Qualitative Theory of Dynamical Systems (2023) 22:127
https://doi.org/10.1007/s12346-023-00826-8

Breather Wave Solutions for the (3+1)-D Generalized Shallow Water Wave Equation with Variable Coefficients

Lafta Abed Dawod[1] · Mehrdad Lakestani[1] · Jalil Manafian[1]

Received: 4 April 2023 / Accepted: 16 June 2023 / Published online: 29 June 2023

Abstract

The equation of the shallow water wave in oceanography and atmospheric science is extended to (3+1) dimensions, which is a known equation. To achieve this, an illustrative example of the VC generalized shallow water wave equation is provided to demonstrate the feasibility and reliability of the used procedure in this study. It is shown that Hirota bilinear method is an important scheme. So, it has plenty of classes of rational solutions by selecting the interaction breather and dark soliton solutions and homoclinic breather wave solutions. Here many types of rough and breather solutions are obtained. The mentioned equation is transformed into the Hirota bilinear form with help of the Hirota direct method. In this process, the Hirota bilinear operator plays a significant role. Based on the Hirota bilinear form, the breather wave forms solutions of the equation are obtained. Meanwhile, the figures of the breather wave forms solutions and periodic wave solutions are plotted. The trajectory solutions of the traveling waves are shown explicitly and graphically. The effect of the free parameters on the behavior of the acquired figures to a few of the obtained solutions for two nonlinear rational exact cases was also discussed. In addition to addressing a scientific explanation of the analytical work, the results are graphically presented to make it simple to recognize the dynamical aspects. Many new types of traveling-wave solutions are revealed, including the breather wave, the dark kink singular, the periodic solitary singular and the singular soliton solutions. By comparing the proposed method with the other existing methods, the results show that the execution of this method is concise, simple, and straightforward.

✉ Jalil Manafian
j_manafianheris@tabrizu.ac.ir

Lafta Abed Dawod
laftaalsalihi@yahoo.com

Mehrdad Lakestani
lakestani@tabrizu.ac.ir

1 Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

Keywords Hirota bilinear form · Homoclinic breather · Periodic wave · Free parameters

1 Introduction

Nonlinear shallow water wave (NSWW) equations generally are explained through flow under a pressure surface, and it transpires everywhere, closely in the domain of oceanography and atmospheric sciences; however, there is the horizontal structure of an atmosphere and the evolution of an incompressible fluid. The shallow water wave have used to many applications due to their importance in Rossby waves [1, 2], fluid dynamics [3, 4] and other subjects [5–7]. Alsu et al. [8] studied novel exact solutions of the (2+1)-dimensions extended shallow water wave equation by the Hirota bi-linear approach to find Lumps and interactions, fission and fusion phenomena in multi solitons. Third-order dispersive evolution equations were widely adopted to model one-dimensional long waves and have extensive applications in fluid mechanics, plasma physics and nonlinear optics by using complete integrability, Lax pairs and bi-Hamiltonian structure [9]. Solitons, lump solutions and lump-periodic solutions were obtained for the generalized (2+1)-dimensional shallow water wave equation via the Hirota bilinear method [10]. Authors of [11] used the the Hirota bilinear method to equation of the shallow water wave in oceanography and atmospheric science with variable coefficients and obtained interaction between the lump and soliton solutions. In [12], the solitary wave solutions by using the rogue wave and semi-inverse variational principle schemes to the generalized (2+1)-dimensional shallow water wave equation have been studied. M lump and interaction between M lump and N stripe solutions to the third-order evolution equation arising in the shallow water have been investigated [13].

In the last decades, researchers have developed numerous methods. One method is study on the nonlinear evolution equations. The study on the nonlinear evolution equations (NLEEs) has caused people's attention in the last century. NLEEs have important applications in many subject areas, for example, theoretical physics, the multiple heaving wave energy [14], floating parabolic breakwaters [15], water environmental pollution [16], the Kudryashov's quintuple self-phase modulation [17], particle physics, crystal physics, signal processing, chemistry, astronomy and so on. These fields are closely related to mathematics. The exact solutions of NLEEs have an important significance for applications in these fields, because researchers can explain so many natural phenomena through these exact solutions. Scholars and researchers have put forward a large number of methods to obtain solutions of NLEEs, including an integrated simulation–optimization framework [18], the Hirota's bilinear method [19], the deep learning algorithm [20], the binary offset carrier signal capture algorithm [21], from van der Waals equation [22], free heat transfer properties of flat panel solar collectors [23], optimal bidding and offering strategies [24], a hybrid convolutional neural network [25], deep learning and modified african vulture optimization algorithm [26], optimal modeling of combined cooling [27], optimal economic scheduling method [28], a new combined energy system [29], a hybrid convolutional neural network and extreme [30], a robust optimization technique [31], a robust optimization based optimal method [32], a hybrid forecast engine based intelligent algorithm method [33], and so forth.

Here, the (3+1)-D variable coefficients generalized SWW equation is examined to use in ocean waves [34, 35] in the following

$$\alpha_1(t)\Phi_{yt} + \alpha_3(t)\Phi_x\Phi_{xy} + \alpha_3(t)\Phi_y\Phi_{xx} + \alpha_2(t)\Phi_{xxxy} + \alpha_4(t)\Phi_{xz} = 0, \quad (1.1)$$

in which $\alpha_4(t)$ relates to the perturbed effects and $\alpha_2(t)$ and $\alpha_3(t)$ point to the dispersion and nonlinearity, respectively. Also, $\alpha_s(t)(s = 1, \ldots, 4)$ is differentiable real function. Applications in weather simulations, tidal waves, tsunami prediction can be arisen in [36] with inserting $\alpha_1(t) = \alpha_2(t) = 1, \alpha_3(t) = -3, \alpha_4(t) = -1$. Furthermore, by putting $\alpha_1(t) = 2, \alpha_2(t) = 1, \alpha_3(t) = 3, \alpha_4(t) = -3$, the Eq. (1.1) will be replaced to (3+1)-dimensional Jimbo–Miwa equation [37]. Also, Eq. (1.1) can be substituted to the (3+1)-D constant coefficients generalized SWW equation [38] as follows

$$\Phi_{yt} - 3\Phi_x\Phi_{xy} - 3\Phi_y\Phi_{xx} + \Phi_{xxxy} - \Phi_{xz} = 0, \quad (1.2)$$

when $\alpha_1(t) = 1, \alpha_3(t) = -3, \alpha_2(t) = 1, \alpha_4(t) = -1$. Huang and co-authors [39, 40] investigated bilinear bäcklund transformation to Eq. (1.1) and obtained the soliton and periodic wave solutions.

Bright soliton, kink wave solution and traveling wave solutions were generated with the advantages of the generalized exponential rational function method to the $(1 + 2)$-dimensional Zoomeron equation [41].

Among these methods, Hirota direct method is so prompt and effective. With this method, NLEEs are transformed into Hirota bilinear forms [42]. This form is so beautiful and beneficial to obtaining exact solutions, especially a kind of important solutions-soliton solutions [43–45]. In addition, lump solutions, breather solutions of NLEEs can be constructed. Lately, Ma developed this method, the proposed generalized bilinear differential operators, NLEEs can be transformed into generalized bilinear forms through this operators [46].

A new method named bilinear neural network and the corresponding tensor formula were proposed to obtain the exact analytical solutions of nonlinear partial differential equations [47]. Bilinear neural network method was introduced to solve the explicit solution of a generalized breaking soliton equation [48]. Based on bilinear neural network method, the generalized lump solution, classical lump solution and the novel analytical solution were constructed to the (2+1)-dimensional Caudrey–Dodd–Gibbon–Kotera–Sawada-like equation [49].

The Hamiltonian amplitude equation with the properties of truncated M-fractional derivative with different forms like, bright, dark, singular, combined and complex solitons were extracted by using the modified Sardar sub-equation method [50]. The three-component coupled nonlinear Schrödinger equation the optical solitons in fiber optics was studied by the extended direct algebraic method [51]. The double-chain DNA dynamical system was investigated using the the new auxiliary equation method and the extended sinh-Gordon equation expansion method [52]. The optical solitons

to the Biswas–Arshed equation with third-order dispersion and self-steepening coefficients that communicate pulse propagation in fiber optics were obtained by using the extended Fan sub-equation method [53].
The exact solutions to one-dimensional long water wave propagation in a nonlinear medium simulated by the fractional-order modified equal-width equation were obtained [54]. The solitary dynamics including rogue wave, periodic wave solution, periodic combined singular soliton, combined singular soliton, and periodic wave solutions of the Biswas–Arshed model without self-phase modulation were investigated [55].
The analytical solutions of some nonlinear time-fractional partial differential equations were investigated by the direct algebraic method [56]. Analytical solutions and physical interpretations for the shallow water wave system were found using the modified expansion method [57]. The modulation instability was applied to study stability of the solutions of of (2+1)-Kadomtsev–Petviashvili equation [58].
The unified method to retrieve optical soliton solutions of the Biswas–Arshed model with the Kerr law nonlinearity was used [59]. The (2+1)-dimensional Benjamin–Bona–Mahony–Burgers model was considered and reduced to bilinear form by using the Hirota bilinear scheme to construct lump waves and collision of lump with periodic waves [60]. By the help of the Hirota bilinear formation and a trial function the novel collision solutions between the lump and kinky waves of the (3+1)-D Jimbo–Miwa-like model were studied [61]. The presence of stable kink soliton and kinky-periodic rogue wave solutions; unstable singular kink wave solutions of the biological dynamical models as a Cahn–Allen model and a diffusive predator–prey model was studied by modified simple equation scheme [62].
The dust ion-acoustic solitary wave in an unmagnetized collisional dusty plasma modeled by the damped modified Korteweg–de Vries equation by applying reductive perturbation technique was investigated [64]. The exact travelling and solitary wave solutions of the Kudryashov–Sinelshchikov equation by implementing the modified mathematical method were constructed [65]. The nonlinear longitudinal wave equation which involves mathematical physics with dispersal produced by the phenomena of transverse Poisson's effect in a magneto-electro-elastic circular rod was investigated [66]. Also, the solitary wave solutions of generalized Kadomtsev–Petviashvili (KP) modified equal width equation with the help of modification form of extended auxiliary equation mapping method were obtained [67]. The exact traveling and solitary wave solutions of nonlinear diffusion reaction equation with quadratic and cubic nonlinearities were created by implementing a modified mathematical method [?].

This work successfully used the Hirota bilinear approach [34, 35] with Hirota direct method to the (3+1)-D generalized shallow water wave (GSWW) equation with variable coefficients for obtaining spatiotemporal breather soliton solutions and exact extended breather wave solutions. As a main result of this paper, theoretical analysis to get to exact solutions by bell polynomial method for a GSWW equation with distributed coefficients is investigated.
In the present work, we have constituted a used model in ocean waves by bilinear method. Due to the advantages of the logarithmic form of function in Hirota bilinear scheme, we have considered a binary bell polynomials to get to bilinear equation with logarithmic transformation in our study.

There are four sections in the paper, and the structure of this article is as follows: In Sect. 2, the basic concept related to Hirota bilinear operator is introduced and the Hirota bilinear form is derived. Meanwhile, with the help of Hirota analysis approach, we prove that the breather wave solutions can be derived from the extended breather wave solutions in Sect. 3. In Sect. 5, we give some conclusions.

2 Binary Bell Polynomials

Consider Ref. [68], take $b = b(x_1, x_2, \ldots, x_n)$ be a C^∞ with the below issues:

$$A_{n_1x_1,\ldots,n_jx_j}(b) \equiv A_{n_1,\ldots,n_j}(b_{l_1x_1,\ldots,l_jx_j}) = e^{-b}\partial_{x_1}^{n_1}\cdots\partial_{x_j}^{n_j}e^b, \tag{2.1}$$

and

$$b_{l_1x_1,\ldots,l_jx_j} = \partial_{x_1}^{l_1}\cdots\partial_{x_j}^{l_j}b, \quad b_{0x_i} \equiv b, \; l_1 = 0, \ldots, n_1; \ldots; l_j = 0, \ldots, n_j,$$

and we have

$$\begin{aligned}
&A_1(b) = b_x, \quad A_2(b) = b_{2x} + b_x^2,\\
&A_3(b) = b_{3x} + 3b_xb_{2x} + b_x^3, \ldots, \quad b = b(x, t),\\
&A_{x,t}(b) = b_{x,t} + b_xb_t, \quad A_{2x,t}(b) = b_{2x,t} + b_{2x}b_t + 2b_{x,t}b_x + b_x^2b_t, \ldots.
\end{aligned} \tag{2.2}$$

The multi form of Bell polynomials is stated as

$$\begin{aligned}
\sum_{n_1x_1,\ldots,n_jx_j}(\mu_1, \mu_2) &= \Upsilon_{n_1,\ldots,n_j}(b)\big|_{b_{l_1x_1,\ldots,l_jx_j}}\\
&= \begin{cases} \mu_{1l_1x_1,\ldots,l_jx_j}, & l_1 + l_2 + \cdots + l_j, \text{ is odd}\\ \mu_{2l_1x_1,\ldots,l_jx_j}, & l_1 + l_2 + \cdots + l_j, \text{ is even.}\end{cases}
\end{aligned} \tag{2.3}$$

The following issues are as

$$\begin{aligned}
&\sum_x(\mu_1) = \mu_{1x}, \quad \sum_{2x}(\mu_1, \mu_2) = \mu_{22x} + \mu_{1x}^2,\\
&\sum_{x,t}(\mu_1, \mu_2) = \mu_{2x,t} + \mu_{1x}\mu_{1t}, \ldots.
\end{aligned} \tag{2.4}$$

