Qualitative Theory of Dynamical Systems (2024) 23:36
https://doi.org/10.1007/s12346-023-00887-9

Normalized Solutions of Schrödinger Equations with Combined Nonlinearities

Ting-ting Dai[1] · Zeng-qi Ou[1] · Ying Lv[1]

Received: 28 June 2023 / Accepted: 2 October 2023 / Published online: 31 October 2023

Abstract
In this paper, we study the nonlinear Schrödinger equation with L^2-norm constraint

$$\begin{cases} -\Delta u = \lambda u + |u|^{p-2}u + h(x)|u|^{q-2}u \ \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = c^2, \end{cases}$$

where $c > 0$, $N \geq 3$, $1 \leq q < 2 < p < 2 + \frac{4}{N}$, $h \in L^{\frac{2}{2-q}}(\mathbb{R}^N)$ and $\lambda \in \mathbb{R}$ is Lagrange multiplier, which appears due to the mass constraint $|u|_2 = c$. We use barycentric functions and minimax method to prove that for any $c > 0$, there exists a positive solution $u \in H^1(\mathbb{R}^N)$ for some $\lambda < 0$.

Keywords Normalized solution · Deformation lemma · Barycentric functions · Brouwer degree

1 Introduction and Main Results

In this paper, we study the existence of solutions for the following elliptic problem with L^2-norm constraint

Supported by Natural Science Foundation of Chongqing, China (cstc2020jcyjjqX0029) and Southwest University Pilot Program (SWU-XDPY22015).

✉ Ying Lv
ly0904@swu.edu.cn

Ting-ting Dai
dtt19971214@163.com

Zeng-qi Ou
ouzengq707@sina.com

1 School of Mathematics and Statistics, Southwest University, Chongqing 400715, People's Republic of China

$$\begin{cases} -\Delta u = \lambda u + |u|^{p-2}u + h(x)|u|^{q-2}u \ \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = c^2, \end{cases} \tag{1.1}$$

where $c > 0$, $N \geq 3$, $1 \leq q < 2 < p < 2 + \frac{4}{N}$, $h \in L^{\frac{2}{2-q}}(\mathbb{R}^N)$ and $\lambda \in \mathbb{R}$ is Lagrange multiplier, which appears due to the mass constraint $|u|_2 = c$.

The energy functional of (1.1) is defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx - \frac{1}{q}\int_{\mathbb{R}^N} h(x)|u|^q dx, \quad \forall u \in H^1(\mathbb{R}^N).$$

For $c > 0$, we define

$$S_c := \{u \in H^1(\mathbb{R}^N) : |u|_2 = c\}.$$

In the last decade, the existence and the properties of the solutions to the nonhomogeneous problem

$$\begin{cases} -\Delta u = \lambda u + g(x, u) \ \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{1.2}$$

has been studied by many peoples. When $g(x, u) = a(x) f(u)$, Lehrer and Maia [22] studied (1.2) via Pohozăev manifold, where $N \geq 3$, $\lambda < 0$, f is asymptotically linear at infinity and a satisfies suitable conditions. The authors obtained the existence of high energy solutions. When $g(x, u) = f(x, u) + h(x)$, (1.2) turns to

$$\begin{cases} -\Delta u = \lambda u + f(x, u) + h(x) \ \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.3}$$

For the homogeneous case, i.e. $h(x) = 0$ (which means 0 is a trivial solution of (1.3)) has been studied extensively (see e.g. [3, 10, 21, 23]). For the nonhomogeneous case ($h(x) \not\equiv 0$), this problem without trivial solutions and presents specific mathematical difficulties. When $f(x, u) = a(x)|u|^{p-2}u$, Adachi and Tanaka [1] obtained the existence of at least four positive solutions under the assumptions: $0 < a(x) \leq a^{\infty} = \lim_{|x|\to\infty} a(x)$, $h \in H^{-1}(\mathbb{R}^N)$ is nonnegative and satisfying $||h||_{H^{-1}(\mathbb{R}^N)}$ sufficiently small. Zhu [29] obtained the existence of two positive solutions for the following problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{p-2}u + h(x) \ \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.4}$$

where $h(x) \in L^2(\mathbb{R}^N)$, $h(x) > 0$, and $1 < p < \frac{N+2}{N-2}(N \geq 3)$, $1 < p < +\infty(N = 2)$. In [11, 15], the authors studied the Sobolev subcritical perturbation problem with fixed frequency and proved that this problem has at least one positive solution when the perturbation is small enough (h may be in different spaces). Moreover, some authors also considered the qualitative and asymptotic analysis of solutions to some related elliptic problem, we refer to [25–28] and the references therein.

In this paper, we consider the normalized solutions to the nonhomogeneous elliptic equations (1.1). In what follows, we recall some basic facts concerning the existence of normalized solutions for nonlinear Schrödinger equations in $\mathbb{R}^N$. It is well known that the following problem

$$- \Delta u = \lambda u + |u|^{p-2}u, \; u > 0 \text{ in } \mathbb{R}^N \tag{1.5}$$

has a unique solution(up to a translation), which is radial, radially decreasing. In addition, there are two exponents which play a crucial role on the existence and profile of the solutions for (1.5) with L^2-norm constraint: in addition to the Sobolev critical exponent $p = 2^*$, we have the mass-critical exponent $p = 2 + \frac{4}{N}$. If $2 < p < 2 + \frac{4}{N}$ (mass-subcritical regime), then the energy functional associated to (1.5) is bounded from below on the L^2-sphere S_c, while if $p \geq 2 + \frac{4}{N}$ (mass-critical or supercritical regime) this is not true and one is forced to search for critical points that are not global minima. For mass-supercritical case, Jeanjean [18] considered the problem

$$\begin{cases} -\Delta u = \lambda u + f(u) \;\; \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = c^2, \end{cases} \tag{1.6}$$

where $N \geq 1$, f is mass supercritical and Sobolev subcritical. A model nonlinearity is $f(u) = \sum_{j=1}^{k} |u|^{p_j - 2}u$ with $2 + \frac{4}{N} < p_j < 2^*$ for all j. Jeanjean obtained a radial solution $(u, \lambda) \in H^1_{rad}(\mathbb{R}^N) \times \mathbb{R}^+$ of (1.6) by a mountain pass argument for I on $S_c \cap H^1_{rad}(\mathbb{R}^N)$. In [2] the authors obtained the existence of infinitely many solutions of (1.6) under the same assumptions as in [18]. For mass-subcritical case, Hirata and Tanaka [14] employed a version of symmetric mountain pass argument on $H^1_r(\mathbb{R}^N)$ to derive the existence and multiplicity of normalized solution for problem (1.6). Here $H^1_r(\mathbb{R}^N)$ denotes the space of radial H^1-functions on $\mathbb{R}^N$. Recently, Jeanjean and Lu developed a new minimax theorem with index theory in [19], and used that theorem to give another proof of the result due to Hirata and Tanaka. This has been done in the recent paper [9] by Chen and Zou who considered the problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{p-2}u + h(x) \;\; \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = c^2. \end{cases}$$

For the mass-subcritical case $2 < p < 2 + \frac{4}{N}$, they proved that there exists a global minimizer with negative energy for arbitrarily positive perturbation. Secondly, for the mass-supercritical case $2 + \frac{4}{N} < p < 2^*$ where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = +\infty$ if $N = 1, 2$, when h is a small radial positive function, they proved that the existence of a mountain pass solution with positive energy. We would like to mention that recently various results have been obtained for normalized solutions, we refer to [4–7, 13, 16] and the references therein.

