.5 \end{bmatrix}, \tag{17}$$

and

$$Q^{\sigma} = \begin{bmatrix} 0.5 & b^{+}_{\sigma(1)\sigma(2)} & \cdots & b^{+}_{\sigma(1)\sigma(n)} \\ b^{-}_{\sigma(2)\sigma(1)} & 0.5 & \cdots & b^{+}_{\sigma(2)\sigma(n)} \\ \vdots & \vdots & \ddots & \vdots \\ b^{-}_{\sigma(n)\sigma(1)} & b^{-}_{\sigma(n)\sigma(2)} & \cdots & 0.5 \end{bmatrix}. \tag{18}$$

Then the following relations are satisfied

$$p^{\sigma}_{ij} \cdot p^{\sigma}_{jk} \cdot p^{\sigma}_{ki} = p^{\sigma}_{ji} \cdot p^{\sigma}_{ik} \cdot p^{\sigma}_{kj}, \tag{19}$$

$$q^{\sigma}_{ij} \cdot q^{\sigma}_{jk} \cdot q^{\sigma}_{ki} = q^{\sigma}_{ji} \cdot q^{\sigma}_{ik} \cdot q^{\sigma}_{kj}. \tag{20}$$

Clearly, we can construct a quasi IARM as

$$\bar{B}'^{\sigma} = \begin{bmatrix} [0.5, 0.5] & \left[b^{-}_{\sigma(1)\sigma(2)}, b^{+}_{\sigma(1)\sigma(2)}\right] & \cdots & \left[b^{-}_{\sigma(1)\sigma(n)}, b^{+}_{\sigma(1)\sigma(n)}\right] \\ \left[b^{+}_{\sigma(2)\sigma(1)}, b^{-}_{\sigma(2)\sigma(1)}\right] & [0.5, 0.5] & \cdots & \left[b^{-}_{\sigma(2)\sigma(n)}, b^{+}_{\sigma(2)\sigma(n)}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \left[b^{+}_{\sigma(n)\sigma(1)}, b^{-}_{\sigma(n)\sigma(1)}\right] & \left[b^{+}_{\sigma(n)\sigma(2)}, b^{-}_{\sigma(n)\sigma(2)}\right] & \cdots & [0.5, 0.5] \end{bmatrix}.$$

In virtue of (19) and (20), it is noted that the entries in $\bar{B}'^{\sigma}$ satisfy the relation $\bar{b}'^{\sigma}_{ij} \cdot \bar{b}'^{\sigma}_{jk} \cdot \bar{b}'^{\sigma}_{ki} = \bar{b}'^{\sigma}_{ji} \cdot \bar{b}'^{\sigma}_{ik} \cdot \bar{b}'^{\sigma}_{kj}$ under the multiplication operations of interval numbers.

Inversely, if $\bar{B} = \left(\bar{b}_{ij}\right)_{n\times n}$ is of multiplicative consistency by using Definition 8, there is a quasi IARM $\bar{B}' = \left(\bar{b}'_{ij}\right)_{n\times n}$ satisfying the relation of $\bar{b}'_{ij} \cdot \bar{b}'_{jk} \cdot \bar{b}'_{ki} = \bar{b}'_{ji} \cdot \bar{b}'_{ik} \cdot \bar{b}'_{kj}$ for all $i, k, j = 1, 2, \ldots, n$. It is supposed that

$$\bar{B}' = \begin{bmatrix} [0.5, 0.5] & \left[b^{+}_{12}, b^{-}_{12}\right] & \cdots & \left[b^{+}_{1n}, b^{-}_{1n}\right] \\ \left[b^{-}_{21}, b^{+}_{21}\right] & [0.5, 0.5] & \cdots & \left[b^{+}_{2n}, b^{-}_{2n}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \left[b^{-}_{n1}, b^{+}_{n1}\right] & \left[b^{-}_{n2}, b^{+}_{n2}\right] & \cdots & [0.5, 0.5] \end{bmatrix}.$$

When another quasi IVPR $\tilde{B}'$ satisfies the condition of multiplicative consistency, there is a permutation σ such that $\bar{B}'$ is obtained from $\tilde{B}'$. The application of $\bar{b}'_{ij} \cdot \bar{b}'_{jk} \cdot \bar{b}'_{ki} = \bar{b}'_{ji} \cdot \bar{b}'_{ik} \cdot \bar{b}'_{kj}$ leads to two ARMs with multiplicative consistency as follows

$$P = \begin{bmatrix} 0.5 & b^{-}_{12} & \cdots & b^{-}_{1n} \\ b^{+}_{21} & 0.5 & \cdots & b^{-}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b^{+}_{n1} & b^{+}_{n2} & \cdots & 0.5 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0.5 & b^{+}_{12} & \cdots & b^{+}_{1n} \\ b^{-}_{21} & 0.5 & \cdots & b^{+}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b^{-}_{n1} & b^{-}_{n2} & \cdots & 0.5 \end{bmatrix}.$$

This means that the matrix $\bar{B} = \left(\bar{b}_{ij}\right)_{n\times n}$ is of multiplicative approximation-consistency according to Definition 11. The proof of the theorem is completed. □

For example, we consider the IARM as follows:

$$\bar{B}_3 = \left[\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & [1/2, 1/2] & [1/3, 3/4] & [1/4, 3/4] \\ x_2 & [1/4, 2/3] & [1/2, 1/2] & [2/5, 1/2] \\ x_3 & [1/4, 3/4] & [1/2, 3/5] & [1/2, 1/2] \end{array}\right].$$

A quasi IARM can be constructed as

$$\bar{B}'_3 = \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & [1/2, 1/2] & [1/3, 3/4] & [1/4, 3/4] \\ x_2 & [2/3, 1/4] & [1/2, 1/2] & [2/5, 1/2] \\ x_3 & [3/4, 1/4] & [3/5, 1/2] & [1/2, 1/2] \end{array}.$$

It is found that the entries in $\bar{B}'_3$ satisfy the condition of multiplicative consistency, meaning that $\bar{B}_3$ is of multiplicative consistency according to Definition 8. In addition, we can construct two ARMs as follows:

$$P_3 = \begin{bmatrix} 1/2 & 1/3 & 1/4 \\ 2/3 & 1/2 & 2/5 \\ 3/4 & 3/5 & 1/2 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 1/2 & 3/4 & 3/4 \\ 1/4 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/2 \end{bmatrix}.$$

It is seen that the matrices P_3 and Q_3 are of multiplicative consistency, meaning that the matrix $\bar{B}_3$ is of multiplicative approximation-consistency according to Definition 11.