Proposition 2.1 *Suppose* $\rho_1 = \ln(\Theta_1/\Theta_2)$, $\rho_2 = \ln(\Theta_1\Theta_2)$, *then the cases between binary Bell polynomials and Hirota operator will be appeared as*

$$\sum_{n_1x_1,\ldots,n_jx_j}(\rho_1,\rho_2)\Bigg|_{\rho_1=\ln(\Theta_1/\Theta_2),\quad \rho_2=\ln(\Theta_1\Theta_2)} = (\Theta_1\Theta_2)^{-1}D_{x_1}^{n_1}\cdots D_{x_j}^{n_j}\Theta_1\Theta_2, \tag{2.5}$$

with Hirota operator

$$\prod_{i=1}^{j} D_{x_i}^{n_i} g.\,\eta = \prod_{i=1}^{j}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_i'}\right)^{n_i}\Theta_1(x_1,\ldots,x_j)\Theta_2(x_1',\ldots,x_j')\Bigg|_{x_1=x_1',\ldots,x_j=x_j'}. \tag{2.6}$$

Proposition 2.2 *Take* $\Xi(\gamma)=\sum_i \delta_i \mathfrak{P}_{l_1x_1,\ldots,l_jx_j}=0$ *and* $\rho_1=\ln(\Theta_1/\Theta_2),\quad \rho_1=\ln(\Theta_1\Theta_2)$, *we have*

$$\begin{cases}\sum_i \delta_{1i}\Upsilon_{n_1x_1,\ldots,n_jx_j}(\rho_1,\rho_2)=0,\\ \sum_i \delta_{1i}\Upsilon_{l_1x_1,\ldots,l_jx_j}(\rho_1,\rho_2)=0,\end{cases} \tag{2.7}$$

in which the below relations are correct

$$\mathfrak{S}(\gamma',\gamma)=\mathfrak{S}(\gamma')-\mathfrak{S}(\gamma)=\mathfrak{S}(\rho_2+\rho_1)-\mathfrak{S}(\rho_2-\rho_1)=0. \tag{2.8}$$

The generalized form of $\Upsilon_{n_1x_1,\ldots,n_jx_j}(\xi)$ is presented as

$$\begin{aligned}(\Theta_1\Theta_2)^{-1}D_{x_1}^{n_1}\cdots D_{x_j}^{n_j}\Theta_1\Theta_2 &= \Sigma_{n_1x_1,\ldots,n_jx_j}(\rho_1,\rho_2)\big|_{\rho_1=\ln(\Theta_1/\Theta_2),\quad \rho_2=\ln(\Theta_1\Theta_2)}\\ &= \Sigma_{n_1x_1,\ldots,n_jx_j}(\rho_1,\rho_1+\gamma)\big|_{\rho_1=\ln(\Theta_1/\Theta_2),\quad \gamma=\ln(\Theta_1\Theta_2)}\\ &= \sum_{k_1}^{n_1}\cdots\sum_{k_j}^{n_j}\prod_{i=1}^{j}\binom{n_i}{k_i}\mathfrak{P}_{k_1x_1,\ldots,k_jx_j}(\gamma)\Upsilon_{(n_1-k_1)x_1,\ldots,(n_j-k_j)x_j}(\rho_1).\end{aligned} \tag{2.9}$$

The Cole–Hopf relation in the following is given

$$\Upsilon_{k_1x_1,\ldots,k_jx_j}(\rho_1=\ln(\varphi))=\frac{\varphi_{n_1x_1,\ldots,n_jx_j}}{\varphi}, \tag{2.10}$$

$$\begin{aligned}&(\Theta_1\Theta_2)^{-1}D_{x_1}^{n_1}\cdots D_{x_j}^{n_j}\Theta_1\Theta_2\big|_{\Theta_2=\exp(\gamma/2),\quad \Theta_1/\Theta_2=\varphi}\\ &\quad=\varphi^{-1}\sum_{k_1}^{n_1}\cdots\sum_{k_j}^{n_j}\prod_{l=1}^{j}\binom{n_l}{k_l}\mathfrak{P}_{k_1x_1,\ldots,k_lx_l}(\gamma)\varphi_{(n_1-k_1)x_1,\ldots,(n_d-k_l)x_l},\end{aligned} \tag{2.11}$$

with

$$\Upsilon_t(\rho_1)=\frac{\varphi_t}{\varphi},\quad \Upsilon_{2x}(\rho_1,\beta)=\gamma_{2x}+\frac{\varphi_{2x}}{\varphi},$$

$$\Upsilon_{2x,y}(\rho_1, \rho_2) = \frac{\gamma_{2x}\varphi_y}{\varphi} + \frac{2\gamma_{x,y}\varphi_x}{\varphi} + \frac{\varphi_{2x,y}}{\varphi}. \tag{2.12}$$

Inserting $\Phi = c(t)f_x + \Phi_0$ into Eq. (1.1), one gives $c(t) = 2\alpha_0$. Hence, to more investigation the following theorem is taken.

Theorem 2.3 *By the below issues, one gets*

$$f = \ln(h) \Leftrightarrow \Phi = 2\alpha_0 \ln(h(x, y, z, t))_x - \omega(z), \tag{2.13}$$

in which $\alpha_2 = \alpha_0\alpha_3(t)/3$ and α_0 is an nonzero constant and $\omega(z)$ is an free function into Eq. (1.1)*, the (3+1)-D VC generalized SWW equation can be expressed in the following*

$$\begin{aligned}\Re(h) &= \alpha_1(t)(hh_{yt} - h_yh_t) + \alpha_4(t)(hh_{xz} - h_xh_z)\\ &\quad + \alpha_2(t)(hh_{xxxy} - 3h_xh_{xxy} + 3h_{xy}h_{xx} - h_yh_{xxx})\\ &= \frac{1}{2}\left(\alpha_1(t)D_yD_t + \alpha_4(t)D_xD_z + \alpha_2(t)D_x^3D_y\right)h.h = 0,\end{aligned} \tag{2.14}$$

where $h = h(x, y, z, t)$ and $f = f(x, y, z, t)$.

3 Two Types Solutions for VCSWW Equation

Two subsections containing the extended breather solutions and rational breather wave solutions are studied.

3.1 Extended Breather Wave Solutions

The following function for the extended breather wave solutions is offered as

$$\begin{aligned}&h = \mathrm{e}^{a_1} + \mathrm{e}^{a_2} + \tanh(a_3) + \tan(a_4),\\ &a_s = m_sx + n_s(y) + p_sz + q_s(t), \quad s = 1, 2, 3, 4.\end{aligned} \tag{3.1}$$

Afterwards, the amounts $m_s, n_s(y), p_s(y), (s = 1 : 2)$ will be determined. By making use of Eq. (3.1) into (2.14) and considering the coefficients the each powers of $\exp(a_s)$, $s = 1, 2$ and $\tanh(a_3)$, $\tan(a_4)$ to zero, reach a system of equations (algebraic) for $m_s, n_s(y), p_s, q_s(t), (s = 1 : 4)$. Based on the (3.1) the solution $\Phi = 2\alpha_0(\ln h)_x$ can be appeared as below:

$$\begin{aligned}&\Phi = \frac{m_1\mathrm{e}^{a_1} + m_2\mathrm{e}^{a_2} + \left(1 - (\tanh(a_3))^2\right)m_3 + \left(1 + (\tan(a_4))^2\right)m_4}{\mathrm{e}^{a_1} + \mathrm{e}^{a_2} + \tanh(a_3) + \tan(a_4) + q_5(t)},\\ &a_s = m_sx + n_s(y) + p_sz + q_s(t), \quad s = 1, 2, 3, 4.\end{aligned}$$

3.1.1 Solution I

$$
\begin{aligned}
&m_s = m_s, \quad s = 1, 2, 3, 4, \quad n_s = p_s = 0, \quad s = 1, 2, 3, 4,\\
&q_i(t) = q_i(t), \quad i = 1, 2, 3, 4, \quad q_5(t) = C_5,
\end{aligned} \tag{3.2}
$$

in which $m_l, l = 1, 2, 3, 4$ are free constants and $q_l(t), l = 1, 2, 3, 4$ are parametric functions. The first solution of extended breather wave solution is obtained as

$$
\begin{aligned}
&\Phi_1 = 2\,\frac{\alpha_0\left(m_1 \mathrm{e}^{m_1x+q_1(t)} + m_2\mathrm{e}^{m_2x+q_2(t)} + \left(\omega^-(x,t)\right)m_3 + \left(\omega^+(x,t)\right)m_4\right)}{\mathrm{e}^{m_1x+q_1(t)} + \mathrm{e}^{m_2x+q_2(t)} + \tanh\left(m_3x + q_3\left(t\right)\right) + \tan\left(m_4x + q_4\left(t\right)\right) + q_5\left(t\right)},\\
&\omega^{\pm}(x,t) = 1 \pm \left(\tanh\left(m_3x + q_3\left(t\right)\right)\right)^2, \quad a_1 = m_lx + q_lt, \quad l = 1, 2, 3, 4.
\end{aligned} \tag{3.3}
$$

3.1.2 Solutions II

$$
\begin{aligned}
&m_l = n_l = 0, \quad l = 1, 3, 4, \quad m_2 = m_2, \quad n_2 = n_2, \quad p_i = p_i,\\
&i = 1, 2, 3, 4, \quad q_5(t) = C_5,\\
&q_i\left(t\right) = \int -\frac{\alpha_4\left(t\right)m_2p_i}{\alpha_1\left(t\right)n_2}\,\mathrm{d}t, \quad i = 1, 3, 4,\\
&q_2\left(t\right) = \int -\frac{m_2\left(\alpha_2\left(t\right)m_2{}^2n_2 + \alpha_4\left(t\right)p_2\right)}{\alpha_1\left(t\right)n_2}\,\mathrm{d}t,
\end{aligned} \tag{3.4}
$$

in which m_2, n_2 are free constants and $p_s, q_s(t), s = 1, 2, 3, 4$ are parametric functions. The rational extended breather wave solution is derived in the following form

$$
\begin{aligned}
&\Phi_2 = \frac{2\,\alpha_0 m_2F_1}{\mathrm{e}^{p_1z+\int-\frac{\alpha_4(t)m_2p_1}{\alpha_1(t)n_2}\,\mathrm{d}t} + F_1 + \tanh\left(p_3z + \int-\frac{\alpha_4(t)m_2p_3}{\alpha_1(t)n_2}\,\mathrm{d}t\,\Lambda\right) + \tan\left(p_4z + \int-\frac{\alpha_4(t)m_2p_4}{\alpha_1(t)n_2}\,\mathrm{d}t\right) + C_5},\\
&F_1 = \mathrm{e}^{m_2x+n_2y+p_2z+\int-\frac{m_2\left(\alpha_2(t)m_2{}^2n_2+\alpha_4(t)p_2\right)}{\alpha_1(t)n_2}\,\mathrm{d}t}.
\end{aligned} \tag{3.5}
$$

Figs. 1, 2 and 3, offer the behavior of interaction between breather and dark soliton solution with two wave solution in which the selected values for Φ_2 are given in the following

$$
\begin{aligned}
&m_2 = 2, n_2 = 1, p_1 = 3, p_2 = 2, p_3 = 3, p_4 = 4, C_5 = \alpha_0 = 1,\\
&\alpha_1(t) = t, \quad \alpha_2(t) = t, \quad \alpha_4(t) = -t^2, \quad x = y = 1,
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
&m_2 = p_2 = 2, \quad p_1 = p_3 = 3, \quad p_4 = 4, \quad n_2 = C_5 = \alpha_0 = 1,\\
&\alpha_1(t) = \cos(t), \quad \alpha_2(t) = \cos(2t), \quad \alpha_4(t) = \sin(2t), \quad x = y = 1,
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
&m_2 = p_2 = 2, \quad p_1 = p_3 = 3, \quad p_4 = 4, \quad C_5 = n_2 = \alpha_0 = 1,\\
&\alpha_1(t) = t^2, \alpha_2(t) = t^3, \quad \alpha_4(t) = t^2, \quad x = y = 1,
\end{aligned} \tag{3.8}
$$

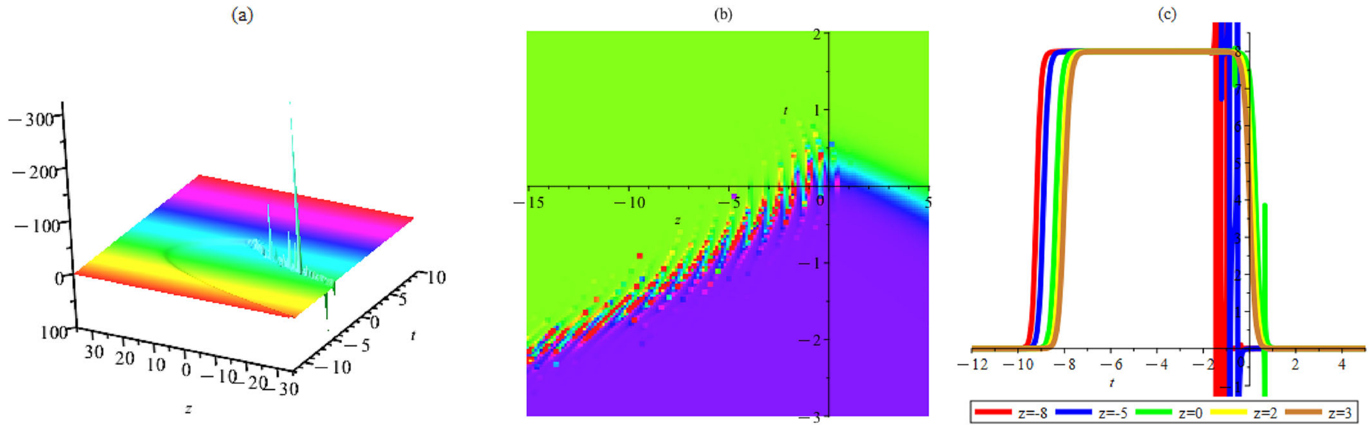

Fig. 1 Plots of the extended breather wave solutions to GSWW equation with distributed coefficients (3.6) (Σ_2) with $\alpha_1(t) = t, \alpha_2(t) = t, \alpha_4(t) = -t^2$; Density map in the (z, t)-plane; Along plane: $x = y = 1$