In this paper, we suppose that

(h_1) $h \in L^{\frac{2}{2-q}}(\mathbb{R}^N)$, $h(x) \le 0$, $h(x) < 0$ on a set with positive measure and

$$|h|_{\frac{2}{2-q}} < \frac{(\eta - 1)m_c^\infty}{c^q},$$

where m_c^∞ and η are defined in (2.10) and Lemma 3.1, respectively.

In light of the above discussion and mainly motivated by the results in [8, 30], we focus our attention on problem (1.1) and establish the existence of positive high energy solutions. It seems this is the first contribution to the high energy solution for problem (1.1). We aim to establish the following result:

Theorem 1.1 *Let $N \ge 3$, $1 \le q < 2 < p < 2 + \frac{4}{N}$ and h satisfies (h_1). Then for any $c > 0$, problem (1.1) has a positive solution $u \in S_c$ for some $\lambda < 0$.*

Remark 1.2

(i) To the best of our knowledge, it seems only [9] studied the normalized solution for such a perturbed equation. In [9], Chen and Zou proved that there exists a ground state normalized solution with negative energy for arbitrarily positive perturbation($h(x) \ge 0$). In this case, inequality $c(a+b) < c(a) + c_\infty(b)$ plays a crucial role in proving the convergence of this minimizing sequence. In this paper, we assume that $h(x) \le 0$, in this case, we do not need to prove that inequality $c(a+b) < c(a) + c_\infty(b)$ holds. In fact, we obtain the convergence of non-negative Palais-Smale sequences of $I|_{S_c}$ by a local compactness result(see Lemma 3.2).

(ii) The question of finding normalized solutions is already interesting for scalar equations and provides features and difficulties that are not present in the fixed frequency problem. And thus the existence of normalized solutions becomes nontrivial and many techniques developed for the fixed frequency problem can not be applied directly. A series of theories and tools related to fixed frequency problem have been developed, such as fixed point theory, bifurcation, the Lyapunov-Schmidt reduction, Nehari manifold method, mountain pass theory and many other linking theories. However, for the fixed mass problem, the normalization constraint certainly brings too much trouble in mathematical treatment. Comparing to the fixed frequency problem, the fixed mass problem possesses the following technical difficulties when dealing with it in the variational framework:

(a) One can not use the usual Nehari manifold method since the frequency is unknown.

(b) The existence of bounded Palais-Smale sequences requires new arguments.

(c) The Lagrange multipliers have to be controlled.

(d) The embedding $H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ is not compact. For the fixed frequency problem, usually a nontrivial weak limit is also a solution. However, for the fixed mass problem, even the weak limit is nontrivial, the constraint condition may be not satisfied.

(e) For the the general mass subcritical problem, we only need to prove the convergence of the minimizing sequence to obtain a solution to the problem. But the perturbation term $h(x)|u|^{q-2}u$ with $h(x) \leq 0$ makes it different from the general mass subcritical case. In this paper, we can not search for the minimizer of I in S_c since it does not exist. This fact is proved in Sect.2. Secondly, it is hard to get the range of energy levels corresponding to pre-compact Palais-Smale sequences of $I|_{S_c}$ since the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $2 < p < 2^*$ is not compact. Finally, we need to find a solution in a high energy level.

(iii) To prove Theorem 1.1, we follow the approach in [8, 30]. Firstly, we can show that problem (1.1) does not have a ground state solution(see Lemma 2.3). Hence, we need to find a solution in the high energy level. To this aim, we use Splitting Lemma and carefully analyse the relation between λ and the energy levels of non-negative Palais-Smale sequences of $I|_{S_c}$ to get the compactness of such sequences with energy levels close to the infimum of I in S_c(see Lemma 3.1). Finally, we prove our main result by topological methods.

This work is organized as follows. In Sect. 2, we introduce the variational formulation of (1.1), some notations and show that the problem (1.1) has no ground state solution. In Sect. 3, we establish a compactness result for some non-negative Palais-Smale sequences, which are essential to carry out the proof of our main theorem. Finally in Sect. 4, we prove Theorem 1.1.

2 Preliminaries and Nonexistence Result

In this section, we introduce some notations, some known results which will be used in this paper and prove that (1.1) has no ground state solution for any $c > 0$.

Let $H^1(\mathbb{R}^N)$ be a Sobolev space with the standard norm

$$||u||^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |u|^2 dx, \quad \forall u \in H^1(\mathbb{R}^N).$$

Moreover, throughout this paper, we will use the notation $|\cdot|_s = |\cdot|_{L^s(\mathbb{R}^N)},\ s \in [1, \infty)$.

Let

$$I^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx, \quad \forall u \in H^1(\mathbb{R}^N).$$

For $\lambda < 0$, $u \in H^1(\mathbb{R}^N)$, we define

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{-\lambda}{2}\int_{\mathbb{R}^N} |u|^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx - \frac{1}{q}\int_{\mathbb{R}^N} h(x)|u|^q dx$$

and

$$I_\lambda^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{-\lambda}{2}\int_{\mathbb{R}^N} |u|^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx.$$

In what follows, we recall the Gagliardo-Nirenberg inequality (see [24]). For any $2 < \alpha < 2^*$, there exists a sharp constant $C(N, \alpha) > 0$ such that

$$|u|_\alpha \le C(N,\alpha)|\nabla u|_2^{\gamma_\alpha}|u|_2^{1-\gamma_\alpha}, \text{ where } \gamma_\alpha := \frac{N(\alpha-2)}{2\alpha}. \tag{2.1}$$