In addition, by considering Definitions 9 and 10, we obtain the following result:

Theorem 4 *It is assumed that $\bar{B} = (\bar{b}_{ij})_{n\times n}$ is an IARM. The method of defining additive consistency of $\bar{B} = (\bar{b}_{ij})_{n\times n}$ from Definition 9 is equivalent to that of additive approximation-consistency in terms of Definition 10.*

Proof Suppose that $\bar{B} = (\bar{b}_{ij})_{n\times n}$ is of additive approximation-consistency (Definition 10). There is a permutation σ such that two matrices P^σ and Q^σ defined by using (16) are additively consistent. That is, by applying (17) and (18), it follows

$$p^\sigma_{ij} + p^\sigma_{jk} + p^\sigma_{ki} = p^\sigma_{ji} + p^\sigma_{ik} + p^\sigma_{kj}, \tag{21}$$

$$q^\sigma_{ij} + q^\sigma_{jk} + q^\sigma_{ki} = q^\sigma_{ji} + q^\sigma_{ik} + q^\sigma_{kj}. \tag{22}$$

By constructing the quasi IARM through P^σ and Q^σ as $\bar{B}'^\sigma$ in Theorem 3, it is found that one has the relation of $\bar{b}'^\sigma_{ij} + \bar{b}'^\sigma_{jk} + \bar{b}'^\sigma_{ki} = \bar{b}'^\sigma_{ji} + \bar{b}'^\sigma_{ik} + \bar{b}'^\sigma_{kj}$ by using (21) and (22). This means that $\bar{B} = (\bar{b}_{ij})_{n\times n}$ is additively consistent as shown in Definition 9.

On the contrary, it is assumed that $\bar{B} = (\bar{b}_{ij})_{n\times n}$ is of additive consistency by using Definition 9. Similar to Theorem 3, a quasi IARM is constructed and two ARMs with additive consistency can be given. That is, $\bar{B} = (\bar{b}_{ij})_{n\times n}$ is of additive approximation-consistency according to Definition 10. □

4.3 A further finding

Furthermore, we consider an IARM $\bar{B} = (\bar{b}_{ij})_{n\times n}$ with multiplicative consistency according to Definition 8. Then there is an associated multiplicative consistent quasi-IARM $\bar{B}' = \left(\bar{b}'_{ij}\right)_{n\times n}$. It is stated that the condition $\bar{b}'_{ij} \cdot \bar{b}'_{jk} \cdot \bar{b}'_{ki} = \bar{b}'_{ji} \cdot \bar{b}'_{ik} \cdot \bar{b}'_{kj}$

for each triple (i, j, k) corresponds to one of the following four cases (Meng et al. 2017c):

$$\bar{b}_{ij} \cdot \bar{b}_{jk} \cdot \bar{b}_{ki} = \bar{b}^{\circ}_{ji} \cdot \bar{b}^{\circ}_{ik} \cdot \bar{b}^{\circ}_{kj}, \tag{23}$$

$$\bar{b}^{\circ}_{ij} \cdot \bar{b}_{jk} \cdot \bar{b}_{ki} = \bar{b}_{ji} \cdot \bar{b}^{\circ}_{ik} \cdot \bar{b}^{\circ}_{kj}, \tag{24}$$

$$\bar{b}_{ij} \cdot \bar{b}^{\circ}_{jk} \cdot \bar{b}_{ki} = \bar{b}^{\circ}_{ji} \cdot \bar{b}^{\circ}_{ik} \cdot \bar{b}_{kj}, \tag{25}$$

$$\bar{b}_{ij} \cdot \bar{b}_{jk} \cdot \bar{b}^{\circ}_{ki} = \bar{b}^{\circ}_{ji} \cdot \bar{b}_{ik} \cdot \bar{b}^{\circ}_{kj}. \tag{26}$$

Let us first analyze the relation (23) and obtain the following result:

Theorem 5 *Let $\bar{B} = \left(\bar{b}_{ij}\right)_{n\times n}$ be an IARM with multiplicative consistency (Definition 8). The relation (23) is satisfied if and only if the interval matrix $\bar{B}$ degenerates to an ARM.*

Proof When the matrix $\bar{B} = \left(\bar{b}_{ij}\right)_{n\times n}$ is an IARM with multiplicative consistency according to Definition 8, there is an associated multiplicative consistent quasi-IARM $\bar{B}' = \left(\bar{b}'_{ij}\right)_{n\times n}$ satisfying $\bar{b}'_{ij} \cdot \bar{b}'_{jk} \cdot \bar{b}'_{ki} = \bar{b}'_{ji} \cdot \bar{b}'_{ik} \cdot \bar{b}'_{kj}$ for all $i, k, j = 1, 2, \ldots, n$. If the relation (23) is satisfied, we have the following relations:

$$\begin{cases} b^{-}_{ij}b^{-}_{jk}b^{-}_{ki} = b^{+}_{ji}b^{+}_{kj}b^{+}_{ik}, \\ b^{+}_{ij}b^{+}_{jk}b^{+}_{ki} = b^{-}_{ji}b^{-}_{kj}b^{-}_{ik}, \end{cases} \tag{27}$$

for $\forall i, j = 1, 2, \ldots, n$. Moreover, since $b^{-}_{ij} \leq b^{+}_{ij}$, it follows

$$\begin{cases} b^{-}_{ij}b^{-}_{jk}b^{-}_{ki} \leq b^{+}_{ij}b^{+}_{jk}b^{+}_{ki}, \\ b^{+}_{ji}b^{+}_{kj}b^{+}_{ik} \geq b^{-}_{ji}b^{-}_{kj}b^{-}_{ik}. \end{cases} \tag{28}$$

In other words, only when $b^{-}_{ij} = b^{+}_{ij}, \forall i, j = 1, 2, \ldots, n$, the relations in (27) are satisfied. This implies that the interval-valued matrix $\bar{B} = \left(\bar{b}_{ij}\right)_{n\times n}$ degenerates to an ARM.

