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ORIGINAL PAPER

Dynamic model of a supply chain network with sticky price

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Abstract This article aims at studying the strategic behaviours of competing firms with sticky retail pricing for the product. We base our study in the context of a supply chain network comprising multiple manufacturers and retailers. The manufacturers are involved in the production of a homogeneous product while the retailers purchase the product and sell it to consumers in the end markets. The retail price of the product is sticky. A differential variational inequality model is proposed to handle the multiple agents and their independent behaviours. Furthermore, the existence and uniqueness of the solution to the dynamic supply chain network with sticky price are shown. A numerical example is provided to illustrate the model and the computational results of equilibrium behaviour are presented. This paper contributes to literature by introducing price stickiness into supply chain networks and developing a differential variational inequality model to analyze the dynamic strategies of firms in the decentralized supply chain network.

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1 Introduction

Economists focus on price as a mechanism for the efficient allocation of resources. Hence, new classical economists always assume that prices can adjust immediately in response to changing demand and supply conditions. However, in recent years, a lot of literatures have found that the prices of goods and services fail to respond to the forces of demand and supply. Specifically, Carlton (1986) finds that the degree of price stickiness in many industries is significantly different, e.g. the average period of price adjustment for household appliances is 6 months while the corresponding period for chemicals is 18 months. At present, there have been several theories to explain why price is unchanging or rigid, such as the theory of menu costs, judging quality by price, and asymmetric responses with costly search. Whether or not price rigidity is efficient, one common conclusion drawn from the models with price rigidity is that markets with rigid prices behave very differently from markets with flexible prices.

This article aims at studying the strategic behaviour of competing firms with sticky retail pricing for the product. We base our study in the context of a supply chain network comprising multiple manufacturers and retailers that engage with each other in a non-cooperative manner. Specifically, the manufacturers are involved in the production of a homogeneous product while the retailers purchase the product from the manufacturers and sell it to consumers in the end market. We assume that the nominal retail price of the product does not adjust immediately to demand and supply conditions. Simply put, the retail price of the product is sticky. A differential variational inequality model is proposed to handle the multiple agents and their independent behaviours. Furthermore, the existence and uniqueness of the solution to the dynamic supply chain network with sticky price are shown in this study. A numerical example is provided to illustrate the model and the computational results of equilibrium behaviour are presented. This paper contributes to literature by introducing price stickiness into a supply chain network and developing a differential variational inequality model to analyze the dynamic strategies of firms in the decentralized network.

Indeed, the study of supply chains has been a major research theme in the last two decades, drawing significant interest from both industry and academia. A supply chain is a coordinated system of organizations, people, activities, information, and resources involved in producing and delivering a product from suppliers to consumers. Notably, the advanced information systems today have enabled activity coordination in a large and complex supply chain that connects global resources and global markets. This opportunity urges operational researchers and management scientists to provide a comprehensive and sound theory for understanding the complex system as well as to provide a solution to a defined problem. Thus, supply chain management has been the subject of a growing body of literatures, such as

Boyaci and Gallego (2004), Cachon and Fisher (2000), Ha et al. (2011), Li et al. (2010), and Zhang and Zhao (2010).

However, a majority of literatures on supply chain management have focussed on intra-chain issues, such as supplier selection, distribution network design, production coordination, and inventory management. These studies focus on a certain part of the system and provide a solution to a local issue of interest confined to the sub-network. The sub-network is usually very limited in scope and simplified in structure, e.g. a two-tier single chain representing the coordination between one supplier and two retailers.

In contrast to these microscopic models, a new stream of studies attempt to provide a global understanding of the interactions among the supply chain agents from a macroscopic point of view. The main characteristics of these studies are a general network model comprising three or more tiers with a generalized number of agents at each tier. One example of this is Nagurney et al. (2002a), which developed a supply chain network equilibrium model consisting of three tiers of decision-makers on the network, namely manufacturers, retailers, and consumers. They established that the governing equilibrium conditions that reflected the optimality conditions of the decision-makers along with the market equilibrium conditions could be formulated and studied in a unified manner as a finite-dimensional variational inequality problem. Such a modelling approach was subsequently extended by Nagurney et al. (2002a, b) to include electronic commerce in the form of business to business and business to consumer transactions and by Dong et al. (2002) to introduce multi-criteria decision-making into supply chain network equilibrium modelling and computations. Recently, several literatures have emerged on this topic including Cheng and Wu (2006), Nagurney and Li (2015), Nagurney et al. (2013, 2014), Wakolbinger et al. (2014), and Yu and Nagurney (2013). However, most of these studies assume that the retail price is accurately determined by the demand and supply conditions of the market. In this paper, we assume that the market retail price is rigid, i.e. the nominal retail price of the product does not adjust immediately to the demand and supply conditions. Specifically, the retail price is adjusted according to two important factors. The first factor is the difference between the price determined by demand and supply conditions and the nominal retail price; the second factor is the adjustment coefficient determined by the product type. Then, the process of adjusting sticky retail prices in this study is expressed as an ordinary differential equation. Consequently, the decisions of manufacturers and retailers generally lead to effects over time . Hence, we develop a differential variational inequality model to analyze the independent behaviours of decision makers and the influences of sticky price.

Although some studies including Cojocaru et al. (2005), Daniele (2003a, b, 2006), Nagurney (2006), Nagurney and Pan (2006), and Nagurney et al. (2007) also introduced evolutionary variational inequality to address the time-dependent equilibrium problems in supply chain networks, the term evolutionary variational inequality implies a variational inequality in the Hilbert space $L^p([0,T],R^m)$ that involves no time derivative. Hence, this research differs from such studies.

Since the sticky price and the decisions of manufacturers and retailers generally lead to effects over time, the agents included in this study actually engage in dynamic or differential games, which form a particularly complex and fruitful branch of game theory. To date, there have been several studies investigating supply chain management in a differential game. Examples include Jøgensen et al. (2000) who apply a Stackelberg differential game to model the interaction between the manufacturer and the retailer in a decentralized channel and He et al. (2009) who modelled a single manufacturer and single retailer supply chain as a stochastic Stackelberg differential game. However, most of these studies have been established only for the supply chain that has an identical and simplified structure such as one-manufacturer one-retailer. In fact, differential games mainly investigate interactive decision making over time. Specifically, the differential Nash game with multiple, homogeneous decision-makers where each player solves an optimal control problem simultaneously may be formulated as a differential variational inequality. At present, many studies employ differential variational inequalities to analyze complex problems with multiple decision-makers engaged in a differential Nash game, including Friesz et al. (1993, 2006, 2011), and Mookherjee and Friesz (2008). In addition, many other applications of differential inequality may be found in Friesz (2010) and the references therein. However, these studies do not consider price stickiness in supply chain networks. In this study, we introduce price stickiness in supply chain networks, and develop a differential variational inequality model to analyze the dynamic strategies of firms in the decentralized supply chain network with multiple manufacturers and retailers.

The rest of this paper is organized as follows. In Sect. 2, we present the basic definition of differential variational inequality. In Sect. 3, we propose a differential variational inequality formulation for the differential Nash equilibrium of the supply chain with sticky prices. Section 4 addresses the existence and uniqueness of the solution to the dynamic model. In Sect. 5, an algorithm is presented to solve the differential variational inequality problem, while numerical examples are presented for illustrative purposes in Sect. 6. Finally, we provide concluding remarks in Sect. 7.