It is well-known that the following problem

$$\begin{cases} -\Delta u = -u + u^{p-1} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(0) = \max_{\mathbb{R}^N} u \end{cases} \tag{2.2}$$

has a unique solution w_p in $H^1(\mathbb{R}^N)$, which is radial, radially decreasing and belongs to $C^2(\mathbb{R}^N)$ (see [20]). For $\lambda < 0$, we denote

$$w_{\lambda,p}(x) := (-\lambda)^{\frac{1}{p-2}} w_p(\sqrt{-\lambda}x), \quad \forall x \in \mathbb{R}^N. \tag{2.3}$$

Then $w_{\lambda,p}$ is the unique solution(up to a translation) of

$$-\Delta u = \lambda u + u^{p-1}, \quad u > 0 \quad \text{in } \mathbb{R}^N. \tag{2.4}$$

Moreover, there exists $c > 0$ (see [21] and the references therein) such that

$$\begin{aligned} & w_{\lambda,p}(x)|x|^{\frac{N-1}{2}} e^{\sqrt{-\lambda}|x|} \to c \quad \text{as } |x| \to \infty, \\ & w'_{\lambda,p}(r) r^{\frac{N-1}{2}} e^{\sqrt{-\lambda}r} \to -c\sqrt{-\lambda} \quad \text{as } r = |x| \to +\infty. \end{aligned} \tag{2.5}$$

Lemma 2.1 (Splitting Lemma). *For any $\lambda < 0$, let $\{u_n\} \subseteq H^1(\mathbb{R}^N)$ be a non-negative Palais-Smale sequence of I_λ. Then up to a subsequence, there exists a number $l \in \mathbb{N} \cup \{0\}$, a non-negative function $u_0 \in H^1(\mathbb{R}^N)$, l sequences of points $\{y_n^i\} \subseteq \mathbb{R}^N$ for $1 \le i \le l$ such that $|y_n^i| \to +\infty$ as $n \to \infty$ and*

$$u_n = u_0 + \sum_{i=1}^{l} w_{\lambda,p}(\cdot + y_n^i) + o_n(1) \quad \text{in } H^1(\mathbb{R}^N). \tag{2.6}$$

Furthermore, u_0 is a weak solution of

$$-\Delta u = \lambda u + u^{p-1} + h(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{2.7}$$

and

$$|u_n|_2^2 = |u_0|_2^2 + l|w_{\lambda,p}|_2^2 + o_n(1), \tag{2.8}$$

$$I_\lambda(u_n) = I_\lambda(u_0) + l I_\lambda^\infty(w_{\lambda,p}) + o_n(1). \tag{2.9}$$

The proof of Lemma 2.1 can be found in [[8], Lemma 3.1]. The difference is that [8] deals with external domains, not with $\mathbb{R}^N$. However, in combination with condition (h_1), the proof is similar.

We denote

$$m_c := \inf_{u \in S_c} I(u), \quad m_c^\infty := \inf_{u \in S_c} I^\infty(u). \tag{2.10}$$

Lemma 2.2 (Lemma 3.1, [30]). *For any $c > 0$, $m_c^\infty \in (-\infty, 0)$. Moreover, if $\{u_n\} \subseteq S_c$ is a non-negative minimizing sequence for I^∞ in S_c, then up to a subsequence, there exists $\{y_n\} \subseteq \mathbb{R}^N$ such that*

$$u_n(\cdot - y_n) \to w_{\lambda,p} \quad \text{strongly in } H^1(\mathbb{R}^N) \ \text{ as } n \to \infty,$$

where $\lambda < 0$ is determined by

$$c^2 = (-\lambda)^{s_p} |w_p|_2^2, \quad s_p := \frac{2}{p-2} - \frac{N}{2}. \tag{2.11}$$

In particular, m_c^∞ is attained by the function $w_{\lambda,p}$ and can be expressed as

$$m_c^\infty = \left(\frac{c^2}{|w_p|_2^2} \right)^{\frac{s_p+1}{s_p}} I^\infty(w_p). \tag{2.12}$$

We define the map $\Pi : \mathbb{R}^N \to S_c$

$$\Pi[y] = w_{\lambda,p}(\cdot - y).$$

Lemma 2.3 *$m_c = m_c^\infty \in (-\infty, 0)$ and m_c could not be attained.*

Proof For any $u \in H^1(\mathbb{R}^N)\backslash\{0\}$, by (h_1), we have $I^\infty(u) \le I(u)$, then $m_c^\infty \le m_c$. Next, we prove that $m_c \le m_c^\infty$. Considering $\{y_n\} \subset \mathbb{R}^N$, $|y_n| \to \infty$, we have $\Pi[y_n] \in S_c$. Moreover, by $h \in L^{\frac{2}{2-q}}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} h(x) |w_{\lambda,p}(x - y_n)|^q dx \to 0 \text{ as } n \to \infty,$$

which implies that

$$m_c \le \lim_{n\to\infty} I(\Pi[y_n]) = \lim_{n\to\infty} I^\infty(\Pi[y_n]) = m_c^\infty.$$

Therefore, $m_c = m_c^\infty \in (-\infty, 0)$.

We suppose that there exists a critical point $u \in H^1(\mathbb{R}^N)$ of I at level m_c. By a direct calculation,

$$m_c^\infty \le I^\infty(u) = I(u) + \frac{1}{q} \int_{\mathbb{R}^N} h(x)|u|^q dx = m_c + \frac{1}{q} \int_{\mathbb{R}^N} h(x)|u|^q dx,$$

then,

$$\int_{\mathbb{R}^N} h(x)|u|^q dx = 0, \tag{2.13}$$

that is, $I^\infty(u) = m_c^\infty$. By Lemma 2.2, we have $u = w_{\lambda,p}$ up to a translation, where $\lambda < 0$ is determined by (2.11). Since $w_{\lambda,p} > 0$ in $\mathbb{R}^N$, we deduce that $u > 0$ in $\mathbb{R}^N$. Moreover, by (h_1), we have

$$\int_{\mathbb{R}^N} h(x)|u|^q dx < 0,$$

which contradicts to (2.13). □

3 Compactness Result

In this section, we prove the compactness of some specific Palais-Smale sequence of I in S_c.

Lemma 3.1 *Let $N \geq 3$, $1 \leq q < 2 < p < 2 + \frac{4}{N}$, $c > 0$. Then, there exists a positive constant $\eta = \eta(c) \in (2^{-1/s_p}, 1)$ depending on c such that if $\{u_n\} \subseteq S_c$ is a nonnegative Palais-Smale sequence of $I|_{S_c}$ at level d with $m_c^\infty < d < \eta m_c^\infty$, then up to a subsequence, there exists $u_0 \in S_c$ such that*

$$u_n \to u_0 \quad \text{strongly in } H^1(\mathbb{R}^N) \text{ as } n \to \infty.$$

Furthermore, u_0 is a positive solution of (1.1) for some $\lambda < 0$.

