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ORIGINAL PAPER

Interval-valued least square prenucleolus of interval-valued cooperative games and a simplified method

Deng-Feng Li[1] · **Yin-Fang Ye**[1]

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Abstract The aim of this paper is to propose the concept of the interval-valued least square prenucleolus of interval-valued cooperative games and develop a direct and an effective simplified method for solving a special subclass of interval-valued cooperative games. In this method, through adding some conditions, the least square prenucleolus of cooperative games is proved to be a monotonic and non-decreasing function of coalitions' values. Hence, the interval-valued least square prenucleolus of coalition size monotonicity-like interval-valued cooperative games can directly obtained via determining its lower and upper bounds by using the lower and upper bounds of the interval-valued coalitions' payoffs, respectively. Thus, the proposed method may overcome the issues resulted from the Moore's interval subtraction and the partial subtraction operator. Examples are used to illustrate the proposed method and comparison analysis is conducted to show its applicability and superiority. Moreover, some important properties of the interval-valued least square prenucleolus of coalition size monotonicity-like interval-valued cooperative games are discussed.

Keywords Game theory · Interval-valued cooperative game · Least square prenucleolus · Interval computing

1 Introduction

Game theory is engaged in competing and strategic interaction among players or subjects in finance, management, business, economics, engineering management, and environment (Owen 1982; Nishizaki and Sakawa 2001; Li 2016). It includes

✉ Deng-Feng Li
lidengfeng@fzu.edu.cn

[1] School of Economics and Management, Fuzhou University, No. 2, Xueyuan Road, Daxue New District, Fuzhou District, Fuzhou 350108, Fujian, China

two main branches: cooperative games (Owen 1982; Li 2003) and non-cooperative games (Dubois and Prade 1980; Bector and Chandra 2005). Numerous researches on non-cooperative games have conducted (Bector and Chandra 2005). Therefore, we focus our attention on cooperative games with transferable utility, which are often called cooperative games for short.

Crisp cooperative games use real numbers to express the values of coalitions of players (Shapley 1953; Schmeidler 1969). They have been widely studied (von Neumann and Morgenstern 1944; Driessen 1988). However, in real cases, coalitions' values cannot be expressed with real numbers because of uncertainty and information imprecision. Presently, much research work uses intervals to estimate coalitions' values and establishes the so-called interval-valued cooperative games (Branzei et al. 2010; Alparslan Gök et al. 2010; Mallozzi et al. 2011). The difference between interval-valued cooperative games and crisp cooperative games is that researchers utilize intervals to express the coalitions' values in the former rather than real numbers. Hence, in real situations, if the lower and upper bounds of all potential values resulted from cooperation are easily known a priori, then the game situations seem to be suitable for modeling as interval-valued cooperative games (Li 2016).

Presently, there has been increasing research on interval-valued cooperative games. Branzei et al. (2010) updated the results about interval-valued cooperative games and reviewed various existing and potential applications of interval-valued cooperative games in management situations. Branzei et al. (2003) considered interval-valued bankruptcy games arised from bankruptcy situations with interval claims, proposed two interval-valued Shapley-like values and studied their interrelations via using the interval arithmetic operations (Moore 1979). Mallozzi et al. (2011) introduced a core-like of cooperative games with coalitions' values represented by fuzzy intervals (Mares 2001) and a balanced-like condition which is proven to be necessary but not sufficient to assure its non-emptiness. Han et al. (2012) proposed the interval-valued core and the interval-valued Shapley-like value of interval-valued cooperative games by defining new order relation of intervals. Branzei et al. (2011) extended the interval-valued core of interval-valued cooperative games based on the interval-valued square dominance core and interval-valued dominance core. Alparslan Gök et al. (2011) also discussed the interval-valued core, the interval-valued dominance core, and the interval-valued stable sets of interval-valued cooperative games. Alparslan Gök et al. (2009) defined the Weber set and the Shapley value for a suitable class of interval-valued cooperative games and established their relations with the interval-valued core for convex interval-valued cooperative games. Li (2016) proposed several important concepts of interval-valued solutions such as the interval-valued Shapley value, the interval-valued solidarity value as well as the interval-valued Banzhaf value and their simplified methods. Li (2016) also established an effective non-linear programming method for computing interval-valued cores of interval-valued cooperative games. However, most of the aforementioned works except from Li (2016) used the partial subtraction operator or the Moore's interval subtraction (Moore 1979) which usually enlarges uncertainty of the resulted interval.

The least square prenucleolus (Ruiz et al. 1996) is one of the important solutions of crisp cooperative games. The main purpose of this paper is to extend it to interval-valued cooperative games. More precisely, through adding some conditions, we proved that the least square prenucleolus of cooperative games is a monotonic and non-decreasing function of coalitions' values. Hereby we can directly and explicitly obtain the interval-valued least square prenucleolus by determining its lower and upper bounds, respectively. Moreover, we can prove that the interval-valued least square prenucleolus possess some useful and important properties.

The rest of this paper is organized as follows. In the next section, we briefly review the concepts and notations of cooperative games, intervals, and interval-valued cooperative games. Section 3 investigates the interval-valued least square prenucleolus of interval-valued cooperative games satisfying Eq. (5), which are called coalition size monotonicity-like interval-valued cooperative games for short. In Sect. 4, the proposed method is illustrated with two real examples and compared with other methods by using the Moore's interval subtraction and the partial subtraction operator. Section 5 discusses some important properties of the interval-valued least square prenucleolus of coalition size monotonicity-like interval-valued cooperative games. Conclusion is made in Sect. 6.

2 Some basic concepts and notations

In the following, we review some basic concepts and notations of cooperative games, intervals, and interval-valued cooperative games (Li 2016).

2.1 The concepts of cooperative games

Let $N = \{1, 2, \ldots, n\}$ be the set of players i $(i = 1, 2, \ldots, n)$, where n is a positive integer, and $n \geq 2$. Any subset S of the set N, i.e., $S \subseteq N$, is called a coalition. N is referred to as the grand coalition. $\emptyset$ is called an empty coalition, i.e., an empty set of players. Usually, we denote the set of coalitions of players in the set N by 2^N.

Denote the set of real numbers by R. A n-person cooperative game is an ordered-pair $\langle N, \upsilon \rangle$, where $\upsilon\colon 2^N \mapsto \mathrm{R}$ is the characteristic function which assigns a value $\upsilon(S)$ to the coalition $S \in 2^N$, and $\upsilon(\emptyset) = 0$. $\upsilon(S)$ is called the value of the coalition S. It can be interpreted as the maximal worth (or profit, reward, cost savings) that the players of the coalition S can obtain when they cooperate. In the sequent, the n-person cooperative game $\langle N, \upsilon \rangle$ usually is referred to as the cooperative game υ for short. The set of n-person cooperative games is denoted by G^n. In the sequent, we usually write $\upsilon(S \cup i)$, $\upsilon(S \backslash i)$, $\upsilon(i)$, and $v(i,j)$ instead of $\upsilon(S \cup \{i\})$, $\upsilon(S \backslash \{i\})$, $\upsilon(\{i\})$, and $\upsilon(\{i,j\})$, respectively.