On the other hand, if all the entries in the matrix $\bar{B}$ satisfy $b^{-}_{ij} = b^{+}_{ij} = b_{ij}$ $(\forall i, j = 1, 2, \ldots, n)$. We have the following results:

$$\begin{cases} \bar{b}_{ij} \cdot \bar{b}_{jk} \cdot \bar{b}_{ki} = b^{-}_{ij}b^{-}_{jk}b^{-}_{ki} = b^{+}_{ij}b^{+}_{jk}b^{+}_{ki} = b_{ij}b_{jk}b_{ki}, \\ \bar{b}^{\circ}_{ji} \cdot \bar{b}^{\circ}_{ik} \cdot \bar{b}^{\circ}_{kj} = b^{+}_{ji}b^{+}_{ik}b^{+}_{kj} = b^{-}_{ji}b^{-}_{ik}b^{-}_{kj} = b_{ji}b_{ik}b_{kj}. \end{cases} \tag{29}$$

Under the assumption of multiplicative consistency (Tanino 1984), one has $b_{ij}b_{jk}b_{ki} = b_{ji}b_{ik}b_{kj}$, meaning that $\bar{b}_{ij} \cdot \bar{b}_{jk} \cdot \bar{b}_{ki} = \bar{b}^{\circ}_{ji} \cdot \bar{b}^{\circ}_{ik} \cdot \bar{b}^{\circ}_{kj}$. The proof is completed. □

It is seen from Theorem 4 that the relation (23) is only related to an ARM with multiplicative consistency. In general, it is not suitable to state that a consistent IARM could satisfy (23). Additionally, one can see from the findings in Liu et al. (2018b) that when the boundary matrices are consistent for any permutation, the interval-valued matrix $\bar{B} = \left(\bar{b}_{ij}\right)_{n\times n}$ degenerates to an ARM. That is to say, the relation (23) is equivalent to the particular case that the two boundary matrices $P^\sigma = (p_{ij}^\sigma)_{n\times n}$ and $Q^\sigma = (q_{ij}^\sigma)_{n\times n}$ are of multiplicative consistency for any permutation σ.

Second, we consider the cases of (24)–(26). As stated in Meng et al. (2017c), the multiplicative consistency of IARMs given in Definition 8 is independent of the permutations of alternatives. When introducing a permutation σ, the relations (24)–(26) can be rewritten as the following three cases, respectively:

$$\bar{b}^{\circ}_{\sigma(i)\sigma(j)} \cdot \bar{b}_{\sigma(j)\sigma(k)} \cdot \bar{b}_{\sigma(k)\sigma(i)} = \bar{b}_{\sigma(j)\sigma(i)} \cdot \bar{b}^{\circ}_{\sigma(i)\sigma(k)} \cdot \bar{b}^{\circ}_{\sigma(k)\sigma(j)}, \tag{30}$$

$$\bar{b}_{\sigma(i)\sigma(j)} \cdot \bar{b}^{\circ}_{\sigma(j)\sigma(k)} \cdot \bar{b}_{\sigma(k)\sigma(i)} = \bar{b}^{\circ}_{\sigma(j)\sigma(i)} \cdot \bar{b}^{\circ}_{\sigma(i)\sigma(k)} \cdot \bar{b}_{\sigma(k)\sigma(j)}, \tag{31}$$

$$\bar{b}_{\sigma(i)\sigma(j)} \cdot \bar{b}_{\sigma(j)\sigma(k)} \cdot \bar{b}^{\circ}_{\sigma(k)\sigma(i)} = \bar{b}^{\circ}_{\sigma(j)\sigma(i)} \cdot \bar{b}_{\sigma(i)\sigma(k)} \cdot \bar{b}^{\circ}_{\sigma(k)\sigma(j)}. \tag{32}$$

Moreover, the relations (30)–(32) correspond to the following ones respectively:

$$\begin{cases} b^+_{\sigma(i)\sigma(j)} b^-_{\sigma(j)\sigma(k)} b^-_{\sigma(k)\sigma(i)} = b^-_{\sigma(j)\sigma(i)} b^+_{\sigma(i)\sigma(k)} b^+_{\sigma(k)\sigma(j)}, \\ b^+_{\sigma(i)\sigma(j)} b^+_{\sigma(j)\sigma(k)} b^+_{\sigma(k)\sigma(i)} = b^+_{\sigma(j)\sigma(i)} b^-_{\sigma(i)\sigma(k)} b^-_{\sigma(k)\sigma(j)}, \end{cases} \tag{33}$$

$$\begin{cases} b^-_{\sigma(i)\sigma(j)} b^+_{\sigma(j)\sigma(k)} b^-_{\sigma(k)\sigma(i)} = b^+_{\sigma(j)\sigma(i)} b^+_{\sigma(i)\sigma(k)} b^-_{\sigma(k)\sigma(j)}, \\ b^+_{\sigma(i)\sigma(j)} b^-_{\sigma(j)\sigma(k)} b^+_{\sigma(k)\sigma(i)} = b^-_{\sigma(j)\sigma(i)} b^-_{\sigma(i)\sigma(k)} b^+_{\sigma(k)\sigma(j)}, \end{cases} \tag{34}$$

$$\begin{cases} b^-_{\sigma(i)\sigma(j)} b^-_{\sigma(j)\sigma(k)} b^+_{\sigma(k)\sigma(i)} = b^+_{\sigma(j)\sigma(i)} b^-_{\sigma(i)\sigma(k)} b^+_{\sigma(k)\sigma(j)}, \\ b^+_{\sigma(i)\sigma(j)} b^+_{\sigma(j)\sigma(k)} b^-_{\sigma(k)\sigma(i)} = b^-_{\sigma(j)\sigma(i)} b^+_{\sigma(i)\sigma(k)} b^-_{\sigma(k)\sigma(j)}. \end{cases} \tag{35}$$

On the other hand, according to Theorem 3, $\bar{B} = \left(\bar{b}_{ij}\right)_{n\times n}$ is also with multiplicative approximation-consistency. This means that there is a permutation $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$ such that P^σ and Q^σ are of multiplicative consistency. It is further found that for boundary matrices P^σ and Q^σ with multiplicative consistency, we have the following results.