Proof Since $2 < p < 2 + \frac{4}{N}$, we have $s_p > 0$, then $0 < 2^{-1/s_p} < 1$. By Lemma 2.2, we get

$$m_c^\infty < \eta m_c^\infty < 2^{-1/s_p} m_c^\infty < 0. \tag{3.1}$$

Let $\{u_n\} \subseteq S_c$ be a nonnegative Palais-Smale sequence of $I|_{S_c}$ at level d, where $d \in (m_c^\infty, 2^{-1/s_p} m_c^\infty)$. By $I(u_n) \to d < 0$ as $n \to \infty$ and

$$\begin{aligned} I(u_n) &= \frac{1}{2}|\nabla u_n|_2^2 - \frac{1}{p}|u_n|_p^p - \frac{1}{q}\int_{\mathbb{R}^N} h(x)|u_n|^q dx \\ &\geq \frac{1}{2}|\nabla u_n|_2^2 - \frac{1}{p}C^p(N,p)c^{p(1-\gamma_p)}|\nabla u_n|^{p\gamma_p} - |h|_{\frac{2}{2-q}} c^q, \end{aligned}$$

which implies that I is coercive in S_c. Then $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Moreover, $(I|_{S_c})'(u_n) = o_n(1)$. By the Lagrange multiplier rule, there exists $\{\lambda_n\} \subseteq \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} \nabla u_n \nabla \psi dx - \lambda_n \int_{\mathbb{R}^N} u_n \psi dx - \int_{\mathbb{R}^N} u_n^{p-1}\psi dx - \int_{\mathbb{R}^N} h(x) u_n^{q-1}\psi dx$$

$$= o_n(1)\|\psi\|, \quad \forall \psi \in H^1(\mathbb{R}^N). \tag{3.2}$$

By $\{u_n\} \subseteq S_c$ is bounded in $H^1(\mathbb{R}^N)$, then from (3.2), we have

$$-\lambda_n c^2 = |u_n|_p^p - |\nabla u_n|_2^2 + \int_{\mathbb{R}^N} h(x)|u_n|^q dx + o_n(1)||u_n||, \tag{3.3}$$

which implies that $\{\lambda_n\}$ is bounded, up to a subsequence, there exists $\lambda \in \mathbb{R}$ such that $\lambda_n \to \lambda$ as $n \to \infty$. Furthermore, by $p > 2$ and $I(u_n) \to d < 0$ as $n \to \infty$, we have

$$\begin{aligned}
-\lambda_n c^2 =& |u_n|_p^p - |\nabla u_n|_2^2 + \int_{\mathbb{R}^N} h(x)|u_n|^q dx + o_n(1) \\
=& - pI(u_n) + \frac{p-2}{2}|\nabla u_n|_2^2 + (1-p)\int_{\mathbb{R}^N} h(x)|u_n|^q dx + o(1) \\
\geq& - pd + o(1).
\end{aligned}$$

Letting $n \to \infty$, we get

$$-\lambda \geq \frac{-pd}{c^2} > 0. \tag{3.4}$$

Thus $\lambda < 0$. By (3.2) and $\{u_n\} \subseteq H^1(\mathbb{R}^N)$ is bounded, we see that $\{u_n\}$ is a Palais-Smale sequence of I_λ. By Lemma 2.1, up to a subsequence, there exists an integer $l \geq 0$, a non-negative function $u_0 \in H^1(\mathbb{R}^N)$, l sequences $\{y_n^i\} \subseteq \mathbb{R}^N$ for $1 \leq i \leq l$ such that $|y_n^i| \to +\infty$ as $n \to \infty$ and

$$u_n = u_0 + \sum_{i=1}^{l} w_{\lambda,p}(\cdot + y_n^i) + o_n(1) \quad \text{in } H^1(\mathbb{R}^N). \tag{3.5}$$

In addition, u_0 is a solution of

$$-\Delta u = \lambda u + u^{p-1} + h(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{3.6}$$

and

$$c^2 = |u_0|_2^2 + l|w_{\lambda,p}|_2^2, \tag{3.7}$$

$$I_\lambda(u_n) = I_\lambda(u_0) + lI_\lambda^\infty(w_{\lambda,p}) + o_n(1). \tag{3.8}$$

Since

$$I_\lambda(v) = I(v) + \frac{-\lambda}{2}|v|_2^2 \quad \text{for } v \in H^1(\mathbb{R}^N)$$

and

$$I_\lambda^\infty(v) = I^\infty(v) + \frac{-\lambda}{2}|v|_2^2 \quad \text{for } v \in H^1(\mathbb{R}^N),$$

by (3.7), (3.8), Lemmas 2.2 and 2.3, we have

$$
\begin{aligned}
d &= I(u_0) + l m^{\infty}_{|w_{\lambda,p}|_2} \\
&\geq m^{\infty}_{|u_0|_2} + l m^{\infty}_{|w_{\lambda,p}|_2} \\
&= C\left[(c^2 - l|w_{\lambda,p}|_2^2)^{\frac{s_p+1}{s_p}} + l(|w_{\lambda,p}|_2^2)^{\frac{s_p+1}{s_p}} \right],
\end{aligned}
\tag{3.9}
$$

where

$$
C := (|w_p|_2^2)^{-\frac{s_p+1}{s_p}} I^{\infty}(w_p) < 0.
$$

Next, we will divide into three steps to prove that there exists $\eta = \eta(c) \in (2^{-1/s_p}, 1)$ such that if $m_c^{\infty} < d < \eta m_c^{\infty}$, then $l = 0$.

Step 1. If $m_c^{\infty} < d < 2^{-1/s_p} m_c^{\infty}$, then $|w_{\lambda,p}|_2^2 \geq kc^2$, where

$$
k := \frac{1}{2}\left[\frac{-pI^{\infty}(w_p)}{|w_p|_2^2} \right]^{s_p} \in \left(0, \frac{1}{2}\right).
$$

In fact, by (3.4) and Lemma 2.2, we have

$$
\begin{aligned}
|w_{\lambda,p}|_2^2 = (-\lambda)^{s_p} |w_p|_2^2 &\geq \left(\frac{-pd}{c^2}\right)^{s_p} |w_p|_2^2 \\
&\geq \left(\frac{-p2^{-1/s_p} m_c^{\infty}}{c^2}\right)^{s_p} |w_p|_2^2 \\
&= kc^2.
\end{aligned}
$$

Moreover, w_p is a solution of (2.2), then we have

$$
|\nabla w_p|_2^2 + |w_p|_2^2 = |w_p|_p^p.
$$

Therefore,

$$
I_{-1}^{\infty}(w_p) = \left(\frac{1}{2} - \frac{1}{p}\right)\left(|\nabla w_p|_2^2 + |w_p|_2^2\right) > \left(\frac{1}{2} - \frac{1}{p}\right)|w_p|_2^2.
$$

Note that $I^{\infty}(w_p) = I_{-1}^{\infty}(w_p) - \frac{1}{2}|w_p|^2 < 0$, we have

$$
0 < \frac{-pI^{\infty}(w_p)}{|w_p|_2^2} < 1.
$$

Since $s_p > 0$, we deduce that $k \in \left(0, \frac{1}{2}\right)$.

