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Normalized Ground States and Multiple Solutions for Nonautonomous Fractional Schrödinger Equations

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Abstract

In this paper, we consider the following fractional Schrödinger equations with prescribed L^2-norm constraint:

$$\begin{cases} (-\Delta)^s u = \lambda u + h(\varepsilon x) f(u) \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases}$$

where $0 < s < 1$, $N \geq 3$, $a, \varepsilon > 0$, $h \in C(\mathbb{R}^N, \mathbb{R}^+)$ and $f \in C(\mathbb{R}, \mathbb{R})$. In the mass subcritical case but under general assumptions on f, we prove the multiplicity of normalized solutions to this problem. Specifically, we show that the number of normalized solutions is at least the number of global maximum points of h when ε is small enough. Before that, without any restrictions on ε and the number of global maximum points, the existence of normalized ground states can be determined. In this sense, by studying the relationship between $h_0 := \inf_{x\in\mathbb{R}^N} h(x)$ and $h_\infty := \lim_{|x|\to\infty} h(x)$, we establish new results on the existence of normalized ground states for nonautonomous elliptic equations.

Keywords Fractional Schrödinger equation · Ground states · Multiple normalized solutions

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1 Introduction and Main Results

We are concerned with the following fractional Schrödinger equations with prescribed L^2-norm constraint:

$$\begin{cases} (-\Delta)^s u = \lambda u + h(\varepsilon x) f(u) \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases} \tag{1.1}$$

where $0 < s < 1$, $N \geq 3$, $a, \varepsilon > 0$, $h \in C(\mathbb{R}^N, \mathbb{R}^+)$, $f \in C(\mathbb{R}, \mathbb{R})$ and $\lambda \in \mathbb{R}$ appears as an unknown Lagrange multiplier.

In particular, $(-\Delta)^s$ is the fractional Laplacian operator defined as

$$(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \ \forall u \in \mathcal{S}(\mathbb{R}^N),$$

where $\mathcal{S}(\mathbb{R}^N)$ denotes the Schwartz space of rapidly decreasing smooth functions, P.V. stands for the principle value of the integral and $C_{N,s}$ is some positive normalization constant [20]. Moreover, the operator $(-\Delta)^s$ can be seen as the infinitesimal generators of Lévy stable diffusion processes, see [4] for example. Of course, this operator also arises in several areas such as optimization, finance, phase transitions, stratified materials, crystal dislocation, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, materials science, and water waves, see for instance [8, 13, 20, 22, 26] for an introduction to these topics and their applications.

When $s \nearrow 1^-$ and $h = 1$, problem (1.1) reduces to the following class of elliptic problems

$$\begin{cases} -\Delta u = \lambda u + f(u) \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2. \end{cases} \tag{1.2}$$

The above problem has been studied by JeanJean in [14], where the author proved that the existence of normalized solutions in purely L^2-supercritical case, i.e., $f(u) = |u|^{p-2}u, 2+4/N < p < 2^* := 2N/(N-2)$. Recently, Soave [24] carefully analyzes the cases when the combined power nonlinearities in (1.2) are of mixed type, that is,

$$f(u) = \mu|u|^{q-2}u + |u|^{p-2}u, \ 2 < q \leq 2 + \frac{4}{N} \leq p < 2^*.$$

After that, a great attention has been paid to problem (1.2), we refer the reader to [2, 5, 15, 25, 31] and the related results mentioned there. As far as the nonautonomous Schrödinger equations are concerned, most of the current researches add an additional term $V(x)u$ compared to problem (1.2), and then prove the existence of normalized solutions under the appropriate assumptions on potential V, see for instance, [19, 28] and the references therein. In particular, if $h \neq 1$, Alves [1] was concerned with the existence of multiple normalized solutions to the following nonautonomous Schrödinger equations

$$\begin{cases} -\Delta u = \lambda u + h(\varepsilon x) f(u) \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases} \tag{1.3}$$

where $a, \varepsilon > 0$ and $\lambda \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier. The nonlinearity f is a continuous function with a L^2-subcritical growth and satisfies the following assumptions:

- (f_1) f is odd and there is $q \in (2, 2 + \frac{4}{N})$ such that $\lim_{s \to 0} \frac{|f(s)|}{|s|^{q-1}} = \alpha > 0$;
- (f_2) there exist constant $c_1, c_2 > 0$ and $p \in (2, 2 + \frac{4}{N})$ such that

$$|f(s)| \leq c_1 + c_2 |s|^{p-1}, \ \forall s \in \mathbb{R};$$

- (f_3) there exists $q_1 \in (2, 2 + \frac{4}{N})$ such that $f(s)/s^{q_1 - 1}$ is an increasing function of s on $(0, \infty)$.

Moreover, the function h satisfies the following conditions:

- (h_1) $h \in C(\mathbb{R}^N)$ and $0 < h_0 := \inf_{x \in \mathbb{R}^N} h(x) \leq \max_{x \in \mathbb{R}^N} h(x) := h_{max}$;
- (h_2) $h_\infty := \lim_{|x| \to \infty} h(x) < h_{max}$;
- (h_3) $h^{-1}(\{h_{max}\}) = \{a_1, a_2, \cdots, a_l\}$ with $a_1 = 0$ and $a_j \neq a_i$ if $j \neq i$.

Based on the above assumptions, it is proved that the number of normalized solutions is at least the number of global maximum points of h when ε is small enough. In addition, it is worth noting that the condition (h_3) was introduced in [10] to prove the multiplicity of positive and nodal solutions of problem (1.3) without mass constraint.

In the sequel, we turn our attention to the case of $s \in (0, 1)$. In this regard, it is well known that when dealing with problem (1.1) with $f(u) = |u|^{p-2}u$ and $h = 1$, the L^2*-critical exponent*

$$\bar{p} := 2 + \frac{4s}{N}$$

plays a special role. From the variational point of view, if the problem is purely L^2-subcritical, i.e., $2 < p < \bar{p}$, then the functional of (1.1) is bounded from below on mass constraint manifold. In the L^2-supercritical case, i.e., $\bar{p} < p < 2_s^* := 2N/(N-2s)$, on the contrary, the functional is unbounded below. For more details, we refer to [3, 17, 29, 30, 32] and the references therein. Compared with the research about nonautonomous Schrödinger equations or autonomous fractional Schrödinger equations, there are few works concerning the existence of normalized solutions for nonautonomous elliptic equations in the fractional setting. Indeed, the corresponding results are presented in [11, 16, 21], which have been studied the fractional Schrödinger equation with potential.

In light of the above discussion and mainly motivated by the results in [1], we focus our attention on problem (1.1) and establish the existence of normalized ground states and multiple solutions in the nonautonomous fractional setting, which have not been studied in the existing literature and are also not a simple extension of the results in [1]. Moreover, in this paper the assumptions related to f are presented as below:

(F_1) $f \in C(\mathbb{R}, \mathbb{R})$ is odd and $\lim_{t\to 0} \frac{|f(t)|}{|t|^{q-1}} = \alpha > 0$ for some $q \in (2, 2+\frac{4s}{N})$;

(F_2) there exist constant $C > 0$ and $p \in (2, 2+\frac{4s}{N})$ such that

$$|f(t)| \le C(1 + |t|^{p-1}),\ \forall t \in \mathbb{R};$$

(F_3) there exists $q_1 \in (2, 2+\frac{4s}{N})$ such that

$$0 < q_1 F(t) \le f(t)t \text{ for all } t \in \mathbb{R}\backslash\{0\}.$$

Here, the conditions (F_1)-(F_2) are fractional versions of (f_1)-(f_2). For convenience, we replace monotonicity condition (f_3) with the Ambrosetti-Rabinowitz condition (F_3). Then, we give two examples that f satisfies the above assumptions: the one is

$$f(t) = \alpha|t|^{q-2}t + |t|^{p-2}t,\ \forall t \in \mathbb{R},$$

where $2 < q_1 \le q < p < 2 + 4s/N$; the other one is

$$f(t) = \alpha|t|^{q-2}t + |t|^{r-2}t\ln(1+|t|),\ \forall t \in \mathbb{R},$$

where $2 < q_1 \le q < r < p < 2 + 4s/N$.

