Qualitative Theory of Dynamical Systems (2023) 22:84
https://doi.org/10.1007/s12346-023-00791-2

A Note on Topological Average Shadowing Property Via Uniformity

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Received: 20 August 2022 / Accepted: 6 April 2023 / Published online: 15 April 2023

Abstract

We study topological ergodic shadowing, topological $\underline{d}$ shadowing and topological average shadowing property for a continuous map on a uniform space and show that they are equivalent for a uniformly continuous map with topological shadowing on a compact uniform space. Furthermore, we prove that topological average shadowing property with Lyapunov stability implies topological ergodicity.

Keywords Topological average shadowing · Topologically chain mixing · Topologically ergodic · Topological ergodic shadowing · Lyapunov stability · Uniform space · p-adic numbers

Mathematics Subject Classification 54H20

1 Introduction

A discrete dynamical system usually consists of a compact metric space X and a continuous map $f : X \to X$. Many properties of interest in such systems are defined in purely topological terms and others are defined in terms of the metric. The theory of shadowing provides tools for finding real orbits nearby to pseudo-orbits. The motivation comes from computer simulations, where we always have a numerical error when calculating an orbit, no matter how small, but at the same time, we want to be sure that what we see on the computer's screen is a good approximation of the genuine orbit of the system.

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Depending on the methods one uses for approximation, there are various kinds of shadowing properties [18]. In [10] the authors introduced the concept of sub-shadowing property formulated in terms of “large” subsets of $\mathbb{N}$, to answer the following question: Given an ergodic pseudo-orbit, when can we find a real orbit which is close to it part of the time? The main idea of the definition of ergodic pseudo-orbit is to use the concepts of *lower density* and *upper density* for elements of Furstenburg families to measure the set of error times in a pseudo-orbit. Ahmadi et al. [3] introduced the notion of exponential shadowing property where one-step errors tend to zero with an exponential rate and showed that it is a persistent property near a hyperbolic set of a dynamical system on a compact manifold. While computing approximate trajectories, it makes sense to consider errors which are small on average, since controlling them in each iteration may be impossible. The notion of average pseudo-orbit introduced by Blank [7].

On the other hand, some dynamical systems may have a dynamical property with respect to one metric, but not with respect to another metric that induces the same topology. For example, a dynamical system on a non-compact metric space may be expansive or have the shadowing property with respect to one metric, but not with respect to another metric that induces the same topology (for example, in [9] authors introduced a dynamical system on $\mathbb{R}^2$ which is expansive with respect to some metric and is not with respect to another metric. Moreover, both metrics induce the same topology on $\mathbb{R}^2$. In [17] authors introduced a dynamical system on $\mathbb{R}$ which is expansive and has the shadowing property in classical sense with respect to a metric. However it is not expansive and does not have the shadowing property in classical sense with respect to the another metric). Many dynamical properties can be defined inherently in Hausdorff spaces. There are two ways to do this, either in terms of finite open covers or in terms of uniformities (In compact metric spaces they coincide exactly with the standard definitions [12]). More precisely, let X be a compact Hausdorff space and $f : X \to X$ be a continuous map. Then f has (Hausdorff) shadowing property if for every finite open cover α of X, there exists a finite open cover β of X such that for any sequence $\{U_n\} \subset \beta$ with $f(U_j) \cap U_{j+1} \neq \emptyset$ for all $j \geq 0$, there exists a point $z \in X$ such that $f^j(z) \in U_j$ for all $j \geq 0$. Another way to define dynamical properties in the absence of metric is to use the uniform structures. This structure enables us to control the errors in a pseudo-orbit.

Using this structure, Shah et al. [21] introduced the notion of distributional chaos on uniform spaces. Then, Ahmadi [2] proved that if a dynamical system is topologically chain transitive with topological shadowing property, then it is topologically ergodic. Wu et al. [26] introduced the topological concepts of weak uniformity, uniform rigidity, multi-sensitivity and obtained some equivalent characterizations of uniform rigidity. Then, they [27] proved that a point transitive dynamical system in a Hausdorff uniform space is either almost (Banach) mean equicontinuous or (Banach) mean sensitive. Ahmadi et al. [4] generalized concepts of entropy points, expansivity and shadowing property for dynamical systems to uniform spaces and obtained a relation between topological shadowing property and positive uniform entropy. Park et al. [19] proved the spectral decomposition theorem to flows that are expansive and have the shadowing property on uniform spaces. Shirazi et al. [22] introduced specification property (stroboscopical property) for dynamical systems on uniform space. Yan et al. [29] proved that if a dynamical system on a compact uniform space is expansive and has shadowing property, then it is topologically stable. Yadav et al. [28] proved that a dynamical system on a uniformly locally compact Hausdorff uniform space with topologically weak specification property and a pair of distal points is topologically distributionally chaotic. Recently we [20] introduced the notion of topological average shadowing property for continuous maps on uniform spaces and proved that this property implies chain transitivity. In fact, we proved the following two theorems.