Let $x_i(\upsilon) \in \mathrm{R}$ be a payoff (or value) which is allocated to the player $i \in N$ when he/she participates in the cooperative game $\upsilon \in G^n$ under the condition that the grand coalition N is reached. Then, $\boldsymbol{x}(\upsilon) = (x_1(\upsilon), x_2(\upsilon), \ldots, x_n(\upsilon))^{\mathrm{T}}$ is a payoff vector of n players, where the symbol "T" is a transpose of a vector or matrix. The efficiency of a payoff vector $\boldsymbol{x}(\upsilon)$ can be expressed as $\sum_{i=1}^{n} x_i(\upsilon) = \upsilon(N)$.

2.2 Intervals and their arithmetic operations

Denote $\bar{a} = [a_L, a_R] = \{a | a \in \mathrm{R}, a_L \leq a \leq a_R\}$, which is called an interval, where R is the set of real numbers stated as the above. $a_L \in \mathrm{R}$ and $a_R \in \mathrm{R}$ are called the lower bound and the upper bound of the interval $\bar{a}$, respectively. Let $\bar{\mathrm{R}}$ be the set of intervals on the set R (Li 2016). Obviously, intervals are a generalization of real numbers. That is to say, real numbers are a special case of intervals (Moore 1979; Li 2011).

In the following, we give some interval arithmetic operations such as the equality, the addition, and the scalar multiplication as follows (Moore 1979; Li 2011).

Definition 1 Let $\bar{a} = [a_L, a_R]$ and $\bar{b} = [b_L, b_R]$ be two intervals on the set $\bar{\mathrm{R}}$, and $\gamma \in \mathrm{R}$ is any real number. The interval arithmetic operations are given as follows:

1. Interval equality: $\bar{a} = \bar{b}$ if and only if $a_L = b_L$ and $a_R = b_R$;
2. Interval addition or sum: $\bar{a} + \bar{b} = [a_L + b_L, a_R + b_R]$;
3. Interval's scalar multiplication:

$$\gamma\bar{a} = \begin{cases} [\gamma a_L, \gamma a_R] & \text{if } \gamma \geq 0 \\ [\gamma a_R, \gamma a_L] & \text{if } \gamma < 0 \end{cases}$$

Obviously, the above interval arithmetic operations are an extension of those of real numbers.

2.3 Interval-valued cooperative games

A n-person interval-valued cooperative game $\bar{\upsilon}$ is an ordered-pair $\langle N, \bar{\upsilon}\rangle$, where $N = \{1, 2, \ldots, n\}$ is the set of players and $\bar{\upsilon}$ is the interval-valued characteristic function of coalitions of players, and $\bar{\upsilon}(\emptyset) = [0, 0]$. Note that usually $\bar{\upsilon}(\emptyset)$ is simply written as $\bar{\upsilon}(\emptyset) = 0$ according to the notation of intervals. Stated as earlier, $\emptyset$ is an empty set. Generally, for any coalition $S \subseteq N$, $\bar{\upsilon}(S)$ is denoted by the interval $\bar{\upsilon}(S) = [\upsilon_L(S), \upsilon_R(S)]$, where $\upsilon_L(S) \leq \upsilon_R(S)$. We usually write $\bar{\upsilon}(S \backslash i)$, $\bar{\upsilon}(S \cup i)$, $\bar{\upsilon}(i)$, and $\bar{\upsilon}(i,j)$ instead of $\bar{\upsilon}(S \backslash \{i\})$, $\bar{\upsilon}(S \cup \{i\})$, $\bar{\upsilon}(\{i\})$, and $\bar{\upsilon}(\{i,j\})$, respectively. In the sequent, a n-person interval-valued cooperative game $\langle N, \bar{\upsilon}\rangle$ is simply called the interval-valued cooperative game $\bar{\upsilon}$. The set of n-person interval-valued cooperative games $\bar{\upsilon}$ is denoted by $\bar{G}^n$.

For any interval-valued cooperative games $\bar{\upsilon} \in \bar{G}^n$ and $\bar{v} \in \bar{G}^n$, according to the case (2) of Definition 1, $\bar{\upsilon} + \bar{v}$ is defined as an interval-valued cooperative game with the interval-valued characteristic function $\bar{\upsilon} + \bar{v}$, where $(\bar{\upsilon} + \bar{v})(S) = \bar{\upsilon}(S) + \bar{v}(S)$ for any coalition $S \subseteq N$, i.e., $(\bar{\upsilon} + \bar{v})(S) = [\upsilon_L(S) + v_L(S), \upsilon_R(S) + v_R(S)]$.

Usually, $\bar{\upsilon} + \bar{v}$ is called the sum of the interval-valued cooperative games $\bar{\upsilon} \in \bar{G}^n$ and $\bar{v} \in \bar{G}^n$. Obviously, $\bar{\upsilon} + \bar{v}$ is also an interval-valued cooperative game belonging to $\bar{G}^n$, i.e., $(\bar{\upsilon} + \bar{v}) \in \bar{G}^n$.

For any interval-valued cooperative game $\bar{\upsilon} \in \bar{G}^n$, it is easy to see that each player should receive an interval-valued payoff from the cooperation due to the fact that each coalition's value is an interval. Let $\bar{x}_i(\bar{\upsilon}) = [x_{Li}(\bar{\upsilon}), x_{Ri}(\bar{\upsilon})]$ be the interval-valued payoff which is allocated to the player $i \in N$ under the cooperation that the grand coalition is reached. Denote $\bar{\boldsymbol{x}}(\bar{\upsilon}) = (\bar{x}_1(\bar{\upsilon}), \bar{x}_2(\bar{\upsilon}), \ldots, \bar{x}_n(\bar{\upsilon}))^{\mathrm{T}}$, which is the vector of the interval-valued payoffs for all n players in the grand coalition N.

In a similar way to the definition of the efficiency stated as in Sect. 2.1, for an interval-valued cooperative game $\bar{\upsilon} \in \bar{G}^n$, the efficiency of an interval-valued payoff vector $\bar{\boldsymbol{x}}(\bar{\upsilon})$ can be expressed as $\sum_{i=1}^n \bar{x}_i(\bar{\upsilon}) = \bar{\upsilon}(N)$, i.e., $\sum_{i=1}^n x_{Li}(\bar{\upsilon}) = \upsilon_L(N)$ and $\sum_{i=1}^n x_{Ri}(\bar{\upsilon}) = \upsilon_R(N)$.