Before stating our main results, we present some necessary notations. The fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined for any $s \in (0, 1)$ as

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N)\},$$

which is a Hilbert space endowed with scalar product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v\mathrm{d}x + \int_{\mathbb{R}^N} uv\mathrm{d}x,\ \forall u, v \in H^s(\mathbb{R}^N)$$

and the norm is given by

$$\|u\|^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x + \int_{\mathbb{R}^N} |u|^2\mathrm{d}x,\ \forall u \in H^s(\mathbb{R}^N)$$

with

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.$$

The usual norm in the Lebesgue space $L^r(\mathbb{R}^N)$ is denoted by $|u|_r$ with $2 \le r \le 2_s^*$. From [20], $H^s(\mathbb{R}^N)$ is continuously embedded into $L^r(\mathbb{R}^N)$ for any $2 \le r \le 2_s^*$ and compactly embedded into $L^r_{loc}(\mathbb{R}^N)$ for every $1 \le r < 2_s^*$. Naturally, associated to problem (1.1), the energy functional $I_\varepsilon : H^s(\mathbb{R}^N) \to \mathbb{R}$ is of the form

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon x)F(u)\mathrm{d}x,\ \forall u \in H^s(\mathbb{R}^N).$$

It is standard to verify that $I_\varepsilon \in C^1(H^s(\mathbb{R}^N), \mathbb{R})$. Meanwhile, for $a > 0$, we define

$$S(a) := \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2 \right\}$$

and

$$m_{\varepsilon,a^2} := \inf_{u \in S(a)} I_\varepsilon(u).$$

Moreover, the ground states of (1.1) on $S(a)$ is defined as follows:

Definition 1.1 We say that $\tilde{u}$ is a ground state of (1.1) on $S(a)$ if it is a solution to (1.1) having minimal energy among all the solutions which belongs to $S(a)$:

$$I_\varepsilon|'_{S(a)}(\tilde{u}) = 0 \text{ and } I_\varepsilon(\tilde{u}) = \inf \left\{ I_\varepsilon(u) : I_\varepsilon|'_{S(a)}(u) = 0 \text{ and } u \in S(a) \right\}.$$

Now, we are in the position to state our main results. Note that (h_1)-(h_2) imply the inequality $0 < h_0 \le h_\infty < h_{max}$ holds. It is necessary to consider two different situations, namely,

$$h_0 = h_\infty \tag{1.4}$$

or the other $h_0 < h_\infty$. For example, set

$$h(x) = \begin{cases} \tilde{h}(x), & 0 \le |x| \le |a_l|, \\ \frac{k_0}{1+|x-a_l|} + 2 - k_0, & |x| > |a_l|, \end{cases}$$

where

$$\tilde{h}(x) = \frac{1}{1 + |x - a_1||x - a_2||x - a_3| \cdots |x - a_l|} + 1,$$
$$0 = |a_1| < |a_2| < \cdots < |a_l| \text{ and } 2 - k_0 > \inf_{0 \le |x| \le |a_l|} \tilde{h}(x).$$

It is obvious that h satisfies $(h_1) - (h_3)$ but the identity (1.4) dose not hold. Moreover, $\tilde{h}(x)$ satisfies $(h_1) - (h_3)$ and (1.4). As we will see, whether the identity (1.4) holds true or not is directly related to the restriction on parameter ε in problem (1.1). Indeed, we can establish the existence of the normalized ground states for any $\varepsilon > 0$ if (1.4) holds.

Theorem 1.2 *Assume that $(F_1) - (F_3)$, $(h_1) - (h_2)$ and (1.4) hold. Then problem (1.1) has a positive ground state solution $\tilde{u} \in S(a)$ for any $\varepsilon > 0$ and the corresponding Lagrange multiplier $\tilde{\lambda} < 0$.*

Remark 1.3 In particular, the condition (h_2) ensures that h is not a constant function. In this sense, we establish new results on the existence of normalized ground states for nonautonomous elliptic equations. Of course, the proof processes of Theorem 1.2 allow h to be a constant function, which extends the results in [17] to the general nonlinearity in the L^2-subcritical sense.

We stress that in order to determine the inequality $m_{\varepsilon,a^2} \leq m_{\infty,a^2}$, the identity (1.4) plays an essential role, see Lemma 2.6. Once the inequality $m_{\varepsilon,a^2} \leq m_{\infty,a^2}$ is established, the relative compactness of all the minimizing sequences for m_{ε,a^2} can be verified in Proposition 2.7. If the condition (1.4) in Theorem 1.2 is not satisfied, it is necessary to impose restriction on parameter ε to estimate the relationship between m_{ε,a^2} and m_{∞,a^2}. Indeed, the property that 0 is a maximum point of h in (h_3) condition determines this point, see the last part of Sect. 2.

Theorem 1.4 *Assume that* $(F_1) - (F_3)$ *and* $(h_1) - (h_3)$ *hold. Then there exists* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$*, the results of Theorem* 1.2 *still hold true.*

To discuss the multiplicity of normalized solutions for problem (1.1), the condition (h_3) is pivotal. Indeed, our result shows how the "shape" of the graph of h affects the number of normalized solutions.

Theorem 1.5 *Assume that* $(F_1) - (F_3)$ *and* $(h_1) - (h_3)$ *hold. Then there exists* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$*, problem (1.1) possesses at least* l *couple* $(u^i, \lambda^i) \in H^s(\mathbb{R}^N) \times \mathbb{R}$ *of weak solutions with* $\int_{\mathbb{R}^3} |u^i|^2 \mathrm{d}x = a^2$*,* $I_\varepsilon(u^i) < 0$ *and the corresponding Lagrange multipliers* $\lambda^i < 0$ *for* $i = 1, 2, \cdots, l$.

The remainder of this paper is organized as follows. In Sect. 2, we establish the strong subadditivity inequality and complete the proof of Theorems 1.2 and 1.4. Section 3 is devoted to accomplishing the proof of Theorem 1.5 if the assumption (h_3) holds.

2 Existence of Normalized Ground States

In this section, we establish the existence of normalized ground states for problem (1.1), namely, Theorems 1.2 and 1.4 can be accomplished. First of all, the properties of functional I_ε and m_{ε,a^2} are as follows. Meanwhile, the letter C will be used to denote a suitable positive constant, whose value can change from line to line.

Lemma 2.1 *Assume that* $(F_1) - (F_3)$ *and* $(h_1) - (h_2)$ *hold, then the functional* I_ε *is coercive and bounded from below on* $S(a)$.