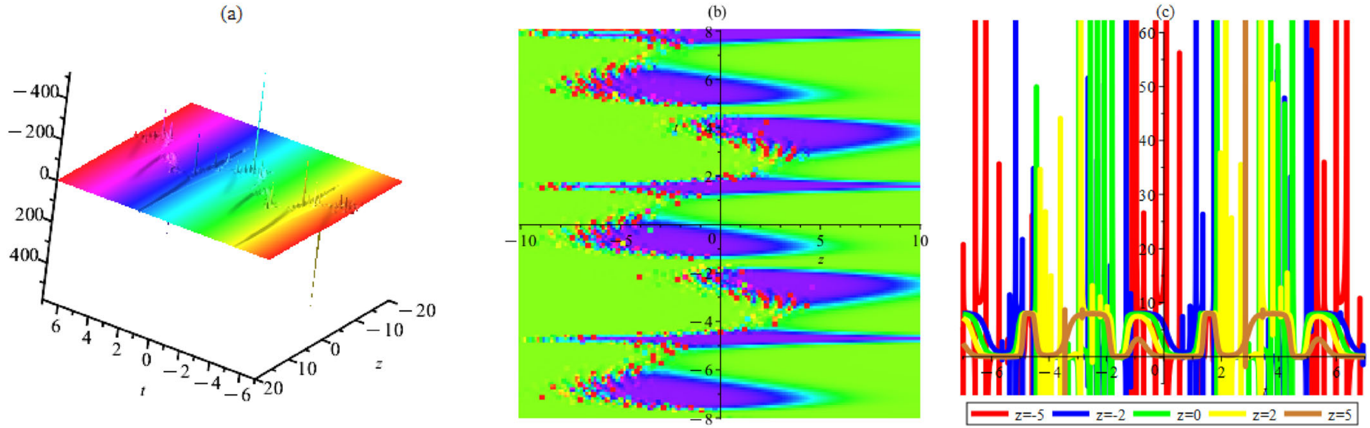

Fig. 2 Plots of the extended breather wave solutions to GSWW equation with distributed coefficients (3.7) (Σ_2) with $\alpha_1(t) = \cos(t), \alpha_2(t) = \cos(2t), \alpha_4(t) = \sin(2t)$; Density map in the (z, t)-plane; Along plane: $x = y = 1$

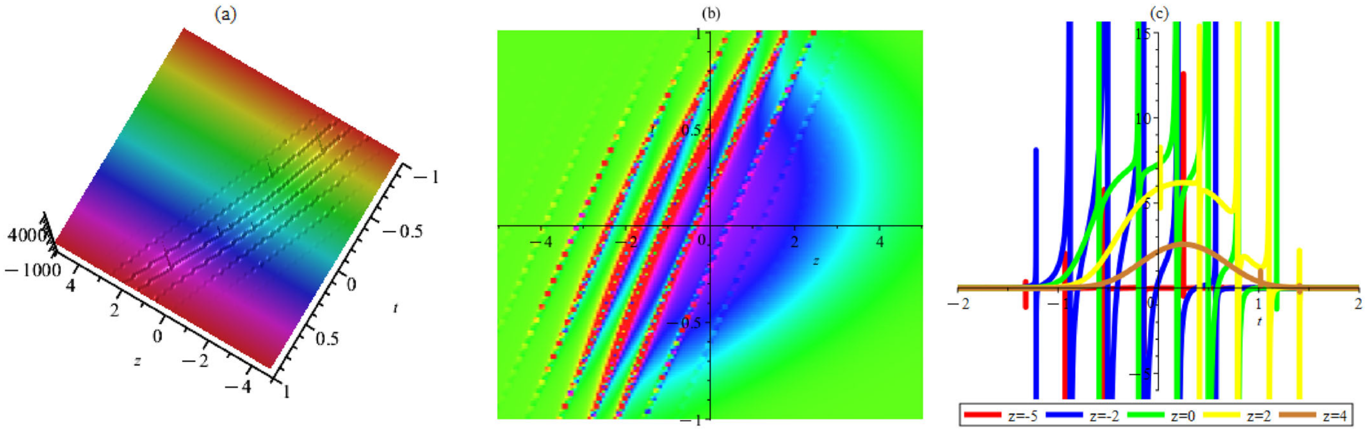

Fig. 3 Plots of the extended breather wave solutions to GSWW equation with distributed coefficients (3.8) (Σ_2) with $\alpha_1(t) = t^2$, $\alpha_2(t) = t^3$, $\alpha_4(t) = t^2$; Density map in the (z, t)-plane; Along plane: $x = y = 1$

in Eq. (3.5). By using the above selected parameters the property interaction between breather and dark soliton solution is shown in Figs. 1, 2 and 3 with graphs to various times. It shown a kind of interaction between two exponential waves, hyperbolic wave and trigonometric waves.

3.1.3 Solution III

$$\begin{aligned} &m_l = n_l = 0, \quad l = 1, 4, \quad m_2 = m_2, \quad n_2 = n_2, \quad m_3 = m_3, \\ &n_3 = n_3, \quad p_i = p_i, \quad i = 1, 2, 3, 4, \quad q_5(t) = C_5, \\ q_i\,(t) &= \int -\frac{m_2\left(\alpha_2\,(t)\,m_2{}^2 n_2 + \alpha_4\,(t)\,p_i\right)}{\alpha_1\,(t)\,n_2}\,\mathrm{d}t, \quad i = 1, 2, \\ q_i\,(t) &= \int -\frac{\alpha_4\,(t)\,m_2 p_i}{\alpha_1\,(t)\,n_2}\,\mathrm{d}t, \quad i = 1, 2, \end{aligned} \tag{3.9}$$

in which m_2, n_2, m_3, n_3 are free values and $p_l, l = 1, 2, 3, 4$ are parametric functions. It derives the rational breather solution in the following way

$$\begin{aligned} \Phi_3 &= \frac{2\,\alpha_0 m_2 F_1}{\mathrm{e}^{p_1 z + \int -\frac{\alpha_4(t) m_2 p_1}{\alpha_1(t) n_2}\,\mathrm{d}t} + F_1 + \tanh\left(p_3 z + \int -\frac{\alpha_4(t) m_2 p_3}{\alpha_1(t) n_2}\,\mathrm{d}t\right) + \tan\left(p_4 z + \int -\frac{\alpha_4(t) m_2 p_4}{\alpha_1(t) n_2}\,\mathrm{d}t\right) + C_5}, \\ F_1 &= \mathrm{e}^{m_2 x + n_2 y + p_2 z + \int -\frac{m_2\left(\alpha_2(t) m_2{}^2 n_2 + \alpha_4(t) p_2\right)}{\alpha_1(t) n_2}\,\mathrm{d}t}. \end{aligned} \tag{3.10}$$

Figs. 4, 5and 6, communicates interaction between breather and dark soliton solution with two wave solution when solution Φ_3 are given with the below selected parameters

$$\begin{aligned} &m_2 = 2, \quad n_2 = 1, \quad p_2 = 2, p_1 = p_3 = 3, \quad p_4 = 4, \quad C_5 = 1, \quad \alpha_0 = 2, \\ &\alpha_1(t) = t, \quad \alpha_2(t) = t, \quad \alpha_4(t) = -t^2, \quad x = y = 1, \end{aligned} \tag{3.11}$$

$$\begin{aligned} &m_2 = \alpha_0 = p_2 = 2, \quad p_1 = p_3 = 3, \quad p_4 = 4, \quad \alpha_1(t) = \cos(t), \\ &\alpha_2(t) = \cos(2t), \quad \alpha_4(t) = \sin(2t), \quad n_2 = C_5 = x = y = 1, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &m_2 = 2, \quad n_2 = 1, \quad p_1 = 3, \quad p_2 = 2, \quad p_3 = 3, \quad p_4 = 4, \quad C_5 = 1, \\ &\alpha_0 = 2, \quad \alpha_1(t) = t^2, \quad \alpha_2(t) = t^3, \quad \alpha_4(t) = t^2, \quad x = y = 1, \end{aligned} \tag{3.13}$$

in Eq. (3.10). By using the above amounts the property interaction between breather and dark soliton solution is presented in Figs. 4, 5 and 6 with plots with various times.

3.1.4 Solutions IV

$$\begin{aligned} &m_l = n_l = 0, \quad l = 2, 3, 4, \quad m_1 = m_1, \quad n_1 = n_1, \quad p_i = p_i, \\ &i = 1, 2, 3, 4, \quad q_5(t) = C_5, \\ q_1\,(t) &= \int -\frac{m_1\left(\alpha_2\,(t)\,m_1{}^2 n_1 + \alpha_4\,(t)\,p_1\right)}{\alpha_1\,(t)\,n_1}\,\mathrm{d}t, \\ q_i\,(t) &= \int -\frac{\alpha_4\,(t)\,m_1 p_i}{\alpha_1\,(t)\,n_1}\,\mathrm{d}t, \quad i = 2, 3, 4, \end{aligned} \tag{3.14}$$

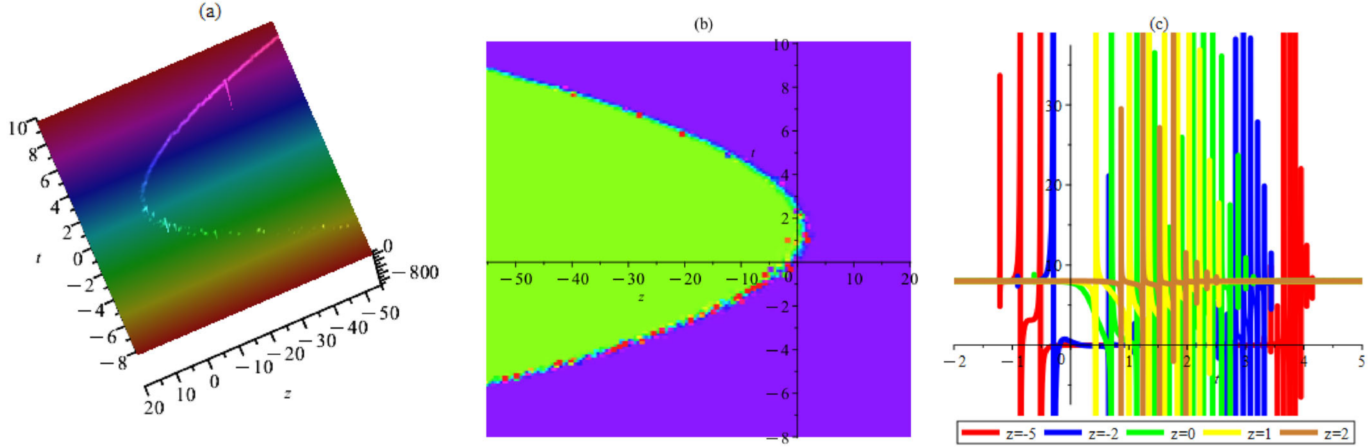

Fig. 4 Plots of the extended breather wave solutions to GSWW equation with distributed coefficients (3.11) (Σ_3) with $\alpha_1(t) = t, \alpha_2(t) = t, \alpha_4(t) = -t^2$; Density map in the (z, t)-plane; Along plane: $x = y = 1$

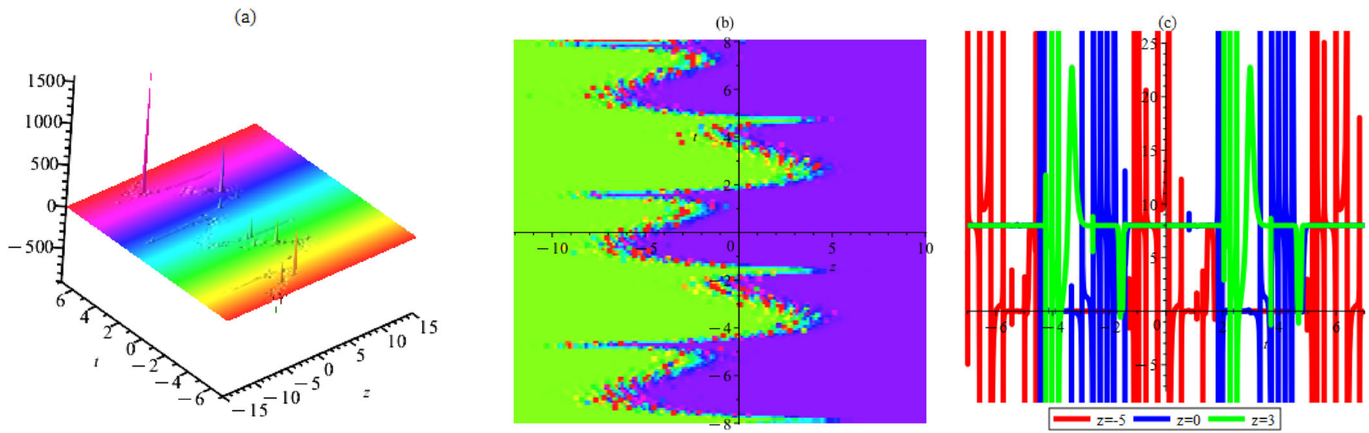

Fig. 5 Plots of the extended breather wave solutions to GSWW equation with distributed coefficients (3.12) (Σ_3) with $\alpha_1(t) = \cos(t), \alpha_2(t) = \cos(2t), \alpha_4(t) = \sin(2t)$; Density map in the (z, t)-plane; Along plane: $x = y = 1$