3 Interval-valued least square prenucleolus

For an arbitrary cooperative game $\upsilon \in G^n$ stated as in the previous Sect. 2.1, we can define its least square prenucleolus as $\boldsymbol{x}^*(\upsilon) = (x_1^*(\upsilon), x_2^*(\upsilon), \ldots x_n^*(\upsilon))^{\mathrm{T}}$, whose components are given as follows (Ruiz et al. 1996):

$$x_i^*(\upsilon) = \frac{\upsilon(N)}{n} + \frac{\sum_{S:i\in S}(n-s)\upsilon(S) - \sum_{S:i\notin S} s\upsilon(S)}{n2^{n-2}} \quad (i = 1, 2, \ldots, n), \tag{1}$$

respectively, where s denotes the cardinality of the coalition S, i.e., $s = |S|$.

For any interval-valued cooperative game $\bar{\upsilon} \in \bar{G}^n$, we can define an associated cooperative game $\upsilon(\alpha) \in G^n$, where the set of players still is $N = \{1, 2, \ldots, n\}$ and the characteristic function $\upsilon(\alpha)$ of coalitions of players is defined as follows:

$$\upsilon(\alpha)(S) = (1-\alpha)\upsilon_L(S) + \alpha\upsilon_R(S) \quad (S \subseteq N) \tag{2}$$

and $\upsilon(\alpha)(\emptyset) = 0$. The parameter $\alpha \in [0, 1]$ is any real number, which may be interpreted as an attitude factor (Yager 2004).

According to Eq. (1), we can easily obtain the least square prenucleolus $\boldsymbol{x}^*(\upsilon(\alpha)) = (x_1^*(\upsilon(\alpha)), x_2^*(\upsilon(\alpha)), \ldots x_n^*(\upsilon(\alpha)))^{\mathrm{T}}$ of the cooperative game $\upsilon(\alpha) \in G^n$, where

$$x_i^*(\upsilon(\alpha)) = \frac{\upsilon(\alpha)(N)}{n} + \frac{\sum_{S:i\in S}(n-s)\upsilon(\alpha)(S) - \sum_{S:i\notin S} s\upsilon(\alpha)(S)}{n2^{n-2}} \quad (i = 1, 2, \ldots, n), \tag{3}$$

i.e.,

$$\begin{aligned} x_i^*(\upsilon(\alpha)) = &\frac{(1-\alpha)\upsilon_L(N) + \alpha\upsilon_R(N)}{n} \\ &+ \frac{\sum_{S:i\in S}(n-s)[(1-\alpha)\upsilon_L(S) + \alpha\upsilon_R(S)] - \sum_{S:i\notin S} s[(1-\alpha)\upsilon_L(S) + \alpha\upsilon_R(S)]}{n2^{n-2}} \\ &(i = 1, 2, \ldots, n). \end{aligned} \tag{4}$$

Obviously, $x_i^*(\upsilon(\alpha))$ $(i = 1, 2, \ldots, n)$ is a continuous function of the parameter $\alpha \in [0, 1]$.

Theorem 1 *For any interval-valued cooperative game $\bar{\upsilon} \in \bar{G}^n$, if the following system of inequalities*

$$\upsilon_R(N) - \upsilon_L(N) \geq \frac{\sum_{S:i \notin S} s(\upsilon_R(S) - \upsilon_L(S)) - \sum_{S:i \in S}(n-s)(\upsilon_R(S) - \upsilon_L(S))}{2^{n-2}} \quad (i = 1, 2, \ldots, n) \tag{5}$$

is satisfied, then the least square prenucleolus $\boldsymbol{x}^*(\upsilon(\alpha))$ of the cooperative game $\upsilon(\alpha) \in G^n$ is a monotonic and non-decreasing function of the parameter $\alpha \in [0, 1]$.

Proof For any $\alpha \in [0, 1]$ and $\alpha' \in [0, 1]$, according to Eq. (4), we have

$$\begin{aligned} x_i^*(\upsilon(\alpha)) - x_i^*(\upsilon(\alpha')) &= \frac{(\alpha-\alpha')(\upsilon_R(N) - \upsilon_L(N))}{n} \\ &+ \frac{\sum_{S:i \in S}(n-s)[(\alpha-\alpha')(\upsilon_R(S) - \upsilon_L(S)] - \sum_{S:i \notin S} s[(\alpha-\alpha')(\upsilon_R(S) - \upsilon_L(S))]}{n2^{n-2}} \\ &= \frac{(\alpha-\alpha')}{n}\left[(\upsilon_R(N) - \upsilon_L(N)) + \frac{\sum_{S:i \in S}(n-s)(\upsilon_R(S) - \upsilon_L(S)) - \sum_{S:i \notin S} s(\upsilon_R(S) - \upsilon_L(S))}{2^{n-2}}\right], \end{aligned}$$

where $i = 1, 2, \ldots, n$.

If $\alpha \geq \alpha'$, then combining with the assumption, i.e., Eq. (5), we have

$$x_i^*(\upsilon(\alpha)) - x_i^*(\upsilon(\alpha')) \geq 0 \quad (i = 1, 2, \ldots, n),$$

i.e., $x_i^*(\upsilon(\alpha)) \geq x_i^*(\upsilon(\alpha'))$ $(i = 1, 2, \ldots, n)$, which mean that $x_i^*(\upsilon(\alpha))(i = 1, 2, \ldots, n)$ are monotonic and non-decreasing functions of the parameter $\alpha \in [0, 1]$. Thus, we have completed the proof of Theorem 1.

An interval-valued cooperative game $\bar{\upsilon} \in \bar{G}^n$ is called the coalition size monotonicity-like if it satisfies Eq. (5). Actually, Eq. (5) can be rewritten as follows:

$$\upsilon_R(N) - \upsilon_L(N) \geq \frac{\sum_{j \in N}\left[\sum_{S:j \in S}(\upsilon_R(S) - \upsilon_L(S)) - \sum_{S:i \in S}(\upsilon_R(S) - \upsilon_L(S))\right]}{2^{n-2}} \quad (i = 1, 2, \ldots, n),$$

which may be interpreted as that for any player $i = 1, 2, \ldots, n$, the length of the grand coalition's value (i.e., interval) is not smaller than the average-like difference of the length of the values (i.e., intervals) of all coalitions with player j $(j = 1, 2, \ldots, n)$ and all coalitions with player $i \in S$.