Step 2. Let

$$k_1 := k^{(s_p+1)/s_p} + (1-k)^{(s_p+1)/s_p} \in (2^{-1/s_p}, 1).$$

If $m_c^\infty < d < k_1 m_c^\infty$, then $l \le 1$. Indeed, for $l \ge 1$, we consider

$$F_l(t) := (c^2 - lt)^{\frac{s_p+1}{s_p}} + lt^{\frac{s_p+1}{s_p}}, \quad \forall\, 0 \le t \le \frac{c^2}{l}.$$

Here we point out that $F_l(t)$ is actually the transformation form of (3.9). Evidently, $F_l \in C^1([0, \frac{c^2}{l}])$ and

$$F_l'(t) = \frac{s_p+1}{s_p} l \left[t^{1/s_p} - (c^2 - lt)^{1/s_p} \right], \quad \forall\, t \in \left[0, \frac{c^2}{l}\right].$$

Let $t_l := \frac{c^2}{l+1}$, then F_l is strictly decreasing in $[0, t_l]$ and strictly increasing in $[t_l, \frac{c^2}{l}]$. Thus,

$$\min_{t\in[0,c^2/l]} F_l(t) = F_l(t_l) = (l+1)^{-\frac{1}{s_p}} (c^2)^{\frac{s_p+1}{s_p}} > 0.$$

Combining Step 1 and (3.7), we have $l \le k^{-1}$. Let

$$h(l) := (1-lk)^{(s_p+1)/s_p} + lk^{(s_p+1)/s_p}, \quad \forall l \in [1, k^{-1}].$$

It is obvious that $h \in C^1([1, k^{-1}])$ and

$$h'(l) = k^{(s_p+1)/s_p} - k\frac{s_p+1}{s_p}(1-lk)^{\frac{1}{s_p}}, \quad \forall l \in [1, k^{-1}].$$

Let

$$l_0 := \frac{1}{k} - \left(\frac{s_p}{s_p+1}\right)^{s_p} \in (1, k^{-1}).$$

Then h is strictly decreasing in $[1, l_0]$ and strictly increasing in $[l_0, k^{-1}]$. If $l \ge 2$, by (3.9), we have

$$\begin{aligned} d &\ge C \max\{F_l(kc^2), F_l(c^2/l)\} = \max\{h(l), l^{-1/s_p}\} m_c^\infty \\ &\ge \max\{h(1), h(k^{-1}), 2^{-1/s_p}\} m_c^\infty \\ &= k_1 m_c^\infty, \end{aligned}$$

contradicting to $d < k_1 m_c^\infty$. Hence $l \le 1$.

Step 3. There exists $\eta = \eta(c) \in [k_1, 1)$ such that if $m_c^\infty < d < \eta m_c^\infty$, then $l = 0$.

By Step 2, we have $l \leq 1$. If $u_0 \equiv 0$, then by (3.7) and (3.9), we have $d = 0$ or $d = m_c^\infty$, which contradicts to $m_c^\infty < d < 0$. Hence, $u_0 \not\equiv 0$. From (3.7), we have

$$|\nabla u_0|_2^2 - \lambda |u_0|_2^2 = |u_0|_p^p + \int_{\mathbb{R}^N} h(x)|u_0|^q dx. \tag{3.10}$$

Combining Sobolev inequality and (3.10), there exists $C_1 > 0$ independent of u_0 and λ such that

$$\begin{aligned} \min\{1, -\lambda\}(|\nabla u_0|_2^2 + |u_0|_2^2) &\leq |u_0|_p^p + \int_{\mathbb{R}^N} h(x)|u_0|^q dx \\ &\leq C_1(|\nabla u_0|_2^2 + |u_0|_2^2)^{\frac{p}{2}}. \end{aligned}$$

By $u_0 \not\equiv 0$, $\lambda < 0$ and $p > 2$, we have

$$|\nabla u_0|_2^2 + |u_0|_2^2 \geq \left(\frac{\min\{1, -\lambda\}}{C_1}\right)^{\frac{2}{p-2}} > 0. \tag{3.11}$$

On the other hand, by (3.10), (2.1) and $\lambda < 0$, we have

$$|\nabla u_0|_2^2 \leq |u_0|_p^p \leq C(N, p)|\nabla u_0|_2^{p\gamma_p}|u_0|_2^{p(1-\gamma_p)}.$$

Since $2 < p < 2 + \frac{4}{N}$, that is, $0 < p\gamma_p < 2$. Then,

$$|\nabla u_0|_2 \leq C^{\frac{p}{2-p\gamma_p}}(N, p)|u_0|_2^{\frac{p(1-\gamma_p)}{2-p\gamma_p}}. \tag{3.12}$$

Thus, together with (3.4), (3.11), (3.12) and $d < 2^{-1/s_p} m_c^\infty$, it follows that

$$\begin{aligned} &C^{\frac{p}{2-p\gamma_p}}(N, p)|u_0|_2^{\frac{p(1-\gamma_p)}{2-p\gamma_p}}|u_0|_2^2 \\ &\geq \left(\frac{1}{C_1}\min\left\{1, -p2^{-1/s_p}\frac{m_c^\infty}{c^2}\right\}\right)^{\frac{2}{p-2}} > 0, \end{aligned} \tag{3.13}$$

which implies that there exists $C(c) > 0$ such that

$$|u_0|_2^2 \geq C(c).$$

Denote

$$\eta(c) := \begin{cases} k_1, & \text{if } c^2 \leq C(c), \\ \max\left\{k_1, \frac{F_1(c^2 - C(c))}{(c^2)^{\frac{s_p+1}{s_p}}}\right\} & \text{if } c^2 > C(c). \end{cases} \tag{3.14}$$

If $c^2 \le C(c)$, together with $|u_0|_2^2 \ge C(c)$ and (3.7), we find that $|u_0|_2^2 = c^2$ and $l = 0$.