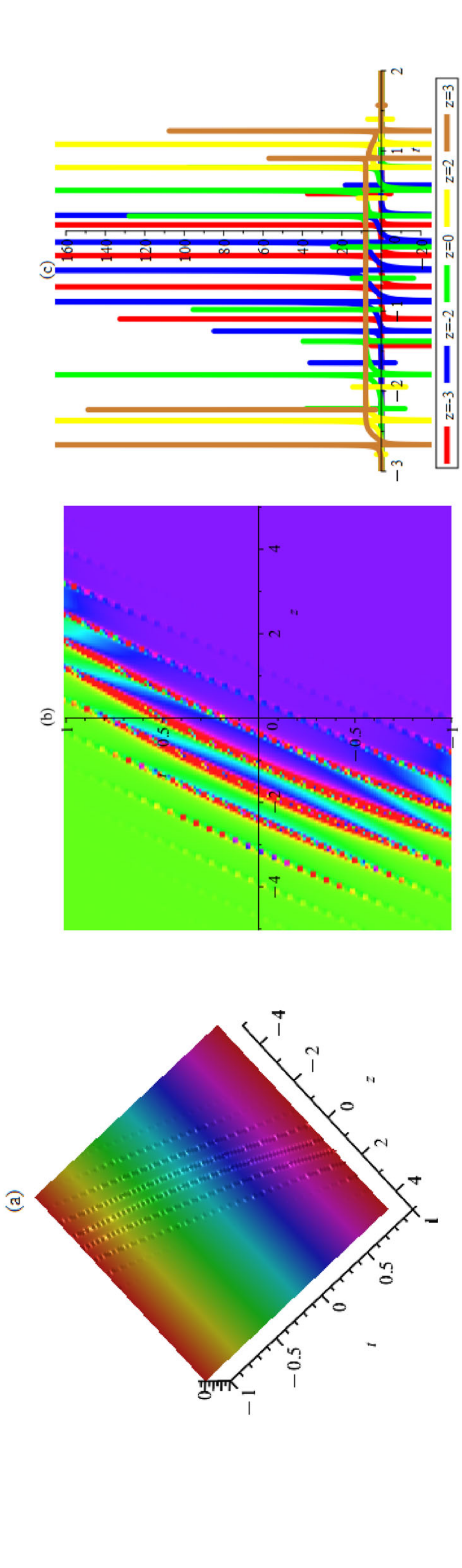

Fig. 6 Plots of the extended breather wave solutions to GSWW equation with distributed coefficients (3.13) (Σ_3) with $\alpha_1(t) = t^2, \alpha_2(t) = t^3, \alpha_4(t) = t^2$; Density map in the (z, t)-plane; Along plane: $x = y = 1$

in which m_1, n_1 are constants and $p_l, l = 1, 2, 3, 4$ the parametric functions. It derives the following rational breather solution

$$\Phi_4 = 2\,\alpha_0 m_2 F_1 \left(F_1 + \mathrm{e}^{p_2 z + \int -\frac{\alpha_4(t) m_1 p_2}{\alpha_1(t) n_1}\,\mathrm{d}t} + \tanh\left(\omega_3(z,t)\right) + \tan\left(\omega_4(z,t)\right) + C_5 \right)^{-1},$$
$$\omega_i(z,t) = p_i z - \int \frac{\alpha_4\,(t)\, m_1 p_i}{\alpha_1\,(t)\, n_1}\,\mathrm{d}t, \quad i = 3, 4,$$
$$F_1 = \mathrm{e}^{m_2 x + n_2 y + p_1 z + \int -\frac{m_1\left(\alpha_2(t) {m_1}^2 n_1 + \alpha_4(t) p_1\right)}{\alpha_1(t) n_1}\,\mathrm{d}t}. \tag{3.15}$$

3.1.5 Solutions V

$$m_l = 0, \;\; l = 2, 3, 4, \;\; m_1 = m_1, \;\; n_l = n_l, \;\; l = 1, 2, 3, 4, \quad q_1(t) = q_1(t),$$
$$q_i(t) = C_i, \quad i = 2, 3, 4, 5,$$
$$p_i = -\frac{n_i\left(\alpha_2\,(t)\, {m_1}^3 + \alpha_1\,(t)\, \frac{\mathrm{d}}{\mathrm{d}t} q_1\,(t)\right)}{\alpha_4\,(t)\, m_1}, \quad i = 1, 2, 3, 4, \tag{3.16}$$

in which $n_l, l = 1, 2, 3, 4$ are constants. The exact solution is given as follows

$$\Phi_5 = \frac{2\,\alpha_0 m_1 F_1}{F_1 + F_2 + \tanh\left(\omega_3(y,z,t)\right) + \tan\left(\omega_4(y,z,t)\right) + C_5},$$
$$\omega_i(y,z,t) = n_i y - \frac{n_i\left(\alpha_2\,(t)\, {m_1}^3 + \alpha_1\,(t)\, \frac{\mathrm{d}}{\mathrm{d}t} q_1\,(t)\right) z}{\alpha_4\,(t)\, m_1} + C_i, \quad i = 3, 4,$$
$$F_1 = \mathrm{e}^{m_1 x + n_1 y - \frac{n_1\left(\alpha_2(t) {m_1}^3 + \alpha_1(t) \frac{\mathrm{d}}{\mathrm{d}t} q_1(t)\right) z}{\alpha_4(t) m_1} + q_1(t)},$$
$$F_2 = \mathrm{e}^{n_2 y - \frac{n_2\left(\alpha_2(t) {m_1}^3 + \alpha_1(t) \frac{\mathrm{d}}{\mathrm{d}t} q_1(t)\right) z}{\alpha_4(t) m_1} + C_2}. \tag{3.17}$$

Figs. 7, 8 and 9, express the behavior of interaction between breather and dark soliton solution with two wave solution to soltion Φ_5 are given the following selected parameters

$$m_1 = n_2 = \alpha_0 = 2, \quad n_1 = n_3 = 3, \quad n_4 = 4, \quad C_i = 1, \quad i = 2, 3, 4, 5,$$
$$q_1(t) = \alpha_1(t) = \alpha_2(t) = t, \quad \alpha_4(t) = 2, \quad x = y = 1, \tag{3.18}$$
$$m_1 = 2, \quad n_1 = 3, \quad n_2 = 2, \quad n_3 = 3, \quad n_4 = 4, \quad C_i = 1, \quad i = 2, 3, 4, 5,$$
$$q_1(t) = t, \quad \alpha_0 = 2,$$
$$\alpha_1(t) = \cos(t), \quad \alpha_2(t) = \sin(2t), \quad \alpha_4(t) = 2 + \cos(2t), \quad x = y = 1, \tag{3.19}$$
$$m_1 = n_2 = \alpha_0 = \alpha_4(t) = 2, \quad n_1 = n_3 = 3, \quad n_4 = 4, \quad q_1(t) = t,$$
$$\alpha_1(t) = t^2, \quad \alpha_2(t) = t^3, \quad C_i = x = y = 1, \quad i = 2, \ldots, 5, \tag{3.20}$$

in Eq. (3.17). By utilizing the above parameters including trigonometric for time-variable functions as presented in Figs. 7, 8 and 9 with various times.

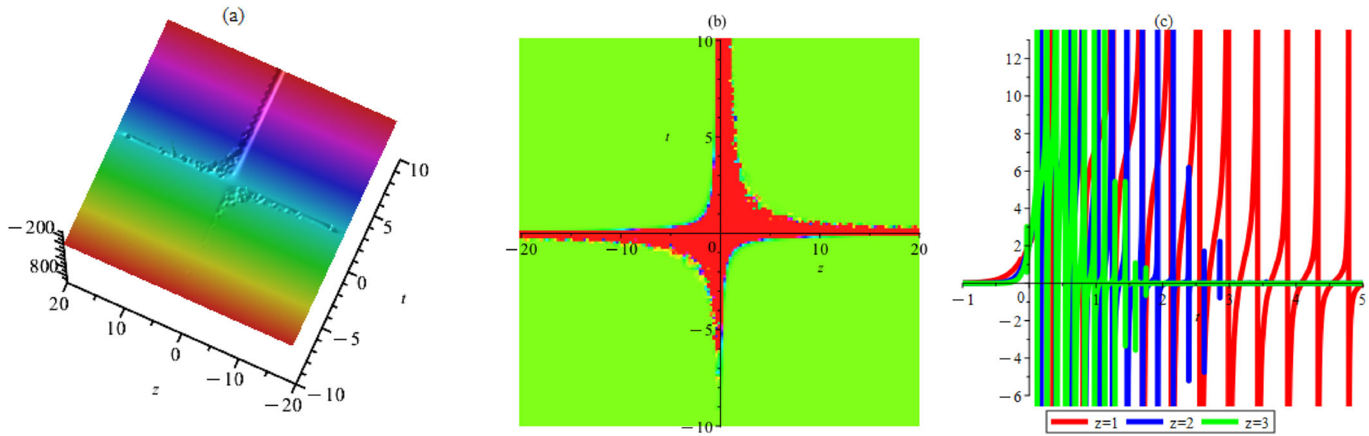

Fig. 7 Plots of the extended breather wave solutions to GSWW equation with distributed coefficients (3.18) (Σ_5) with $\alpha_0 = 2, \alpha_1(t) = t, \alpha_2(t) = t, \alpha_4(t) = 2$; Density map in the (z, t)-plane; Along plane: $x = y = 1$

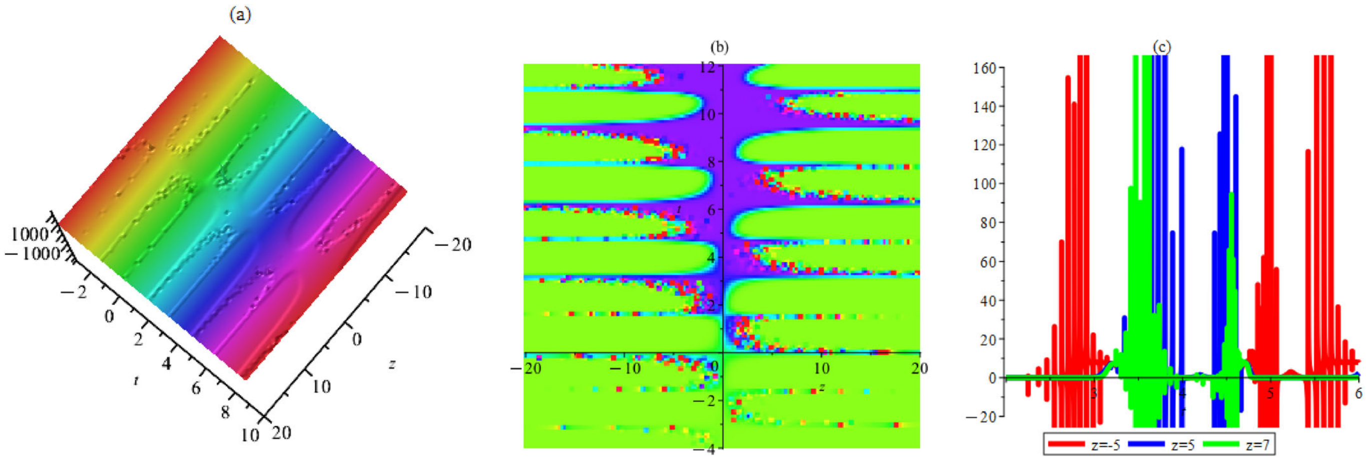

Fig. 8 Plots of the extended breather wave solutions to GSWW equation with distributed coefficients (3.19) (Σ_5) with $\alpha_1(t) = \cos(t)$, $\alpha_2(t) = \sin(2t)$, $\alpha_4(t) = 2 + \cos(2t)$; Density map in the (z, t)-plane; Along plane: $x = y = 1$

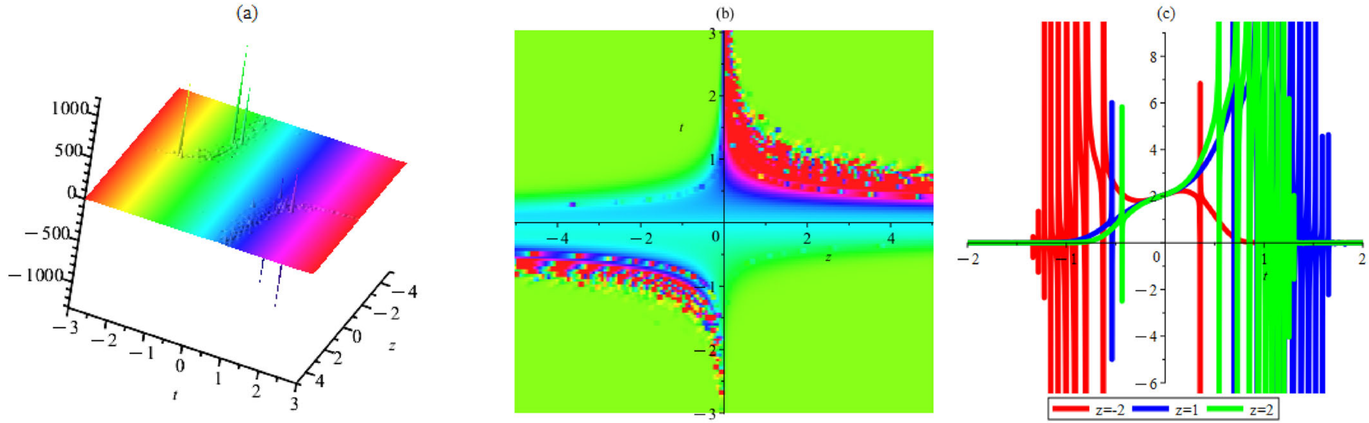

Fig. 9 Plots of the extended breather wave solutions to GSWW equation with distributed coefficients (3.20) (Σ_5) with $\alpha_0 = 2, \alpha_1(t) = t^2, \alpha_2(t) = t^3, \alpha_4(t) = 2$; Density map in the (z, t)-plane; Along plane: $x = y = 1$