2 Definition of differential variational inequality

The Differential Variational Inequality (*DVI*) comprises two major components: an ordinary differential equation (*ODE*) and a variational inequality (*VI*). The reader is referred to Pang and Stewart (2008) for an extensive treatment of the problem. Specially, the *VI* may be defined as:

Definition 1 For given $\Phi : R^m \to R^m$ and a nonempty closed convex set K in R^m, the $VI(K, \Phi)$, is to find a vector $u^* \in K$ such that:

$$(u - u^*)^T \Phi(u^*) \geq 0, \quad \forall u \in K \tag{1}$$

Let $SOL(K, \Phi)$ denote the solution set of this problem. The formal definition of differential variational inequality is:

Definition 2 (Pang and Stewart 2008) Let $f : R^{1+n+m} \to R^n$ and $F : R^{1+n+m} \to R^m$ be two continuous vector functions. Let K be a nonempty closed convex set in R^m. Let $\Gamma : R^{2n} \to R^n$ be a boundary function and $T > 0$ be the terminal time. Then, the *DVI* defined by the triplet of functions of f, F, and Γ, the set K, and the scalar T, is to find time dependent trajectories $x(t)$ and $u(t)$ for $t \in [0, T]$ that satisfy conditions (2) and (3) whereby:

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)) \\ u(t) &\in SOL(K, F(t, x(t), u(t))) \end{aligned} \tag{2}$$

$$\Gamma(x(0), x(T)) = 0 \tag{3}$$

where x is an absolutely continuous function on [0, T], and u is an integrable function on [0, T]. Moreover, $u(t) \in SOL(K, F(t, x(t), u(t)))$ means that for any continuous $\tilde{u}(t) : [0, T] \to K$ satisfies

$$\int_0^T (\tilde{u}(t) - u(t))^T F(t, x(t), u(t)) dt \geq 0 \tag{4}$$

According to Proposition 23.2 in Zeidler (1990), the space of continuous functions $C([0, T]; R^m)$ is dense in $L^p([0, T]; R^m)$. Thus, if condition (4) holds for all continuous $\tilde{u}(t) \in C([0, T]; R^m)$, it also holds for all $\tilde{u}(t) \in L^p([0, T]; R^m)$.

3 The dynamic equilibrium model of the supply chain network with sticky price

We consider a supply chain network with multiple manufacturers and retailers. Specifically, m manufacturers are involved in the production of a homogeneous product, which may then be purchased by n retailers in different geographical locations, who, in turn, make the product available to consumers. The manufacturing firms are located at the top tier of nodes in the network while the retailers are located at the bottom tier. For convenience, we denote a typical manufacturer by i, where $i = 1, 2, \ldots, m$, and a typical retailer by j, where $j = 1, 2, \ldots, n$. We first focus on the differential Nash equilibrium of the non-cooperative, competing manufacturers, and then turn to the retailers. The complete dynamic equilibrium model is constructed along with the differential variational inequality formulation of the governing equilibrium conditions.

3.1 The equilibrium conditions for the manufacturers

Let $q_{ij}(t)$ denote the amount of the product shipped (or transacted) between manufacturer i and retailer j at time t. We group the product shipments between the manufacturers and the retailers into an mn-dimensional column vector q, where

$q \in (L^2([0,T]))^{m\times n}$. $Q_i(t)$ denotes the production output of manufacturer i at time t, such that $Q_i(t) \geq 0$. If we group all $Q_i(t)$ to a column vector $Q(t)$, then, $Q \in (L^2([0,T]))^m$ where $L^2([0,T])$ is the space of square-integrable functions. We assume that each manufacturer i is faced with a production cost function f_i, which generally, may depend on the production outputs of other manufacturers, that is $f_i = f_i(Q(t))$. Let ρ_{ij} denote the price charged for the product by manufacturer i to retailer j (i.e. the supply price), and we assume ρ_{ij} is a function of q_{ij}. Furthermore, let $I_i(t)$ denote the inventory of the product at time t and φ_i denote the inventory cost function of manufacturer i. We assume that φ_i is continuous and convex in $I_i(t)$. Without loss of generality, we assume that the retailer bears the transaction cost, which includes the cost of shipping the product.

Obviously, the profit of manufacturer i is equal to the price that the manufacturer charges for the product multiplied by the total quantity purchased by all retailers minus the production cost and inventory cost in [0, T], where T is the terminal decision time. Consequently, we can express the criterion of present value maximization for manufacturer i between [0, T] as the following optimal control problem:

$$\max \theta_i = \int_0^T e^{-rt}\left[-f_i(Q(t)) - \varphi_i(I_i(t)) + \sum_{j=1}^n \rho_{ij}\big(q_{ij}(t)\big)q_{ij}(t)\right]dt \tag{5}$$

subject to

$$\begin{aligned} \frac{dI_i(t)}{dt} &= Q_i(t) - \sum_{j=1}^n q_{ij}(t) \\ I_i(0) &= I_i^0 \\ Q_i(t) &\geq 0 \end{aligned} \tag{6}$$

where $Q_i(t)$ and $I_i(t)$ are the control variable and the state variable of the above optimal control problem, respectively. The discount rate is represented by r, and $\frac{dI_i(t)}{dt} = Q_i(t) - \sum_{j=1}^n q_{ij}(t)$ expresses the inventory dynamics of firm i. If we group $I_i(t)$ of all manufacturers into an m-column vector I, i.e. $I = (I_1, I_2, \ldots, I_m)^T$, then $I(Q,q) \in (L^2([0,T]))^m \times (L^2([0,T]))^{m\times n} \rightarrow (\mathcal{H}^1([0,T]))^m$, where $(\mathcal{H}^1([0,T]))^m$ is a Sobolev space for the real interval $[0,T] \in R_+^m$, the detail of Sobolev space can be referred Adams and Fournier (2003). I_i^0 is the initial inventory of manufacturer i at time $t = 0$, and we assume that I^0 is fixed and known to all manufacturers.

In this study, we assume the manufacturers compete in a non-cooperative differential Nash game, which states that each manufacturer will determine the optimal production trajectory to maximize his profit, given the optimal production trajectories of his competitors. Hence, manufacturer i's problem is to determine, for each fixed but arbitrary tuple $Q_{-i}(t)$ and $I_{-i}(t)$ of other manufacturers strategies, where $Q_{-i} = (Q_1, Q_2, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_m)^T$ and $I_{-i} = (I_1, I_2, \ldots, I_{i-1}, I_{i+1}, \ldots, I_m)^T$, an optimal strategy that solves the optimal control problem in Eqs. (5) and (6). A differential Nash equilibrium solution is a pair of trajectories $(Q^*(t), I^*(t))$ such that for every manufacturer i, $i = 1, 2, \ldots, m$, $(Q_i^*(t), I_i^*(t))$ solves manufacturer i's optimal control problem simultaneously, given that all manufacturer i's competitors play their Nash strategies $(Q_{-i}^*(t), I_{-i}^*(t))$. Consequently, the equilibrium of the differential Nash game between manufacturers may be formulated by a differential variational inequality. Furthermore, the equilibrium conditions between these manufacturers may be expressed by the following theorem.