Proof According to $(F_1) - (F_2)$, there is $C > 0$ such that

$$|F(t)| \leq C(|t|^q + |t|^p), \ \forall t \in \mathbb{R}. \tag{2.1}$$

Then, by the fractional Gagliardo-Nirenberg inequality [6, Appendix B.1]

$$\int_{\mathbb{R}^N} |u|^r \mathrm{d}x \leq C_{s,N,r}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\right)^{\frac{N(r-2)}{4s}} \left(\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x\right)^{\frac{r}{2}-\frac{N(r-2)}{4s}}, \ \forall r \in (2, 2_s^*), \tag{2.2}$$

we can conclude that

$$\begin{aligned} I_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon x)F(u)\mathrm{d}x \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - Ch_{max}\int_{\mathbb{R}^N} |u|^q \mathrm{d}x - Ch_{max}\int_{\mathbb{R}^N} |u|^p \mathrm{d}x \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - CC_{s,N,q}h_{max}a^{q-\frac{N(q-2)}{2s}}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\right)^{\frac{N(q-2)}{4s}} \\ &\quad - CC_{s,N,p}h_{max}a^{p-\frac{N(p-2)}{2s}}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\right)^{\frac{N(p-2)}{4s}}. \end{aligned}$$

As $q, p \in (2, 2+\frac{4s}{N})$, we derive $N(q-2)/4s < 1$ and $N(p-2)/4s < 1$, which imply that the functional I_ε is coercive and bounded from below on $S(a)$. □

Lemma 2.2 *Assume that* $(F_1)-(F_3)$ *and* $(h_1)-(h_2)$ *hold, for any* $a > 0$*, the following statements hold true:*

- (i) $m_{\varepsilon,a^2} < 0$*;*
- (ii) *let* $\{u_n\} \subset S(a)$ *be a minimizing sequence for* m_{ε,a^2}*, then there exist a constant* $\tilde{\eta} > 0$ *and* $n_0 \in \mathbb{N}$ *such that*

$$\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x > \tilde{\eta}$$

for all $n > n_0$*.*

Proof (i) Given $u \in S(a)$, we define

$$H(u,\tau)(x) := e^{\frac{N\tau}{2}}u(e^\tau x) \text{ for } x \in \mathbb{R}^N \text{ and } \tau \in \mathbb{R}.$$

A direct computation provides

$$\int_{\mathbb{R}^N} |H(u,\tau)(x)|^2 \mathrm{d}x = a^2$$

and

$$\int_{\mathbb{R}^N} F(H(u,\tau)(x))\mathrm{d}x = e^{-N\tau}\int_{\mathbb{R}^N} F(e^{\frac{N\tau}{2}}u(x))\mathrm{d}x.$$

It follows from the assumption (F_1) that $\lim_{t\to 0}\frac{qF(t)}{t^q} = \alpha > 0$, then there is a $\delta > 0$ such that

$$\frac{qF(t)}{t^q} \geq \frac{\alpha}{2}, \ \forall t \in [0, \delta]. \tag{2.3}$$

Note that $H^s(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in (2, 2+\frac{4s}{N})$. Thus, in view of (2.3), for $\tau \ll -1$, we have

$$\begin{aligned}
I_\varepsilon(H(u,\tau)) &= \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} H(u,\tau)|^2 dx - \int_{\mathbb{R}^N} h(\varepsilon x) F(e^{\frac{N\tau}{2}} u(e^\tau x))\mathrm{d}x \\
&\leq \frac{1}{2}e^{2s\tau}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx - h_0 e^{-N\tau}\int_{\mathbb{R}^N} F(e^{\frac{N\tau}{2}} u(x))\mathrm{d}x \\
&\leq \frac{1}{2}e^{2s\tau}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \frac{\alpha h_0}{2q} e^{\frac{(q-2)N\tau}{2}}\int_{\mathbb{R}^N} |u|^q \mathrm{d}x \\
&< 0,
\end{aligned}$$

which shows that $m_{\varepsilon,a^2} < 0$.

(ii) Arguing by contradiction suppose that there exists a subsequence of $\{u_n\}$ with respect to m_{ε,a^2}, still denoted by itself, such that

$$\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x \to 0 \text{ as } n \to \infty.$$

By (i), we conclude that

$$0 > m_{\varepsilon,a^2} + o_n(1) = I_\varepsilon(u_n) \geq -\int_{\mathbb{R}^N} h(\varepsilon x) F(u_n)\mathrm{d}x \geq -h_{max}\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x,$$

which is a contradiction. Thus, the proof is completed. □

Lemma 2.3 *Assume that* $(F_1) - (F_3)$ *and* $(h_1) - (h_2)$ *hold, then* m_{ε,a^2} *is continuous on* $(0, \infty)$ *with regard to* a.

Proof For any $a > 0$, let $a_n > 0$ and $a_n \to a$. Let $\{u_n\} \subset S(a_n)$ such that $I_\varepsilon(u_n) < m_{\varepsilon,a_n^2} + \frac{1}{n}$ for every $n \in \mathbb{N}$. Then Lemma 2.1 implies that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Moreover, the fact $\{\frac{a}{a_n}u_n\} \subset S(a)$ that

$$\begin{aligned}
m_{\varepsilon,a^2} &\leq I_\varepsilon\left(\frac{a}{a_n}u_n\right) \\
&= \frac{a^2}{2a_n^2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon x) F\left(\frac{a}{a_n}u_n\right)\mathrm{d}x \\
&= I_\varepsilon(u_n) + o_n(1) \leq m_{\varepsilon,a_n^2} + o_n(1).
\end{aligned}$$

Analogously, considering a minimizing sequence $\{v_n\} \subset S(a)$, we have $\{\frac{a_n}{a}v_n\} \subset S(a_n)$ and

$$m_{\varepsilon,a_n^2} \leq I_\varepsilon\left(\frac{a_n}{a}v_n\right) = I_\varepsilon(v_n) + o_n(1) \leq m_{\varepsilon,a^2} + o_n(1).$$

Therefore, m_{ε,a^2} is continuous on $(0, \infty)$ with regard to a. □

Lemma 2.4 *Assume that $(F_1) - (F_3)$ and $(h_1) - (h_2)$ hold, then for $0 < a_1 < a_2$,*

$$\frac{a_1^2}{a_2^2} m_{\varepsilon,a_2^2} < m_{\varepsilon,a_1^2}.$$

Proof Let $\{u_n\} \subset S(a_1)$ be a minimizing sequence for m_{ε,a_1^2} and let $\xi = \frac{a_2}{a_1}$. Then $\xi > 1$ and $\{\xi u_n\} \subset S(a_2)$. Obviously, from (F_3), the function $t \mapsto \frac{F(t)}{t^{q_1}}$ is increasing on $(0, \infty)$. Therefore, we have

$$F(rt) \geq r^{q_1} F(t) \text{ for all } t > 0 \text{ and } r \geq 1.$$

Moreover, we conclude that

$$\begin{aligned} m_{\varepsilon,a_2^2} \leq I_\varepsilon(\xi u_n) &= \xi^2 I_\varepsilon(u_n) + \xi^2 \int_{\mathbb{R}^N} h(\varepsilon x) F(u_n) \mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon x) F(\xi u_n) \mathrm{d}x \\ &\leq \xi^2 I_\varepsilon(u_n) + (\xi^2 - \xi^{q_1}) \int_{\mathbb{R}^N} h(\varepsilon x) F(u_n) \mathrm{d}x \\ &\leq \xi^2 I_\varepsilon(u_n) + (\xi^2 - \xi^{q_1}) h_0 \int_{\mathbb{R}^N} F(u_n) \mathrm{d}x. \end{aligned}$$

Using Lemma 2.2$-(ii)$ and the fact $\xi^2 - \xi^q < 0$, we obtain

$$m_{\varepsilon,a_2^2} \leq \xi^2 I_\varepsilon(u_n) + (\xi^2 - \xi^{q_1}) h_0 \tilde{\eta}$$

for $n \gg 1$. Letting $n \to \infty$, it follows that

$$m_{\varepsilon,a_2^2} \leq \xi^2 m_{\varepsilon,a_1^2} + (\xi^2 - \xi^{q_1}) h_0 \tilde{\eta} < \xi^2 m_{\varepsilon,a_1^2},$$

that is,

$$\frac{a_1^2}{a_2^2} m_{\varepsilon,a_2^2} < m_{\varepsilon,a_1^2}.$$

□

Corollary 2.5 *For $0 < a_1 < a_2$, the strong subadditivity inequality*

$$m_{\varepsilon,a_2^2} < m_{\varepsilon,a_1^2} + m_{\varepsilon,(a_2^2 - a_1^2)}$$

holds.