Therefore, for any coalition size monotonicity-like interval-valued cooperative game $\bar{\upsilon} \in \bar{G}^n$, i.e., it satisfies Eq. (5), then it is directly derived from Theorem 1 and Eq. (4) that the lower and upper bounds of the components (intervals) $\bar{x}_i^*(\bar{\upsilon})$ $(i = 1, 2, \ldots, n)$ of the interval-valued least square prenucleolus $\bar{\boldsymbol{x}}^*(\bar{\upsilon}) = (\bar{x}_1^*(\bar{\upsilon}), \bar{x}_2^*(\bar{\upsilon}), \ldots \bar{x}_n^*(\bar{\upsilon}))^{\mathrm{T}}$ are given as follows:

$$x^*_{Li}(\bar{\upsilon}) = x^*_i(\upsilon(0)) = \frac{\upsilon_L(N)}{n} + \frac{\sum_{S:i\in S}(n-s)\upsilon_L(S) - \sum_{S:i\notin S} s\upsilon_L(S)}{n2^{n-2}} \quad (i=1,2,\ldots,n) \tag{6}$$

and

$$x^*_{Ri}(\bar{\upsilon}) = x^*_i(\upsilon(1)) = \frac{\upsilon_R(N)}{n} + \frac{\sum_{S:i\in S}(n-s)\upsilon_R(S) - \sum_{S:i\notin S} s\upsilon_R(S)}{n2^{n-2}} \quad (i=1,2,\ldots,n). \tag{7}$$

Thus, $\bar{x}^*_i(\bar{\upsilon})$ for the players i $(i=1,2,\ldots,n)$ in the coalition size monotonicity-like interval-valued cooperative game $\bar{\upsilon}\in\bar{G}^n$ are directly and explicitly expressed as follows:

$$\bar{x}^*_i(\bar{\upsilon}) = \left[\frac{\upsilon_L(N)}{n} + \frac{\sum_{S:i\in S}(n-s)\upsilon_L(S) - \sum_{S:i\notin S} s\upsilon_L(S)}{n2^{n-2}}, \frac{\upsilon_R(N)}{n} + \frac{\sum_{S:i\in S}(n-s)\upsilon_R(S) - \sum_{S:i\notin S} s\upsilon_R(S)}{n2^{n-2}}\right]$$

or equivalently,

$$\bar{x}^*_i(\bar{\upsilon}) = \left[\frac{\upsilon_L(N)}{n} + \frac{n\sum_{S:i\in S}\upsilon_L(S) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_L(S)}{n2^{n-2}}, \frac{\upsilon_R(N)}{n} + \frac{n\sum_{S:i\in S}\upsilon_R(S) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_R(S)}{n2^{n-2}}\right]. \tag{8}$$

To better understand Eq. (8), we define the average difference contributions of all coalitions with player i and all coalitions with player j $(j=1,2,\ldots,n)$ as follows:

$$C^{\mathrm{AD}}_{iL}(\bar{\upsilon}) = \frac{n\sum_{S:i\in S}\upsilon_L(S) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_L(S)}{n} \quad (i=1,2,\ldots,n)$$

and

$$C^{\mathrm{AD}}_{iR}(\bar{\upsilon}) = \frac{n\sum_{S:i\in S}\upsilon_R(S) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_R(S)}{n} \quad (i=1,2,\ldots,n).$$

Hence,

$$\frac{n\sum_{S:i\in S}\upsilon_L(S) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_L(S)}{n2^{n-2}} = \frac{1}{2^{n-2}}C^{\mathrm{AD}}_{iL}(\bar{\upsilon})$$

and

$$\frac{n\sum_{S:i\in S}\upsilon_R(S) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_R(S)}{n2^{n-2}} = \frac{1}{2^{n-2}}C^{\mathrm{AD}}_{iR}(\bar{\upsilon})$$

may be regarded as the lower and upper bound weighted average difference contributions of all coalitions with player i and all coalitions with player j $(j=1,2,\ldots,n)$. Then, Eq. (8) can be rewritten as follows:

$$\bar{x}_i^*(\bar{\upsilon}) = \left[\frac{\upsilon_L(N)}{n} + \frac{1}{2^{n-2}} C_{iL}^{\mathrm{AD}}(\bar{\upsilon}), \frac{\upsilon_R(N)}{n} + \frac{1}{2^{n-2}} C_{iR}^{\mathrm{AD}}(\bar{\upsilon})\right] \quad (i = 1, 2, \ldots, n).$$

Thus, the lower bound of the components (intervals) $\bar{x}_i^*(\bar{\upsilon})$ $(i = 1, 2, \ldots, n)$ of the interval-valued least square prenucleolus $\bar{\boldsymbol{x}}^*(\bar{\upsilon})$ can be obtained by first assigning the worth $\upsilon_L(N)$ equally among all n players to player i and then distributing the lower bound average difference contributions of all coalitions with player i and all coalitions with player j $(j = 1, 2, \ldots, n)$ of the lower bounds of the interval-valued coalitions' payoffs. Analogously, we can obtain the upper bounds of $\bar{x}_i^*(\bar{\upsilon})$ $(i = 1, 2, \ldots, n)$ for player i.

4 Computational results of real numerical examples and analysis

4.1 Two real numerical examples

Example 1 The Chinese National 13th Five-Year Plan encourages green travel to promote the development of low-carbon transportation. The government strongly supports the purchase of new energy vehicles, in order to implement the new energy vehicle promotion plan and improve industrialization level of electric vehicles. Therefore, it is predicted that the demand of new energy vehicles will continue to rise in the future. Hence, the research and development and the manufacture of new energy vehicles will become a top priority for car companies. Suppose that there are three car companies (i.e., players) 1, 2, and 3, who have the ability to produce separately. The set of players are denoted by $N' = \{1, 2, 3\}$. They plan to work together in order to manufacture a new kind of new energy vehicle. Due to the uncertain information in real situation, they cannot precisely forecast their profits (i.e., values). They can only estimate ranges of their profits. In this case, we can regard the optimal allocation problem of profits for the car companies as an interval-valued cooperative game. Thus, if they manufacture the new energy vehicle by themselves, then their profits are expressed with the intervals $\bar{\upsilon}'(1) = [0,\ 2]$, $\bar{\upsilon}'(2) = [1,\ 2.5]$, and $\bar{\upsilon}'(3) = [1.5,\ 2.5]$, respectively. Similarly, if any two car companies cooperatively manufacture the new energy vehicle, then their profits are expressed with the intervals $\bar{\upsilon}'(1, 2) = [3,\ 5]$, $\bar{\upsilon}'(1, 3) = [2.5,\ 6]$, and $\bar{\upsilon}'(2, 3) = [5,\ 8]$, respectively. If all three car companies (i.e., the grand coalition N') cooperatively manufacture the new energy vehicle, then the profit is expressed with the interval $\bar{\upsilon}'(1, 2, 3) = [7.5,\ 10]$. Now, we want to compute the interval-valued least square prenucleolus of the interval-valued cooperative game $\bar{\upsilon}' \in \bar{G}^3$.