Theorem 1 *For every manufacturer i, $i = 1, 2, \ldots, m$, if the production cost function f_i and the inventory cost function φ_i are continuous and convex in Q_i and I_i respectively, a tuple $(Q^*(t), I^*(t))^T$ is the differential Nash equilibrium between manufacturers if and only if $(Q^*(t), I^*(t))^T$ is determined by the following differential variational inequality:*

$$
\begin{aligned}
\dot{\lambda}^*(t) &= \left(-e^{-rt} \frac{d\varphi_i(I_i^*(t))}{dI_i} : i = 1, 2, \ldots, m \right) \\
\dot{I}^*(t) &= \left(Q_i^*(t) - \sum_{j=1}^{n} q_{ij}(t) : i = 1, 2, \ldots, m \right) \\
\sum_{i=1}^{m} \int_0^T \left[e^{-rt} \frac{\partial f_i(Q^*(t))}{\partial Q_i} + \lambda_i^*(t) \right] &\left[Q_i(t) - Q_i^*(t) \right] dt \geq 0, \quad \forall Q \in \Omega_1 \\
I^*(0) &= I^0
\end{aligned}
\tag{7}
$$

where $\Omega_1 = \{(Q, I)$: (6) hold for every i, $i = 1, 2, \ldots, m\}$, λ_i is the adjoint variable to differential equation in (6).

Proofs of all the theorems are given in "Appendix".

Furthermore, the dynamic games that we considered between the manufacturers are known as open-loop games. An open-loop game is one in which the initial information is perfect and complete solution trajectories from the start time may be calculated without reliance on any feedback (Yeung and Petrosyan 2000). Consequently, the solution of the differential variational inequality (7) $Q(t)^*$ is an open-loop solution of the differential Nash game.

3.2 The equilibrium model of the retailers

Retailers purchase the product from manufacturers and sell it to consumers in the end markets. In this study, we assume that retailers are based in different geographical locations. We group all $q_{ij}(t)$, which is the amount of the product that is purchased by retailer j from manufacturers i, for $i = 1, 2, \ldots, m$, as a column vector q_j, i.e. $q_j(t) = (q_{1j}(t), q_{2j}(t), \ldots, q_{mj}(t))^T$. In this study, we further assume that the transaction costs are borne by the retailer. Let c_{ij} denote the transaction cost of retailer i when purchasing the product from manufacturer j; then, c_{ij} is a function of $q_{ij}(t)$. Furthermore, the retailer j incurs, what we term, a handling cost, which may include the display and storage costs associated with the product. We denote this cost by c_j and, in the simplest case, c_j would be a function of $\sum_{i=1}^{m} q_{ij}(t)$, i.e. the holding cost of a retailer is a function of the quantity of the product he has obtained from the various manufacturers.

Let $p_j(t)$ represent the product price that the consumers are charged by retailer j for the product at time t. Grouping all retail prices p_j into an n-dimensional column vector p, we obtain $p(q) \in (L^2[0,T])^{m\times n} \rightarrow (\mathcal{H}^1([0,T]))^n$. In this study, we assume that the retail price for the product is sticky, i.e. the retail price of the product fails to respond to the forces of demand and supply. Specifically, we assume that the adjustment process of the sticky retail price may be expressed by an ordinary differential equation as following:

$$\frac{dp_j(t)}{dt} = k\left[a_j(t) - b_j\sum_{i=1}^{m} q_{ij}(t) - \omega_j \sum_{i=1}^{m}\sum_{j'=1,j'\neq j}^{n} q_{ij'}(t) - p_j(t)\right] \tag{8}$$

where $a_j(t)$ is the market base of retailer j in the market. The retailer with a larger $a_j(t)$ has a relative advantage in accessing customers due to a better brand, position, reputation, quality, and so on. b_j is the sensitivity coefficient of product supply on the retail price in market j, ω_j is the sensitivity coefficient of other retailers' product supply on the retail price in market j, and $b_j > \omega_j$ for $j = 1, 2, \ldots, n$. The sticky retail price is adjusted according to two important factors, the difference between the price that is determined by demand and supply conditions, i.e. $a_j(t) - b_j\sum_{i=1}^{m} q_{ij}(t) - \omega_j\sum_{i=1}^{m}\sum_{j'=1,j'\neq j}^{n} q_{ij'}(t)$, and the actual retail price $p_j(t)$ at time t, and the adjustment coefficient k, which is determined by the product type. Note that $k \in [0,1]$, and a bigger k represents that the retail price is changing faster. Specifically, the value of k is always depended on the type of the product and not change over time. $k = 1$ means that the retail price of the product is perfectly supply elastic, while $k = 0$ means that the retail price is complete rigid. Furthermore, if $a_j(t) - b_j\sum_{i=1}^{m} q_{ij}(t) - \omega_j\sum_{i=1}^{m}\sum_{j'=1,j'\neq j}^{n} q_{ij'}(t) > p_j(t)$, it means that the retail price is on the low side at time t and the retailer j would likely raise the retail price. Conversely, if $a_j(t) - b_j\sum_{i=1}^{m} q_{ij}(t) - \omega_j\sum_{i=1}^{m}\sum_{j'=1,j'\neq j}^{n} q_{ij'}(t) < p_j(t)$, then retailer j would reduce the retail price. Consequently, the differential Eq. (8) depicts the adjustment process of the sticky retail price.

The profit of retailer j is equal to the price that the retailer charges the customer multiplied by the total quantity supplied minus the transaction cost, the handling cost, and the payout to the manufacturers. Hence, retailer j would determine the ordering quantity from manufacturers, i.e. $q_{ij}(t)$, such that he maximizes the present value of his profit. Consequently, the optimal control problem of retailer j may be expressed as follows:

$$\begin{aligned} \max \theta_j = \int_0^T e^{-rt}\Bigg[& p_j(t)\sum_{i=1}^{m} q_{ij}(t) - \sum_{i=1}^{m} c_{ij}(q_{ij}(t)) - c_j\left(\sum_{i=1}^{m} q_{ij}(t)\right) \\ & - \sum_{i=1}^{m} \rho_{ij}(q_{ij}(t))q_{ij}(t)\Bigg] dt \end{aligned} \tag{9}$$

Subject to

$$\sum_{j=1}^{n} q_{ij}(t) \leq Q_i(t) + I_i(t), i = 1, 2, \ldots, m$$

$$\frac{dp_j(t)}{dt} = k\left[a_j(t) - b_j \sum_{i=1}^{m} q_{ij}(t) - \omega_j \sum_{i=1}^{m} \sum_{j'=1, j' \neq j}^{n} q_{ij'}(t) - p_j(t)\right] \tag{10}$$

$$p_j(0) = p^0$$

where the condition $\sum_{j=1}^{n} q_{ij}(t) \leq Q_i(t) + I_i(t)$ implies that the quantity ordered by all retailers of manufacturer i cannot exceed the quantity that he holds. Although backorder is very common in inventory management in many industries, the case is not considered in this paper. p_j^0 is the initial price in the demand market j at $t = 0$; we assume p^0 is fixed and known to all retailers. $p_j(t)$ is a state variable of retailer j's optimal control problem. We assume that the retailers are engaged in a non-cooperative, differential Nash game whereby each retailer will determine his optimal ordering trajectory such that he maximizes his profit given the optimal trajectories of his competitors. Similar to manufacturers, the equilibrium conditions between these retailers may be expressed by the following theorem.