Theorem 1.1 [20] *Let $(X, \mathscr{U})$ be a compact uniform space. Let f be a continuous map from X onto itself. If (X, f) has the topological average shadowing property, then it is topologically chain mixing.*

Theorem 1.2 [20] *Let $(X, \mathscr{U})$ be a compact uniform space. Let f be a continuous map from X onto itself. If f has the topological average shadowing property and the minimal points of f are dense in X, then f is topologically totally strongly ergodic.*

For more results on shadowing properties and chain transitivity, one is referred to [1, 5, 11, 13, 25] and references therein.
Here we study the transitivity properties of dynamical systems with topological average shadowing property and we show that topological average shadowing with Lyapunov stability implies topological ergodicity. Finally we prove that for a dynamical system with shadowing property the topological versions of transitivity, chain transitivity, mixing, ergodic shadowing, specification property and average shadowing property are equivalent.

2 Preliminaries

A uniform structure on X is a family $\mathscr{U}$ of subsets of $X \times X$ with the following properties.

(U1) If $U \in \mathscr{U}$ and $V \supset U$ then $V \in \mathscr{U}$;
(U2) If $U, V \in \mathscr{U}$ then $U \cap V \in \mathscr{U}$;
(U1) Every set $U \in \mathscr{U}$ contains the diagonal $\Delta_X = \{(x, x) : x \in X\}$;
(U2) If $U \in \mathscr{U}$, then $U^{-1} = \{(y, x) : (x, y) \in U\} \in \mathscr{U}$;
(U3) for any $U \in \mathscr{U}$ there exists $V \in \mathscr{U}$ such that $V \circ V \subset U$, where $V \circ V = \{(x, y) : \exists z \in X \text{ with } (x, z) \in V, (z, y) \in V\}$.

The set X with a uniformity $\mathscr{U}$ on it is called *uniform space* and denoted by $(X, \mathscr{U})$. Each element of $\mathscr{U}$ is called *entourage* of X. An entourage E is said to be *symmetric* if $E = E^{-1}$. If $x \in X$ and $E \in \mathscr{U}$, then the set $E[x] = \{y \in X : (x, y) \in E\}$ is called the *cross-section* of E at a point x. If $\tau_{\mathscr{U}} = \{A \subset X : \forall a \in A, \exists E \in \mathscr{U}, \textit{ such that } E[a] \subset A\}$, then $\tau_{\mathscr{U}}$ is a topology on X which is called the *uniform topology* on X.

A map $f : X \to X$ is called *uniformly continuous* if $f^{-1}(E) \in \mathscr{U}$ for any $E \in \mathscr{U}$. In this paper, by a *dynamical system* (briefly, we mean a pair (X, f), where X is a uniform space and $f : X \longrightarrow X$ is a uniformly continuous map.

Let (X, f) be a dynamical system. Let $D \in \mathscr{U}$. A *D-chain* of length n is a sequence $\xi = \{x_i\}_{i=0}^{n}$ such that $(f(x_i), x_{i+1}) \in D$ for $i = 0, \ldots, n-1$. An infinite D-chain is called a *D-pseudo-orbit*. A D-pseudo-orbit $\xi = \{x_i\}_{i=0}^{\infty}$ is *E-shadowed* by a point $z \in X$ if $(f^i(z), x_i) \in E$ for all $i \in \mathbb{N}_0$. For $z \in X$, a sequence $\xi = \{x_i\}_{i=0}^{\infty}$ and an entourage E we denote

$$\begin{aligned}\Lambda(\xi, f, E) &:= \{i \in \mathbb{N}_0 : (x_{i+1}, f(x_i)) \in E\},\\ \Lambda^c(\xi, f, E) &:= \{i \in \mathbb{N}_0 : (x_{i+1}, f(x_i)) \notin E\},\\ \Lambda(\xi, z, f, E) &:= \{i \in \mathbb{N}_0 : (x_i, f^i(z)) \in E\},\\ \Lambda^c(\xi, z, f, E) &:= \{i \in \mathbb{N}_0 : (x_i, f^i(z)) \notin E\}.\end{aligned}$$