Case 1: When $i < k < j$ or $j < k < i$, it follows

$$b^+_{\sigma(i)\sigma(j)} b^-_{\sigma(j)\sigma(k)} b^-_{\sigma(k)\sigma(i)} = b^-_{\sigma(j)\sigma(i)} b^+_{\sigma(i)\sigma(k)} b^+_{\sigma(k)\sigma(j)},$$
$$b^-_{\sigma(i)\sigma(j)} b^+_{\sigma(j)\sigma(k)} b^+_{\sigma(k)\sigma(i)} = b^+_{\sigma(j)\sigma(i)} b^-_{\sigma(i)\sigma(k)} b^-_{\sigma(k)\sigma(j)}.$$

Case 2: When $j < i < k$ or $k < i < j$, one has

$$b^{+}_{\sigma(i)\sigma(j)}b^{-}_{\sigma(j)\sigma(k)}b^{+}_{\sigma(k)\sigma(i)} = b^{-}_{\sigma(j)\sigma(i)}b^{+}_{\sigma(i)\sigma(k)}b^{-}_{\sigma(k)\sigma(j)},$$
$$b^{-}_{\sigma(i)\sigma(j)}b^{+}_{\sigma(j)\sigma(k)}b^{-}_{\sigma(k)\sigma(i)} = b^{+}_{\sigma(j)\sigma(i)}b^{-}_{\sigma(i)\sigma(k)}b^{+}_{\sigma(k)\sigma(j)}.$$

Case 3: When $i < j < k$ or $k < j < i$, it gives

$$b^{+}_{\sigma(i)\sigma(j)}b^{+}_{\sigma(j)\sigma(k)}b^{-}_{\sigma(k)\sigma(i)} = b^{-}_{\sigma(j)\sigma(i)}b^{-}_{\sigma(i)\sigma(k)}b^{+}_{\sigma(k)\sigma(j)},$$
$$b^{-}_{\sigma(i)\sigma(j)}b^{-}_{\sigma(j)\sigma(k)}b^{+}_{\sigma(k)\sigma(i)} = b^{+}_{\sigma(j)\sigma(i)}b^{+}_{\sigma(i)\sigma(k)}b^{-}_{\sigma(k)\sigma(j)}.$$

The observations show that the relations (30)–(32) correspond to Cases 1–3, respectively.

For instance, we consider the following IVPR (Meng et al. 2017c):

$$\bar{B}_4 = \left[\begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline x_1 & [0.5, 0.5] & [0.427, 0.706] & [0.544, 0.698] & [0.439, 0.59] \\ x_2 & [0.294, 0.573] & [0.5, 0.5] & [0.49, 0.615] & [0.374, 0.512,] \\ x_3 & [0.302, 0.456] & [0.385, 0.51] & [0.5, 0.5] & [0.384, 0.396] \\ x_4 & [0.41, 0.561] & [0.488, 0.626] & [0.604, 0.616] & [0.5, 0.5] \end{array}\right].$$

A quasi IARM is constructed by introducing the permutation $\sigma = (1, 4, 3, 2)$ as follows:

$$\bar{B}'_4 = \left[\begin{array}{c|cccc} & x_1 & x_4 & x_3 & x_2 \\ \hline x_1 & [0.5, 0.5] & [0.439, 0.59] & [0.544, 0.698] & [0.427, 0.706] \\ x_4 & [0.561, 0.41] & [0.5, 0.5] & [0.604, 0.616] & [0.488, 0.626] \\ x_3 & [0.456, 0.302] & [0.396, 0.384] & [0.5, 0.5] & [0.385, 0.51] \\ x_2 & [0.573, 0.294] & [0.512, 0.374] & [0.615, 0.49] & [0.5, 0.5] \end{array}\right].$$

One can see that the relation (12) is satisfied for $\bar{B}'_4$, then $\bar{B}_4$ is of multiplicative consistency by using Definition 8. Correspondingly, the boundary matrices P_4^{σ} and Q_4^{σ} are given as follows:

$$P_4^{\sigma} = \left[\begin{array}{c|cccc} & x_1 & x_4 & x_3 & x_2 \\ \hline x_1 & 0.5 & 0.439 & 0.544 & 0.427 \\ x_4 & 0.61 & 0.5 & 0.604 & 0.488 \\ x_3 & 0.456 & 0.396 & 0.5 & 0.385 \\ x_2 & 0.573 & 0.512 & 0.615 & 0.5 \end{array}\right],$$

$$Q_4^{\sigma} = \left[\begin{array}{c|cccc} & x_1 & x_4 & x_3 & x_2 \\ \hline x_1 & 0.5 & 0.59 & 0.698 & 0.706 \\ x_4 & 0.41 & 0.5 & 0.616 & 0.626 \\ x_3 & 0.302 & 0.384 & 0.5 & 0.51 \\ x_2 & 0.294 & 0.374 & 0.49 & 0.5 \end{array}\right].$$