Theorem 2 *For every retailer j, $j = 1, 2, \ldots, n$, if c_{ij}, c_j, and ρ_{ij} are continuous concave functions of q_{ij}, a tuple $(q^*(t), p^*(t))^T$ is the differential Nash equilibrium between retailers if and only if $(q^*(t), p^*(t))^T$ is determined by the following differential variational inequality:*

$$\dot{\beta}^*(t) = \left(e^{-rt} \sum_{i=1}^{m} q_{ij}^*(t) + \beta_j^*(t)k : j = 1, 2, \ldots, n\right)$$

$$\dot{p}^*(t) = \left(k\left[a_j(t) - b_j \sum_{i=1}^{m} q_{ij}^*(t) - \omega_j \sum_{i=1}^{m} \sum_{j'=1, j' \neq j}^{n} q_{ij'}^*(t) - p_j^*(t)\right] : j = 1, 2, \ldots, n\right)$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \int_0^T \left[e^{-rt}\left[-p_j^*(t) + \frac{\partial c_{ij}\left(q_{ij}^*(t)\right)}{\partial q_{ij}} + \frac{\partial c_j\left(\sum_{i=1}^{m} q_{ij}^*(t)\right)}{\partial q_{ij}} + \frac{\partial \rho_{ij}\left(q_{ij}^*(t)\right)}{\partial q_{ij}} q_{ij}^*(t) + \rho_{ij}\left(q_{ij}^*(t)\right)\right] + \beta_j^*(t)k\right] \times [q_{ij}(t) - q_{ij}^*(t)]dt \geq 0, \forall q \in \Omega_2$$

$$p(0) = p^0$$

$$\beta(0) \quad \mathit{free} \tag{11}$$

where $\Omega_2 = \{(q, p) : (10)$ hold for every i and j, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$, β is a column vector of adjoint variables for all retailers' optimal control problems, that is $\beta = (\beta_1, \beta_2, \ldots, \beta_n)^T$.

In the equilibrium conditions of the supply chain network model, the manufacturers and the retailers must satisfy the differential variational inequalities in Eqs. (7) and (11), simultaneously. Similar to the theory proposed in Nagurney et al. (2002a), we now state this explicitly in the following definition:

Definition 3 The equilibrium state of the supply chain network with sticky price is one where the production quantity and the product flows between the distinct tiers of the decision-makers satisfy the sum of the optimality conditions in Eqs. (7) and (11).

Consequently, we may express the equilibrium conditions of the supply chain network that engages in a differential Nash game with sticky price as the following differential variational inequality:

Theorem 3 *The supply chain network with sticky price is in equilibrium if and only if the following differential variational inequality is satisfied:*

$$\begin{aligned}
&\dot{\lambda}^*(t) = \left(-e^{-rt}\frac{d\varphi_i\left(I_i^*(t)\right)}{dI_i} : i = 1,2,\ldots,m\right)\\
&\dot{\beta}^*(t) = \left(e^{-rt}\sum_{i=1}^{m} q_{ij}^*(t) + \beta_j^*(t)k : j = 1,2,\ldots,n\right)\\
&\dot{I}^*(t) = \left(Q_i^*(t) - \sum_{j=1}^{n} q_{ij}^*(t) : i = 1,2,\ldots,m\right)\\
&\dot{p}^*(t) = \left(k\left[a_j(t) - b_j\sum_{i=1}^{m} q_{ij}^*(t) - \omega_j\sum_{i=1}^{m}\sum_{j'=1,j'\neq j}^{n} q_{ij'}^*(t) - p_j^*(t)\right] : j = 1,2,\ldots,n\right)\\
&\qquad \sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T\left[e^{-rt}\left(-p_j^*(t) + \frac{\partial c_{ij}\left(q_{ij}^*(t)\right)}{\partial q_{ij}} + \frac{\partial c_j(\sum_{i=1}^{m} q_{ij}^*(t))}{\partial q_{ij}} + \frac{\partial \rho_{ij}(q_{ij}^*(t))}{\partial q_{ij}} q_{ij}^*(t)\right.\right.\\
&\qquad \left.\left. + \rho_{ij}(q_{ij}^*(t))\right) + \beta_j^*(t)k\right] \times \left[q_{ij}(t) - q_{ij}^*(t)\right]dt\\
&\qquad + \sum_{i=1}^{m}\int_0^T\left[e^{-rt}\frac{\partial f_i(Q^*(t))}{\partial Q_i} + \lambda_i^*(t)\right]\left[Q_i(t) - Q_i^*(t)\right]dt \geq 0,\quad \forall (Q,q)^T \in \Omega\\
&\qquad I(0) = I^0\\
&\qquad p(0) = p^0\\
&\qquad \lambda(0)\ \ free\\
&\qquad \beta(0)\ \ free
\end{aligned} \tag{12}$$

where $\Omega = \Omega_1 \times \Omega_2$.

The sufficient condition in the proof of Theorem 3 has a sound economic interpretation, i.e. since this research does not consider the relationship hierarchy between manufacturer and retailer, the manufacturers and retailers are considered to have homogeneous engagements in the non-cooperative, differential Nash game to maximize their own profits simultaneously. If $(Q^*(t), q^*(t), I^*(t), p^*(t))^T$ satisfy the differential variational inequality in Eq. (12), then $Q^*(t)$ and $q^*(t)$ are the equilibrium strategies for manufacturers and retailers, respectively. Consequently, $Q^*(t)$ and $q^*(t)$ must satisfy Eqs. (7) and (11) simultaneously; otherwise, some manufacturer or retailer would have the incentive to change his own strategy.

The variables in the differential variational inequality problem are: the product output of manufacturers $Q(t)$, product inventory $I(t)$, product shipments from the manufacturers to the retailers $q(t)$, and the retail product prices $p(t)$. The equilibrium strategy of manufacturers and retailers, in turn, is to find the time dependent trajectories $(Q^*(t), q^*(t), I^*(t), p^*(t))$ that satisfy the differential variational inequality for $t \in [0, T]$.