Using this notions a sequence $\xi = \{x_i\}_{i=0}^{\infty}$ is called an *ergodic D-pseudo-orbit* if $\overline{d}(\Lambda^c(\xi, f, D)) = \limsup_{n\to\infty} \frac{|\Lambda^c(\xi, f, D)\cap\{1,2,\ldots,n\}|}{n} = 0$. An ergodic D-pseudo-orbit $\xi = \{x_i\}_{i=0}^{\infty}$ is *ergodicaly E-shadowed* by a point $z \in X$ if $\overline{d}(\Lambda^c(\xi, z, f, E)) = \limsup_{n\to\infty} \frac{|\Lambda^c(\xi,z,f,E)\cap\{1,2,\ldots,n\}|}{n} = 0$. Denote by $\Sigma_{\mathscr{U}}$ the family of all sequences $\mathcal{E} = \{E_i\}_{i=0}^{\infty}$ of entourages in $\mathscr{U}$ with $E_0 = X \times X$, such that $E_i \supset E_{i+1}$ for all $i \in \mathbb{N}_0$. For $z \in X$, a sequence $\mathcal{E} = \{E_i\}_{i=0}^{\infty} \in \Sigma_{\mathscr{U}}$, a map $f : X \to X$ and a sequence $\xi = \{x_i\}_{i=0}^{\infty}$ in X we define

$$\begin{aligned}\mathcal{A}_n(\xi, f, \mathcal{E}) &= \mathcal{A}_n(\xi, f, \{E_i\}_{i=0}^{\infty})\\ &= \inf\left\{\sum_{j=0}^{n} \frac{1}{2^{\sigma(j)}} : \quad (x_{j+1}, f(x_j)) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\right\}; \quad n \in \mathbb{N}\\ &= \sum_{j=0}^{n} \inf\left\{\frac{1}{2^{\sigma(j)}} : \quad (x_{j+1}, f(x_j)) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^j\right\}; \quad n \in \mathbb{N},\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}_n(\xi, z, f, \mathcal{E}) &= \mathcal{A}_n(\xi, z, f, \{E_i\}_{i=0}^{\infty})\\ &= \inf\left\{\sum_{j=0}^{n} \frac{1}{2^{\sigma(j)}} : \quad (x_j, f^j(z)) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\right\},\end{aligned}$$

where $\mathbb{N}_0^n$ is the set of all maps from $\{0, 1, \ldots, n\}$ to $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $\mathcal{D} \in \Sigma_{\mathscr{U}}$, a *topological average $\mathcal{D}$-pseudo-orbit* of f is a sequence $\xi = \{x_i\}_{i=0}^{\infty}$ in X such that $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\xi, f, \mathcal{D}) = 0$. Let $\mathcal{E} = \{E_i\}_{i=0}^{\infty} \in \Sigma_{\mathscr{U}}$. We say that the sequence $\xi = \{x_i\}_{i=0}^{\infty}$ is *$\mathcal{E}$-shadowed on average by some point* $z \in X$, if $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\xi, z, f, \mathcal{E}) = 0$.

Let (X, f) be a dynamical system. For any nonempty open subset U of X and a pint $x \in X$, we define $N_f(x, U) = \{i \in \mathbb{N} : f^i(x) \in U\}$. A point $x \in X$ is called *minimal* (or *almost periodic*) if $N_f(x, U)$ is syndetic for every neighborhood U of x (a subset $S \subset \mathbb{N}$ is called *syndetic* if there exists a natural number k such that $S \cap \{n, n+1, n+2, \cdots, n+k\} \neq \emptyset$ for all $n \in \mathbb{N}$). (X, f) is said to be *topologically chain transitive* if, for any entourage D of X and any two points $x, y \in X$, there exists a D-chain from x to y. (X, f) is said to be *topologically chain mixing* if, for any two points $x, y \in X$ and any entourage D of X, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, there exists a D-chain from x to y of length n.

For any two nonempty open subsets U and V of X, define $N_f(U, V) = \{i \in \mathbb{N} : U \cap f^{-i}(V) \neq \emptyset\}$, then (X, f) is said to be *topologically transitive* if $N_f(U, V)$ is a non-empty set for any pair of nonempty open subsets U and V of X. (X, f) is said to be *topologically totally transitive* if f^n is topologically transitive for any $n \in \mathbb{N}$ and it is said to be *topologically ergodic* if $\overline{d}(N_f(U, V)) = \limsup_{n\to\infty} \frac{|N_f(U,V)\cap\{1,2,...,n\}|}{n} > 0$ for any pair of nonempty open subsets U and V of X. Also (X, f) is said to be *topologically strongly ergodic* if $N_f(U, V)$ is a syndetic set for any pair of nonempty open subsets U and V of X. (X, f) is *topologically totally strongly ergodic* if f^n is topologically ergodic for any $n \in \mathbb{N}$.

A dynamical system (X, f) has *topological shadowing property* [9] if for every entourage E of X, there exists an entourage D such that every D-pseudo-orbit is E-shadowed by some point in X. (X, f) has ergodic shadowing if for every entourage E there is an entourage D such that every ergodic D-pseudo orbit is ergodicaly E-shadowed by some point in X. (X, f) has $\underline{d}$-shadowing if for every entourage E there is an entourage D such that every ergodic D-pseudo orbit is E-shadowed by some true orbit $\{f^i(z)\}_{i=0}^{\infty}$ in such a way that $\underline{d}(\Lambda(\xi, z, f, D)) = \liminf_{n\to\infty} \frac{|\Lambda(\xi,f,D)\cap\{1,2,...,n\}|}{n} > 0$. We say that (X, f) has the *topological average shadowing property*, if for every $\mathcal{E} = \{E_i\}_{i=0}^{\infty} \in \Sigma_{\mathscr{U}}$, there exists $\mathcal{D} = \{D_i\}_{i=0}^{\infty} \in \Sigma_{\mathscr{U}}$ such that every topological average $\mathcal{D}$-pseudo-orbit is $\mathcal{E}$-shadowed on average by some point of X.