After some computations, we can determine that the relation (32) is equivalent to the one of Case 3. As an example, we compute $p^{\sigma}_{12}p^{\sigma}_{23}p^{\sigma}_{31} = p^{\sigma}_{21}p^{\sigma}_{13}p^{\sigma}_{32} \approx 0.1209$ and $q^{\sigma}_{12}q^{\sigma}_{23}q^{\sigma}_{31} = q^{\sigma}_{21}q^{\sigma}_{13}q^{\sigma}_{32} \approx 0.1098$. Since $\sigma(1) = 1, \sigma(2) = 4, \sigma(3) = 3, \sigma(4) = 2$ and for $i = 1 < j = 2 < k = 3$, Case 3 is satisfied. That is, we have

$$b^{+}_{\sigma(1)\sigma(2)}b^{+}_{\sigma(2)\sigma(3)}b^{-}_{\sigma(3)\sigma(1)} = b^{-}_{\sigma(2)\sigma(1)}b^{-}_{\sigma(1)\sigma(3)}b^{+}_{\sigma(3)\sigma(2)},$$
$$b^{-}_{\sigma(1)\sigma(2)}b^{-}_{\sigma(2)\sigma(3)}b^{+}_{\sigma(3)\sigma(1)} = b^{+}_{\sigma(2)\sigma(1)}b^{+}_{\sigma(1)\sigma(3)}b^{-}_{\sigma(3)\sigma(2)},$$

and obtain

$$b^{+}_{14}b^{+}_{43}b^{-}_{31} = b^{-}_{41}b^{-}_{13}b^{+}_{32} \approx 0.1098,$$
$$b^{-}_{14}b^{-}_{43}b^{+}_{31} = b^{+}_{41}b^{+}_{13}b^{-}_{32} \approx 0.1209.$$

Theorems 2–4 show that the effects of two methods for defining the consistency of IVPRs are equivalent. In the next section, we further discuss the idea behind the two approaches for defining the consistency of IVPRs and the consistency indexes of IVPRs.

5 Further comments and comparison

It is observed from the above analysis that the two approaches of defining the consistency of IVPRs are equivalent. The observations can be extended to analyze the existing consistency definitions of the preference relations with triangular fuzzy numbers (Liu et al. 2017a, 2018a; Meng et al. 2017b). The similar analysis has been omitted due to the direct extensions of the obtained results. It should be pointed out that the concept of consistency definition is an attempt to capture the strict transitivity of a comparison matrix (Saaty 1980; Tanino 1984), which is the source of the consistency definitions of IVPRs. As compared to consistency definition, the concept of consistency index is to quantify the inconsistency degree of a preference relation (Saaty 1980; Herrera-Viedma et al. 2007). A consistent preference relation corresponds to a particular value of its consistency index. For example, the consistency index (*CI*) and consistency ratio (*CR*) were defined by Saaty (1980) to measure the inconsistency degrees of MRMs. When the *CI* or *CR* of a matrix equals to zero, the matrix is consistent. The *CI* or *CR* of any inconsistent MRM is bigger than zero. A consistency measure of ARMs (*cl*) was defined by Herrera-Viedma et al. (2007). A consistent ARM corresponds to $cl = 1$ and the values of *cl* for all inconsistent ARMs are less than 1. The above observation reveal that if and only a preference relation is consistent, the value of the defined consistency index is a particular one. That is, a consistent preference relation is only a particular case with a fixed value of the defined consistency index. Hence, it is of much interest to further compare the ideas underlying the consistency definitions and consistency indexes, respectively.

5.1 The underlying ideas of the consistency definitions

In what follows, we further compare the idea behind the two approaches defining consistency of IVPRs in Definitions 5, 8, 9 and approximate consistency of IVPRs in Definitions 6, 10, 11, respectively. In order to analyze Definitions 5, 8, 9, it is inevitable to consider the generalized interval numbers and their arithmetic operations. One should ask what is the meaning of imaginary and quasi interval numbers.

For example, the meaning of the interval [0.2, 0.7] is the set of real numbers $\{x|0.2 \leq x \leq 0.7\}$. The interval number [0.7, 0.2] is imaginary or quasi and what is its meaning? Obviously, there is not any explanation for the definition of imaginary and quasi interval numbers (Meng et al. 2017a, 2016; Meng and Tan 2017; Meng et al. 2017c). The introduction of quasi-positive IVPRs is only to satisfy the mathematical relations of consistent judgements (Meng et al. 2017c; Meng and Tan 2017; Meng et al. 2016). Moreover, it is seen that the imaginary and quasi interval $[0.7, 0.2]$ denotes the interval $[0.7, +\infty) \cup (-\infty, 0.2]$(Zhai 1998; Moorse 1966). While the interval $\bar{a}_{ij} = [0.2, 0.7]$ stands for the preference intensity of the alternative x_i over the alternative x_j, the imaginary and quasi interval number $\bar{a}_{ij}^{\circ} = [0.7, 0.2] = [0.7, +\infty) \cup (-\infty, 0.2]$ means that the preference intensity is not in [0.2, 0.7]. Clearly, the explanation about the interval [0.7, 0.2] is not feasible for the preference intensity of decision makers on alternatives. The multiplicative consistency defined in Definitions 5 and 8 is not related to the multiplicative transitivity of interval numbers. Moreover, when the judgements of decision makers are expressed as IARMs, the additive consistency defined in Definition 9 becomes independent of the additive transitivity of interval numbers. On the other hand, the approximate consistency of IVPRs defined in Definitions 6, 10 and 11 are related to the boundary matrices and regardless of the arithmetic operations of interval numbers. The idea behind the approximate consistency is that the fuzzy-valued judgements are inconsistent in nature and the reciprocal properties of IVPRs should be considered. The boundary values of interval numbers should be carefully estimated when providing comparison ratios of alternatives. The permutations of alternatives reflect the randomness experienced by decision makers in comparing alternatives. The comparison shows that the more required properties of IVPRs have been incorporated in the concept of approximate consistency than the consistency based on the quasi-positive intervals. In addition, it is worth noting that in order to characterize the consistency of fuzzy-valued comparison matrices, an axiomatic approach has been proposed for triangular fuzzy multiplicative reciprocal matrices in Liu et al. (2017a) and IMRMs in Wang et al. (2019).

5.2 The ideas behind the consistency indexes of IVPRs

In general, the consistency indexes for measuring the inconsistency degree of IVPRs have been studied widely (Dong et al. 2015, 2016; Li et al. 2019; Liu et al. 2018c). In what follows, some reviews are reported by comparing with the consistency definitions of IVPRs.