4 Qualitative properties

In this section, we provide some qualitative properties of the solution to the differential variational inequality in Eq. (12). In particular, we prove the existence and uniqueness of the solution to the dynamic supply chain network with sticky price. For convenience, we let

$$\begin{aligned}
x(t) &= (\lambda(t), \beta(t), I(t), p(t))^T \qquad\qquad u(t) = (Q(t), q(t))^T \\
f_1 &= \left(-e^{-rt}\frac{d\varphi_i(I_i(t))}{dI_i} : i = 1, 2\ldots, m\right) \\
f_2 &= \left(e^{-rt}\sum_{i=1}^{m} q_{ij}(t) + \beta_j(t)k : j = 1, 2, \ldots, n\right) \\
f_3 &= \left(Q_i(t) - \sum_{j=1}^{n} q_{ij}(t) : i = 1, 2, \ldots, m\right) \\
f_4 &= \left(k\left[a_j(t) - b_j\sum_{i=1}^{m} q_{ij}(t) - \omega_j\sum_{i=1}^{m}\sum_{j'=1, j'\neq j}^{n} q_{ij'}(t) - p_j(t)\right] : j = 1, 2, \ldots, n\right) \\
F_1 &= (\nabla_{Q_i} H_i(t, Q_i, I_i, \lambda_i; Q_{-i}, I_{-i}) : i = 1, 2, \ldots, m) \\
F_2 &= \left(\nabla_{q_{ij}} H_j(t, q_j, p_j, \beta_j; q_{-j}, p_{-j}) : i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\right)
\end{aligned}$$

and

$$f(t, x(t), u(t)) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \quad F(t, x(t), u(t)) = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \quad x^0 = \begin{pmatrix} \lambda^0 \\ \beta^0 \\ I^0 \\ p^0 \end{pmatrix}$$

Consequently, the differential variational inequality (12) may be rewritten as

$$\begin{aligned}
\dot{x}(t) &= f(t, x(t), u(t)) \\
u^*(t) &\in SOL(\Omega, F(t, x(t), u(t))) \\
x(0) &= x^0
\end{aligned} \tag{13}$$

Under the assumption of regularity conditions, Friesz (2010) proved the existence of the solution of the differential variational inequality (13); specifically, the regularity conditions are as per the following definition:

Definition 4 (Friesz 2010) We consider the differential variational inequality (13) to be regular if:

(1) $u \in \Omega \subset (L^2[0,T])^{m+m\times n}$, and Ω is convex.
(2) for every $u \in \Omega$, $x(u,t) : (L^2[0,T])^{m+m\times n} \to (\mathcal{H}^1[0,T])^{m+n+m+n}$ exists and is unique, strongly continuous, and G-differentiable.
(3) F is continuous with respect to x and u.
(4) f is continuously differentiable with respect to x and u.
(5) x^0 are known and fixed.

The motivation behind this definition of regularity is to analyze traditional optimal control problems from the perspective of infinite-dimensional mathematical programming. According to Theorem 6.5 in Friesz (2010), when the differential variational inequality is regular, in accordance with Definition 4, and Ω is compact, the differential variational inequality (13) has a solution. Although, we do not emphasize the compactness of Ω in this study, the following reasonable assumption could guarantee the compactness of the set.

Assumption 1 Assume that there exist positive constants M, such that:

$$\frac{\partial \theta_i}{\partial Q_i} < 0, \quad \forall Q \geq M, \quad i = 1, 2, \ldots, m$$

The above assumption is reasonable from an economics perspective whereby when the production output of the manufacturer is large enough such that $Q \geq M$, then due to the limitations of resources and production capacity, we could expect the marginal profit to negative. Consequently, the manufacturer will ensure that his production output does not exceed M in order to maximize his profit. Hence, we are ready to state the following existence result:

Theorem 4 (Existence) *When both regularity, in the sense of Definition 4, and Assumption 1 hold, the differential variational inequality* (13) *has a solution.*

Specifically, since $SOL(\Omega, F(t, x(t), \bullet))$ might have multiple solutions, consequently, substitution of solutions u into the differential equation would give differential inclusions, that is $\dot{x}(t) \in \mathrm{F}\,(t,x) \equiv f(t,x,SOL(\Omega,F(t,x,\bullet)))$. At present, the detailed analysis of these systems already pose considerable technical challenges, and a number of issues generally complicate their theory and computation. A special case of the differential variational inequality (13) is that the $DVI(\Omega, F(t, x(t), \bullet))$ has a unique solution, say $u(t, x)$, that is a Lipschitz continuous function of (t, x). In this case, provided that f is Lipschitz continuous, we arrive at the familiar domain of an ordinary differential equation with a Lipschitz right-hand side whereby the differential variational inequality (13) has a unique solution. Consequently, we spell out the conditions that guarantee a unique solution of the $DVI(\Omega, F(t, x(t), \bullet))$, as follows:

Theorem 5 (Uniqueness) *Under the following two conditions, there exists a Lipschitz continuous function* $u : [0, T] \times X \to \Omega$ *such that for each pair* $(t, x) \in [0, T] \times X$, *u(x, t) is the unique solution of* $DVI(\Omega, F(t, x(t), \bullet))$.

(1) $F(t, x, \bullet)$ is *a continuous, strictly monotone function on* Ω *with a modulus that is independent of (t, x), i.e. there exists a constant* $\eta_F > 0$, *such that*

$$(u - u')^T \Big(F(t, x, u) - F(t, x, u') \Big) \geq \eta_F \|u - u'\|^2$$

for all $(t, x) \in [0, T] \times X$, $u \equiv (u_v)_{v=1}^N \in \Omega$, and $u' \equiv (u'_v)_{v=1}^N \in \Omega$.

(2) $F(\bullet, \bullet, \bullet)$ *is Lipschitz continuous with a constant, i.e. there exists a constant*

$$\|F(t, x, u) - F(t', x', u')\| \leq L_F \Big[\|t - t'\| + \|x - x'\| + \|u - u'\| \Big]$$

for all $(t, x) \in [0, T] \times X$, $(t', x') \in [0, T] \times X$, *and* $u \in \Omega$.

Accordingly, we may obtain the following theorem for the differential variational inequality established in this paper.

Theorem 6 *For every* $i (i = 1, 2, \ldots, m)$ *and* $j (j = 1, 2, \ldots, n)$, *if (1)* $\frac{\partial f_i(Q(t))}{\partial Q_i}$ *is strongly monotone for* Q_i, *and* $\frac{\partial c_{ij}(q_{ij}(t))}{\partial q_{ij}}$, $\frac{\partial c_j(\sum_{i=1}^m q_{ij}(t))}{\partial q_{ij}}$, $\frac{\partial \rho_{ij}(q_{ij}(t))}{\partial q_{ij}} q_{ij}(t)$, *and* $\rho_{ij}(q_{ij}(t))$ *are continuous and strongly monotone for* q_{ij}; *(2) the above functions and* $a_j(t)$ *are Lipschitz continuous with* t; *and (3)* $\frac{d\varphi_i(I_i(t))}{dI_i}$ *is Lipschitz continuous with* (t, I_i), *then the differential variational inequality* (12) *has a unique equilibrium solution* $(Q^*(t), q^*(t), I^*(t), p^*(t))$.

5 The Algorithm

Currently, there have been many studies investigating the methods to solve the differential variational inequality. In this study, we employ the *DVI*-specific approach that is proposed by Pang and Stewart (2008) to solve the *DVI* in Eq. (13). The algorithm may be summarized as follows:

Step 1. Divide the time interval [0, *T*] into $N_h + 1$ subintervals, and $0 = t_{h,0} < t_{h,1} < \cdots < t_{h,N_H+1} = T$.
Step 2. Initialization: set x^0, and $\vartheta \in (0, 1)$.
Step 3. Computation: solve the following *ODE* and variational inequality simultaneously:

$$\begin{cases} x^{h,i+1} = x^{h,i} + hf\left(t_{h,i+1}, \vartheta x^{h,i} + (1-\vartheta)x^{h,i+1}, u^{h,i+1}\right) \\ u^{h,i+1} \in SOL\left(\Omega, F(t_{h,i+1}, x^{h,i}, \bullet)\right) \\ x^{h,0} = x^0 \end{cases} \tag{14}$$

The details of the *DVI*-specific approach and its convergence result may be obtained from Pang and Stewart (2008). Specifically, in the *DVI*-specific approach, the variational inequality problem may be solved independent of the first equation (assuming $x^{h,i}$ is known). Furthermore, many algorithms including the Project algorithm (Solodov and Svaiter 1999) and the Euler algorithm (Nagurney and Zhang 1996) may solve this problem. The method to solve the variational inequality problem may be outlined as follows:

Step 1. Initialization: Set $u^0 \in \Omega$, α and ϵ;
Step 2. Computation: Solve the following convex quadratic programming problem at iteration τ:

$$u^{\tau+1} = arg \min_{u \in \Omega} \frac{1}{2} u^T u - (u^\tau + \alpha F(u^\tau))^T u$$

Step 3. Convergence verification: If $\|u^{\tau+1} - u^\tau\| \leq \epsilon$, then stop; otherwise, set $\tau := \tau + 1$, and go to Step 2.