If $c^2 > C(c)$, then $0 < c^2 - C(c) < c^2$ and hence

$$0 < F_1(c^2 - C(c)) < \max\{F_1(0), F_1(c^2)\} = (c^2)^{\frac{sp+1}{sp}},$$

which implies that $0 < \frac{F_1(c^2 - C(c))}{(c^2)^{\frac{sp+1}{sp}}} < 1$. Therefore, $\eta(c) \in [k_1, 1)$. If $l = 1$, then it follows from Step 1 and (3.7), we have

$$kc^2 \le |w_{\lambda,p}|_2^2 \le c^2 - C(c).$$

Then, by (3.9), we deduce that

$$d \ge C \max\{F_1(kc^2), F_1(c^2 - C(c))\} \ge \eta(c) m_c^\infty,$$

which contradicts to $d < \eta(c) m_c^\infty$. Hence $l = 0$. □

4 Proof of Theorem 1.1

In this section, let $c > 0$, $N \ge 3$ and $1 \le q < 2 < p < 2 + \frac{4}{N}$. We focus on the proof of Theorem 1.1.

Lemma 4.1 *Assume that h satisfy (h_1). Then*

$$\sup_{y \in \mathbb{R}^N} I(\Pi[y]) < \eta m_c^\infty, \tag{4.1}$$

where $\eta = \eta(c) \in (0, 1)$ is defined in (3.14).

Proof By (h_1), we have

$$\begin{aligned} I(\Pi[y]) &= I^\infty(\Pi[y]) + (I(\Pi[y]) - I^\infty(\Pi[y])) \\ &= m_c^\infty + \int_{\mathbb{R}^N} h(x)|w_{\lambda,p}|^q dx \\ &\le m_c^\infty + |h|_{\frac{2}{2-q}} |w_{\lambda,p}|_2^q \\ &= m_c^\infty + |h|_{\frac{2}{2-q}} c^q \\ &< \eta m_c^\infty. \end{aligned}$$

□

Next, we define the barycentre of a function $u \in H^1(\mathbb{R}^N)\backslash\{0\}$. Let

$$\mu(u)(x) = \frac{1}{|B_r|} \int_{B_r(x)} |u(y)| dy,$$

with $\mu(u) \in L^\infty(\mathbb{R}^N)$ and μ is a continuous function. Subsequently, take

$$\hat{u}(x) = \left[\mu(u)(x) - \frac{1}{2}\max \mu(u)\right]^+.$$

It follows that $\hat{u} \in C_0(\mathbb{R}^N)$. Now, we define the barycenter of u by

$$\beta(u) = \frac{1}{|\hat{u}|_{L^1}} \int_{\mathbb{R}^N} x\hat{u}(x)dx \in \mathbb{R}^N.$$

Since $\hat{u}$ has compact support, by definition, $\beta(u)$ is well defined. The function β satisfies the following properties:

(a) β is a continuous function in $H^1(\mathbb{R}^N)\backslash\{0\}$.
(b) If u is radial, then $\beta(u) = 0$.
(c) $\beta(tu) = \beta(u)$.
(d) Given $y \in \mathbb{R}^N$ and defining $u_y(x) := u(x-y)$, then $\beta(u_y) = \beta(u) + y$.
Let $\mathcal{P}$ be the cone of non-negative functions of $H^1(\mathbb{R}^N)$. Define

$$\mathcal{M} := \{u \in \mathcal{P} \cap S_c : \beta(u) = 0\}.$$

Moreover, by $w_{\lambda,p} \in \mathcal{M}$, we have $\mathcal{M} \neq \emptyset$. Therefore, we are allowed to define $b := \inf_{u\in\mathcal{M}} I(u)$.

Lemma 4.2 *There holds* $b > m_c^\infty$.

Proof By the definition of b, we have $b \geq m_c = m_c^\infty$. To reach the conclusion, we argue by contradiction. Indeed, suppose $b = m_c^\infty$, then there is a sequence $\{u_n\} \subseteq \mathcal{P} \cap S_c$ such that

$$\beta(u_n) = 0, \forall n \geq 1 \text{ and } I(u_n) \to m_c^\infty \text{ as } n \to \infty.$$

Thus, by Lemma 2.2, there exits a sequence $\{y_n\} \subseteq \mathbb{R}^N$ such that

$$u_n(\cdot - y_n) \to w_{\lambda,p} \quad \text{as } n \to \infty \text{ in } H^1(\mathbb{R}^N). \tag{4.2}$$

Then, we have

$$y_n = \beta(u_n) + y_n = \beta(u_n(x-y_n)) \to \beta(w_{\lambda,p}) = 0 \text{ as } n \to \infty,$$

that is, $\lim_{n\to\infty} y_n = 0$. From (4.2), we have $u_n \to w_{\lambda,p}$ in $H^1(\mathbb{R}^N)$ and $I(u_n) \to I(w_{\lambda,p}) = m_c$, which contradicts to Lemma 2.3. □

Condition (h_1) implies that $I^\infty(\Pi[y]) < I(\Pi[y])$, for any $y \in \mathbb{R}^N$. By Lemma 4.2, $b > m_c^\infty$. By the definition of $\Pi[y]$ and Lemma 2.3, we have $I(\Pi[y]) \to m_c^\infty$ as $|y| \to \infty$. Then there exists $\bar{R} > 0$ such that

$$m_c^\infty < \max_{|y|=\bar{R}} I(\Pi[y]) < \frac{b + m_c^\infty}{2} \tag{4.3}$$

for any $R \geq \bar{R}$. Next, we define a set $\Sigma \subset \mathcal{P} \subset H^1(\mathbb{R}^N)$ as follows:

$$\Sigma := \{\Pi(y) : |y| \leq \bar{R}\}.$$

Let

$$\mathcal{H} = \left\{ h \in C(\mathcal{P} \cap S_c, \mathcal{P} \cap S_c) : h(u) = u, \forall u \in \mathcal{P} \cap S_c \text{ with } I(u) < \frac{b + m_c^\infty}{2} \right\}$$

and

$$\Gamma = \{A \subset \mathcal{P} \cap S_c : A = h(\Sigma), h \in \mathcal{H}\}.$$