3.1.6 Solutions VI

$$
\begin{aligned}
& m_l=0,\ \ l=3,4,\ \ m_1=m_1,\ \ m_2=m_2,\ \ n_l=n_l,\ \ l=1,2,3,4,\quad q_1(t)=q_1(t),\\
& q_2(t)=\frac{q_1(t)\,m_2}{m_1}+\frac{m_2}{m_1}\int-\frac{\alpha_2(t)\,m_1m_2{}^2}{\alpha_1(t)}+\frac{\alpha_2(t)\,m_1{}^3}{\alpha_1(t)}\,\mathrm{d}t,\\
& q_i(t)=C_i,\quad i=3,4,5,\\
& p_i=-\frac{n_2\left(\alpha_2(t)\,m_1{}^3+\alpha_1(t)\,\frac{\mathrm{d}}{\mathrm{d}t}q_1(t)\right)z}{\alpha_4(t)\,m_1},\quad i=1,2,\\
& p_i=-\frac{n_i\left(\alpha_2(t)\,m_1{}^3+\alpha_1(t)\,\frac{\mathrm{d}}{\mathrm{d}t}q_1(t)\right)z}{\alpha_4(t)\,m_1},\quad i=3,4,
\end{aligned}
\tag{3.21}
$$

in which $m_1, m_2, n_l, l=1,2,3,4$ are parametric functions of t. The novel solution for the extended breather solution is given as follows

$$
\begin{aligned}
& \Phi_6=\frac{2\,\alpha_0\,(m_1F_1+m_2F_2)}{F_1+F_2+\tanh(\omega_3(y,z,t))+\tan(\omega_4(y,z,t))+C_5},\\
& \omega_i(y,z,t)=n_iy-\frac{n_i\left(\alpha_2(t)\,m_1{}^3+\alpha_1(t)\,\frac{\mathrm{d}}{\mathrm{d}t}q_1(t)\right)z}{\alpha_4(t)\,m_1}+C_i,\quad i=3,4\\
& F_1=\mathrm{e}^{m_1x+n_2y-\frac{n_2\left(\alpha_2(t)m_1{}^3+\alpha_1(t)\frac{\mathrm{d}}{\mathrm{d}t}q_1(t)\right)z}{\alpha_4(t)m_1}+q_1(t)},\\
& F_2=\mathrm{e}^{m_2x+n_2y-\frac{n_2\left(\alpha_2(t)m_1{}^3+\alpha_1(t)\frac{\mathrm{d}}{\mathrm{d}t}q_1(t)\right)z}{\alpha_4(t)m_1}+\frac{q_1(t)m_2}{m_1}+\frac{m_2}{m_1}\int-\frac{\alpha_2(t)m_1m_2{}^2}{\alpha_1(t)}+\frac{\alpha_2(t)m_1{}^3}{\alpha_1(t)}\,\mathrm{d}t}.
\end{aligned}
\tag{3.22}
$$

3.1.7 Solutions VII

$$
\begin{aligned}
& m_l=0,\ \ l=3,4,\ \ m_1=m_1,\ \ m_2=m_2,\ \ n_l=n_l,\ \ l=1,2,3,4,\\
& q_1(t)=q_1(t),\ \ q_2(t)=q_2(t),\\
& q_i(t)=C_i,\quad i=3,4,5,\quad p_i=-\frac{n_i\left(\alpha_2(t)\,m_2{}^3+\alpha_1(t)\,\frac{\mathrm{d}}{\mathrm{d}t}q_1(t)\right)}{\alpha_4(t)\,m_2},\\
& i=1,2,3,4,
\end{aligned}
\tag{3.23}
$$

in which $m_1, m_2, n_l, l=1,2,3,4$ are parametric functions of t. The novel solution for the extended breather solution is given as follows

$$
\begin{aligned}
\Phi_7 &= \frac{2\,\alpha_0\,(m_2F_1+m_2F_2)}{F_1+F_2+\tanh\left(n_3y-\frac{n_3\left(\alpha_2(t)m_2{}^3+\alpha_1(t)\frac{\mathrm{d}}{\mathrm{d}t}q_1(t)\right)z}{\alpha_4(t)m_2}+C_3\right)+\tan\left(n_4y-\frac{n_4(\alpha_2(t)m_2{}^3+\alpha_1(t)S_1)z}{\alpha_4(t)m_2}+C_4\right)+C_5},\\
F_1 &= \mathrm{e}^{m_2x+n_1y-\frac{n_1\left(\alpha_2(t)m_2{}^3+\alpha_1(t)\frac{\mathrm{d}}{\mathrm{d}t}q_1(t)\right)z}{\alpha_4(t)m_2}+q_1(t)},\quad F_2=\mathrm{e}^{m_2x+n_2y-\frac{n_2\left(\alpha_2(t)m_2{}^3+\alpha_1(t)\frac{\mathrm{d}}{\mathrm{d}t}q_1(t)\right)z}{\alpha_4(t)m_2}+q_2(t)}.
\end{aligned}
\tag{3.24}
$$

3.1.8 Solutions VIII

$$
m_l=0,\ \ l=1,3,4,\ \ m_2=m_2,\ \ n_l=n_l,\ \ l=1,2,3,4,
$$

$$
\begin{aligned}
&q_2(t)=q_2(t),\quad q_i(t)=C_i,\quad i=1,3,4,5,\\
&p_i=-\frac{n_i\left(\alpha_2(t)\,{m_2}^3+\alpha_1(t)\,\frac{\mathrm{d}}{\mathrm{d}t}q_2(t)\right)z}{\alpha_4(t)\,m_2},\quad i=1,2,3,4,
\end{aligned}
\tag{3.25}
$$

in which $m_2, n_l, l = 1, 2, 3, 4$ parametric functions of t. The novel solution for the extended breather solution is given as follows

$$
\begin{aligned}
\Phi_8&=\frac{2\,\alpha_0 m_2 F_1}{F_2+F_1+\tanh\left(\omega_3(y,z,t)\right)+\tan\left(\omega_4(y,z,t)\right)+C_5},\\
\omega_i(y,z,t)&=n_i y-\frac{n_i\left(\alpha_2(t)\,{m_2}^3+\alpha_1(t)\,\frac{\mathrm{d}}{\mathrm{d}t}q_2(t)\right)z}{\alpha_4(t)\,m_2}+C_i,\quad i=3,4,\\
F_1&=\mathrm{e}^{m_2x+n_2y-\frac{n_2\left(\alpha_2(t)m_2{}^3+\alpha_1(t)\frac{\mathrm{d}}{\mathrm{d}t}q_2(t)\right)z}{\alpha_4(t)m_2}+q_2(t)},\\
F_2&=\mathrm{e}^{n_1y-\frac{n_1\left(\alpha_2(t)m_2{}^3+\alpha_1(t)\frac{\mathrm{d}}{\mathrm{d}t}q_2(t)\right)z}{\alpha_4(t)m_2}+C_1}.
\end{aligned}
\tag{3.26}
$$

Figs. 10, 11and 12, express the behavior of the obtained solution to Φ_8 are given with the following selected parameters

$$
\begin{aligned}
&m_2=2, n_1=1, n_2=3, n_3=2, n_4=3, C_i=1,\quad i=1,3,4,5,\quad q_2(t)=t,\quad \alpha_0=2,\\
&\alpha_1(t)=\frac{1}{1+t^2}, \alpha_2(t)=\frac{4}{1+t^2}, \alpha_4(t)=\frac{4t}{1+t^2}, x=y=1,
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
&m_2=2, n_1=1, n_2=3, n_3=2, n_4=3, C_i=1,\quad i=1,3,4,5,\quad q_2(t)=t,\quad \alpha_0=2,\\
&\alpha_1(t)=\cos(t), \alpha_2(t)=\sin(2t), \alpha_4(t)=2+\cos(2t), x=y=1,
\end{aligned}
\tag{3.28}
$$

$$
\begin{aligned}
&m_2=2, n_1=1, n_2=3, n_3=2, n_4=3, C_i=1,\quad i=1,3,4,5,\quad q_2(t)=t,\quad \alpha_0=2,\\
&\alpha_1(t)=\sinh(t), \alpha_2(t)=\sinh(2t), \alpha_4(t)=\cosh(2t), x=y=1,
\end{aligned}
\tag{3.29}
$$

in Eq. (3.26). By employing the above amounts the the reached property are studied in Figs. 10, 11 and 12 with plots along with various times.

3.2 Rational Breather Wave

As last part of this section, the rational breather solutions and rational breather wave solutions are investigated.

$$
\begin{aligned}
&h=\cosh(a_1)+\lambda\cos(a_2)+\cosh(a_3)+q_4(t),\\
&a_s=m_sx+n_sy+p_sz+q_s(t),\quad s=1,2,3,4.
\end{aligned}
\tag{3.30}
$$

Also, the amounts $m_s, n_s, p_s, q_s(t), (s = 1, 2, 3, 4)$ will be discovered. Inserting Eq. (3.30) into (2.14) get the coefficients powers of $\cosh(a_j), \cos(a_2), \cosh(a_3), j = 1, 2, 3, 4$ and $\sinh(a_j), \sin(a_2), \sinh(a_3), j = 1, 2, 3, 4$ to zero, conclude a system of equations to obtain the parameters $m_j, n_j, p_j, q_j(t), (j = 1, 2, 3, 4)$. Based on the (3.30) the solution $\Phi = 2\alpha_0(\ln h)_x$ can be appeared as follow:

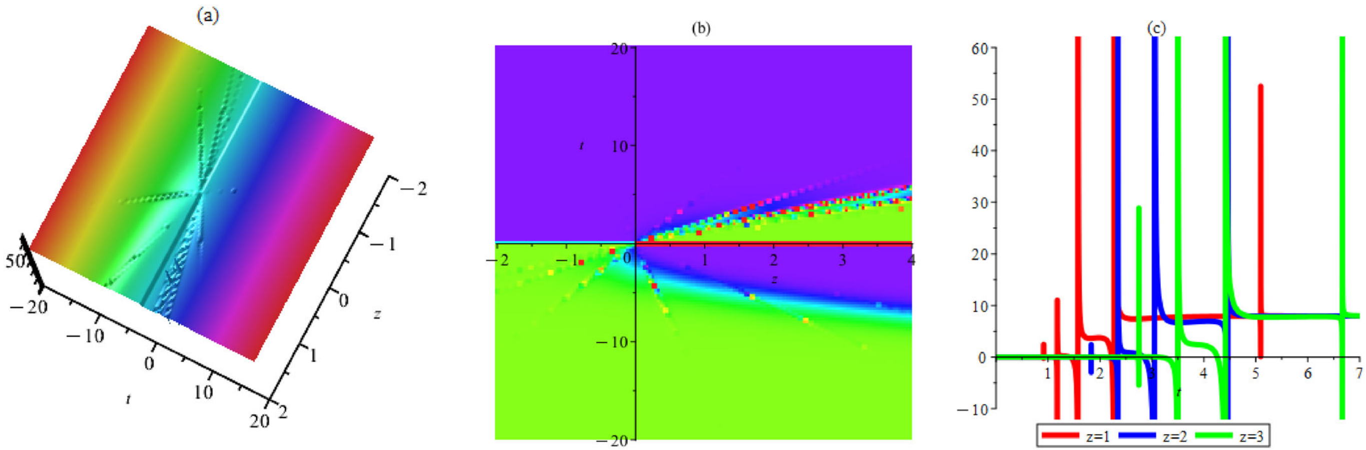

Fig. 10 Plots of the extended breather wave solutions to GSWW equation with distributed coefficients (3.27) (Σ_8) with $\alpha_1(t) = \frac{1}{1+t^2}$, $\alpha_2(t) = \frac{4}{1+t^2}$, $\alpha_4(t) = \frac{4t}{1+t^2}$; Density map in the (z, t)-plane; Along plane: $x = y = 1$

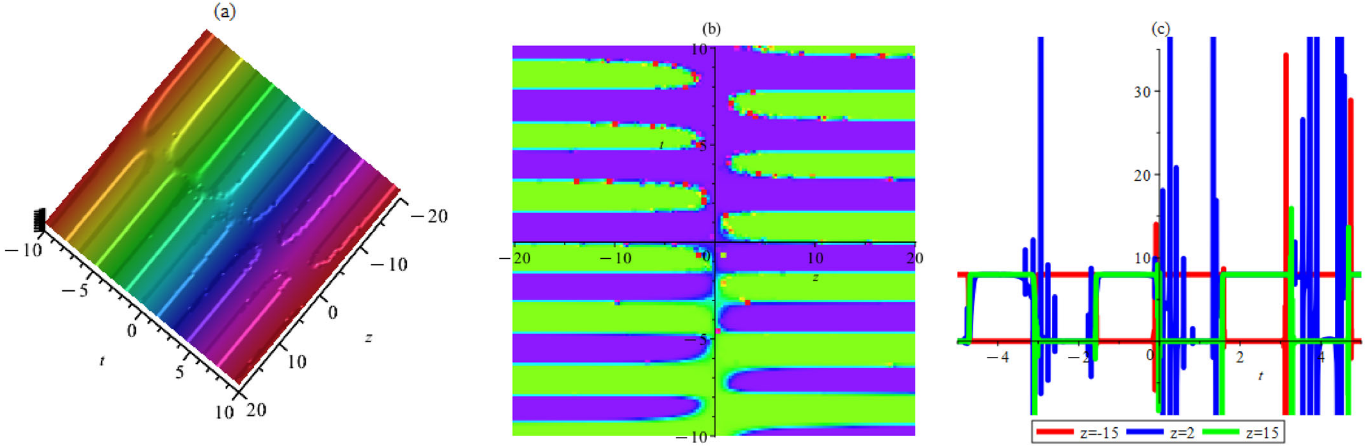

Fig. 11 Plots of the extended breather wave solutions to GSWW equation with distributed coefficients (3.28) (Σ_8) with $\alpha_1(t) = \cos(t)$, $\alpha_2(t) = \sin(2t)$, $\alpha_4(t) = 2 + \cos(2t)$; Density map in the (z, t)-plane; Along plane: $x = y = 1$