Proof According to Lemma 2.4, it is clear that

$$\frac{a_1^2}{a_2^2} m_{\varepsilon,a_2^2} < m_{\varepsilon,a_1^2} \text{ and } \frac{a_2^2 - a_1^2}{a_2^2} m_{\varepsilon,a_2^2} < m_{\varepsilon,(a_2^2-a_1^2)},$$

which imply

$$m_{\varepsilon,a_2^2} = \frac{a_1^2}{a_2^2} m_{\varepsilon,a_2^2} + \frac{a_2^2 - a_1^2}{a_2^2} m_{\varepsilon,a_2^2} < m_{\varepsilon,a_1^2} + m_{\varepsilon,(a_2^2-a_1^2)}.$$

□

In what follows, we need to consider the following two functionals:

$$I_{max}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - h_{max} \int_{\mathbb{R}^N} F(u)\mathrm{d}x$$

and

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - h_\infty \int_{\mathbb{R}^N} F(u)\mathrm{d}x, \ \forall u \in H^s(\mathbb{R}^N).$$

Obviously, $I_{max}, I_\infty \in C^1(H^s(\mathbb{R}^N), \mathbb{R})$. Moreover, $I_{max}(u)$ and $I_\infty(u)$ correspond to the energy functional of problem (1.1) when $h = h_{max}$ and $h = h_\infty$, respectively. Then, we define

$$m_{max,a^2} := \inf_{u \in S(a)} I_{max}(u)$$

and

$$m_{\infty,a^2} := \inf_{u \in S(a)} I_\infty(u).$$

According to the above arguments, if $h = h_{max}$ and $h = h_\infty$, Lemmas 2.1–2.4 still hold. Moreover, using standard arguments or similar to Proposition 2.7, it is easy to prove that $m_{max,a^2} < 0$ and $m_{\infty,a^2} < 0$ are achieved.

Lemma 2.6 *Assume that $(F_1)-(F_3)$, $(h_1)-(h_2)$ and (1.4) hold, then $m_{\varepsilon,a^2} \le m_{\infty,a^2}$ for any $a > 0$.*

Proof Let $a > 0$ and $\{u_n\} \subset S(a)$ such that $I_\infty(u_n) \to m_{\infty,a^2}$. From (1.4), it follows that

$$m_{\varepsilon,a^2} \le I_\varepsilon(u_n) \le I_\infty(u_n) = m_{\infty,a^2} + o_n(1),$$

which shows that $m_{\varepsilon,a^2} \le m_{\infty,a^2}$ for any $a > 0$. □

Proposition 2.7 *Assume that* $(F_1)-(F_3)$, $(h_1)-(h_2)$ *and* (1.4) *hold. Let* $\{u_n\} \subset S(a)$ *such that* $I_\varepsilon(u_n) \to m_{\varepsilon,a^2}$, *then the sequence* $\{u_n\}$ *is relatively compact in* $H^s(\mathbb{R}^N)$ *up to translations and* $m_{\varepsilon,a^2} < 0$ *is achieved for each* $a > 0$.

Proof In view of Lemma 2.1, the sequence $\{u_n\}$ is bounded. Then there exists $u \in H^s(\mathbb{R}^N)$ such that, up to a subsequence,

$$u_n \rightharpoonup u \text{ in } H^s(\mathbb{R}^N),\ u_n \to u \text{ in } L^r_{loc}(\mathbb{R}^N) \text{ for } 1 \le r < 2^*_s \text{ and } u_n \to u \ a.e. \text{ in } \mathbb{R}^N. \tag{2.4}$$

Now, we claim that

$$I_\varepsilon(u_n) = I_\varepsilon(u_n - u) + I_\varepsilon(u) + o_n(1). \tag{2.5}$$

It follows from (2.4) that

$$\int_{\mathbb{R}^{2N}} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} u \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\mathrm{d}y + o_n(1).$$

Thus, we conclude that

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} (u_n - u)|^2 \mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\mathrm{d}y \\ &= 2\int_{\mathbb{R}^{2N}} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} u \mathrm{d}x\mathrm{d}y - 2\int_{\mathbb{R}^{2N}} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x = o_n(1).\end{aligned}$$

On the other hand, in view of (2.4), Brézis-Lieb Lemma [7] and [33, Lemma 2.2], we know that

$$|u_n|_2^2 = |u_n - u|_2^2 - |u|_2^2 + o_n(1)$$

and

$$\int_{\mathbb{R}^N} h(\varepsilon x) F(u_n) \mathrm{d}x = \int_{\mathbb{R}^N} h(\varepsilon x) F(u_n - u) \mathrm{d}x + \int_{\mathbb{R}^N} h(\varepsilon x) F(u) \mathrm{d}x + o_n(1).$$

Hence the claim is true. In what follows, it is necessary to consider the following two cases:

Case 1 $|u|_2 = b \in (0, a]$. If $b \in (0, a)$, let $v_n = u_n - u$, $d_n = |v_n|_2$ and supposing that $d_n \to d$, we get $a^2 = b^2 + d^2$. From $d_n \in (0, a)$ and (2.5), we have

$$m_{\varepsilon,a^2} + o_n(1) = I_\varepsilon(u_n) = I_\varepsilon(v_n) + I_\varepsilon(u) + o_n(1) \ge m_{\varepsilon,d_n^2} + m_{\varepsilon,b^2} + o_n(1).$$

Moreover, Lemma 2.4 indicates that

$$m_{\varepsilon,a^2} + o_n(1) \ge \frac{d_n^2}{a^2} m_{\varepsilon,a^2} + m_{\varepsilon,b^2} + o_n(1).$$

Letting $n \to \infty$, we deduce that

$$m_{\varepsilon,a^2} \geq \frac{d^2}{a^2} m_{\varepsilon,a^2} + m_{\varepsilon,b^2} > \frac{d^2}{a^2} m_{\varepsilon,a^2} + \frac{b^2}{a^2} m_{\varepsilon,a^2} = m_{\varepsilon,a^2},$$

which is absurd. This shows that $|u|_2 = a$. In addition, $(F_1)-(F_2)$, (h_1) and the Lebesgue's dominated convergence theorem ensure that

$$\int_{\mathbb{R}^N} h(\varepsilon x) F(u_n) \mathrm{d}x \to \int_{\mathbb{R}^N} h(\varepsilon x) F(u) \mathrm{d}x \text{ as } n \to \infty.$$

Therefore,

$$m_{\varepsilon,a^2} = \lim_{n\to\infty} I_\varepsilon(u_n) \geq I_\varepsilon(u) \geq m_{\varepsilon,a^2},$$

i.e., $I_\varepsilon(u) = m_{\varepsilon,a^2}$ and $|(-\Delta)^{\frac{s}{2}} u_n|_2^2 \to |(-\Delta)^{\frac{s}{2}} u|_2^2$ as $n \to \infty$. Thus, m_{ε,a^2} is achieved by $u \in S(a)$.