Using the above values (i.e., intervals) of the coalitions $S \subseteq N'$, we directly have

$$\upsilon'_R(N') - \upsilon'_L(N') = 10 - 7.5 = \frac{5}{2},$$

$$\frac{\sum_{S:1\notin S} s(\upsilon'_R(S) - \upsilon'_L(S)) - \sum_{S:1\in S}(3-s)(\upsilon'_R(S) - \upsilon'_L(S))}{2^{3-2}}$$
$$= \frac{(1.5+1+6)-(4+2+3.5)}{2} = -\frac{1}{2},$$

$$\frac{\sum_{S:2\notin S} s(\upsilon'_R(S) - \upsilon'_L(S)) - \sum_{S:2\in S}(3-s)(\upsilon'_R(S) - \upsilon'_L(S))}{2^{3-2}}$$
$$= \frac{(2+1+7)-(3+2+3)}{2} = 1,$$

and

$$\frac{\sum_{S:3\notin S} s(\upsilon'_R(S) - \upsilon'_L(S)) - \sum_{S:3\in S}(3-s)(\upsilon'_R(S) - \upsilon'_L(S))}{2^{3-2}}$$
$$= \frac{(2+1.5+4)-(2+3.5+3)}{2} = -\frac{1}{2}$$

.Hereby, we have

$$\upsilon'_R(N') - \upsilon'_L(N') > \frac{\sum_{S:1\notin S} s(\upsilon'_R(S) - \upsilon'_L(S)) - \sum_{S:1\in S}(3-s)(\upsilon'_R(S) - \upsilon'_L(S))}{2},$$
$$\upsilon'_R(N') - \upsilon'_L(N') > \frac{\sum_{S:2\notin S} s(\upsilon'_R(S) - \upsilon'_L(S)) - \sum_{S:2\in S}(3-s)(\upsilon'_R(S) - \upsilon'_L(S))}{2},$$

and

$$\upsilon'_R(N') - \upsilon'_L(N') > \frac{\sum_{S:3\notin S} s(\upsilon'_R(S) - \upsilon'_L(S)) - \sum_{S:3\in S}(3-s)(\upsilon'_R(S) - \upsilon'_L(S))}{2}.$$

i.e., the interval-valued cooperative game $\bar{\upsilon}' \in \bar{G}^3$ satisfies Eq. (5). In other words, it is a coalition size monotonicity-like interval-valued cooperative game. Thus, according to Eq. (8), we can easily obtain the interval-valued payoffs of the players $i \in N'$ in the interval-valued cooperative game $\bar{\upsilon}' \in \bar{G}^3$ as follows:

$$\bar{x}_1^*(\bar{\upsilon}') = \left[\frac{\upsilon'_L(N)}{3} + \frac{\sum_{S:1\in S}(3-s)\upsilon'_L(S) - \sum_{S:1\notin S} s\upsilon'_L(S)}{3\times 2^{3-2}}, \frac{\upsilon'_R(N)}{3}\right.$$
$$\left. + \frac{\sum_{S:1\in S}(3-s)\upsilon'_R(S) - \sum_{S:1\notin S} s\upsilon'_R(S)}{3\times 2^{3-2}}\right]$$
$$= \left[\frac{7.5}{3} + \frac{(0+3+2.5)-(1+1.5+10)}{6}, \frac{10}{3}\right.$$
$$\left. + \frac{(4+5+6)-(2.5+2.5+16)}{6}\right] = \left[\frac{4}{3}, \frac{7}{3}\right],$$

$$\begin{aligned}\bar{x}_2^*(\bar{v}') &= \left[\frac{v'_L(N)}{3} + \frac{\sum_{S:2\in S}(3-s)v'_L(S) - \sum_{S:2\notin S} sv'_L(S)}{3\times 2^{3-2}}, \frac{v'_R(N)}{3}\right.\\ &\quad \left.+ \frac{\sum_{S:2\in S}(3-s)v'_R(S) - \sum_{S:2\notin S} sv'_R(S)}{3\times 2^{3-2}}\right]\\ &= \left[\frac{7.5}{3} + \frac{(2+3+5)-(0+1.5+5)}{6}, \frac{10}{3} + \frac{(5+5+8)-(2+2.5+12)}{6}\right]\\ &= \left[\frac{37}{12}, \frac{43}{12}\right],\end{aligned}$$

and

$$\begin{aligned}\bar{x}_3^*(\bar{v}') &= \left[\frac{v'_L(N)}{3} + \frac{\sum_{S:3\in S}(3-s)v'_L(S) - \sum_{S:3\notin S} sv'_L(S)}{3\times 2^{3-2}}, \frac{v'_R(N)}{3}\right.\\ &\quad \left.+ \frac{\sum_{S:3\in S}(3-s)v'_R(S) - \sum_{S:3\notin S} sv'_R(S)}{3\times 2^{3-2}}\right]\\ &= \left[\frac{7.5}{3} + \frac{(3+2.5+5)-(0+1+6)}{6}, \frac{10}{3}\right.\\ &\quad \left.+ \frac{(5+6+8)-(2+2.5+10)}{6}\right] = \left[\frac{37}{12}, \frac{49}{12}\right],\end{aligned}$$

respectively. Therefore, we obtain the interval-valued least square prenucleolus of the coalition size monotonicity-like interval-valued cooperative game $\bar{v}' \in \bar{G}^3$ as follows:

$$\bar{\boldsymbol{x}}^*(\bar{v}') = \left(\left[\frac{4}{3}, \frac{7}{3}\right], \left[\frac{37}{12}, \frac{43}{12}\right], \left[\frac{37}{12}, \frac{49}{12}\right]\right)^{\mathrm{T}}.$$

As stated earlier, Eq. (5) plays an important role in the interval-valued least square prenucleolus given by Eq. (8) (or Eqs. (6) and (7)) for any interval-valued cooperative game. In other words, if Eq. (5) is not satisfied, then the interval-valued least square prenucleolus given by Eq. (8) is not always reasonable and correct.