We say that (X, f) has the *topological weak specification* if for every entourage E there is a positive integer M such that for any finite sequence of points $x_1, x_2, ..., x_n$ in X, and any non-negative integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ with $a_{j+1} - b_j > M$ for $1 \leq j \leq (n-1)$, there is $y \in X$ such that $(f^i(y), f^i(x_j)) \in E$ for $a_j \leq i \leq b_j$, $1 \leq j \leq n$. (X, f) has the *topological pseudo orbital specification* if for every entourage E there exists an entourage D and a positive integer M such that for any non-negative integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ with $a_{j+1} - b_j > M$ for $1 \leq j \leq (n-1)$ and any D-chain $\xi_1, \xi_2, \ldots, \xi_n$ where $\xi_j = \{x_{(i,j)}\}_{a_j \leq i \leq b_j}$ for $1 \leq j \leq n$, there is $z \in X$ such that $(f^i(z), x_{(i,j)}) \in E$ for $a_j \leq i \leq b_j$, $1 \leq j \leq n$.

3 Main Results

The following lemma is the crucial step in the proof of our main results.

Lemma 3.0.1 *Let $\mathcal{D} = \{D_i\}_{i=0}^{\infty} \in \Sigma_{\mathscr{U}}$. Let f be a map. If $\xi = \{x_i\}_{i=0}^{\infty}$ be a sequence in X, then the following prperties are equivalent.*

1. $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\xi, f, \mathcal{D}) = 0$.
2. *There exists a subset $\mathbb{J} \subset \mathbb{N}$ with zero density such that for any $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $(f(x_{j-1}), x_j) \in D_k$ for all $j \in \mathbb{J}^c \cap \{N, N+1, N+2, \cdots\}$.*

Proof (1 $\Rightarrow$ 2) Suppose that $\lim_{n\to\infty}\frac{1}{n}\mathcal{A}_n(\xi, f, \mathcal{D}) = 0$. Then

$$\lim_{n\to\infty}\frac{1}{n}\inf\left\{\sum_{j=1}^{n}\frac{1}{2^{\sigma(j)}}: \quad (f(x_{j-1}), x_j)\in D_{\sigma(j)}, \sigma\in\mathbb{N}_0^n\right\} = 0$$

Hence

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\inf\left\{\frac{1}{2^{\sigma(j)}}: \quad (f(x_{j-1}), x_j)\in D_{\sigma(j)}, \sigma\in\mathbb{N}_0^n\right\} = 0,$$

therefore

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\inf\left\{\frac{1}{2^{\sigma(j)}}: \quad (f(x_{j-1}), x_j)\in D_{\sigma(j)}, \sigma\in\mathbb{N}_0^j\right\} = 0.$$

By [23, Theorem 1.20] there exists $\mathbb{J}\subset\mathbb{N}$ with density zero such that

$$\lim_{\substack{j\to\infty\\ j\notin\mathbb{J}}}\inf\left\{\frac{1}{2^{\sigma(j)}}: \quad (f(x_{j-1}), x_j)\in D_{\sigma(j)}, \sigma\in\mathbb{N}_0^j\right\} = 0.$$

Hence for any $k\in\mathbb{N}$, there exists $N\in\mathbb{N}$ such that $\frac{1}{2^{\sigma(j)}} < \frac{1}{2^k}$ for all $j\in\mathbb{J}^c\cap\{N, N+1, N+2, \cdots\}$. This implies that $(f(x_{j-1}), x_j)\in D_k$ for all $j\in\mathbb{J}^c\cap\{N, N+1, N+2, \cdots\}$.

(2 $\Rightarrow$ 1) Given $\epsilon > 0$ there exists $k\in\mathbb{N}$ such that $\frac{1}{2^k} < \frac{\epsilon}{3}$. Therefore there exists a subset $\mathbb{J}\subset\mathbb{N}$ with zero density and $N\in\mathbb{N}$ such that $(f(x_{j-1}), x_j)\in D_k$ for all $j\in\mathbb{J}^c\cap\{N, N+1, N+2, \cdots\}$. Choose $M > N$ such that $\frac{N}{M} < \frac{\epsilon}{3}$ and $\frac{1}{n}|\mathbb{J}\cap\{1, 2, \cdots, n\}| < \frac{\epsilon}{3}$ for all $n\geq M$. For any $n\geq M$ we obtain