Case 2 $u = 0$. Namely, $u_n \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$, then $u_n \to 0$ in $L^w_{loc}(\mathbb{R}^N)$ for $1 \leq w < 2_s^*$ and $u_n \to 0$ a.e. in $\mathbb{R}^N$. Combining with $(F_1)-(F_2)$ and (h_1), it is easy to check that

$$\int_{\mathbb{R}^N} [h(\varepsilon x) - h_\infty] F(u_n) \mathrm{d}x = o_n(1). \tag{2.6}$$

Hence,

$$I_\varepsilon(u_n) = I_\infty(u_n) + o_n(1) = m_{\varepsilon,a^2} + o_n(1). \tag{2.7}$$

Next, we claim that there is a constant $\delta > 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^2 \mathrm{d}x > \delta. \tag{2.8}$$

Otherwise, by using [12, Lemma 2.2], one has $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2_s^*$ and (2.1) ensures that $\int_{\mathbb{R}^N} F(u_n) \mathrm{d}x \to 0$, which contradicts Lemma 2.2-(ii). Therefore, there exists $|y_n| \to \infty$ such that

$$\int_{B_r(y_n)} |u_n|^2 \mathrm{d}x > \delta.$$

In view of (2.8), letting the sequence of translations $\tilde{u}_n(x) := u_n(x + y_n)$, we may assume that there exists $\tilde{u} \in H^s(\mathbb{R}^N) \backslash \{0\}$ such that, up to a subsequence,

$$\begin{cases} \tilde{u}_n \rightharpoonup \tilde{u} \text{ in } H^s(\mathbb{R}^N); \\ \tilde{u}_n \to \tilde{u} \text{ in } L^w_{loc}(\mathbb{R}^N), \ \forall w \in [1, 2_s^*); \\ \tilde{u}_n \to \tilde{u} \ a.e. \text{ in } \mathbb{R}^N. \end{cases} \tag{2.9}$$

Noting that $|y_n| \to \infty$ implies (2.6) holds, then it is obvious that

$$I_\infty(\tilde{u}_n) = m_{\varepsilon,a^2} + o_n(1). \tag{2.10}$$

In addition, similar to (2.5), we have $\lim_{n\to\infty} I_\infty(\tilde{u}_n) = I_\infty(\tilde{u}) + \lim_{n\to\infty} I_\infty(\tilde{u}_n - \tilde{u})$. Then it follows from (2.9), (2.10), Lemma 2.3 and Lemma 2.6 that

$$\begin{aligned} m_{\infty,a^2} \geq m_{\varepsilon,a^2} &= \lim_{n\to\infty} I_\infty(\tilde{u}_n) \\ &= I_\infty(\tilde{u}) + \lim_{n\to\infty} I_\infty(\tilde{u}_n - \tilde{u}) \\ &\geq m_{\infty,(|\tilde{u}|_2^2)} + \lim_{n\to\infty} m_{\infty,(|\tilde{u}_n-\tilde{u}|_2^2)} \\ &= m_{\infty,(|\tilde{u}|_2^2)} + m_{\infty,(a^2-|\tilde{u}|_2^2)}. \end{aligned} \tag{2.11}$$

This proves $|\tilde{u}|_2^2 = a^2$. In fact, if $|\tilde{u}|_2^2 < a^2$, in accordance with Corollary 2.5, we have

$$m_{\infty,(|\tilde{u}|_2^2)} + m_{\infty,(a^2-|\tilde{u}|_2^2)} > m_{\infty,a^2},$$

which contradicts (2.11). Hence, we have $\tilde{u}_n \to \tilde{u}$ in $L^w(\mathbb{R}^N)$ for $2 \leq w < 2_s^*$. Moreover, from the weakly lower semicontinuous, we conclude

$$m_{\varepsilon,a^2} = \lim_{n\to\infty} I_\varepsilon(\tilde{u}_n) \geq I_\varepsilon(\tilde{u}) \geq m_{\varepsilon,a^2},$$

which verifies m_{ε,a^2} is achieved. □

Proof of Theorem 1.2 In view of Lemma 2.1, there exists a bounded minimizing sequence $\{u_n\} \subset S(a)$ with respect to m_{ε,a^2}, that is $I_\varepsilon(u_n) \to m_{\varepsilon,a^2}$ as $n \to \infty$. Then, by using Proposition 2.7, there exists $\tilde{u} \in S(a)$ such that $I_\varepsilon(\tilde{u}) = m_{\varepsilon,a^2}$. Now, we prove that $\tilde{u}$ can be chosen to be positive. Indeed, by formula (A.11) in [23], we have

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}|\tilde{u}||^2 \mathrm{d}x \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\tilde{u}|^2 \mathrm{d}x.$$

Hence, we infer that $|\tilde{u}| \in S(a)$ and $I_\varepsilon(|\tilde{u}|) = m_{\varepsilon,a^2}$, i.e., $\tilde{u}$ can be replaced by $|\tilde{u}|$. For the convenience, it is still denoted by $\tilde{u}$. Moreover, the strong maximum principle [9] yields that $\tilde{u}(x) > 0$ for all $x \in \mathbb{R}^N$. Corresponding to $\tilde{u}$, in view of (F_3) and $m_{\varepsilon,a^2} < 0$, there exists a Lagrange multiplier $\tilde{\lambda} \in \mathbb{R}$ such that

$$\begin{aligned} \tilde{\lambda} a^2 &= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\tilde{u}|^2 \mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon x) f(\tilde{u})\tilde{u} \mathrm{d}x \\ &= 2m_{\varepsilon,a^2} + 2\int_{\mathbb{R}^N} h(\varepsilon x) F(\tilde{u}) \mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon x) f(\tilde{u})\tilde{u} \mathrm{d}x \\ &< 2m_{\varepsilon,a^2} < 0. \end{aligned}$$

□

Up to now, we are in the position to establish the proof of Theorem 1.4. Indeed, if $h_0 \leq h_\infty$, the preview Lemma 2.6 cannot hold, which depends heavily on $h_0 = h_\infty$. In this regard, we can overcome this difficulty by choosing ε small enough such that the following strict inequality

$$m_{\varepsilon,a^2} < m_{\infty,a^2}$$

holds. Once the inequality above is established, repeating the process in the proof of Proposition 2.7 and Theorem 1.2, the proof of Theorem 1.4 is complete.

Proof of Theorem 1.4 According to the description above Lemma 2.6, there exists $u_\infty \in S(a)$ satisfying $I_\infty(u_\infty) = m_{\infty,a^2}$. In view of (h_2), we have

$$m_{max,a^2} \leq I_{max}(u_\infty) < I_\infty(u_\infty) = m_{\infty,a^2}. \tag{2.12}$$

On the other hand, there exists $u_{max} \in S(a)$ with $I_{max}(u_{max}) = m_{max,a^2}$. Then,

$$m_{\varepsilon,a^2} \leq I_\varepsilon(u_{max}) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_{max}|^2 \mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon x)F(u_{max})\mathrm{d}x. \tag{2.13}$$

Now, we claim that there exists $\varepsilon_0 > 0$ such that

$$m_{\varepsilon,a^2} < m_{\infty,a^2} \tag{2.14}$$

for all $\varepsilon \in (0, \varepsilon_0)$. Indeed, letting $\varepsilon \to 0^+$ in (2.13) and using (h_3), (2.12), we deduce

$$\lim_{\varepsilon\to 0^+} \sup m_{\varepsilon,a^2} \leq \lim_{\varepsilon\to 0^+} I_\varepsilon(u_{max}) = I_{max}(u_{max}) = m_{max,a^2} < m_{\infty,a^2}.$$

Hence, the claim is true and the proof is complete. □

3 Multiple Normalized Solutions

In this section, we establish the existence of multiple normalized solutions based on the (h_3) condition, and prove Theorem 1.5. Meanwhile, decreasing if necessary ε_0, we always assume that $\varepsilon \in (0, \varepsilon_0)$, which not only to ensure that (2.14) holds. Moreover, by (2.12), we have

$$m_{max,a^2} < m_{\infty,a^2} < 0.$$

Then, we fix $0 < \rho_1 = \frac{1}{2}(m_{\infty,a^2} - m_{max,a^2})$ and establish the following two lemmas that will be used to prove the $(PS)_c$ condition for I_ε restricted to $S(a)$ at some levels.