Example 2 Let us consider a slightly modified version $\bar{v}'' \in \bar{G}^3$ of the interval-valued cooperative $\bar{v}' \in \bar{G}^3$ given in Example 1. The only difference between the interval-valued cooperative games $\bar{v}'' \in \bar{G}^3$ and $\bar{v}' \in \bar{G}^3$ is that $\bar{v}'(N') = [7.5,\ 10]$ is modified as $\bar{v}''(N') = [7.5,\ 8.25]$, where $N' = \{1, 2, 3\}$. Namely, the interval-valued characteristic function of the interval-valued cooperative game $\bar{v}'' \in \bar{G}^3$ is given as follows: $\bar{v}''(N') = [7.5,\ 8.25]$ and $\bar{v}''(S) = \bar{v}'(S)$ for all other coalitions $S \subset N'$. We try to discuss the interval-valued least square prenucleolus of the interval-valued cooperative game $\bar{v}'' \in \bar{G}^3$.

Using the above values (i.e., intervals) of the coalitions $S \subseteq N'$, we directly have

$$v''_R(N') - v''_L(N') = 8.25 - 7.5 = \frac{3}{4}.$$

Hereby, we have

$$v''_R(N') - v''_L(N') < \frac{\sum_{S:2\notin S} s(v''_R(S) - v''_L(S)) - \sum_{S:2\in S} (3-s)(v''_R(S) - v''_L(S))}{2} = 1,$$

i.e., the interval-valued cooperative game $\bar{v}'' \in \bar{G}^3$ does not satisfy Eq. (5). But, if Eqs. (6) and (7) were used, then we can obtain the lower and upper bounds of the interval-valued payoffs of the player 2 in the interval-valued cooperative game $\bar{v}'' \in \bar{G}^3$ as follows:

$$\begin{aligned} x^*_{L2}(\bar{v}'') &= \frac{v''_L(N')}{3} + \frac{\sum_{S:2\in S} (3-s)v''_L(S) - \sum_{S:2\notin S} sv''_L(S)}{3 \times 2^{3-2}} \\ &= \frac{7.5}{3} + \frac{(2+3+5) - (0+1.5+5)}{6} = \frac{37}{12} \end{aligned}$$

and

$$\begin{aligned} x^*_{R2}(\bar{v}'') &= \frac{v''_R(N')}{3} + \frac{\sum_{S:2\in S} (3-s)v''_R(S) - \sum_{S:2\notin S} sv''(S)}{3 \times 2^{3-2}} \\ &= \frac{8.25}{3} + \frac{(5+5+8) - (2+2.5+12)}{6} = \frac{36}{12}. \end{aligned}$$

Clearly, the above results are irrational due to $x^*_{L2}(\bar{v}'') = \frac{37}{12} > x^*_{R2}(\bar{v}'') = \frac{36}{12}$ from the notation of intervals stated as in the previous Sect. 2.2.

4.2 Computational results obtained by using the interval subtraction

In order to make comparison, we consider using the Moore's interval subtraction (Moore 1979) and the partial subtraction operator (Branzei et al. 2010) to solve the above numerical example 1.According to the above values of the coalitions $S \subseteq N'$, namely, $\bar{v}'(1) = [0,\ 2]$, $\bar{v}'(2) = [1,\ 2.5]$, $\bar{v}'(3) = [1.5,\ 2.5]$, $\bar{v}'(1,2) = [3,\ 5]$, $\bar{v}'(1,3) = [2.5,\ 6]$, $\bar{v}'(2,3) = [5,\ 8]$, and $\bar{v}'(N') = [7.5,\ 10]$, and using the Moore's interval subtraction (Moore 1979), i.e., $\bar{a} - \bar{b} = [a_L - b_R, a_R - b_L]$, we directly have

$$\begin{aligned} \bar{x}^{*M}_1(\bar{v}') &= \frac{\bar{v}'(N)}{3} + \frac{\sum_{S:1\in S} (3-s)\bar{v}'(S) - \sum_{S:1\notin S} s\bar{v}'(S)}{3 \times 2^{3-2}} \\ &= \frac{\bar{v}'(1,2,3)}{3} + \frac{2\bar{v}'(1) + \bar{v}'(1,2) + \bar{v}'(1,3) - (\bar{v}'(2) + \bar{v}'(3) + 2\bar{v}'(2,3))}{6} \\ &= \frac{[7.5,\ 10]}{3} + \frac{[5.5, 15] - [12.5, 21]}{6} \\ &= \left[-\frac{1}{12}, \frac{45}{12}\right]. \end{aligned}$$

However, the above result is irrational due to the lower bound $-\frac{1}{12} < 0$ from the realistic meaning of the profit.

Similarly, using the partial subtraction operator (Branzei et al. 2010), i.e., $\bar{a} - \bar{b} = [a_L - b_L, a_R - b_R]$ if $a_R - a_L \geq b_R - b_L$, we easily have

$$\begin{aligned}\bar{x}_2^{*\mathrm{P}}(\bar{\upsilon}') &= \frac{\bar{\upsilon}'(N)}{3} + \frac{\sum_{S:2\in S}(3-s)\bar{\upsilon}'(S) - \sum_{S:2\notin S} s\bar{\upsilon}'(S)}{3\times 2^{3-2}} \\ &= \frac{\bar{\upsilon}'(1,2,3)}{3} + \frac{2\bar{\upsilon}'(2) + \bar{\upsilon}'(1,2) + \bar{\upsilon}'(2,3) - (\bar{\upsilon}'(1) + \bar{\upsilon}'(3) + 2\bar{\upsilon}'(1,3))}{6} \\ &= \frac{[7.5,\ 10]}{3} + \frac{[10,18] - [6.5,16.5]}{6}.\end{aligned}$$

However, in this case, the partial subtraction operator (Branzei et al. 2010) cannot be used to calculate the above $\bar{x}_2^{*\mathrm{P}}(\bar{\upsilon}')$ due to $18 - 10 = 8 < (16.5 - 6.5) = 10$.

Furthermore, it is easy to see that if an interval-valued cooperative game $\bar{\upsilon} \in \bar{G}^n$ satisfies the following system of inequalities

$$\sum_{S:i\in S}[(n-s)(\upsilon_R(S) - \upsilon_L(S))] \geq \sum_{S:i\notin S} s(\upsilon_R(S) - \upsilon_L(S)) \quad (i = 1,2,\ldots,n), \tag{9}$$

then its interval-valued least square prenucleolus $\boldsymbol{x}^*(\bar{\upsilon})$ can be directly obtained by using the partial subtraction operator (Branzei et al. 2010). Similar to the above, if an interval-valued cooperative game $\bar{\upsilon} \in \bar{G}^n$ satisfies Eq. (9), then it is called the sum size monotonicity-like.