$$\begin{aligned}
\frac{1}{n}\mathcal{A}_n(\xi, f, \mathcal{D}) &= \frac{1}{n}\inf\left\{\sum_{j=1}^{n}\frac{1}{2^{\sigma(j)}}: \quad (f(x_{j-1}), x_j)\in D_{\sigma(j)}, \sigma\in\mathbb{N}_0^n\right\}\\
&= \frac{1}{n}\inf\left\{\sum_{j\in\mathbb{J}\cap\{1,2,\cdots,n\}}\frac{1}{2^{\sigma(j)}}: \quad (f(x_{j-1}), x_j)\in D_{\sigma(j)}, \sigma\in\mathbb{N}_0^n\right\}\\
&\quad + \frac{1}{n}\inf\left\{\sum_{j\in\mathbb{J}^c\cap\{1,2,\cdots,N-1\}}\frac{1}{2^{\sigma(j)}}: \quad (f(x_{j-1}), x_j)\in D_{\sigma(j)}, \sigma\in\mathbb{N}_0^n\right\}\\
&\quad + \frac{1}{n}\inf\left\{\sum_{j\in\mathbb{J}^c\cap\{N,N+1,\cdots,n\}}\frac{1}{2^{\sigma(j)}}: \quad (f(x_{j-1}), x_j)\in D_{\sigma(j)}, \sigma\in\mathbb{N}_0^n\right\}\\
&< \frac{1}{n}|\mathbb{J}\cap\{1, 2, \cdots, n\}| + \frac{N}{n} + \frac{1}{n}\frac{1}{2^k}|\mathbb{J}^c\cap\{1, 2, \cdots, n\}| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Therefore $\lim_{n\to\infty}\frac{1}{n}\mathcal{A}_n(\xi, f, \mathcal{D}) = 0$. □

Proposition 3.1 *Let $(X, \mathscr{U})$ be a uniform space. Let f be a continuous map from X onto itself. Then the following are equivalent:*

1. *(X, f) has the topological average shadowing property;*
2. *(X, f^k) has the topological average shadowing property for every $k \in \mathbb{N}$;*
3. *(X, f^k) has the topological average shadowing property for some $k \in \mathbb{N}$.*

Proof $(1 \Rightarrow 2)$ Fix $k > 1$ and let $\mathcal{E} = \{E_i\} \in \Sigma_{\mathscr{U}}$. Since f has the topological average shadowing property, there exists $\mathcal{D} = \{D_i\} \in \Sigma_{\mathscr{U}}$ such that every topological average $\mathcal{D}$-pseudo-orbit is $\mathcal{E}$-shadowed on average by some point in X. Let $\xi = \{x_i\}_{i=0}^{\infty}$ be a topological average $\mathcal{D}$-pseudo-orbit of f^k, that is

$$\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n\left(\xi, f^k, \mathcal{D}\right) = 0.$$

Putting $f(x_i), f^2(x_i), \ldots, f^{k-1}(x_i)$ between x_i and x_{i+1} for all $i \geq 0$, we get a topological average $\mathcal{D}$-pseudo-orbit ξ' for f. Let $y_{lk+j} = f^j(x_l)$, for all $0 \leq j \leq k$ and all $l \geq 0$. Then

$$\lim_{n\to\infty} \frac{1}{n} \inf \left\{ \sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad (f(y_{j-1}), y_j) \in D_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} = 0$$

that is, the sequence $\{y_i\}_{i=0}^{\infty}$ is a topological average $\mathcal{D}$-pseudo-orbit of f. So there is a point $z \in X$ such that

$$\lim_{n\to\infty} \frac{1}{nk} \inf \left\{ \sum_{j=1}^{nk} \frac{1}{2^{\sigma(j)}} : \quad (f^j(z), y_j) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} = 0.$$

Therefore

$$\lim_{n\to\infty} \frac{1}{n} \inf \left\{ \sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad (f^{jk}(z), x_j) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} = 0.$$

Hence f^k has topological average shadowing property.
$(3 \Rightarrow 1)$ Assume that f^k has average shadowing property. Let $\mathcal{E} = \{E_i\} \in \Sigma_{\mathscr{U}}$. Let $G_0 = E_0$ and inductively for each $i \geq 1$ choose $G_i \in \mathscr{U}$ such that $G_i^k \subset E_i$ and $G_i \subset G_{i-1}$. Put $\hat{E}_0 = G_0$ and $\hat{E}_i = \bigcap_{j=1}^{k}(f \times f)^{-j}G_i$ for all $i \geq 1$. If $\hat{\mathcal{E}} = \{\hat{E}_i\}$ then it is easy to see that $\hat{\mathcal{E}} \in \Sigma_{\mathscr{U}}$. Since f^k has topological average shadowing property there exists $\hat{\mathcal{D}} = \{\hat{D}_i\} \in \Sigma_{\mathscr{U}}$ with $\hat{D}_i \subset G_i$ such that any average $\hat{\mathcal{D}}$-pseudo-orbit of f^k can be $\hat{\mathcal{E}}$-shadowed on average by some point of X. Let $H_0 = \hat{D}_0$ and for each $i \geq 1$ choose $H_i \in \mathscr{U}$ such that $H_i^k \subset \hat{D}_i$ and $H_i \subset H_{i-1}$. Let $D_i = \bigcap_{j=1}^{k}(f \times f)^{-j}H_i$, then $\mathcal{D} = \{D_i\} \in \Sigma_{\mathscr{U}}$. Let $\xi = \{x_i\}_{i=0}^{\infty}$ be an average $\mathcal{D}$-pseudo-orbit for f and $n \in \mathbb{N}$. If $\hat{\xi} = \{x_{jk}\}_{j=0}^{\infty}$, and $m \in \mathbb{N}$, then by Lemma 3.0.1