6 Numerical examples

In this section, we present a numerical example to illustrate the model. Haier and Midea are the two largest manufacturers of water heaters in China, accounting for 48.99% of market share in May 2015. Furthermore, the two manufacturers' products that have the same specifications are sometimes deemed homogeneous in the markets. Hence, to keep customer base intact, the price of the water heater is rigid in response to changes in demand and supply conditions, i.e. if the two manufacturers were to keep changing the price of the commodities sold, they would offend heir customers. Consequently, the price of water heaters is sticky. Hence, we propose the following numerical example to illustrate the competition between Haier and Midea in this study. We assume the existence of two retailers that sell water heaters to customers, with terminal time $T = 5$, and discount rate $r = 0.05$. The data for this example was constructed for ease of interpretation. The production cost function $f_i(Q)$ for manufacturer $i (i = 1, 2)$, is given by

$$f_1(Q(t)) = 2.5(Q_1(t))^2 + Q_1(t)Q_2(t) + Q_1(t)$$
$$f_2(Q(t)) = 3(Q_2(t))^2 + Q_1(t)Q_2(t) + 2Q_2(t)$$

The supply price $\rho_{ij}(q_{ij}(t))$ of the product charged by manufacturer i to retailer j (where $i = 1, 2$ and $j = 1, 2$) is:

$$\rho_{11}(q_{11}(t)) = 10 - 2.5q_{11}(t), \quad \rho_{12}(q_{12}(t)) = 10 - 2q_{12}(t);$$
$$\rho_{21}(q_{21}(t)) = 10 - 2q_{21}(t), \quad \rho_{22}(q_{22}(t)) = 10 - 2.5q_{22}(t)$$

The inventory costs for manufacturer $i(i = 1, 2)$ are

$$\varphi_1(I_1(t)) = 2(I_1(t))^2, \varphi_2(I_2(t)) = 2(I_2(t))^2;$$

The initial product inventory for manufacturer $i(i = 1, 2)$ are

$$I_1(0) = I_1^0 = 5, I_2(0) = I_2^0 = 5$$

The transaction cost functions faced by the retailers in association with transacting with the manufacturers are the following:

$$c(q_{11}(t)) = 0.5(q_{11}(t))^2 + 4q_{11}(t), c(q_{12}(t)) = 0.5(q_{12}(t))^2 + 3.5q_{12}(t)$$
$$c(q_{21}(t)) = 0.5(q_{21}(t))^2 + 3q_{21}(t), c(q_{22}(t)) = 0.5(q_{22}(t))^2 + 4q_{22}(t)$$

The handling costs of the retailers are given by:

$$c_1 = 0.5(q_{11}(t) + q_{21}(t))^2, c_2 = 0.5(q_{12}(t) + q_{22}(t))^2;$$

The initial retail prices are given by:

$$p_1(0) = p_1^0 = 50, p_2(0) = p_2^0 = 60$$

The adjustment process of retail prices may be written as:

$$\frac{dp_1(t)}{dt} = 0.5[30 + 2t - 0.2(q_{11}(t) + q_{21}(t)) - p_1(t)]$$
$$\frac{dp_2(t)}{dt} = 0.5[25 + 2t - 0.2(q_{12}(t) + q_{22}(t)) - p_2(t)]$$

We employ the *DVI*-specific approach to solve this numerical example (length of subinterval of time $h = 0.1$) and the Euler algorithm to solve the variational inequality problem in (14) (termination criteria $\varepsilon = 0.01$). The optimal trajectories for $Q(t)$ and $q(t)$, and $I(t)$ and $p(t)$ are shown in Figs. 1 and 2, respectively.

Based on the equilibrium solutions shown in Figs. 1 and 2, we may conclude the following:

First, since the two manufacturers' initial inventory is relatively high, the manufacturers would consciously control the production output to reduce inventory cost. Consequently, the product inventory would be gradually reduced to zero. Second, since the initial retail prices are relatively high, the prices determined by demand and supply conditions, $a_j(t) - b_j \sum_{i=1}^{m} q_{ij}(t)$, are much lower than the actual retail prices, $p_j(t)$, and hence, the optimal trajectories of $p_j(t)$ are decreasing in the first part of terminal time. However, as time progresses, since the market base for the product increases over time, the actual retail prices, $p_j(t)$, will increase slowly. Furthermore, since the production efficiency of manufacturer 1 is higher than that of manufacturer 2, manufacturer 1 enjoys a larger market share for the product.

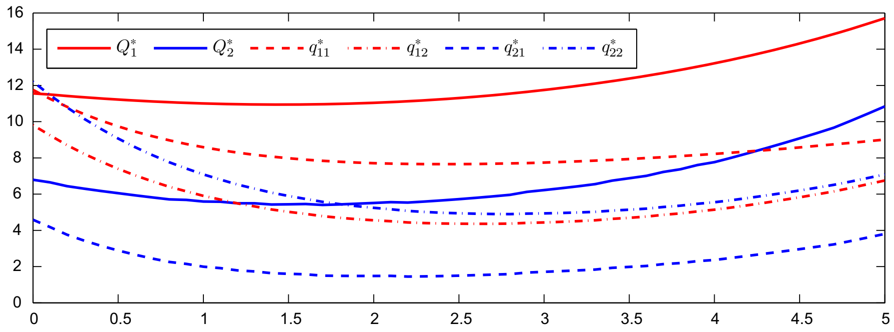

Fig. 1 Optimal trajectories of the control variables, $Q(t)$ and $q(t)$

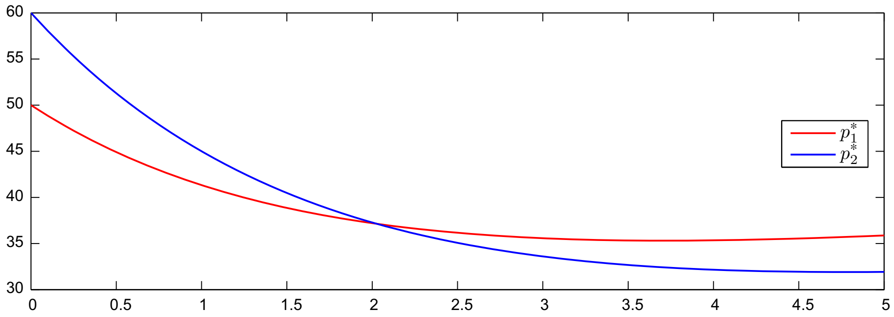

Fig. 2 Optimal trajectories of the state variables, $p(t)$

Furthermore, the effect of adjustment coefficient k on the operational decisions of every enterprise can also be analyzed. Specially, we select three different adjustment coefficient, that is $k = 0$ (when the retail price is complete rigid), $k = 0.5$ (when the retail price is moderate rigid) and $k = 0.8$ (when the retail price is relative elastic). The results can be showed in Figs. 3 and 4.