Lemma 3.1 *Assume that $(F_1) - (F_3)$ and $(h_1) - (h_3)$ hold. Let $\{u_n\} \subset S(a)$ with $I_\varepsilon(u_n) \to m_{\varepsilon,a^2}$ and $m_{\varepsilon,a^2} < m_{max,a^2} + \rho_1 < 0$. If $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$, then $u \neq 0$.*

Proof Assume by contradiction that $u = 0$. By (h_2), for any given $\zeta > 0$, there exists $R > 0$ such that

$$h_\infty \geq h(x) - \zeta \text{ for all } |x| > R. \tag{3.1}$$

Thus,

$$\begin{aligned}
m_{\varepsilon,a^2} + o_n(1) &= I_\varepsilon(u_n) \\
&= I_\infty(u_n) + \int_{\mathbb{R}^N} (h_\infty - h(\varepsilon x))F(u_n)\mathrm{d}x \\
&= I_\infty(u_n) + \int_{B_{R/\varepsilon}(0)} (h_\infty - h(\varepsilon x))F(u_n)\mathrm{d}x + \int_{B^c_{R/\varepsilon}(0)} (h_\infty - h(\varepsilon x))F(u_n)\mathrm{d}x \\
&\geq I_\infty(u_n) + \int_{B_{R/\varepsilon}(0)} (h_\infty - h(\varepsilon x))F(u_n)\mathrm{d}x - \zeta \int_{B^c_{R/\varepsilon}(0)} F(u_n)\mathrm{d}x.
\end{aligned}$$

It follows from Lemma 2.1 that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and $u_n \to 0$ in $L^w(B_{R/\varepsilon}(0))$ for all $w \in [1, 2^*_s)$, by using (2.1), we deduce that

$$m_{\varepsilon,a^2} + o_n(1) \geq I_\infty(u_n) - \zeta C$$

for some $C > 0$. Since $\zeta > 0$ is arbitrary, we conclude that

$$m_{\varepsilon,a^2} \geq m_{\infty,a^2},$$

which contradicts (2.14). Hence, $u \neq 0$. □

Lemma 3.2 *Assume that $(F_1) - (F_3)$ and $(h_1) - (h_3)$ hold. Let $\{u_n\}$ be a $(PS)_c$ sequence for I_ε restricted to $S(a)$ with $c < m_{max,a^2} + \rho_1 < 0$ and $u_n \rightharpoonup u_\varepsilon$ in $H^s(\mathbb{R}^N)$, that is,*

$$I_\varepsilon(u_n) \to c, \ \|I_\varepsilon|'_{S(a)}(u_n)\| \to 0 \text{ as } n \to \infty.$$

If $u_n \nrightarrow u_\varepsilon$ in $H^s(\mathbb{R}^N)$, there is $\beta > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that

$$\lim_{n\to\infty} \inf |u_n - u_\varepsilon|_2^2 \geq \beta.$$

Proof Define the functional $\Phi: H^s(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x,$$

we infer that $S(a) = \Phi^{-1}\left(\{a^2/2\}\right)$. Then, by [27, Proposition 5.12], there exists $\{\lambda_n\} \subset \mathbb{R}$ such that

$$\|I'_\varepsilon(u_n) - \lambda_n \Phi'(u_n)\|_{H^{-s}(\mathbb{R}^N)} \to 0 \text{ as } n \to \infty. \tag{3.2}$$

It is clear that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Then, we get $\{\lambda_n\}$ is bounded. Thus, we can assume that $\lambda_n \to \lambda_\varepsilon$ as $n \to \infty$. Together with (3.2), we get

$$I'_\varepsilon(u_\varepsilon) - \lambda_\varepsilon \Phi'(u_\varepsilon) = 0 \text{ in } H^{-s}(\mathbb{R}^N)$$

and

$$\|I'_\varepsilon(v_n) - \lambda_\varepsilon \Phi'(v_n)\|_{H^{-s}(\mathbb{R}^N)} \to 0 \text{ as } n \to \infty, \tag{3.3}$$

where $v_n = u_n - u_\varepsilon$. Using (F_3), then

$$\begin{aligned} 0 > m_{max,a^2} + \rho_1 &\geq \lim_{n\to\infty} \inf I_\varepsilon(u_n) \\ &= \lim_{n\to\infty} \inf \left[I_\varepsilon(u_n) - \frac{1}{2} I'_\varepsilon(u_n)u_n + \lambda_n a^2 + o_n(1) \right] \geq \lambda_\varepsilon a^2, \end{aligned}$$

which implies that

$$\lim_{\varepsilon\to 0^+} \sup \lambda_\varepsilon \leq \frac{\rho_1 + m_{max,a^2}}{a^2} < 0.$$

Thus, there exists $\lambda_* < 0$ independent of ε such that

$$\lambda_\varepsilon \leq \lambda_* < 0 \text{ for all } \varepsilon \in (0, \varepsilon_0). \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x - \lambda_\varepsilon \int_{\mathbb{R}^N} |v_n|^2 \mathrm{d}x = \int_{\mathbb{R}^N} f(v_n)v_n \mathrm{d}x + o_n(1)$$

and

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x - \lambda_* \int_{\mathbb{R}^N} |v_n|^2 \mathrm{d}x \leq \int_{\mathbb{R}^N} f(v_n)v_n \mathrm{d}x + o_n(1).$$

Combining $(F_1) - (F_2)$ with Young's inequality, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\int_{\mathbb{R}^N} f(v_n)v_n \mathrm{d}x \leq \epsilon \int_{\mathbb{R}^N} |v_n|^2 \mathrm{d}x + C_\epsilon \int_{\mathbb{R}^N} |v_n|^p \mathrm{d}x.$$

This yields that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 \mathrm{d}x + (-\lambda_* - \epsilon) \int_{\mathbb{R}^N} |v_n|^2 \mathrm{d}x \leq C_\epsilon \int_{\mathbb{R}^N} |v_n|^p \mathrm{d}x + o_n(1).$$

Since $u_n \nrightarrow u_\varepsilon$ in $H^s(\mathbb{R}^N)$, namely, $v_n \nrightarrow 0$ in $H^s(\mathbb{R}^N)$, the inequality above verifies that there is $C > 0$ independent of ε such that

$$\lim_{n\to\infty} \inf \int_{\mathbb{R}^N} |v_n|^p \mathrm{d}x \geq C.$$

Since $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$, $\{v_n\}$ is also bounded in $H^s(\mathbb{R}^N)$. Then we assume that $||v_n|| \leq \kappa$ for all $n \in \mathbb{N}$, where $\kappa > 0$ is a constant independent of $\varepsilon \in (0, \varepsilon_0)$. From (2.2), we see that

$$C \leq \lim_{n\to\infty} \inf |v_n|_p^p \leq C_{s,N,p}\kappa^{\frac{N(p-2)}{2s}} \left(\lim_{n\to\infty} \inf |v_n|_2\right)^{p-\frac{N(p-2)}{2s}}.$$

Hence, the proof is complete. □

Lemma 3.3 *Assume that* $(F_1) - (F_3)$ *and* $(h_1) - (h_3)$ *hold. Let*

$$0 < \rho_2 < \min\left\{\frac{1}{2}, \frac{\beta}{a^2}\right\}(m_{\infty,a^2} - m_{max,a^2}) \leq \rho_1, \tag{3.5}$$

then the functional I_ε *satisfies the* $(PS)_c$ *condition restricted to* $S(a)$ *for* $c < m_{max,a^2} + \rho_2$.