Lemma 4.3 *If $A \in \Gamma$, then $A \cap \mathcal{M} \neq \emptyset$.*

Proof We just to show that for every $A \in \Gamma$, there exists $u \in A$ such that $\beta(u) = 0$. It suffices to prove that for every $h \in \mathcal{H}$, there exists $\tilde{y} \in \mathbb{R}^N$ with $|\tilde{y}| \leq \bar{R}$ such that

$$(\beta \circ h \circ \Pi)[\tilde{y}] = 0.$$

For any $h \in \mathcal{H}$, we define

$$\mathcal{J} = \beta \circ h \circ \Pi : \mathbb{R}^N \to \mathbb{R}^N$$

and $\mathcal{F} : [0, 1] \times \bar{B}_{\bar{R}}(0) \to \mathbb{R}^N$ given by

$$\mathcal{F}(t, y) = t\mathcal{J}(y) + (1 - t)y.$$

We claim that $0 \notin \mathcal{F}(t, \partial \bar{B}_{\bar{R}}(0))$. Indeed, for $|y| = \bar{R}$, by (4.3), we have

$$I(\Pi[y]) < \frac{b + m_c^\infty}{2}.$$

Hence,

$$\mathcal{F}(t, y) = t(\beta \circ \Pi)[y] + (1 - t)y$$

and

$$(\mathcal{F}(t, y), y) = t(\beta(\Pi[y]), y) + (1 - t)(y, y).$$

If $t = 0$, then $(\mathcal{F}(t, y), y) = |y|^2 = \bar{R}^2 > 0$. If $t = 1$, then by $\beta(\Pi[y]) = y$, we have $\mathcal{F}(1, y) = (\beta(\Pi[y]), y) = |y|^2 > 0$. If $t \in (0, 1)$, then $(\mathcal{F}(t, y), y) > 0$ since the terms $t, 1 - t, (\beta(\Pi[y]), y)$ and $|y|^2$ are all positive. Then, by the invariance under homotopy of the Brouwer degree, one has

$$\deg(\mathcal{F}(t, \cdot), B_{\bar{R}}(0), 0) = \deg(\mathcal{J}, B_{\bar{R}}(0), 0) = 1, \ \forall t \in [0, 1].$$

Then, there exists $\tilde{y} \in B_{\tilde{R}}(0)$ such that $\mathcal{J}(\tilde{y}) = 0$, that is,

$$\mathcal{J}(\tilde{y}) = (\beta \circ h \circ \Pi)[\tilde{y}] = 0.$$

□

Now, let us denote

$$d := \inf_{A \in \Gamma} \sup_{u \in A} I(u), \tag{4.4}$$

$$\mathcal{K}_d = \{u \in \mathcal{P} \cap S_c : I(u) = d \text{ and } \nabla I|_{S_c}(u) = 0\},$$

and

$$L_\gamma = \{u \in S_c : I(u) \leq \gamma\}$$

for every $\gamma \in \mathbb{R}$.

Proof of Theorem 1.1. We will show that d given by (4.4) is a critical value, that is, $\mathcal{K}_d \neq \emptyset$. First, we claim that

$$m_c^\infty < d < \eta m_c^\infty.$$

In fact, by Lemma 4.3, for each $A \in \Gamma$, there is $\tilde{u} \in A \cap \mathcal{M}$. Hence,

$$b = \inf_{u \in \mathcal{M}} I(u) \leq \inf_{u \in A \cap \mathcal{M}} I(u) \leq I(\tilde{u}) \leq \sup_{u \in A \cap \mathcal{M}} I(u) \leq \sup_{u \in A} I(u).$$

Since $b > m_c^\infty$, from Lemma 4.2, we have

$$m_c^\infty < b \leq \sup_{u \in A} I(u), \ \forall A \in \Gamma.$$

Thus,

$$m_c^\infty < b \leq \inf_{A \in \Gamma} \sup_{u \in A} I(u) = d.$$

By the definition of d,

$$d \leq \sup_{u \in A} I(u), \ \forall A \in \Gamma,$$

it follows that

$$d \leq \sup_{\Pi[y] \in \Sigma} I(h(\Pi[y])), \ \forall h \in \mathcal{H}.$$

Now, taking $h \equiv I$, we find

$$d \leq \sup_{\Pi[y] \in \Sigma} I(\Pi[y]).$$

Hence,

$$d \leq \sup_{|y| \leq \bar{R}} I(\Pi[y]) \leq \sup_{y \in \mathbb{R}^N} I(\Pi[y]).$$

By Lemma 4.1, we have

$$d \leq \sup_{y \in \mathbb{R}^N} I(\Pi[y]) < \eta m_c^\infty. \tag{4.5}$$

Combining (4.4) and (4.5), one has

$$m_c^\infty < d < \eta m_c^\infty.$$

Suppose on the contrary that $\mathcal{K}_d = \emptyset$. Note that

$$\frac{b + m_c^\infty}{2} \leq \frac{d + m_c^\infty}{2} < d < \eta m_c^\infty.$$

By Lemma 3.1 and the deformation lemma, there exists a continuous map

$$\tau : [0, 1] \times S_c \cap \mathcal{P} \to S_c \cap \mathcal{P}$$

and a positive number ϵ_0 such that

(a) $L_{d+\epsilon_0} \backslash L_{d-\epsilon_0} \subset\subset L_{\eta m_c^\infty} \backslash L_{\frac{b+m_c^\infty}{2}}$,
(b) $\tau(t, u) = u, \forall u \in L_{d-\epsilon_0} \cup \{S_c \cap \mathcal{P} \backslash L_{d+\epsilon_0}\}$ and for any $t \in [0, 1]$, and
(c) $\tau(1, L_{d+\frac{\epsilon_0}{2}}) \subset L_{d-\frac{\epsilon_0}{2}}$.

Fix $\tilde{A} \in \Gamma$ such that

$$d \leq \sup_{u \in \tilde{A}} I(u) < d + \frac{\epsilon_0}{2}.$$

Since

$$I(u) < d + \frac{\epsilon_0}{2}, \text{ for any } u \in \tilde{A},$$

it follows that

$$\tilde{A} \subset L_{d+\frac{\epsilon_0}{2}}.$$

Now, by item (c) above, we have

$$I(u) < d - \frac{\epsilon_0}{2}, \text{ for any } u \in \tau(1, \tilde{A}),$$

that is,

$$\sup_{u \in \tau(1,\tilde{A})} I(u) < d - \frac{\epsilon_0}{2}. \tag{4.6}$$

Moreover, we note that $\tau(1,\cdot) \in C(\mathcal{P} \cap S_c, \mathcal{P} \cap S_c)$. By $\tilde{A} \in \Gamma$, there exists $h \in \mathcal{H}$ such that $\tilde{A} = h(\Sigma)$. Then,

$$\tilde{h} = \tau(1,\cdot) \circ h \in C(\mathcal{P} \cap S_c, \mathcal{P} \cap S_c).$$

By the definition of $\mathcal{H}$,

$$h(u) = u, \forall u \in \mathcal{P} \cap S_c \text{ with } I(u) < \frac{b + m_c^\infty}{2}$$

and

$$\tilde{h}(u) = \tau(1, u), \forall u \in \mathcal{P} \cap S_c \text{ with } I(u) < \frac{b + m_c^\infty}{2}.$$