Based on the compare of the equilibrium solutions with different adjustment coefficient, we may conclude the following: According to Fig. 3, with higher production efficiency, the manufacture 1 can reach a higher level of output even though the market price is sticky. Hence, the sticky retail price of product can not change the market share of enterprises. Furthermore, based on Fig. 4, we may conclude that equilibrium price becomes less responsive to market supply as the adjustment coefficient getting bigger, and the result also applied to the manufactures output level. Hence, the stickiness of market price should never be ignored when the manufacture and retailer try to work out his optimal yield. This numerical example illustrates the rationality of the proposed differential variational inequalities for the supply chain network with sticky price.

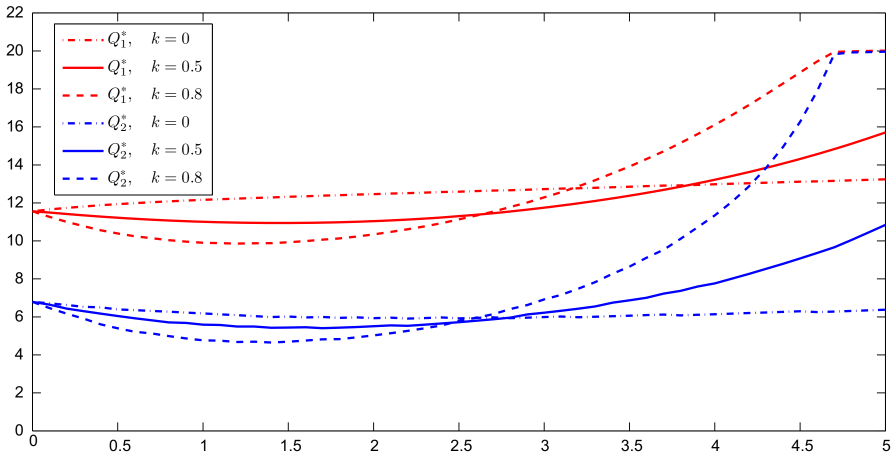

Fig. 3 Optimal trajectories of control variables with different adjustment coefficients k

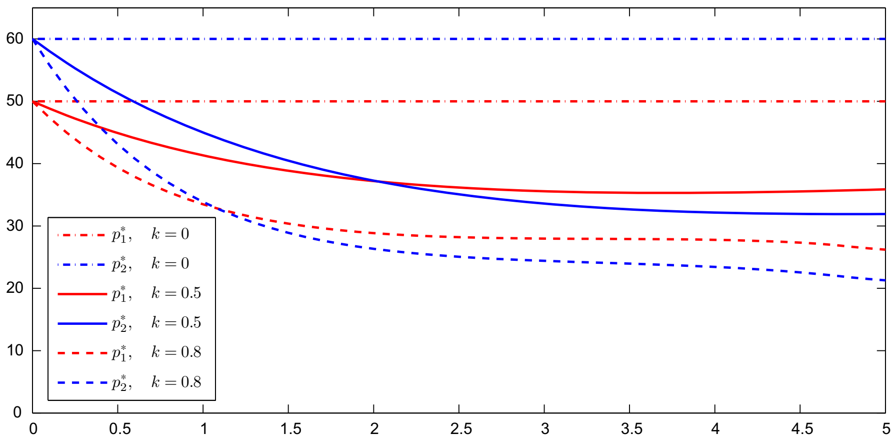

Fig. 4 Optimal trajectories of state variables with different adjustment coefficients k

7 Conclusions

In this study, we have proposed a theoretically rigorous framework for the modelling, qualitative analysis, and computation of solutions to the supply chain network problems within an equilibrium context in the case of sticky pricing. The theoretical analysis is based on differential variational inequality theory.

In particular, we have assumed a supply chain network consisting of competing manufacturers and competing retailers, each of whom seeks to maximize profits. The manufacturers are involved in the production of a homogeneous product. The retailers purchase the product from manufacturers and sell it to consumers in the end market. Specifically, the retail product price in the market is sticky. The differential variational inequality is then employed to derive, under reasonable conditions, the existence of the differential Nash equilibrium solution, as well as to establish its uniqueness. Moreover, a numerical example is presented to illustrate the model.

This study establishes the foundations for decentralized and competitive supply chain network problems in the case of sticky price within an equilibrium framework. Future research may include random demand in the dynamic model.

Appendix

Proof of Theorem 1 The corresponding Hamiltonian for manufacturer i's optimal control problem reflected in Eqs. (5) and (6), $i = 1, 2, \ldots, m$, is

$$\begin{aligned} H_i(t, Q_i, I_i, \lambda_i; Q_{-i}, I_{-i}) = & e^{-rt}\left[f_i(Q(t)) + \varphi_i(I_i(t)) - \sum_{j=1}^{n} \rho_{ij}(q_{ij}(t))q_{ij}(t)\right] \\ & + \lambda_i(t)\left[Q_i(t) - \sum_{j=1}^{n} q_{ij}(t)\right] \end{aligned}$$

where λ_i is the adjoint variable that solves the adjoint differential equation in (6) and satisfies the transversality conditions for the given state and control variables in manufacturer i's optimal control problem. For every manufacturer i, $i = 1, 2, \ldots, m$, if $f_i(Q)$ and $\varphi_i(I_i(t))$ are two continuous and concave functions of Q_i and $I_i(t)$ respectively, then for a given instant in time, $H_i(t, Q_i, I_i, \lambda_i; Q_{-i}, I_{-i})$ is a continuous and concave function of Q_i and I_i. According to the Pontryagin minimum principle, the optimal condition for manufacturer i's optimal control problem is:

$$Q^* = \arg\left\{\min_{Q \in \Omega_1} H_i(t, Q_i, I_i, \lambda_i; Q_{-i}, I_{-i})\right\}$$

For each $t \in [0, T]$, it is now a relatively easy matter to derive the necessary and sufficient conditions of manufacturer i's optimal control problem as:

$$\left[\nabla_{Q_i} H_i(t, Q_i^*, I_i, \lambda_i; Q_{-i}, I_{-i})\right]^T \left(Q_i(t) - Q_i^*(t)\right) \geq 0, \forall Q_i \geq 0$$

where λ_i is determined by:

$$\dot{\lambda}_i(t) = -\nabla_{I_i} H_i(t, Q_i, I_i, \lambda_i; Q_{-i}, I_{-i}) = -e^{-rt}\frac{d\varphi_i(I_i(t))}{dI_i}$$

Since manufacturers are engaged in a non-cooperative Nash game, then solutions to manufacturer's optimal control problems are simultaneously determined. Concatenating together these conditions across all manufacturers, we obtain Theorem 1. □

Proof of Theorem 2 The proof of Theorem 2 is similar to that in Theorem 1, and the detail of the proof is omitted here. □

Proof of Theorem 3 (Necessary Condition) The summation of Eqs. (7) and (11) yields the inequality in Eq. (12).