- Optimistic consistency index of IVPRs

  It is seen from the consistency definitions of consistent IVPRs that an IVPR is of consistency if there is a consistent real-valued matrix whose entries belong to the interval-valued entries (Wang et al. 2005; Xu and Chen 2008; Lan et al. 2012). This kind of consistency definition has been compared by considering the others (Liu et al. 2017b; Krejčí 2017, 2019). It is noted that the mentioned consistency definitions are considered to be the best consistency indexes of IVPRs

(Dong et al. 2016; Li et al. 2019). In order to distinguish the concepts of consistency definition and consistency index analogous to the classic ones (Saaty 1980; Tanino 1984), here the statements related to consistent IVPRs are classified as consistency definitions.

- Pessimistic consistency index of IVPRs

  Using the associated real-valued matrices of IVPRs, the consistency indexes of IVPRs have been proposed by defining the worst derivation degree from a consistent real-valued matrix (Dong et al. 2015, 2016; Li et al. 2019). The underlying idea is to use the consistency property of a real-valued matrix to capture the inconsistency degree of IVPRs. In a sense, the pessimistic consistency index of IVPRs is an improvement of the optimistic consistency definition (Li et al. 2019).

- Average consistency index of IARMs

  It is assumed that the entries of an ARM are randomly distributed in the intervals belonging to an IARM. The average consistency index of IARMs has been proposed in Dong et al. (2016). One can find that if and only if the IARM degenerates to an additive consistent ARM, the value of the average consistency index equals to 1. This means that the average consistency index reflects the derivation degree of IARMs from an additive consistent ARM. As compared to the previous two cases, the average consistency index is based on the whole view of an interval-valued entry.

- Boundary consistency index of IVPRs

  Based on the boundary matrices of IVPRs, the consistency indexes have been proposed (Liu and Zhang 2014; Liu et al. 2014, 2018c, 2020; Wan et al. 2018). In particular, it is found that the permutations of alternatives have been considered in Liu et al. (2018c, 2020) and the consistent MRMs and ARMs are the limiting cases. That is, when the values of the consistency indexes in Liu et al. (2018c, 2020) are chosen as a particular one, the IVPRs degenerate to a consistent MRM or ARM. As compared to the classic consistency indexes (Saaty 1980; Herrera-Viedma et al. 2007), the consistency indexes of IVPRs in Liu et al. (2018c, 2020) satisfy the proposed criterion. Hence, it is concluded that IVPRs and inconsistent real-valued preference relations are all the softened versions of a consistent real-valued comparison matrix. The consistency indexes in Liu et al. (2018c, 2020) can be used to quantify the deviation degree of IVPRs from a consistent real-valued preference relation. This is similar to the situations in Saaty (1980) and Herrera-Viedma et al. (2007) for measuring the deviation degree of an inconsistent MRM or ARM from a consistent one. Following the concept of boundary consistency (Dong et al. 2016; Li et al. 2019), the consistency indexes in Liu et al. (2018c, 2020) can be called the boundary consistency indexes under permutations, since they are based on the two boundary matrices.

One can see from the above discussions that the consistency indexes are always used to capture the derivation degree of a preference relation from a consistent real-valued one. The existing consistency definitions of IVPRs usually do not correspond a particular value of a consistency index of IVPRs. The behind reason is that the fuzzy-valued preference relations are inconsistent in the nature. As compared to the concept of consistency definition, the consistency index exhibits more flexibility to reflect the consistency property of IVPRs. The methods of measuring the consistency of IVPRs can be extended to investigate the consistency definitions and consistency indexes of the other matrices under uncertainty such as the hesitant preference relations (Li et al. 2018, 2019).

6 Illustrative examples and discussion

Although the two approaches to consistency definition of IVPRs are equivalent, the decision making models (Meng et al. 2017a, 2016; Meng and Tan 2017; Meng et al. 2017c) do not consider the effects of permutations of alternatives. In what follows, let us elaborate on an algorithm for addressing decision making problems with IVPRs under the consideration of approximate consistency.

Step 1: Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of alternatives. The decision maker could provide an IVPR $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ or $\bar{B} = \left(\bar{b}_{ij}\right)_{n\times n}$.

Step 2: Checking approximate consistency of $\bar{A}$ or $\bar{B}$ by using Definitions 6, 10 or 11, where the methods in Liu et al. (2017b, 2018b) can be used. If $\bar{A}$ or $\bar{B}$ is not with approximate consistency, they can be adjusted to a new matrix with approximate consistency by using some methods (Xu and Wei 1999; Xu and Da 2003).

Step 3: Deriving interval priority weights of alternatives from IVPRs with approximate consistency, where the methods in Liu et al. (2017b) and Liu et al. (2018b) could be used.

Step 4: According to the possibility degree formula in Liu (2009), the possibility degree matrix is obtained.

Step 5: The ranking of alternatives is generated by using the row-column elimination method in Wang et al. (2005).

Step 6: End.

In what follows, two examples are carried out to illustrate the above algorithm and some comparisons are offered.

Example 1 (Meng and Tan 2017) A clothing manufacturer wants to choose a factory from four alternatives $X = \{x_1, x_2, x_3, x_4\}$. A decision maker provides the IMRM $\bar{A}_3$ on X as:

$$\bar{A}_3 = \begin{bmatrix} [1,1] & [2,3] & [0.6325, 1.5265] & [1,2] \\ \left[1/3, 1/2\right] & [1,1] & \left[1/2, 0.84\right] & [1,1] \\ \left[1/1.5265, 1/0.6325\right] & \left[1/0.84, 2\right] & [1,1] & [1.0815, 5] \\ \left[1/2, 1\right] & [1,1] & \left[1/5, 1/1.0815\right] & [1,1] \end{bmatrix}.$$

Now we check the approximate consistency of $\bar{A}_3$ according to Definition 6. It is found that for all permutation σ, the boundary matrices C^σ and D^σ determined from $\bar{A}_3$ are inconsistent, meaning that $\bar{A}_3$ does not satisfy the condition of approximate consistency. So we randomly choose a permutation of alternatives such as