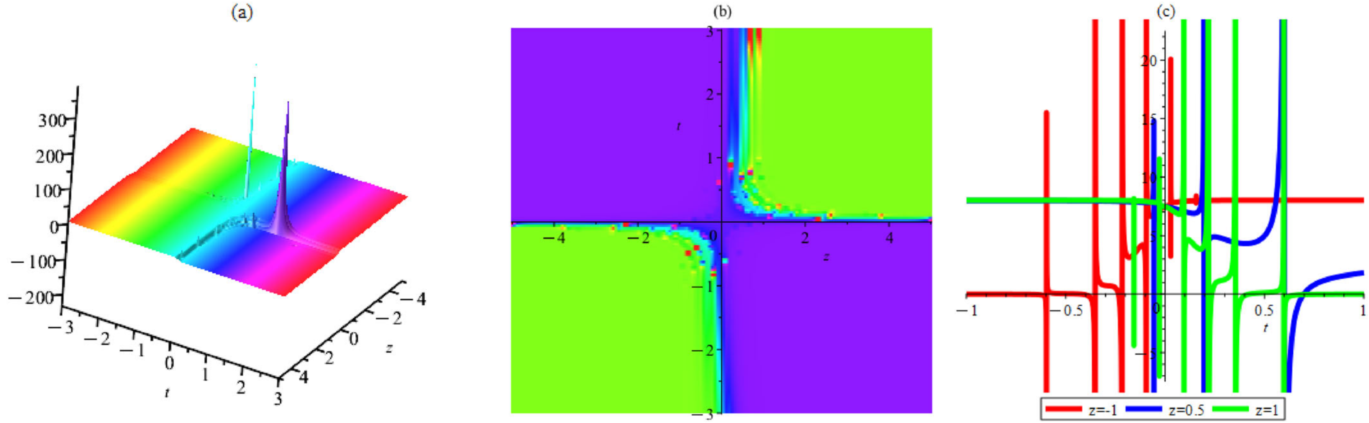

Fig. 12 Plots of the extended breather wave solutions to GSWW equation with distributed coefficients (3.29) (Σ_8) with $\alpha_1(t) = \sinh(t)$, $\alpha_2(t) = \sinh(2t)$, $\alpha_4(t) = \cosh(2t)$; Density map in the (z, t)-plane; Along plane: $x = y = 1$

$$\Phi = 2\,\frac{\begin{array}{r}\alpha_0\left(\sinh\left(m_1x+n_1y+p_1z+q_1\left(t\right)\right)m_1-\lambda\,\sin\left(m_2x+n_2y+p_2z\right.\right.\\ \left.\left.+q_2\left(t\right)\right)m_2+\sinh\left(m_3x+n_3y+p_3z+q_3\left(t\right)\right)m_3\right)\end{array}}{\begin{array}{r}\cosh\left(m_1x+n_1y+p_1z+q_1\left(t\right)\right)+\lambda\,\cos\left(m_2x+n_2y+p_2z+q_2\left(t\right)\right)\\ +\cosh\left(m_3x+n_3y+p_3z+q_3\left(t\right)\right)+q_4\left(t\right)\end{array}}.$$

3.2.1 First Solutions

$$\begin{aligned}&m_j = m_j,\quad j=1,2,3,\quad n_j=p_j=0,\quad j=1,2,3,\quad q_i(t)=q_i(t),\\ &i=1,2,3,4,\end{aligned} \tag{3.31}$$

in which $m_l, l = 1, 2$ are constants and $n_l(y), l = 1, 2, 3, 4$ parametric functions of y. The novel solution for the rational breather solution is given as follows

$$\Phi_1 = 2\,\frac{\alpha_0\left(\sinh\left(m_1x+q_1\left(t\right)\right)m_1-\lambda\,\sin\left(m_2x+q_2\left(t\right)\right)m_2+\sinh\left(m_3x+q_3\left(t\right)\right)m_3\right)}{\cosh\left(m_1x+q_1\left(t\right)\right)+\lambda\,\cos\left(m_2x+q_2\left(t\right)\right)+\cosh\left(m_3x+q_3\left(t\right)\right)+q_4\left(t\right)}. \tag{3.32}$$

Figs. 13 and 14, express the behavior of interaction between a periodic wave and homoclinic wave solution with two wave solution to solution Φ_1 are inserted with the following parameters

$$\begin{aligned}&\lambda=2, m_1=1, m_2=2, m_3=3, \alpha_0=2, q_1(t)=t,\\ &q_2(t)=t^2+2t, q_3(t)=2t, q_4(t)=t^2,\end{aligned} \tag{3.33}$$

$$\begin{aligned}&\lambda=2, m_1=1, m_2=2, m_3=3, \alpha_0=2, q_1(t)=\cos(t),\\ &q_2(t)=\cos(2t), q_3(t)=\sin(2t), q_4(t)=\sin(2t).\end{aligned} \tag{3.34}$$

Some properties of interactions are presented in Figs. 13 and 14 with 3D, 2D, and density plots with different spaces.

3.2.2 Second Solutions

$$\begin{aligned}&m_j=m_j,\quad j=1,2,\quad m_3=n_3=p_3=0,\quad n_1=-\frac{m_1n_2}{m_2},\\ &q_1\left(t\right)=\int\frac{m_1\left(\alpha_2\left(t\right)m_1{}^2n_2+3\,\alpha_2\left(t\right)m_2{}^2n_2-2\,\alpha_4\left(t\right)p_2\right)}{2n_2\alpha_1\left(t\right)}\,\mathrm{d}t,\\ &q_2\left(t\right)=\int\frac{m_2\left(\alpha_2\left(t\right)m_2{}^2n_2-\alpha_4\left(t\right)p_2\right)}{n_2\alpha_1\left(t\right)}\,\mathrm{d}t,\quad q_3\left(t\right)=C_3,\\ &q_4\left(t\right)=C_4,\end{aligned} \tag{3.35}$$

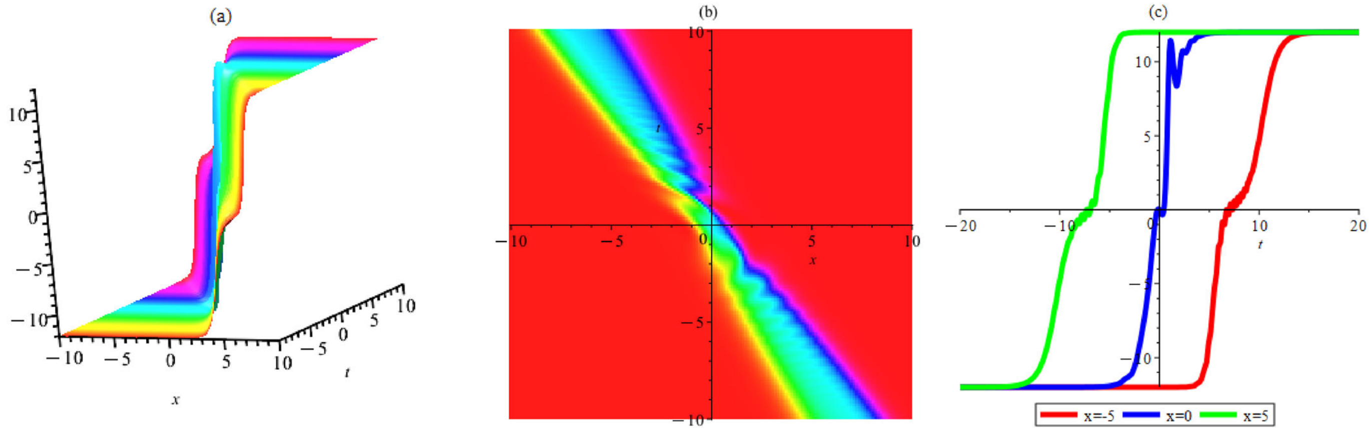

Fig. 13 Plots of the rational breather wave solutions to GSWW equation with distributed coefficients (3.33) (Σ_1) with $q_1(t) = t, q_2(t) = t^2 + 2t, q_3(t) = 2t, q_4(t) = t^2$; Density map in the (x, t)-plane; Along plane: arbitrary (z, y)

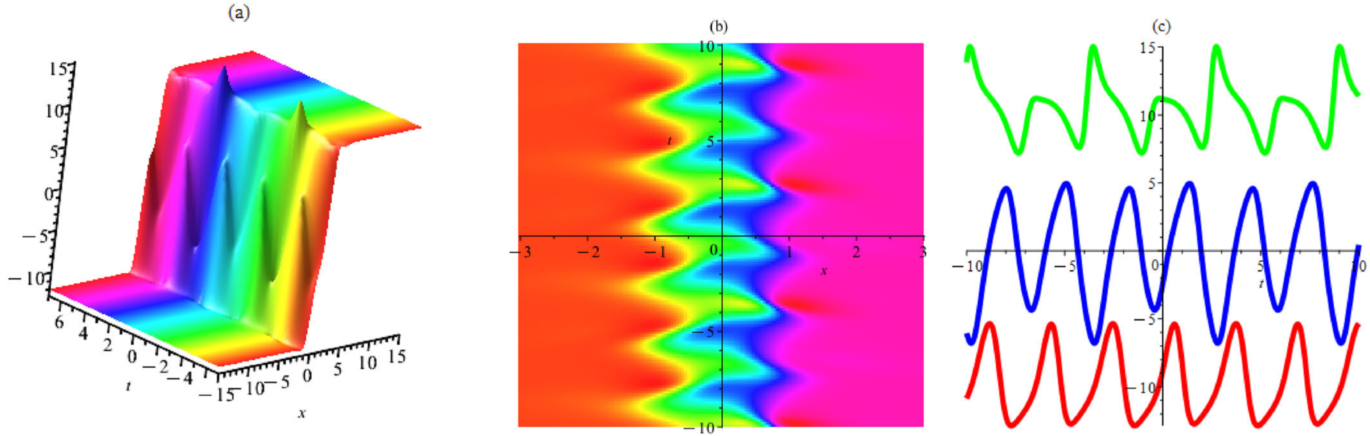

Fig. 14 Plots of the rational breather wave solutions to GSWW equation with distributed coefficients (3.34) (Σ_1) with $q_1(t) = \cos(t), q_2(t) = \cos(2t), q_3(t) = \sin(2t), q_4(t) = \sin(2t)$; Density map in the (x, t)-plane; Along plane: arbitrary (z, y)

where m_l, m_2 and n_2 are constants. The novel solution for the rational breather solution is given as follows

$$\Phi_2 = \frac{2\,\alpha_0\left(\sinh\left(F_1\right)m_1 - \frac{{m_1}^2}{m_2}\sin\left(m_2x + n_2y + p_2z + \int \frac{m_2\left(\alpha_2(t){m_2}^2n_2 - \alpha_4(t)p_2\right)}{n_2\alpha_1(t)}\,\mathrm{d}t\right)\right)}{\cosh\left(F_1\right) + \frac{{m_1}^2}{{m_2}^2}\cos\left(m_2x + n_2y + p_2z + \int \frac{m_2\left(\alpha_2(t){m_2}^2n_2 - \alpha_4(t)p_2\right)}{n_2\alpha_1(t)}\,\mathrm{d}t\right) + \cosh\left(C_3\right) + C_4},$$

$$\begin{aligned} F_1 &= m_1x - \frac{m_1n_2y}{m_2} + 1/2\,\frac{m_1\left(3\,\alpha_2\left(t\right)m_1{}^2n_2 + 3\,\alpha_2\left(t\right)m_2{}^2n_2 - 2\,\alpha_4\left(t\right)p_2\right)z}{m_2\alpha_4\left(t\right)} \\ &\quad + \int 1/2\,\frac{m_1\left(\alpha_2\left(t\right)m_1{}^2n_2 + 3\,\alpha_2\left(t\right)m_2{}^2n_2 - 2\,\alpha_4\left(t\right)p_2\right)}{n_2\alpha_1\left(t\right)}\,\mathrm{d}t. \end{aligned} \tag{3.36}$$

Figs. 15and 16, express the behavior of interaction solutions with two wave solution to Φ_2 are offered by the following parameters

$$\begin{aligned} &m_1 = 1, \quad m_2 = 2, \quad n_2 = 3, \quad p_2 = 4, \quad \alpha_0 = 2, \quad \alpha_1(t) = 1 + t^2, \quad \alpha_2(t) = t, \\ &\alpha_4(t) = 1 + t^2, \quad C_3 = 1, \quad C_4 = 1, \quad z = y = 1, \end{aligned} \tag{3.37}$$

$$\begin{aligned} &m_1 = 1, \quad m_2 = 2, \quad n_2 = 3, \quad p_2 = 4, \quad \alpha_0 = 2, \quad \alpha_1(t) = \exp(t), \\ &\alpha_2(t) = \exp(t), \quad \alpha_4(t) = \exp(2t), \quad C_3 = 1, \quad C_4 = 1, \quad z = y = 1. \end{aligned} \tag{3.38}$$

By considering the property interactions solution as presented in Figs. 15 and 16 with 3D, 2D, and density plots with various spaces.

3.2.3 Third Solutions

$$\begin{aligned} &m_s = m_s, \quad s = 1, 2, \quad m_3 = m_3, \quad n_3 = p_3 = 0, \quad p_1 = \frac{n_1p_2}{n_2}, \quad q_1\left(t\right) = C_1, \\ &q_2\left(t\right) = C_2, \\ &q_3\left(t\right) = \int -\frac{m_3\left(\alpha_2\left(t\right)m_3{}^2n_2 + \alpha_4\left(t\right)p_2\right)}{n_2\alpha_1\left(t\right)}\,\mathrm{d}t, \quad q_4\left(t\right) = C_4, \end{aligned} \tag{3.39}$$

where m_l, m_2 and n_2 free parameters. The novel solution for the rational breather solution is given as follows

$$\Phi_3 = \frac{2\,\alpha_0\sinh\left(m_3x + \int -\frac{m_3\left(\alpha_2(t){m_3}^2n_2 + \alpha_4(t)p_2\right)}{n_2\alpha_1(t)}\,\mathrm{d}t\right)m_3}{\cosh\left(n_1y + \frac{n_1p_2z}{n_2} + C_1\right) + \lambda\cos\left(n_2y + p_2z + C_2\right) + \cosh\left(m_3x + \int -\frac{m_3\left(\alpha_2(t){m_3}^2n_2 + \alpha_4(t)p_2\right)}{n_2\alpha_1(t)}\,\mathrm{d}t\right) + C_4}. \tag{3.40}$$

Figs. 17and 18, show the analysis of treatment of interaction between a periodic wave and homoclinic wave solution with two wave solution where graphs of Φ_3 are given with the below selected parameters

$$\lambda = 1, \quad n_1 = 1, \quad n_2 = 2, \quad m_3 = 3, \quad p_2 = 4, \quad \alpha_0 = 2, \quad \alpha_1(t) = 1 + t^2,$$