Proof We assume that $\{u_n\} \subset S(a)$ is a $(PS)_c$ sequence for I_ε restricted to $S(a)$ with $c < m_{max,a^2} + \rho_2$. From Lemma 2.1, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and let $u_n \rightharpoonup u_\varepsilon$ in $H^s(\mathbb{R}^N)$. Then we know $u_\varepsilon \neq 0$ by Lemma 3.1. Let $v_n = u_n - u_\varepsilon$. If $v_n \to 0$ in $H^s(\mathbb{R}^N)$, the proof is complete. On the contrary, $v_n \nrightarrow 0$ in $H^s(\mathbb{R}^N)$, then $|u_\varepsilon|_2 = b \in (0, a)$ and it follows from Lemma 3.2 that there is $\beta > 0$ independent of ε such that

$$\lim_{n\to\infty} \inf |v_n|_2^2 \geq \beta.$$

Setting $d_n = |v_n|_2 \in (0, a)$ and supposing that $d_n \to d > 0$, we get $d^2 \geq \beta$ and $a^2 = b^2 + d^2$. Then, in view of (2.5), we deduce

$$c + o_n(1) = I_\varepsilon(u_n) = I_\varepsilon(v_n) + I_\varepsilon(u_\varepsilon) + o_n(1) \geq m_{\infty,d_n^2} + m_{max,b^2} + o_n(1).$$

Arguing as in the proof of Lemma 2.4, we obtain

$$\rho_2 + m_{max,a^2} \geq \frac{d_n^2}{a^2} m_{\infty,a^2} + \frac{b^2}{a^2} m_{max,a^2}.$$

Letting $n \to \infty$, we can infer that

$$\rho_2 \geq \frac{d^2}{a^2}(m_{\infty,a^2} - m_{max,a^2}) \geq \frac{\beta}{a^2}(m_{\infty,a^2} - m_{max,a^2}),$$

which contradicts (3.5). □

In the following, let us fix $\rho_0, r_0 > 0$ satisfying:

(1) $\overline{B_{\rho_0}(a_i)} \bigcap \overline{B_{\rho_0}(a_j)} = \emptyset$ for $i \neq j$ and $i, j \in \{1, 2, \cdots, l\}$;
(2) $\bigcup_{i=1}^{l} B_{\rho_0}(a_i) \subset B_{r_0}(0)$;
(3) $K_{\frac{\rho_0}{2}} = \bigcup_{i=1}^{l} \overline{B_{\frac{\rho_0}{2}}(a_i)}$.

We define the function $Q_\varepsilon : H^s(\mathbb{R}^N)\backslash\{0\} \to \mathbb{R}^N$ by

$$Q_\varepsilon(u) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon x)|u|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x},$$

where $\chi : \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$\chi(x) = \begin{cases} x, & \text{if } |x| \leq r_0, \\ r_0 \frac{x}{|x|}, & \text{if } |x| > r_0. \end{cases} \tag{3.6}$$

Lemma 3.4 *Assume that $(F_1)-(F_3)$ and $(h_1)-(h_3)$ hold. There is $\rho_3 \in (0, \rho_2)$ such that if $u \in S(a)$ and $I_\varepsilon(u) \leq m_{max,a^2} + \rho_3$, then*

$$Q_\varepsilon(u) \in K_{\frac{\rho_0}{2}}, \ \forall \varepsilon \in (0, \varepsilon_0).$$

Proof Assume by contradiction that there exist $\rho_n \to 0$, $\varepsilon_n \to 0$ and $\{u_n\} \subset S(a)$ such that

$$I_{\varepsilon_n}(u_n) \leq m_{max,a^2} + \rho_n \tag{3.7}$$

and

$$Q_\varepsilon(u_n) \notin K_{\frac{\rho_0}{2}}.$$

This gives

$$m_{max,a^2} \leq I_{max}(u_n) \leq I_{\varepsilon_n}(u_n) \leq m_{max,a^2} + \rho_n,$$

which shows that

$$\{u_n\} \subset S(a) \text{ and } I_{max}(u_n) \to m_{max,a^2} \text{ as } n \to \infty.$$

In view of Proposition 2.7, we have the following cases:

(i) $u_n \to u$ in $H^s(\mathbb{R}^N)$ for $u \in S(a)$;
(ii) there exists $\{y_n\} \subset \mathbb{R}^N$ with $|y_n| \to \infty$ such that $v_n = u(\cdot + y_n)$ is convergence to $v \in S(a)$ in $H^s(\mathbb{R}^N)$.

If case (i) holds, by Lebesgue's dominated convergence theorem,

$$Q_{\varepsilon_n}(u_n) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon_n x)|u_n|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x} \to \frac{\int_{\mathbb{R}^N} \chi(0)|u|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x} = 0 \in K_{\frac{\rho_0}{2}} \text{ as } n \to \infty,$$

which is a contradiction.

If case (ii) holds, we need to consider $|\varepsilon_n y_n| \to y_0$ for some $y_0 \in \mathbb{R}^N$ or $|\varepsilon_n y_n| \to \infty$. Provided that $|\varepsilon_n y_n| \to y_0$ for some $y_0 \in \mathbb{R}^N$, the $v_n \rightharpoonup v$ in $H^s(\mathbb{R}^N)$ states that

$$\begin{aligned} I_{\varepsilon_n}(u_n) &= \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 - \int_{\mathbb{R}^N} h(\varepsilon_n x) F(u_n) \mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 - \int_{\mathbb{R}^N} h(\varepsilon_n x + \varepsilon_n y_n) F(v_n) \mathrm{d}x \\ &\to I_{h(y_0)}(v) \text{ as } n \to \infty. \end{aligned}$$

Combining (3.7), we find that

$$m_{max,a^2} \geq I_{h(y_0)}(v) \geq m_{h(y_0),a^2}. \tag{3.8}$$

Next, we verify that $h(y_0) = h_{max}$, namely, $y_0 = a_i$ for some $i = 1, 2, \cdots, l$. Supposing $h(y_0) < h_{max}$, it is obvious that $m_{h(y_0),a^2} > m_{max,a^2}$, which contradicts (3.8). From this,

$$\begin{aligned} Q_{\varepsilon_n}(u_n) &= \frac{\int_{\mathbb{R}^N} \chi(\varepsilon_n x)|u_n|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x} = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon_n x + \varepsilon_n y_n)|v_n|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |v_n|^2 \mathrm{d}x} \\ &\to \frac{\int_{\mathbb{R}^N} \chi(y_0)|v|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |v|^2 \mathrm{d}x} = \chi(y_0) = a_i \in K_{\frac{\rho_0}{2}} \text{ as } n \to \infty, \end{aligned}$$

which is absurd.