there exists a subset $\mathbb{I} \subset \mathbb{N}$ with density zero and $N \in \mathbb{N}$ such that $(f(x_{j-1}), x_j) \in D_m$ for all $j \in \mathbb{I}^c \cap \{N, N+1, N+2, \cdots\}$. Define

$$\mathbb{K} = \{i \in \mathbb{N} : \quad \{ik, ik+1, ik+2, \cdots, ik+k-1\} \cap \mathbb{I} \neq \emptyset\}.$$

Then $\mathbb{K}$ has zero density and for any $i \in \mathbb{K}^c \cap \{N, N+1, N+2, \cdots\}$

$$\{ik, ik+1, ik+2, \cdots, ik+k-1\} \subset \mathbb{I}^c \cap \{N, N+1, \cdots\}.$$

Therefore $(f(x_j), x_{j+1}) \in D_m = \bigcap_{j=1}^{k} (f \times f)^{-j} H_m$ for all $j = ik, ik+1, \cdots, ik+k-1$. This implies that

$$\left(f^k(x_{ik}), f^{k-1}(x_{ik+1})\right) \in H_m \left(f^{k-1}(x_{ik+1}), f^{k-2}(x_{ik+2})\right) \in H_m$$
$$\left(f^{k-2}(x_{ik+2}), f^{k-3}(x_{ik+3})\right) \in H_m, \cdots, \left(f^1(x_{ik+k-1}), (x_{ik+k})\right) \in H_m.$$

Therefore $(f^k(x_{ik}), x_{(i+1)k}) \in H_m^k \subset \hat{D}_m$ for any $i \in \mathbb{K}^c \cap \{N, N+1, N+2, \cdots\}$ and $i \in \mathbb{K}^c \cap \{N, N+1, N+2, \cdots\}$. Hence by Lemma 3.0.1 we obtain $\lim_{n\to\infty} \frac{1}{n} \mathcal{A}_n(\hat{\xi}, f^k, \hat{\mathcal{D}}) = 0$. That is $\hat{\xi}$ is an average $(\hat{\mathcal{D}}, f^k)$-pseudo-orbit. Hence there exists z such that

$$\lim_{n\to\infty} \frac{1}{n} \mathcal{A}_n(\hat{\xi}, z, f^k, \hat{\mathcal{E}}) = 0.$$

Note that

$$\lim_{n\to\infty} \frac{1}{n} \mathcal{A}_n(\hat{\xi}, f^k, \hat{\mathcal{D}}) = 0.$$

By Lemma 3.0.1 there are subsets $\mathbb{J} \subset \mathbb{N}$ and $\mathbb{I} \subset \mathbb{N}$ with zero density such that for any $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $(f^{kj}(z), x_{kj}) \in \hat{E}_m = \bigcap_{j=1}^{k} (f \times f)^{-j} G_m$ and $(f(x_{jk}), x_{jk+1}) \in D_m$ for all $j \in \mathbb{I}^c \cap \mathbb{J}^c \cap \{N, N+1, N+2, \cdots\}$. Note that $\mathbb{I} \cup \mathbb{J}$ has zero density and for all $j \in \mathbb{I}^c \cap \mathbb{J}^c \cap \{N, N+1, N+2, \cdots\}$ we obtain

$$\begin{cases} \left(f^{kj+1}(z), f(x_{kj})\right) \in G_m, \\ (f(x_{jk}), x_{jk+1}) \in D_m \subset G_m \end{cases} \Rightarrow \left(f^{kj+1}(z), x_{kj+1}\right) \in G_m^2 \subset E_m$$

$$\begin{cases} \left(f^{kj+2}(z), f^2(x_{kj})\right) \in G_m, \\ \left(f^2(x_{jk}), x_{jk+1}\right) \in H_m \subset G_m \\ (f(x_{jk+1}), x_{jk+2}) \in H_m \subset G_m \end{cases} \Rightarrow \left(f^{kj+2}(z), x_{kj+2}\right) \in G_m^3 \subset E_m$$

$$\vdots$$

$$\left(f^{kj+k-1}(z), x_{kj+k-1}\right) \in G_m^k \subset E_m$$

Hence by Lemma 3.0.1 we have $\lim_{n\to\infty} \frac{1}{n} \mathcal{A}_n(\xi, z, f, \mathcal{E}) = 0$. □