(Sufficient Condition) Suppose the time dependent trajectories $(Q^*(t), q^*(t), I^*(t), p^*(t))^T$ satisfy the differential variational inequality in Eq. (12) for arbitrary $(Q, q)^T \in \Omega$. Clearly $(Q^*(t), q^*(t))^T$ belongs to Ω. By taking $q_{ij}(t) = q^*_{ij}(t)$ for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$, the differential variational inequality in Eq. (12) degenerates to Eq. (7). Then, for arbitrary $Q \in \Omega_1$, $Q^*(t)$ satisfies Eq. (7). Consequently, $Q^*(t)$ is a solution to the differential variational inequality in Eq. (7). Similarly, we can prove that the $q^*(t)$ that satisfies Eq. (12) is also a solution to the differential variational inequality in Eq. (11). □

Proof of Theorem 4 Under the assumption of regularity of the differential variational inequality (13), $x(u, t)$ is well defined and continuous. So $F(t, x(t), u(t))$ is continuous in u. Also, by regularity and Assumption 1, Ω is convex and compact. Consequently, based on Browder fixed-point theorem of multi-valued mappings (Theorem 2 in Browder 1968), the differential variational inequality (13) has a solution. □

Proof of Theorem 5

(1) According to condition (1) of Theorem 5 and Theorem 3.5.15 in Facchinei and Pang (2003) , $VI(\Omega, F(t, x(t), \bullet))$ has a unique solution for any given $(t, x) \in [0, T] \times X$. The details of the proof may be obtained from Facchinei and Pang (2003).

(2) We claim that $u(x, t)$ is continuous in $(t, x) \in [0, T] \times X$. From Proposition 1.5.9 of Facchinei and Pang (2003), for any given (t, x), $u(x, t)$ is a solution of $VI(\Omega, F(t, x(t), \bullet))$ if and only if there exists a vector z such that $g(t, x; z) = \Pi_\Omega(z) - F\big(t, x, \Pi_\Omega(z)\big)$ and $u(t, x) = \Pi_\Omega(z)$, where $\Pi_\Omega(z)$ is the Euclidean projection of z onto Ω. For any given $(t, x), (t', x') \in [0, T] \times X$, we assume u and u' are the solutions of $VI(\Omega, F(t, x, \bullet))$ respectively, and can obtain:

$$\begin{aligned}\| u - u' \| = \| \Pi_\Omega(z) - \Pi_\Omega(z') \| &\le \| z - z' \| = \| g(t, x; z) - g(t', x'; z') \| \\ &= \| g(t, x; z) - g(t, x; z') + g(t, x; z') - g(t', x'; z') \| \\ &\le \| g(t, x; z) - g(t, x; z') \| + \| g(t, x; z') - g(t', x'; z') \|\end{aligned}$$

First,

$$\begin{aligned}&\| g(t, x; z) - g(t, x; z') \|^2 = \| \Pi_\Omega(z) - \Pi_\Omega(z') - F(t, x, \Pi_\Omega(z)) + F(t, x, \Pi_\Omega(z')) \|^2 \\ &\quad = \big\langle \Pi_\Omega(z) - \Pi_\Omega(z') - F(t, x, \Pi_\Omega(z)) + F(t, x, \Pi_\Omega(z')), \\ &\Pi_\Omega(z) - \Pi_\Omega(z') - F(t, x, \Pi_\Omega(z)) + F(t, x, \Pi_\Omega(z'))\big\rangle \\ &\quad = \| \Pi_\Omega(z) - \Pi_\Omega(z') \|^2 + \|F(t, x, \Pi_\Omega(z)) - F(t, x, \Pi_\Omega(z'))\|^2 \\ &\qquad - 2\big\langle \Pi_\Omega(z) - \Pi_\Omega(z'), F(t, x, \Pi_\Omega(z)) - F(t, x, \Pi_\Omega(z'))\big\rangle\end{aligned}$$

where $\langle \bullet, \bullet \rangle$ is the inner product of two vectors. Since $F(t, x, u)$ is strictly monotone in u for any given $(t, x) \in [0, T] \times X$, consequently, there exists a constant $\eta_F > 0$ such that:

$$\left\langle \Pi_{\Omega}(z) - \Pi_{\Omega}(z'), F(t,x,\Pi_{\Omega}(z)) - F(t,x,\Pi_{\Omega}(z')) \right\rangle \geq \eta_F \parallel \Pi_{\Omega}(z) - \Pi_{\Omega}(z') \parallel$$

Hence, according to condition (2) of Theorem 5, there exists a constant $L_F > 0$ such that:

$$\parallel F(t,x,\Pi_{\Omega}(z)) - F(t,x,\Pi_{\Omega}(z')) \parallel^2 \leq L_F^2 \parallel \Pi_{\Omega}(z) - \Pi_{\Omega}(z') \parallel^2 \leq L_F^2 \parallel z - z' \parallel^2$$

Consequently, we obtain:

$$\parallel g(t,x;z) - g(t,x;z') \parallel \leq \sqrt{1 - 2\eta_F + L_F^2} \parallel z - z' \parallel$$

Second, since

$$\begin{aligned} &\parallel g(t,x;z') - g(t',x',;z') \parallel \\ &\quad = \parallel \Pi_{\Omega}(z') - \Pi_{\Omega}(z') - F(t,x,\Pi_{\Omega}(z')) + F(t',x',\Pi_{\Omega}(z')) \parallel \\ &\quad = \parallel F(t,x,\Pi_{\Omega}(z')) - F(t',x',\Pi_{\Omega}(z')) \parallel \leq L_F \left[\parallel t - t' \parallel + \parallel x - x' \parallel \right] \end{aligned}$$

Consequently,

$$\begin{aligned} \parallel u(t,x) - u'(t',x') \parallel \leq \parallel z - z' \parallel &\leq \sqrt{1 - 2\eta_F + L_F^2} \parallel z - z' \parallel \\ &+ L_F \left[\parallel t - t' \parallel + \parallel x - x' \parallel \right] \end{aligned}$$

which implies that:

$$\parallel z - z' \parallel \leq \frac{L_F}{1 - \sqrt{1 - 2\eta_F + L_F^2}} \left[\parallel t - t' \parallel + \parallel x - x' \parallel \right]$$

Furthermore, we obtain:

$$\parallel u(t,x) - u'(t',x') \parallel \leq \frac{L_F}{1 - \sqrt{1 - 2\eta_F + L_F^2}} \left[\parallel t - t' \parallel + \parallel x - x' \parallel \right]$$

This implies that if $(t,x) \to (t',x')$, then $u \to u'$. Hence, $u(x, y)$ is Lipschitz continuous for each pair $(t,x) \in [0,T] \times X$. Hence, Theorem 5 is proved. □

Proof of Theorem 6 Based on the conditions of Theorem 6, it is easy to confirm that $F(t, x, u)$ is strictly monotone of u and Lipschitz continuous of (t, x, u). Hence, Theorem 6 may be derived from Theorem 5. □

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