Actually, for any sum size monotonicity-like interval-valued cooperative game $\bar{\upsilon} \in \bar{G}^n$, i.e., it satisfies Eq. (9), then using the partial subtraction operator (Branzei et al. 2010): $\bar{a} - \bar{b} = [a_L - b_L, a_R - b_R]$ if $a_R - a_L \geq b_R - b_L$, we have

$$\begin{aligned}\bar{x}_i^{*\mathrm{P}}(\bar{\upsilon}) &= \frac{[\upsilon_L(N), \upsilon_R(N)]}{n} + \frac{\sum_{S:i\in S}(n-s)[\upsilon_L(S), \upsilon_R(S)] - \sum_{S:i\notin S} s[\upsilon_L(S), \upsilon_R(S)]}{n2^{n-2}} \\ &= \left[\frac{\upsilon_L(N)}{n} + \frac{\sum_{S:i\in S}(n-s)\upsilon_L(S) - \sum_{S:i\notin S} s\upsilon_L(S)}{n2^{n-2}}, \frac{\upsilon_R(N)}{n}\right. \\ &\quad \left. + \frac{\sum_{S:i\in S}(n-s)\upsilon_R(S) - \sum_{S:i\notin S} s\upsilon_R(S)}{n2^{n-2}}\right],\end{aligned}$$

where $i = 1,2,\ldots,n$.

It is easily seen that the condition given by Eq. (5) is weaker than Eq. (9). That is to say, if Eq. (9) is satisfied, then Eq. (5) is always true. Hence, when calculating the interval-valued least square prenucleolus, we can regard the method by using the partial subtraction operator (Branzei et al. 2010) as a special case of the method proposed in this paper.

5 Some properties of interval-valued least square prenucleolus

In the sequent, we give a theorem which summarizes some useful and important properties of interval-valued least square prenucleolus of coalition size monotonicity-like interval-valued cooperative games.

Players $i \in N$ and $k \in N$ $(i \neq k)$ are said to be symmetric in the interval-valued cooperative game $\bar{\upsilon} \in \bar{G}^n$ if $\bar{\upsilon}(S \cup i) = \bar{\upsilon}(S \cup k)$ for any coalition $S \subseteq N \backslash \{i, k\}$ (Li 2016).

Let σ be any permutation on the set N. For an interval-valued cooperative game $\bar{\upsilon} \in \bar{G}^n$, we can define the interval-valued cooperative game $\bar{\upsilon}^\sigma \in \bar{G}^n$ with interval-valued characteristic function $\bar{\upsilon}^\sigma$, where $\bar{\upsilon}^\sigma(S) = \bar{\upsilon}(\sigma^{-1}(S))$ for any coalition $S \subseteq N$.

Let $\boldsymbol{\sigma}^\# : \mathrm{R}^n \mapsto \mathrm{R}^n$ be a mapping so that $\sigma^\#_{\sigma(i)}(z) = z_i$ for any vector $\boldsymbol{z} = (z_1, z_2, \ldots, z_n)^{\mathrm{T}} \in \mathrm{R}^n$ and $i \in N$, where $\boldsymbol{\sigma}^\#(\boldsymbol{z}) = (\sigma^\#_{\sigma(1)}(\boldsymbol{z}), \sigma^\#_{\sigma(2)}(\boldsymbol{z}), \ldots, \sigma^\#_{\sigma(n)}(\boldsymbol{z}))^{\mathrm{T}}$.

Theorem 2 *For any coalition size monotonicity-like interval-valued cooperative game $\bar{\upsilon} \in \bar{G}^n$, i.e., it satisfies Eq. (5), there always exists a unique interval-valued least square prenucleolus $\bar{\boldsymbol{x}}^*(\bar{\upsilon})$ determined by Eq. (8), which satisfies the following properties:*

1. *efficiency: $\sum_{i=1}^n \bar{x}_i^*(\bar{\upsilon}) = \bar{\upsilon}(N)$,*
2. *additivity: $\bar{\boldsymbol{x}}^*(\bar{\upsilon} + \bar{\nu}) = \bar{\boldsymbol{x}}^*(\bar{\upsilon}) + \bar{\boldsymbol{x}}^*(\bar{\nu})$ for any coalition size monotonicity-like interval-valued cooperative game $\bar{\nu} \in \bar{G}^n$,*
3. *symmetry: $\bar{x}_i^*(\bar{\upsilon}) = \bar{x}_k^*(\bar{\upsilon})$ for any symmetric players $i \in N$ and $k \in N$ $(i \neq k)$,*
4. *anonymity: $\bar{x}^*_{\sigma(i)}(\bar{\upsilon}^\sigma) = \bar{x}_i^*(\bar{\upsilon})$ $(i = 1, 2, \ldots, n)$ for any permutation σ on the set N.*

Proof According to Eq. (8), and combining with Definition 1, we can straightforwardly know that there always exists a unique interval-valued least square prenucleolus $\bar{\boldsymbol{x}}^*(\bar{\upsilon})$, which is determined by Eq. (8).

(1) According to Eq. (8) and Definition 1, we have

$$\begin{aligned}\sum_{i=1}^n \bar{x}_i^*(\bar{\upsilon}) &= \sum_{i=1}^n \left[\frac{\upsilon_L(N)}{n} + \frac{n\sum_{S:i\in S}\upsilon_L(S) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_L(S)}{n2^{n-2}}, \frac{\upsilon_R(N)}{n}\right.\\ &\qquad \left. + \frac{n\sum_{S:i\in S}\upsilon_R(S) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_R(S)}{n2^{n-2}}\right]\\ &= [\upsilon_L(N) + \frac{n\sum_{i=1}^n\sum_{S:i\in S}\upsilon_L(S) - n\sum_{j\in N}\sum_{S:j\in S}\upsilon_L(S)}{n2^{n-2}}, \upsilon_R(N)\\ &\qquad + \frac{n\sum_{i=1}^n\sum_{S:i\in S}\upsilon_R(S) - n\sum_{j\in N}\sum_{S:j\in S}\upsilon_R(S)}{n2^{n-2}}]\\ &= [\upsilon_L(N), \upsilon_R(N)] = \bar{\upsilon}(N)\end{aligned}$$

i.e., $\sum_{i=1}^n \bar{x}_i^*(\bar{\upsilon}) = \bar{\upsilon}(N)$. Therefore, we have proved the efficiency.