$\sigma = (1, 2, 3, 4)$ to adjust C^σ and D^σ to two new matrices with consistency. Then an IMRM $\bar{A}_3^\sigma$ with approximate consistency is obtained as follows:

$$\bar{A}_3^\sigma = \begin{bmatrix} [1,1] & [1.5200, 2.4310] & [0.7721, 1.2134] & [1.0742, 2.8526] \\ [0.4114, 0.6579] & [1,1] & [0.5277, 0.5314] & [0.7380, 1.2344] \\ [0.8242, 1.2951] & [1.8818, 1.9131] & [1,1] & [1.3586, 2.7794] \\ [0.3506, 0.9309] & [0.8107, 1.3550] & [0.3598, 0.7360] & [1,1] \end{bmatrix}.$$

By considering all the permutations (Liu et al. 2017b, 2018b), the priority weights are computed as $\omega_1 = [0.2370, 0.3529]$, $\omega_2 = [0.2364, 0.2638]$, $\omega_3 = [0.2149, 0.2485]$, and $\omega_4 = [0.1722, 0.2649]$. The matrix Pd_1 of possibility degrees is obtained as:

$$Pd_1 = \left[\begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline x_1 & 0.5 & 0.8869 & 0.9830 & 0.9638 \\ x_2 & 0.1131 & 0.5 & 0.8410 & 0.8403 \\ x_3 & 0.0170 & 0.1590 & 0.5 & 0.6419 \\ x_4 & 0.0362 & 0.1597 & 0.3581 & 0.5 \end{array}\right].$$

The ranking of alternatives is determined as $x_1 \overset{88.69\%}{\succ} x_2 \overset{84.10\%}{\succ} x_3 \overset{64.19\%}{\succ} x_4$. It is noted that the ranking of alternatives was given as $x_3 \succ x_4 \succ x_1 \succ x_2$ in Meng and Tan (2017). There is some difference behind the main reason that here the randomness experienced by the decision maker is considered. For the purpose of comparisons, we further recall the adjusted matrix of $\bar{A}_3$ in Meng and Tan (2017) as follows:

$$\bar{A}_3' = \begin{bmatrix} [1,1] & [1.7836, 2.011] & [0.621, 1.5533] & [1.5159, 2.3164] \\ [0.4973, 0.5607] & [1,1] & [0.3402, 0.7511] & [0.733, 1.269] \\ [0.6438, 1.6103] & [1.3314, 2.9394] & [1,1] & [0.9764, 3.7301] \\ [0.4317, 0.6597] & [0.788, 1.3643] & [0.2681, 1.0242] & [1,1] \end{bmatrix}.$$

It is found that $\bar{A}_3'$ is of approximate consistency in terms of Definition 6. The priority weights can be further computed as $\omega_1 = [0.2153, 0.3342]$, $\omega_2 = [0.2133, 0.2812]$, $\omega_3 = [0.2188, 0.2748]$, and $\omega_4 = [0.1620, 0.3003]$, where the method in Liu et al. (2017b) has been used. Then the possibility degree matrix Pd_2 is computed as follows:

$$Pd_2 = \left[\begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline x_1 & 0.5 & 0.7310 & 0.7351 & 0.7803 \\ x_2 & 0.2690 & 0.5 & 0.5066 & 0.6164 \\ x_3 & 0.2649 & 0.4934 & 0.5 & 0.6132 \\ x_4 & 0.2197 & 0.3836 & 0.3868 & 0.5 \end{array}\right].$$

Thereby the ranking of alternatives is $x_1 \overset{73.10\%}{\succ} x_2 \overset{50.66\%}{\succ} x_3 \overset{61.32\%}{\succ} x_4$. The obtained result is in agreement with the previous one. The comparison shows that the proposed algorithm is effective.

Example 2 (Meng et al. 2017c; Xu 2011) In a fire system, there are five critical factors $X = \{x_1, x_2, x_3, x_4, x_5\}$ given as follows:

x_1: Concealment by making use of landform;
x_2: Reduction of the mobility of enemy airplanes;
x_3: Combination with obstacle;
x_4: Cooperation with mutual firepower;
x_5: Air-defense capacity.

In order to evaluate the importance of the factors to the fire deployment, a decision maker provides the following interval-valued matrix:

$$\bar{B}_5 = \left[\begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x_1 & [0.5, 0.5] & [0.5, 0.7] & [0.8, 0.9] & [0.3, 0.5] & [0.3, 0.6] \\ x_2 & [0.3, 0.5] & [0.5, 0.5] & [0.6, 0.7] & [0.5, 0.6] & [0.5, 0.5] \\ x_3 & [0.1, 0.2] & [0.3, 0.4] & [0.5, 0.5] & [0.7, 0.9] & [0.6, 0.7] \\ x_4 & [0.5, 0.7] & [0.4, 0.5] & [0.1, 0.3] & [0.5, 0.5] & [0.5, 0.6] \\ x_5 & [0.4, 0.7] & [0.5, 0.6] & [0.3, 0.4] & [0.4, 0.5] & [0.5, 0.5] \end{array}\right].$$