Remark Note that in this paper, we control the pseudo-orbit errors on average differently from the metric version. One can see this difference in the following two cases:

a) If we consider the identity map on S^1 with the usual topology, then for $\mathcal{D} = \{V_{\frac{1}{i}}\}_{i=0}^{\infty} \in \Sigma_{\mathscr{U}_{S^1}}$ and any $\delta > 0$, the sequence $x_{j+1} = x_j + \delta/2$ with $x_0 = 0$ is an average δ-pseudo orbit which is not a topological average $\mathcal{D}$-pseudo orbit. That is, the d-average pseudo-orbit is different from the topological average pseudo-orbit when the uniform structure is came from the metric d.
b) In [24] the secound author proved that if (X, f) is a dynamical system on a compact metric space. Then, for any $\epsilon > 0$ there exists $\delta > 0$ such that for any ergodic δ-pseudo-orbit $\{x_i\}$ of f, there exists a average ϵ-pseudo-orbit $\{y_i\}$ of f such that $d(\{i : x_i \neq y_i\}) = 0$.
If we consider the identity map on S^1 with the usual topology, then for $\mathcal{E} = \{V_{\frac{1}{i}}\}_{i=0}^{\infty} \in \Sigma_{\mathscr{U}_{S^1}}$ and any $\delta > 0$, the sequence $x_{j+1} = x_j + \delta/2$ with $x_0 = 0$ is a ergodic δ-pseudo-orbit. If $\{y_i\}$ be a sequence with $d(\mathbb{J} = \{i : x_i \neq y_i\}) = 0$, we show that it can not be a topological average $\mathcal{E}$-pseudo-orbit. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\delta}{2} < \frac{1}{N-1}$. Then

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \inf\left\{\sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad (f(y_{j-1}), y_j) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\right\} \\
&= \lim_{n\to\infty} \frac{1}{n} \inf\left\{\sum_{j\in\mathbb{J}\cap\{1,2,\cdots,n\}} \frac{1}{2^{\sigma(j)}} : \quad (f(y_{j-1}), y_j) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\right\} \\
&\quad + \lim_{n\to\infty} \frac{1}{n} \inf\left\{\sum_{j\in\mathbb{J}^c\cap\{1,2,\cdots,n\}} \frac{1}{2^{\sigma(j)}} : \quad (f(y_{j-1}), y_j) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\right\} \\
&= \lim_{n\to\infty} \frac{1}{n} \inf\left\{\sum_{j\in\mathbb{J}\cap\{1,2,\cdots,n\}} \frac{1}{2^{\sigma(j)}} : \quad (f(y_{j-1}), y_j) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\right\} \\
&\quad + \lim_{n\to\infty} \frac{1}{n} \inf\left\{\sum_{j\in\mathbb{J}^c\cap\{1,2,\cdots,n\}} \frac{1}{2^{\sigma(j)}} : \quad (f(x_{j-1}), x_j) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\right\} \\
&= \lim_{n\to\infty} \frac{1}{n} \inf\left\{\sum_{j\in\mathbb{J}\cap\{1,2,\cdots,n\}} \frac{1}{2^{\sigma(j)}} : \quad (f(y_{j-1}), y_j) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\right\} \\
&\quad + \lim_{n\to\infty} \frac{1}{n} |\mathbb{J}^c \cap \{1, 2, \cdots, n\}| \frac{1}{2^{N-1}} \\
&> \frac{1}{2^{N-1}}
\end{aligned}$$

This implies that $\{y_i\}$ is not a topological average $\mathcal{E}$-pseudo-orbit.

The stability theory is one of the most important concepts in mathematics which addresses the stability of trajectories of dynamical systems under small perturbattions of initial conditions. That is the forward orbit of any point closed to an initial condition stay closed to the forward orbit of the initial condition.

Definition 3.2 A point $x \in X$ is said to be a *stable point* of f if for any entourage $E \in \mathscr{U}$ there is an entourage $D \in \mathscr{U}$ such that $f^n(D[x]) \subset E[f^n(x)]$ for every positive integer n. The map f is called *Lyapunov stable* if every point of X is a stable point of f.

An stronger form of transitivity formulated in terms of subsets of $\mathbb{N}$ with positive density, called topological ergodicity. The following theorem generalizes Theorem 3.1 [14] from metric spaces to uniform spaces.