(2) Assume that $\bar{\upsilon} \in \bar{G}^n$ and $\bar{\nu} \in \bar{G}^n$ are coalition size monotonicity-like interval-valued cooperative games. Then, according to Eq. (6), we have

$$\begin{aligned} x^*_{Li}(\bar{\upsilon}+\bar{v}) &= \frac{\upsilon_L(N)+v_L(N)}{n} + \frac{n\sum_{S:i\in S}(\upsilon_L(S)+v_L(S)) - \sum_{j\in N}\sum_{S:j\in S}(\upsilon_L(S)+v_L(S))}{n2^{n-2}} \\ &= \left(\frac{\upsilon_L(N)}{n} + \frac{n\sum_{S:i\in S}\upsilon_L(S) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_L(S)}{n2^{n-2}}\right) \\ &\quad + \left(\frac{v_L(N)}{n} + \frac{n\sum_{S:i\in S}v_L(S) - \sum_{j\in N}\sum_{S:j\in S}v_L(S)}{n2^{n-2}}\right) \\ &= x^*_{Li}(\bar{\upsilon}) + x^*_{Li}(\bar{v}), \end{aligned}$$

i.e., $x^*_{Li}(\bar{\upsilon}+\bar{v}) = x^*_{Li}(\bar{\upsilon}) + x^*_{Li}(\bar{v})$ $(i=1,2,\ldots,n)$.

Analogously, according to Eq. (7), we can easily prove that $x^*_{Ri}(\bar{\upsilon}+\bar{v}) = x^*_{Ri}(\bar{\upsilon}) + x^*_{Ri}(\bar{v})$ $(i=1,2,\ldots,n)$. Combining with the aforementioned conclusion, according to the case (1) of Definition 1, we obtain

$$\bar{x}^*_i(\bar{\upsilon}+\bar{v}) = \bar{x}^*_i(\bar{\upsilon}) + \bar{x}^*_i(\bar{v}) \quad (i=1,2,\ldots,n).$$

Hence, $\bar{\boldsymbol{x}}^*(\bar{\upsilon}+\bar{v}) = \bar{\boldsymbol{x}}^*(\bar{\upsilon}) + \bar{\boldsymbol{x}}^*(\bar{v})$. Thus, we have proved the additivity.

(3) Due to the assumption that $i \in N$ and $k \in N$ $(i \neq k)$ are symmetric players in the coalition size monotonicity-like interval-valued cooperative game $\bar{\upsilon} \in \bar{G}^n$, then we have

$$\bar{\upsilon}(S\cup i) = \bar{\upsilon}(S\cup k)$$

for any coalition $S \subseteq N\backslash\{i,k\}$. Namely, $\upsilon_L(S\cup i) = \upsilon_L(S\cup k)$ and $\upsilon_R(S\cup i) = \upsilon_R(S\cup k)$. Hence, we have $\sum_{S:i\in S}\upsilon_L(S) = \sum_{S:k\in S}\upsilon_L(S)$ and $\sum_{S:i\in S}\upsilon_R(S) = \sum_{S:k\in S}\upsilon_R(S)$. According to Eq. (8), we can easily obtain that

$$\begin{aligned} \bar{x}^*_i(\bar{\upsilon}) &= \left[\frac{\upsilon_L(N)}{n} + \frac{n\sum_{S:i\in S}\upsilon_L(S) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_L(S)}{n2^{n-2}}, \frac{\upsilon_R(N)}{n}\right. \\ &\quad \left. + \frac{n\sum_{S:i\in S}\upsilon_R(S) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_R(S)}{n2^{n-2}}\right] \\ &= \left[\frac{\upsilon_L(N)}{n} + \frac{n\sum_{S:k\in S}\upsilon_L(S) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_L(S)}{n2^{n-2}}, \frac{\upsilon_R(N)}{n}\right. \\ &\quad \left. + \frac{n\sum_{S:k\in S}\upsilon_R(S) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_R(S)}{n2^{n-2}}\right] \\ &= \bar{x}^*_k(\bar{\upsilon}), \end{aligned}$$

i.e., $\bar{x}^*_i(\bar{\upsilon}) = \bar{x}^*_k(\bar{\upsilon})$. Thus, we have proved the symmetry.

(4) According to Eq. (8), we can obtain that

$$\begin{aligned}
\bar{x}^*_{\sigma(i)}(\bar{\upsilon}^\sigma) &= \left[\frac{\upsilon_L^\sigma(N)}{n} + \frac{n\sum_{S:\sigma(i)\in S}\upsilon_L^\sigma(S) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_L^\sigma(S)}{n2^{n-2}},\right.\\
&\left.\frac{\upsilon_R^\sigma(N)}{n} + \frac{n\sum_{S:\sigma(i)\in S}\upsilon_R^\sigma(S) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_R^\sigma(S)}{n2^{n-2}}\right]\\
&= \left[\frac{\upsilon_L(\sigma^{-1}(N))}{n} + \frac{n\sum_{S:\sigma(i)\in S}\upsilon_L(\sigma^{-1}(S)) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_L(\sigma^{-1}(S))}{n2^{n-2}},\right.\\
&\left.\frac{\upsilon_R(\sigma^{-1}(N))}{n} + \frac{n\sum_{S:\sigma(i)\in S}\upsilon_R(\sigma^{-1}(S)) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_R(\sigma^{-1}(S))}{n2^{n-2}}\right]\\
&= \left[\frac{\upsilon_L(N)}{n} + \frac{n\sum_{S:i\in S}\upsilon_L(S) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_L(S)}{n2^{n-2}}, \frac{\upsilon_R(N)}{n}\right.\\
&\left.+\frac{n\sum_{S:i\in S}\upsilon_R(S) - \sum_{j\in N}\sum_{S:j\in S}\upsilon_R(S)}{n2^{n-2}}\right] = \bar{x}^*_i(\bar{\upsilon}),
\end{aligned}$$

i.e., $\bar{x}^*_{\sigma(i)}(\bar{\upsilon}^\sigma) = \bar{x}^*_i(\bar{\upsilon})$ $(i = 1, 2, \ldots, n)$. Namely, $\bar{\boldsymbol{x}}^*(\bar{\upsilon}^\sigma) = \boldsymbol{\sigma}^\#(\bar{\boldsymbol{x}}^*(\bar{\upsilon}))$. Therefore, we have proved Theorem 2.

6 Conclusions

We define the concept of the interval-valued least square prenucleolus of interval-valued cooperative games. The main work of this paper is to develop a simplified method for computing the interval-valued least square prenucleolus of coalition size monotonicity-like interval-valued cooperative games, which satisfy Eq. (5). The method proposed in this paper does not use either the Moore's interval subtraction or the ranking of intervals (or interval comparison) due to the monotonicity of our introduced concept under some conditions, i.e., Eq. (5). Thus, it can effectively overcome the irrational issues resulted from the interval subtraction. It is noted that the method by using the partial subtraction operator is a special case of the method proposed in this paper when the interval-valued least square prenucleolus is calculated. Moreover, we discuss some important and useful properties of the interval-valued least square prenucleolus of coalition size monotonicity-like interval-valued cooperative games, which are also desire for (interval-valued) cooperative games. In the future, we will find other weaker conditions than Eq. (5), which always ensure that a more broad class of interval-valued cooperative games has interval-valued least square prenucleolus determined by Eq. (8).

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