For $|\varepsilon_n y_n| \to \infty$, the similar argument yields that

$$m_{\infty,a^2} \leq m_{max,a^2},$$

which contradicts (2.12). Based on the above discussion, the proof is complete. □

Now, we define the notations as follows:

$$\theta_\varepsilon^i := \{u \in S(a) : |Q_\varepsilon(u) - a_i| \leq \rho_0\}, \ \partial\theta_\varepsilon^i = \{u \in S(a) : |Q_\varepsilon(u) - a_i| = \rho_0\}$$

and

$$\eta_\varepsilon^i := \inf_{u \in \theta_\varepsilon^i} I_\varepsilon(u), \ \hat{\eta}_\varepsilon^i := \inf_{u \in \partial\theta_\varepsilon^i} I_\varepsilon(u).$$

Lemma 3.5 *Assume that* $(F_1)-(F_3)$ *and* $(h_1)-(h_3)$ *hold. Then*

$$\eta_\varepsilon^i < m_{max,a^2} + \rho_3 \text{ and } \eta_\varepsilon^i < \hat{\eta}_\varepsilon^i \text{ for all } \varepsilon \in (0, \varepsilon_0)$$

Proof Let $u \in H^s(\mathbb{R}^N)$ satisfy $I_{max}(u) = m_{max,a^2}$. For $1 \le i \le l$, we define the function $\hat{u}_\varepsilon^i : \mathbb{R}^N \to \mathbb{R}$ by

$$\hat{u}_\varepsilon^i := u\left(x - \frac{a_i}{\varepsilon}\right).$$

It is clear that $\hat{u}_\varepsilon^i \in S(a)$ for all $\varepsilon > 0$, $1 \le i \le l$. By a direct calculation, we deduce that

$$I_\varepsilon(\hat{u}_\varepsilon^i) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon x + a_i) F(u) \mathrm{d}x,$$

which, together with (h_3), signifies that

$$\lim_{\varepsilon \to 0^+} I_\varepsilon(\hat{u}_\varepsilon^i) = I_{h(a_i)}(u) = I_{max}(u) = m_{max,a^2}. \tag{3.9}$$

In light of definition of Q_ε, we have $Q_\varepsilon(\hat{u}_\varepsilon^i) \to a_i$ as $\varepsilon \to 0^+$. It follows that $\hat{u}_\varepsilon^i \in \theta_\varepsilon^i$ for ε sufficiently small. Combining with (3.9), there is $\varepsilon \in (0, \varepsilon_0)$ such that

$$m_{max,a^2} + \rho_3 > I_\varepsilon(\hat{u}_\varepsilon^i)$$

and

$$m_{max,a^2} + \rho_3 > \eta_\varepsilon^i, \tag{3.10}$$

the first conclusion is reached immediately. Therefore, it suffices to verify the remaining one. Notice that if $u \in \partial\theta_\varepsilon^i$, then

$$u \in S(a) \text{ and } |Q_\varepsilon(u) - a_i| = \rho_0 > \frac{\rho_0}{2},$$

which indicates that $Q_\varepsilon(u) \notin K_{\frac{\rho_0}{2}}$. Therefore, by Lemma 3.4, for all $u \in \partial\theta_\varepsilon^i$ and $\varepsilon \in (0, \varepsilon_0)$, we obtain

$$I_\varepsilon(u) > m_{max,a^2} + \rho_3.$$

Then

$$\hat{\eta}_\varepsilon^i = \inf_{u \in \partial\theta_\varepsilon^i} I_\varepsilon(u) \ge m_{max,a^2} + \rho_3. \tag{3.11}$$

Consequently, together with (3.10) and (3.11), the desired result is reached. □

Proof of Theorem 1.5 For every $i \in \{1, 2, 3, \cdots, l\}$, we can use the Ekeland's variational principle [18, Theorem 4.1] to find a sequence $\{u_n^i\} \subset S(a)$ satisfying

$$I_\varepsilon(u_n^i) \to \eta_\varepsilon^i$$

and

$$I_\varepsilon(v) - I_\varepsilon(u_n^i) \geq -\frac{1}{n}||v - u_n^i||, \ \forall v \in \theta_\varepsilon^i \text{ with } v \neq u_n^i.$$

In view of Lemma 3.5, for all $\varepsilon \in (0, \varepsilon_0)$, we have $\eta_\varepsilon^i < \hat{\eta}_\varepsilon^i$. Hence, $u_n^i \in \theta_\varepsilon^i \backslash \partial\theta_\varepsilon^i$ for all n large enough.

Next, we consider the path $\gamma : (-\delta, \delta) \to S(a)$ defined by

$$\gamma(t) = a\frac{u_n^i + tv}{|u_n^i + tv|_2}$$

belongs to $C^1((-\delta, \delta), S(a))$, where $v \in T_{u_n^i}S(a) = \{z \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} u_n^i z \mathrm{d}x = 0\}$. Thus,

$$\gamma(t) \in \theta_\varepsilon^i \backslash \partial\theta_\varepsilon^i, \ \forall t \in (-\delta, \delta), \ \gamma(0) = u_n^i, \ \gamma'(0) = v$$

and

$$I_\varepsilon(\gamma(t)) - I_\varepsilon(u_n^i) \geq -\frac{1}{n}\left\|\gamma(t) - u_n^i\right\|, \forall t \in (-\delta, \delta).$$

Then,

$$\begin{aligned}\frac{I_\varepsilon(\gamma(t)) - I_\varepsilon(\gamma(0))}{t} &= \frac{I_\varepsilon(\gamma(t)) - I_\varepsilon(u_n^i)}{t} \geq -\frac{1}{n}\left\|\frac{\gamma(t) - u_n^i}{t}\right\| \\ &= -\frac{1}{n}\left\|\frac{\gamma(t) - \gamma(0)}{t}\right\|, \ \forall t \in (-\delta, \delta).\end{aligned}$$

Since $I_\varepsilon \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$, letting the limit of $t \to 0^+$, we obtain

$$I'_\varepsilon(u_n^i)v \geq -\frac{1}{n}\|v\|.$$

Furthermore, replacing v by $-v$, we have

$$\sup\left\{|I'_\varepsilon(u_n^i)(v)| : \|v\| \leq 1\right\} \leq \frac{1}{n},$$

which indicates that

$$I_\varepsilon(u_n) \to \eta_\varepsilon^i \text{ and } \|I_\varepsilon|'_{S(a)}(u_n)\| \to 0 \text{ as } n \to \infty,$$

namely, $\{u_n\}$ is a $(PS)_{\eta_\varepsilon^i}$ for I_ε restricted to $S(a)$. From Lemma 3.5, we have $\eta_\varepsilon^i < m_{max,a^2} + \rho_3$. Then combining with Lemma 3.3, there exists u^i such that $u_n^i \to u^i$ in $H^s(\mathbb{R}^N)$. Consequently,

$$u^i \in \eta_\varepsilon^i, \ \ I_\varepsilon(u^i) = \eta_\varepsilon^i \text{ and } I_\varepsilon|'_{S(a)}(u^i) = 0.$$

As

$$Q_\varepsilon(u^i) \in \overline{B_{\rho_0}(a_i)}, \ \ Q_\varepsilon(u^j) \in \overline{B_{\rho_0}(a_j)}$$

and

$$\overline{B_{\rho_0}(a_i)} \cap \overline{B_{\rho_0}(a_j)} = \emptyset \text{ for } i \neq j,$$

we deduce that $u^i \neq u^j$ for $i \neq j$, where $1 \leq i, j \leq l$. Hence, I_ε has at least l nontrivial critical points for all $\varepsilon \in (0, \varepsilon_0)$. Then there exists a Lagrange multiplier λ^i such that

$$\lambda^i a^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u^i|^2 \mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon x) f(u^i) u^i \mathrm{d}x.$$

In view of $I_\varepsilon(u^i) = \eta_\varepsilon^i < 0$ and (F_3), it is obvious that $\lambda^i < 0$ for $i = 1, 2, \cdots, l$. The proof is complete. □

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