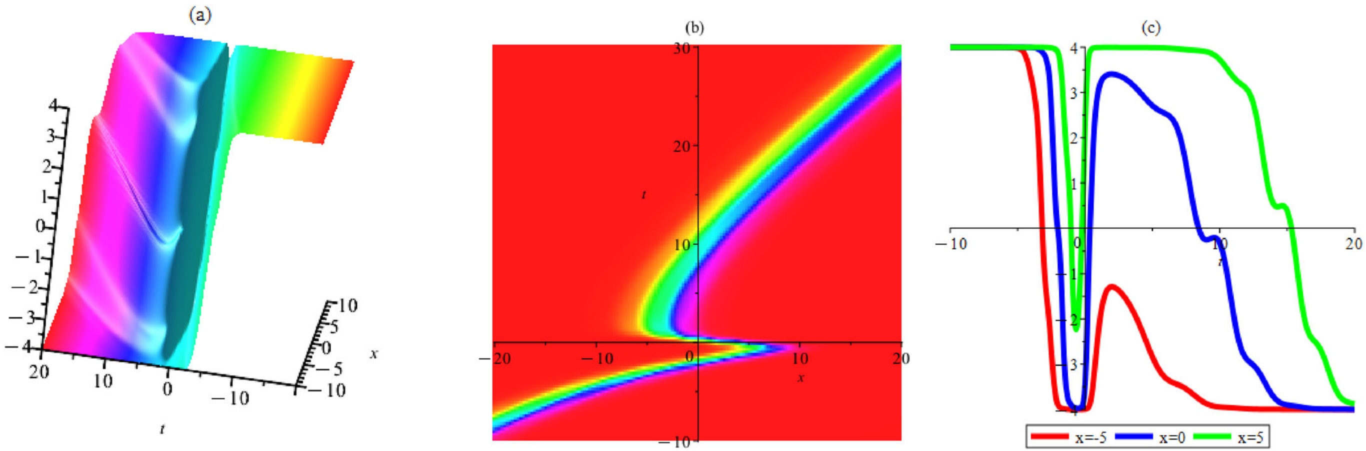

Fig. 15 Plots of the rational breather wave solutions to GSWW equation with distributed coefficients (3.37) (Σ_2) with $\alpha_0 = 2,\ \alpha_1(t) = 1 + t^2, \alpha_2(t) = t, \alpha_4(t) = 1 + t^2$; Density map in the (x, t)-plane; Along plane: $z = y = 1$

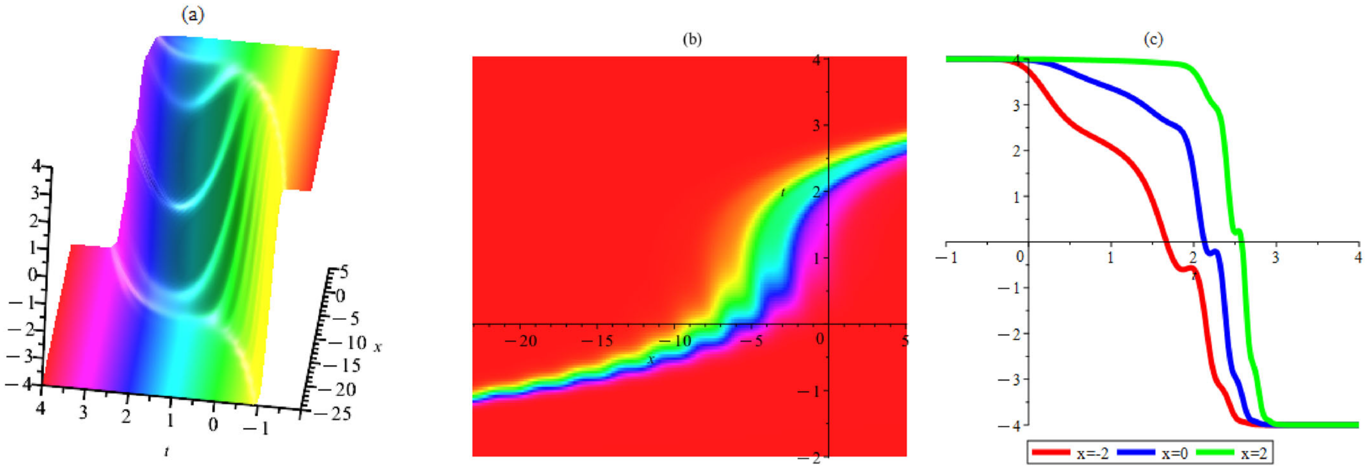

Fig. 16 Plots of the rational breather wave solutions to GSWW equation with distributed coefficients (3.38) (Σ_2) with $\alpha_0 = 2,\ \alpha_1(t) = \exp(t), \alpha_2(t) = \exp(t), \alpha_4(t) = \exp(2t)$; Density map in the (x, t)-plane; Along plane: $z = y = 1$

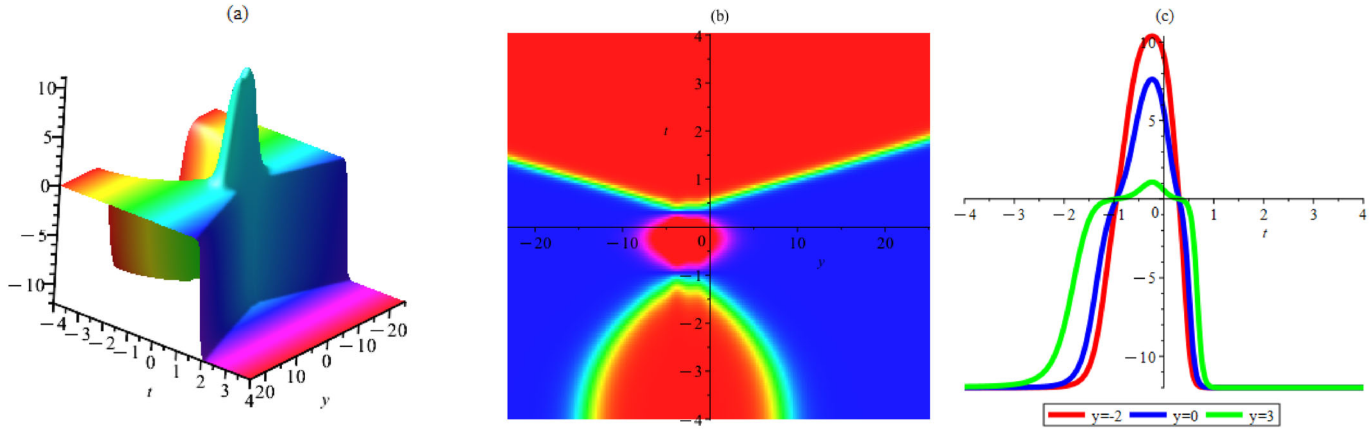

Fig. 17 Plots of the rational breather wave solutions to GSWW equation with distributed coefficients (3.41) (Σ_3) with $\alpha_0 = 2$, $\alpha_1(t) = 1 + t^2, \alpha_2(t) = t, \alpha_4(t) = 1 + t^2$; Density map in the (y, t)-plane; Along plane: $x = y = 1$

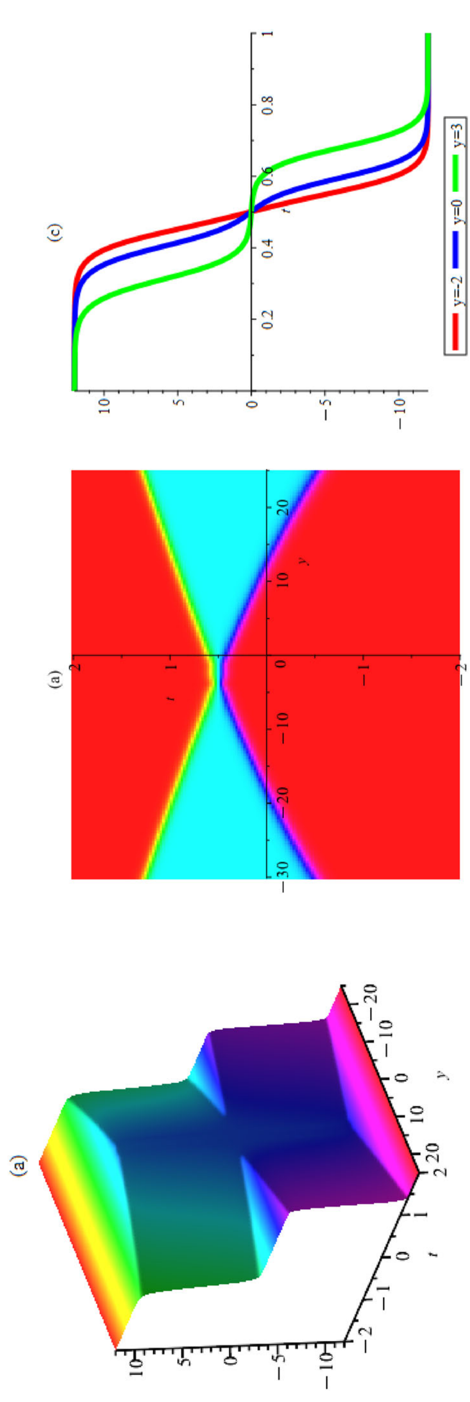

Fig. 18 Plots of the rational breather wave solutions to GSWW equation with distributed coefficients (3.42) (Σ_3) with $\alpha_0 = 2$, $\alpha_1(t) = \cos(t)$, $\alpha_2(t) = \cos(t)$, $\alpha_4(t) = \sin(2t)$; Density map in the (y, t)-plane; Along plane: $x = y = 1$

$$\alpha_2(t) = t, \quad \alpha_4(t) = 1 + t^2, \quad C_1 = 1, \quad C_2 = 1, \quad C_4 = 1, \quad x = y = 1, \tag{3.41}$$

$$\lambda = 1, \quad n_1 = 1, \quad n_2 = 2, \quad m_3 = 3, \quad p_2 = 4, \quad \alpha_0 = 2, \quad \alpha_1(t) = \cos(t),$$
$$\alpha_2(t) = \cos(t), \quad \alpha_4(t) = \sin(2t), \quad C_1 = C_2 = C_4 = 1, \quad x = y = 1, \tag{3.42}$$

in Eq. (3.40). By employing the above values the characteristic interaction between a periodic wave and homoclinic wave solution are expressed in Figs. 17 and 18 with 3D, 2D, and density plots with different spaces. It offers a type of interaction solutions between hyperbolic waves and trigonometric wave.

4 Results and Discussion

In this section, the obtained solutions of the GSWW equation in (3+1)-dimensional with variable coefficients are illustrated by numerical simulation and physical interpretation. In order to describe the outcomes of the proposed technique, graphs of the surface soliton solutions to the governing equation have been displayed. The graphs in Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17 and 18 have been plotted for various values of the parameter $\alpha_i, i = 1..4$ and the other parameters have fixed values. Thus, we have observed and discussed the effects of each parameter $\alpha_i, i = 1..4$ on the stability of the model for the matter dominated era. Perhaps, the used value of one parameter by fixing the values of the other parameters is able to make our model stable. However, by taking suitable values of all the parameters as has been discussed above, the existence of the model can be achieved easily for the matter dominated era.

5 Conclusion

In this work, mainly the (2+1)-dimensional GSWW equation with variable coefficients was investigated. Also, the Hirota bilinear forms, extended breather solutions and rational breather wave solutions respectively were derived. This work was in order to obtain the exact solutions of the GSWW equation, the breather wave solutions and extended breather-wave solutions were studied. Using the Hirota bilinear method, two distinct types of solutions such as combination of exponential and hyperbolic, and also combination of trigonometric, and hyperbolic function solutions were achieved with unknown parameters. By demonstrating that the bilinear approach was highly efficient and suited for discovering the exact soliton solutions of nonlinear models. There were many new types of breather wave solutions that were discovered including the plenty of solutions as breather and extended breather wave solutions. Many other nonlinear evolution equations including coupled ones, were solved using this technique. From the results, we understood the Hirota direct method was really prompt and effective to obtain the exact solutions for some NLPDEs. Along with the scientific derivation for the analytical findings, the outcomes were graphically displayed to help identifying the dynamical aspects of solutions. Theoretical insights reported here may be helpful to future experimental studies. This proposed approach was extended to solve a wide variety of numerous nonlinear evolution equations that arise in mathematical physics since it is noteworthy and efficient. By Maple was shown in which the obtained results were correct. In the future, this research might be expanded to include the stochastic GSWW equation with variable coefficients.

Author Contributions The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Funding This work was not supported by any specific funding.

Data Availability Please contact author for data requests.

Declaration

Conflict of interest Authors declare they have no conflict of interest regarding the publication of the article.