For the purpose of analyzing the IARM $\bar{B}_5$, there are two approaches. One is based on additive approximation-consistency and the other is to use multiplicative approximation-consistency. Since the additive approximation-consistency of an IARM has been considered in Liu et al. (2018b), here the concept of multiplicative approximation-consistency is utilized. According to Definition 11, it is found that $\bar{B}_5$ does not satisfy the requirement of multiplicative approximation-consistency. Furthermore, according to the method in Xu and Da (2003), we adjust P_5 and Q_5 derived from $\bar{B}_5$ to P'_5 and Q'_5 with multiplicative consistency; they are given as follows:

$$P'_5 = \left[\begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x_1 & 0.5 & 0.5025 & 0.5018 & 0.5009 & 0.4999 \\ x_2 & 0.4975 & 0.5 & 0.4990 & 0.4986 & 0.4974 \\ x_3 & 0.4982 & 0.5010 & 0.5 & 0.4999 & 0.4987 \\ x_4 & 0.4991 & 0.5014 & 0.5001 & 0.5 & 0.4990 \\ x_5 & 0.5001 & 0.5026 & 0.5013 & 0.5010 & 0.5 \end{array}\right],$$

$$Q'_5 = \left[\begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x_1 & 0.5 & 0.5087 & 0.5104 & 0.5094 & 0.5109 \\ x_2 & 0.4913 & 0.5 & 0.5005 & 0.5038 & 0.5033 \\ x_3 & 0.4896 & 0.4995 & 0.5 & 0.5100 & 0.5071 \\ x_4 & 0.1906 & 0.4962 & 0.4900 & 0.5 & 0.5015 \\ x_5 & 0.4891 & 0.4967 & 0.4929 & 0.4985 & 0.5 \end{array}\right].$$

By using the method in Liu et al. (2018b), the interval weights are calculated as

$$\omega_1 = [0.1846, 0.2360], \quad \omega_2 = [0.1737, 0.2149],$$
$$\omega_3 = [0.1972, 0.2085], \quad \omega_4 = [0.1851, 0.2148],$$
$$\omega_5 = [0.1735, 0.2183].$$

The possibility degree matrix Pd_3 is determined as

$$Pd_3 = \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x_1 & 0.5 & 0.5665 & 0.6449 & 0.7014 & 0.7802 \\ x_2 & 0.4335 & 0.5 & 0.2925 & 0.3629 & 0.4643 \\ x_3 & 0.3551 & 0.7075 & 0.5 & 0.5976 & 0.6551 \\ x_4 & 0.2986 & 0.6371 & 0.4024 & 0.5 & 0.5904 \\ x_5 & 0.2198 & 0.5357 & 0.3449 & 0.4.96 & 0.5 \end{array}.$$

Hence the ranking of alternatives is given as $x_1 \overset{64.49\%}{\succ} x_3 \overset{59.76\%}{\succ} x_4 \overset{59.04\%}{\succ} x_5 \overset{53.57\%}{\succ} x_2$. On the other hand, the interval matrix $\bar{B}_5$ is also considered to be not of multiplicative consistency in Meng et al. (2017c), which is in accordance with the previous finding. Then we also use the adjusted matrix with multiplicative consistency $\bar{B}_5'$ in Meng et al. (2017c) to compute, where

$$\bar{B}_5' = \begin{bmatrix} [0.5, 0.5] & [0.569, 0.569] & [0.567, 0.614] \\ [0.431, 0.433] & [0.5, 0.5] & [0.500, 547] \\ [0.386, 0.433] & [0.453, 0.5] & [0.5, 0.5] \\ [0.368, 0.371] & [0.433, 0.438] & [0.433, 0.485] \\ [0.373, 0.433] & [0.439, 0.500] & [0.486, 0.500] \end{bmatrix}$$

$$\leftarrow \begin{matrix} [0.629, 0.632] & [0.567, 0.627] \\ [0.562, 0.567] & [0.5, 0.561] \\ [0.7, 0.9] & [0.5, 0.514] \\ [0.515, 0.567] & [0.433, 0.498] \\ [0.502, 0.567] & [0.5, 0.5] \end{matrix}\Bigg],$$

By considering all the permutations, the interval weights are obtained as $\omega_1 = [0.2055, 0.2155]$, $\omega_2 = [0.1967, 0.2033]$, $\omega_3 = [0.1980, 0.2017]$, $\omega_4 = [0.1985, 0.2023]$, and $\omega_5 = [0.1832, 0.1960]$. We further determine the possibility degree matrix Pd_4 as follows:

$$Pd_4 = \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x_1 & 0.5 & 1 & 1 & 1 & 1 \\ x_2 & 0 & 0.5 & 0.5227 & 0.4394 & 1 \\ x_3 & 0 & 0.4773 & 0.5 & 0.3642 & 1 \\ x_4 & 0 & 0.5606 & 0.6358 & 0.5 & 1 \\ x_5 & 0 & 0 & 0 & 0 & 0.5 \end{array}.$$

It is obtained from Pd_4 that the ranking of alternatives is $x_1 \overset{100\%}{\succ} x_4 \overset{56.06\%}{\succ} x_2 \overset{52.27\%}{\succ} x_3 \overset{100\%}{\succ} x_5$. By comparing the two ranking orders of alternatives, we find that they are different. This means that it should be careful when dealing with an IARM without additive or multiplicative consistency.

7 Conclusions

In order to cope with the uncertainty experienced by decision makers in a complex decision making problem, it is suitable to express the judgements as interval-valued preference relations (IVPRs). The consistency of judgements reflects the rationality of decision makers in comparing alternatives. A reasonable ranking of alternatives is also dependent on the consistency degree of preference relations. The latest view is that interval-valued judgements are inconsistent; and this finding is compatible with the undlying idea of fuzzy sets. Here we compare the two approaches to defining consistency of IVPRs and review the two concepts of consistency index and consistency definition. Some observations are given as follows:

- Two approaches to consistency definition of IVPRs are equivalent, where one is based on the concept of imaginary intervals and the other is based on the boundary matrices.
- The concepts of consistency definition and consistency index are all used to measure the consistency of a preference relation. The consistency index has more flexibility than the consistency definition.
- The consistency index could be used to quantify the deviation degree of an inconsistent preference relation from a consistent real-valued one. It is not found that an existing consistency definition corresponds a particular value of a consistency index for IVPRs.

It is seen that the above results are obtained by considering the IVPRs. In fact, there are the other interval-like preference relations such as interval-valued intuitionistic fuzzy preference relations, interval-valued hesitant fuzzy preference relations, interval type-2 fuzzy preference relations and others. The similar analysis could be made in the future works. Moreover, some comparative studies could be made for consistency definitions and consistency indexes of the other preference relations under uncertainty. Axiomatic properties of characterizing the consistency definitions and consistency indexes of fuzzy-valued comparison matrices could be further investigated.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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