By

$$\frac{b + m_c^\infty}{2} < d - \epsilon_0$$

and (b), we have

$$\tilde{h}(u) = \tau(1, u) = u, \forall u \in \mathcal{P} \cap S_c \text{ with } I(u) < \frac{b + m_c^\infty}{2} < d - \epsilon_0,$$

which implies that $\tilde{h} \in \mathcal{H}$. Moreover, $\tau(1, \tilde{A}) \in \Gamma$ since $\tau(1, \tilde{A}) = \tilde{h}(\Sigma)$. Therefore, by the definition of d, we have

$$d \le \sup_{u \in \tau(1,\tilde{A})} I(u),$$

which contradicts (4.6). Consequently, $\mathcal{K}_d \neq \emptyset$ and d is a critical value of functional I on $\mathcal{P} \cap S_c$. By $u \in \mathcal{P} \cap S_c$, we have $u \ge 0$ in $\mathbb{R}^N$. Since $u \neq 0$, it follows from the strong maximum principle in [12] that $u > 0$ in $\mathbb{R}^N$, and thus, $u \in S_c$ is a positive solution of (1.1) for some $\lambda < 0$. The proof is finished. □

Author Contributions T-TD wrote the main manuscript text. All authors reviewed the manuscript.

Declarations

Conflict of interest The authors declare no competing interests.

References

1. Adachi, S., Tanaka, K.: Four positive solutions for the semilinear elliptic equation: $-\Delta u + u = a(x)u^p + f(x)$ in $\mathbb{R}^N$. Calc. Var. Partial Differ. Equ. **11**(1), 63–95 (2000)
2. Bartsch, T., de Valeriola, S.: Normalized solutions of nonlinear Schödinger equations. Arch. Math. **100**(1), 75–83 (2013)
3. Berestyeki, H., Lions, P.L.: Nonlinear scalar field equations, I. Existence of a ground state. Arch. Rational Meek Anal. **82**, 313–345 (1983)
4. Bartsch, T., Jeanjean, L.: Normalized solutions for nonlinear Schrödinger systems. Proc. Roy. Soc. Edinburgh Sect. A. **148**(2), 225–242 (2018)
5. Bartsch, T., Jeanjean, L., Soave, N.: Normalized solutions for a system of coupled cubic Schrödinger equations on $\mathbb{R}^3$. J. Math. Pures Appl. **106**(4), 583–614 (2016)
6. Bartsch, T., Soave, N.: A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems. J. Funct. Anal. **272**(12), 4998–5037 (2017)
7. Bartsch, T., Zhong, X.X., Zou, W.M.: Normalized solutions for a coupled Schrödinger system. Math. Ann. **380**, 1713–1740 (2021)
8. Benci, V., Cerami, G.: Positive solutions of some nonlinear elliptic problems in exterior domains. Arch. Ration. Mech. Anal. **99**(4), 283–300 (1987)
9. Chen, Z., Zou, W.M.: Existence of normalized positive solutions for a class of nonhomogeneous elliptic equations. J. Geom. Anal. **33**, 147 (2023)
10. Ding, W.Y., Ni, W.M.: On the existence of entire solution of a semilinear elliptic equation. Arch. Rational Meek Anal. **91**, 283–308 (1986)
11. Deng, Y.H., Li, Y.: Existence of multiple positive solutions for a semilinear elliptic equation. Adv. Differ. Equ. **2**(3), 361–382 (1997)
12. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer-Verlag, New York (1998)
13. Gou, T., Jeanjean, L.: Multiple positive normalized solutions for nonlinear Schrödinger systems. Nonlinearity. **31**(5), 2319–2346 (2018)
14. Hirata, J., Tanaka, K.: Nonlinear scalar field equations with L^2 constraint: mountain pass and symmetric mountain pass approaches. Adv. Nonlinear Stud. **19**, 263–290 (2019)
15. Hirano, N.: Existence of entire positive solutions for nonhomogeneous elliptic equations. Nonlinear Anal. **29**(8), 889–901 (1997)
16. Ikoma, N., Miyamoto, Y.: Stable standing waves of nonlinear Schrödinger equations with potentialsand general nonlinearities. Calc. Var. Partial Differ. Equ. **59**(2), 1–20 (2020)
17. Ikoma, N., Tanaka, K.: A note on deformation argument for L^2 normalized solutions of nonlinear Schrödinger equations and systems. Adv. Diff. Eq. **24**(11–12), 609–646 (2019)
18. Jeanjean, L.: Existence of solutions with prescribed norm for semilinear elliptic equations. Nonlinear Anal. **28**(10), 1633–1659 (1997)
19. Jeanjean, L., Lu, S.S.: Nonradial normalized solutions for nonlinear scalar field equations. Nonlinearity **32**, 4942–4966 (2019)
20. Kwong, M.K.: Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$. Arch. Ration. Mech. Anal. **105**(3), 243–266 (1989)
21. Lions, P.L.: The concentration-compactness principle in the calculus of variations. The Locally compact case. II. Anna. de l'Inst. Henri Poincaré Anal. Non Linéaire **1**(4), 223–283 (1984)
22. Lehrer, R., Maia, L.: Positive solutions of asymptotically linear equations via Pohozăev manifold. J. Funct. Anal. **266**, 213–246 (2014)
23. Strauss, W.: Existence of solitary waves in higher dimensions. Comm. Math. Phys. **55**, 149–162 (1977)
24. Weinstein, M.I.: Nonlinear Schrödinger equations and sharp interpolation estimates. Commun. Math. Phys. **87**, 567–576 (1982)

25. Zhang, J., Zhang, W.: Semiclassical states for coupled nonlinear Schrödinger system with competing potentials. J. Geom. Anal. **32**, 114 (2022)
26. Zhang, J., Zhang, W., Rădulescu, V.D.: Double phase problems with competing potentials: concentration and multiplication of ground states. Math. Z. **301**, 4037–4078 (2022)
27. Zhang, W., Zhang, J.: Multiplicity and concentration of positive solutions for fractional unbalanced double-phase problems. J. Geom. Anal. **32**, 235 (2022)
28. Zhang, W., Zhang, J., Rădulescu, V.D.: Concentrating solutions for singularly perturbed double phase problems with nonlocal reaction. J. Differ. Equ. **347**, 56–103 (2023)
29. Zhu, X.P.: A perturbation result on positive entire solutions of a semilinear elliptic equation. J. Differ. Equ. **92**, 163–178 (1991)
30. Zhang, Z.X., Zhang, Z.T.: Normalized solutions of mass subcritical Schrödinger equations in exterior domains. NoDEA Nonlinear Differ. Equ. Appl. **29**, 32 (2022)