References

1. Randall, D. A.: The Shallow Water Equations. Technical Report, Atmospheric Science, Colorado State University, Fort Collins (2006)
2. Shinbrot, M.: The shallow water equations. J. Eng. Math. **4**(4), 293–304 (1970)
3. Zhou, B., Ding, K., Wang, J., Wang, L., Jin, P., Tang, T.: Experimental study on the interactions between wave groups in double-wave-group focusing. Phys. Fluids **35**(3), 37118 (2023)
4. Feng, Q., Feng, Z., Su, X., Bai, Y.D., Ding, B.Y.: Design and simulation of human resource allocation model based on double-cycle neural network. Comput. Intell. Neurosci. **2021**, 1–10 (2021)
5. Jiang, S., Zhao, C., Zhu, Y., Wang, C., Du, Y., Lei, W., Wang, L.: A practical and economical ultra-wideband base station placement approach for indoor autonomous driving systems. J. Adv. Trans. **2022**, 1–12 (2022)
6. Zhang, X., Wang, Y., Yang, M., Geng, G.: Toward concurrent video multicast orchestration for caching-assisted mobile networks. IEEE Trans. Veh. Technol. **70**(12), 13205–13220 (2021)
7. Ye, R., Liu, P., Shi, K., Yan, B.: State damping control: a novel simple method of rotor UAV with high performance. IEEE Access **8**, 214346–214357 (2020)
8. Alsufi, N.A., Fatima, N., Noor, A., Gorji, M.R., Alam, M.M.: Lumps and interactions, fission and fusion phenomena in multi solitons of extended shallow water wave equation of (2+1)-dimensions. Chaos Solitons Fract. **170**, 113410 (2023)
9. Zhang, Q., Yan, T., Gao, G.H.: The energy method for high-order invariants in shallow water wave equations. Appl. Math. Lett. **142**, 108626 (2023)
10. Yu, S., Huang, L.: Exact solutions of the generalized (2+1)-dimensional shallow water wave equation. Results Phys. **42**, 106020 (2022)
11. Li, R., Ilhan, O.A., Manafian, J., Mahmoud, K.H., Abotaleb, M., Kadi, A.: A mathematical study of the (3+1)-D variable coefficients generalized shallow water wave equation with its application in the interaction between the lump and soliton solutions. Mathematics **10**(17), 3074 (2022)
12. Gu, Y., Zia, S.M., Isam, M., Manafian, J., Hajar, A., Abotaleb, M.: Bilinear method and semi-inverse variational principle approach to the generalized (2+1)-dimensional shallow water wave equation. Results Phys. **45**, 106213 (2023)
13. Ilhan, O.A., Manafian, J., Alizadeh, A., Mohammed, S.A.: M lump and interaction between M lump and N stripe for the third-order evolution equation arising in the shallow water. Adv. Differ. Equ. **2020**, 207 (2020)
14. Zhou, B., Hu, J., Jin, P., Sun, K., Li, Y., Ning, D.: Power performance and motion response of a floating wind platform and multiple heaving wave energy converters hybrid system. Energy **265**, 126314 (2023)
15. Zhou, B., Zheng, Z., Zhang, Q., Jin, P., Wang, L., Ning, D.: Wave attenuation and amplification by an abreast pair of floating parabolic breakwaters. Energy **271**, 127077 (2023)
16. Hu, Q., Han, E., Wang, W., Wang, X., Li, Z.: The Interactive Relationship Between Water Environmental Pollution and the Quality of Economic Growth in China: An Empirical Research Based on the Dynamic Simultaneous Equations Model. J. Comput. Methods Sci. Eng. **1**, 77–87 (2022)

17. Li, R., Bu Sinnah, Z.A., Shatouri, Z.M., Manafian, J., Aghdaei, M.F., Kadi, A.: Different forms of optical soliton solutions to the Kudryashov's quintuple self-phase modulation with dual-form of generalized nonlocal nonlinearity. Results Phys. **46**, 106293 (2023)
18. Nourani, V., Rouzegari, N., Molajou, A., Baghanam, A.H.: An integrated simulation–optimization framework to optimize the reservoir operation adapted to climate change scenarios. J. Hydrol. **587**, 125018 (2020)
19. Guo, B., Dong, H., Fang, Y.: Lump solutions and interaction solutions for the dimensionally reduced nonlinear evolution equation. Complexity **2019**, 5765061 (2019)
20. Zhen, L.: Research on mathematical teaching innovation strategy and best practices based on deep learning algorithm. J. Commer. Biotech. **27**(3) (2022)
21. Chen, N., Li, H., Li, Y.: Evaluation of binary offset carrier signal capture algorithm for development of the digital health literacy instrument. J. Commer. Biotech. **27**(2) (2022)
22. Della Volpe, C., Siboni, S.: From van der Waals equation to acid-base theory of surfaces: chemical-mathematical journey. Math. J. Rev. Adhesion Adhesives **10**(1), 47–97 (2022)
23. Rikani, A.S.: Numerical analysis of free heat transfer properties of flat panel solar collectors with different geometries. J. Res. Sci. Eng. Technol. **9**(01), 95–116 (2021)
24. Cai, W., Mohammaditab, R., Fathi, G., Wakil, K., Ebadi, A.G., Ghadimi, N.: Optimal bidding and offering strategies of compressed air energy storage: a hybrid robust-stochastic approach. Renew. Energy **143**, 1–8 (2019)
25. Han, E., Ghadimi, N.: Model identification of proton-exchange membrane fuel cells based on a hybrid convolutional neural network and extreme learning machine optimized by improved honey badger algorithm. Sustain. Energy Tech. Assess **52**, 102005 (2022)
26. Zhang, J., Khayatnezhad, M., Ghadimi, N.: Optimal model evaluation of the proton exchange membrane fuel cells based on deep learning and modified African vulture optimization algorithm. Energy Sources A **44**, 287–305 (2022)
27. Chen, L., Huang, H., Tang, P., Yao, D., Yang, H., Ghadimi, N.: Optimal modeling of combined cooling, heating, and power systems using developed African vulture optimization: a case study in watersport complex. Energy Sources A **44**, 4296–4317 (2022)
28. Jiang, W., Wang, X., Huang, H., Zhang, D., Ghadimi, N.: Optimal economic scheduling of microgrids considering renewable energy sources based on energy hub model using demand response and improved water wave optimization algorithm. J Energy Storage **55**, 105311 (2022)
29. Mehrpooya, M., Ghadimi, N., Marefati, M., Ghorbanian, S.A.: Numerical investigation of a new combined energy system includes parabolic dish solar collector, stirling engine and thermoelectric device. Int. J. Energy Res. **45**(11), 16436–16455 (2021)
30. Erfeng, H., Ghadimi, N.: Model identification of proton-exchange membrane fuel cells based on a hybrid convolutional neural network and extreme learning machine optimized by improved honey badger algorithm. Sustain. Energy Tech. Asses **52**, 102005 (2022)
31. Yu, D., Zhang, T., He, G., Nojavan, S., Jermsittiparsert, K., Ghadimi, N.: Energy management of wind-PV-storage-grid based large electricity consumer using robust optimization technique. J. Energy Storage **27**, 101054 (2020)
32. Saeedi, M., Moradi, M., Hosseini, M., Emamifar, A., Ghadimi, N.: Robust optimization based optimal chiller loading under cooling demand uncertainty. Appl. Therm. Eng. **148**, 1081–1091 (2019)
33. Mir, M., Shafieezadeh, M., Heidari, M.A., Ghadimi, N.: Application of hybrid forecast engine based intelligent algorithm and feature selection for wind signal prediction. Evol. Syst. **11**(4), 559–573 (2020)
34. Liu, J.G., Zhu, W.H.: Breather wave solutions for the generalized shallow water wave equation with variable coefficients in the atmosphere, rivers, lakes and oceans. Comput. Math. Appl. **78**, 848–856 (2019)
35. Liu, J.G., Zhu, W.H., He, Y., Lei, Z.Q.: Characteristics of lump solutions to a (3+1)-dimensional variable-coefficient generalized shallow water wave equation in oceanography and atmospheric science. Eur. Phys. J. Plus **134**, 385 (2019)
36. Zayeda, E.M.E., Gepreel, K.A.: The (G'/G)-expansion method for finding traveling wave solutions of nonlinear partial differential equations in mathematical physics. J. Math. Phys. **50**, 013502 (2009)
37. Darvishi, M.T., Najafi, M.: Some new exact solutions of the (3+1)-dimensional breaking soliton equation by the Exp-function method. Int. J. Comput. Math. Sci. **1**, 25C27 (2012)
38. Chen, Y., Liu, R.: Some new nonlinear wave solutions for two (3+1)-dimensional equations. Appl. Math. Comput. **260**, 397–411 (2015)

39. Huang, Q.M., Gao, Y.T., Jia, S.L., Wang, Y.L., Deng, G.F.: Bilinear backlund transformation, soliton and periodic wave solutions for a (3+1)-dimensional variable-coefficient generalized shallow water wave equation. Nonlinear Dyn. **87**(4), 2529–2540 (2017)
40. Huang, Q.M., Gao, Y.T.: Bilinear backlund transformation and dynamic features of the soliton solutions for a variable-coefficient (3+1)-dimensional generalized shallow water wave equation. Mod. Phys. Lett. B **31**(22), 1750126 (2017)
41. Duran, S.: An investigation of the physical dynamics of a traveling wave solution called a bright soliton. Phys. Scr. **96**, 125251 (2021)
42. Zhang, Z., Guo, Q., Li, B., Chen, J.C.: A new class of nonlinear superposition between lump waves and other waves for Kadomtsev–Petviashvili I equation. Commun. Nonlinear Sci. Numer. Simul. **101**, 105866 (2021)
43. Moghadam, R.A., Ebrahimi, S.: Design and analysis of a torsional mode MEMS disk resonator for RF applications. J. Multidiscip. Eng. Sci. Technol. **8**(7), 14300–14303 (2021)
44. Fu, H.H., Guo, Y., Shen, Z., Zhao, J., Xie, Y., Ling, Y., Ouyang, S., Li, S., Zhang, W.: Tuning the shell thickness of core-shell alpha-Fe_2O_3@SiO_2 nanoparticles to promote microwave absorption. Chin. Chem. Lett. **33**(2), 957–962 (2022)
45. Bai, J.L., Wang, X., Zhu, Y., Yuan, G., Wu, S., Qin, F., Yu, X., Ren, L.: Polymer types regulation strategy toward the synthesis of carbonized polymer dots with excitation-wavelength dependent or independent fluorescence. Chin. Chem. Lett. **34**(2) (2022)
46. Ma, W.X.: Generalized bilinear differential equations. Stud. Nonlinear Sci. **2**, 140–144 (2011)
47. Zhang, R.F., Bilige, S.: Bilinear neural network method to obtain the exact analytical solutions of nonlinear partial differential equations and its application to p-gBKP equation. Nonlinear Dyn. **95**, 3041–3048 (2019)
48. Zhang, R.F., Li, M.C., Gan, J,Y., Qing, Li., Lan, Z.Z.: Novel trial functions and rogue waves of generalized breaking soliton equation via bilinear neural network method. Chaos Solitons Fract. **154**, 111692 (2022)
49. Zhang, R.F., Li, M.C., Albishari, M., Zheng, F.C., Lan, Z.Z.: Generalized lump solutions, classical lump solutions and rogue waves of the (2+1)-dimensional Caudrey–Dodd–Gibbon–Kotera–Sawada-like equation. Appl. Math. Comput. **403**, 126201 (2021)
50. Younas, T., Sulaiman, T.A., Ren, J.: Propagation of M-truncated optical pulses in nonlinear optics. Opt. Quan. Electron. **55**, 102 (2023)
51. Younas, T., Sulaiman, T.A., Ren, J.: On the study of optical soliton solutions to the three-component coupled nonlinear Schrödinger equation: applications in fiber optics. Opt. Quan. Electron. **55**, 72 (2023)
52. Younas, T., Ren, J., Akinyemi, L., Rezazadeh, H.: On the multiple explicit exact solutions to the double-chain DNA dynamical system. Math. Methods Appl. Sci. **46**, 6309–6323 (2023)
53. Younas, T., Ren, J.: Construction of optical pulses and other solutions to optical fibers in absence of self-phase modulation. Int. J. Modern Phys. B **36**(32), 2250239 (2022)
54. Bashar, Md.H., Inc, M., Islam, S.M.R., Mahmoud, K.H., Akbar, M.A.: Soliton solutions and fractional effects to the time-fractional modified equal width equation. Alex. Eng. J. **61**, 12539–12547 (2022)
55. Islam, S.M.R., Bashar, Md.H., Arafat, S.M.Y., Wang, H., Roshid, Md.M.: Effect of the free parameters on the Biswas–Arshed model with a unified technique. Chin. J. Phys. **77**, 2501–2519 (2022)
56. Duran, S.: Exact solutions for time-fractional Ramani and Jimbo–Miwa equations by direct algebraic method. Adv. Sci. Eng. Med. **12**, 982–988 (2020)
57. Duran, S., Kaya, D.: Breaking analysis of solitary waves for the shallow water wave system in fluid dynamics. Eur. Phys. J. Plus **136**, 980 (2021)
58. Yokus, A., Isah, M.A.: Stability analysis and solutions of (2+1)-Kadomtsev–Petviashvili equation by homoclinic technique based on Hirota bilinear form. Nonlinear Dyn. **109**, 3029–3040 (2022)
59. Safi Ullah, M., Abdeljabbar, A., Roshid, H.O., Ali, M.Z.: Application of the unified method to solve the Biswas–Arshed model. Results Phys. **42**, 105946 (2022)
60. Safi Ullah, M., Ali, M. Z., Roshid, H. O., Hoque, Md. F.: Collision phenomena among lump, periodic and stripe soliton solutions to a (2+1)-dimensional Benjamin–Bona–Mahony–Burgers Model. Eur. Phys. J. Plus **136**, 370 (2021)
61. Safi Ullah, M., Ahmed, O., Mahbub, Md. A.: Collision phenomena between lump and kink wave solutions to a (3+1)-dimensional Jimbo–Miwa-like model. Partial Differ. Equ. Appl. Math. **5**, 100324 (2022)

62. Safi Ullah, M., Roshid, H.O., Ali, M.Z., Noor, N.F.M.: Novel dynamics of wave solutions for Cahn–Allen and diffusive predator–prey models using MSE scheme. Partial Differ. Equ. Appl. Math. **3**, 100017 (2021)
63. Safi Ullah, M., Seadawy, A.R., Ali, M.Z., Roshid, H.O.: Optical soliton solutions to the Fokas–Lenells model applying the φ^6-model expansion approach. Opt. Quan. Electron. **55**, 495 (2023)
64. Seadawy, A.R., Iqbal, M., Lu, D.: Propagation of kink and anti-kink wave solitons for the nonlinear damped modified Korteweg–de Vries equation arising in ion-acoustic wave in an unmagnetized collisional dusty plasma. Phys. A: Stat. Mech. Appl. **544**, 123560 (2020)
65. Seadawy, A.R., Iqbal, M., Lu, D.: Nonlinear wave solutions of the Kudryashov–Sinelshchikov dynamical equation in mixtures liquid-gas bubbles under the consideration of heat transfer and viscosity. J. Taibah Univ. Sci. **13**, 1060–1072 (2019)
66. Iqbal, M., Seadawy, A.R., Lu, D.: Applications of nonlinear longitudinal wave equation in a magneto-electro-elastic circular rod and new solitary wave solutions. Mod. Phys. Let. B **33**(18), 1950210 (2019)
67. Seadawy, A.R., Iqbal, M., Lu, D.: Applications of propagation of long-wave with dissipation and dispersion in nonlinear media via solitary wave solutions of generalized Kadomtsev-Petviashvili modified equal width dynamical equation. Comput. Math. Appl. **78**, 3620–3632 (2019)
68. Ma, W.X.: Bilinear equations, Bell polynomials and linear superposition principle. J. Phys. Conf. Ser. **411**, 012021 (2013)