Theorem 3.3 *Let X be a uniform compact space. Let f be topological Lyapunov stable map from X onto itself. If f has the topological average shadowing property, then f is topologically ergodic.*

Proof Let U and V be non-empty open subsets of X. Let $x \in U$ and $y \in V$. Let E be an entourage such that $E[x] \subset U$ and $E[y] \subset V$. There is a symmetric entourage $\hat{E}$ such that $\hat{E} \circ \hat{E} \subset E$. Since f is topological Lyapunov stable, for any $x \in X$, there exists $\hat{D}_x$ such that $\hat{D}_x[x] \subset f^{-n}(\hat{E}[f^n(x)])$ for every positive integer n. The family $\{int(\hat{D}_x[x]) : x \in X\}$ constitutes an open covering of X and by [16, Proposition 8.16] there exists an entourage $D \subset \hat{D}$ of X such that each uniform neighborhood $D[y]$, $y \in X$, is contained in $int(\hat{D}_x[x])$ for some x. Thus, $D[y] \subset f^{-n}(\hat{E}[f^n(x)])$ or equivalently $f^n(D[y]) \subset \hat{E}[f^n(x)]$. In particular, $f^n(y) \in \hat{E}[f^n(x)]$ and so $f^n(x) \in \hat{E}[f^n(y)]$. Therefore

$$f^n(D[y]) \subset \hat{E}[f^n(x)] \subset \hat{E} \circ \hat{E}[f^n(y)] \subset E[f^n(y)].$$

Choose $\mathcal{E} = \{E_i\} \in \Sigma_{\mathscr{U}}$ with $E_1 = D$. Since f has topological average shadowing property, there exists $\mathcal{D} = \{D_i\} \in \Sigma_{\mathscr{U}}$ such that every topological average $\mathcal{D}$-pseudo-orbit can be $\mathcal{E}$-shadowed on average by some point of X.
Now, we construct a sequence $\{w_i\}_{i=0}^{\infty}$ as follows.

$$w_0 = x, \quad w_1 = y$$

$$w_2 = x, \quad w_3 = y$$

$$w_4 = x_{-1}, \; w_5 = x, \quad w_6 = y_{-1}\,, w_7 = y$$

$$\vdots$$

$$w_{2^k} = x_{-2^{-k-1}+1}, \; x_{-2^{-k-1}}, \ldots, x_{-1}, x \quad y_{-2^{k-1}+1}, \ldots, y_{-1} \; w_{2^{k+1}-1} = y$$

$$\vdots$$

where $f(x_{-j}) = x_{-j+1}$ for every $j > 0$, $x_0 = x$ and $f(y_{-l}) = y_{-l+1}$ for every $l > 0$, $y_0 = y$.

For $n \geq 2$ we obtain that

$$\inf \left\{ \sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad (f(w_{j-1}), w_j) \in D_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} \leq 2 \log_2 n,$$

therefore $\lim_{n\to\infty} \frac{1}{n} \mathcal{A}_n(\{w_i\}, f, \mathcal{D}) = 0$. That is, $\{w_i\}$ is a topological average $\mathcal{D}$-pseudo-orbit of f. Hence there is a point $w \in X$ such that $\lim_{n\to\infty} \frac{1}{n} \mathcal{A}_n(\{w_i\}, w, f, \mathcal{E}) = 0$.

For $z \in \{x, y\}$, let

$$\mathbb{J}_x = \{i : w_i \in \{x_{-2^{i-1}+1}, x_{-2^{i-1}}, \dots, x_{-1}, x\} \text{ and } \left(f^i(w), w_i\right) \in D\}$$
$$\mathbb{J}_y = \{i : w_i \in \{y_{-2^{i-1}+1}, y_{-2^{i-1}}, \dots, y_{-1}, y\} \text{ and } \left(f^i(w), w_i\right) \in D\}.$$

We have the following claim

Claim 1 $\mathbb{J}_x$ has positive upper density, that is $\overline{d}(\mathbb{J}_x) > 0$.

Proof Suppose on the contrary that $\overline{d}(\mathbb{J}_x) = 0$. Let

$$\mathbb{J}_x^c = \{i : w_i \in \{x_{-2^{i-1}+1}, x_{-2^{i-1}}, \dots, x_{-1}, x\} \text{ and } (f^i(w), w_i) \notin D\}.$$

Then $\overline{d}(\mathbb{J}_x^c) = \frac{1}{2}$. Therefore

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \inf \left\{ \sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad \left(f^j(w), w_j\right) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} \\
&= \lim_{n\to\infty} \frac{1}{n} \left(\inf \left\{ \sum_{j\in\mathbb{J}_x \cap \{0,1,\dots,n-1\}} \frac{1}{2^{\sigma(j)}} : \quad \left(f^j(w), w_j\right) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} \right. \\
&\quad \left. + \inf \left\{ \sum_{j\in\mathbb{J}_x^c \cap \{0,1,\dots,n-1\}} \frac{1}{2^{\sigma(j)}} : \quad \left(f^j(w), w_j\right) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \inf \left\{ \sum_{j\in\mathbb{J}_x \cap \{0,1,\dots,n-1\}} \frac{1}{2^{\sigma(j)}} : \quad \left(f^j(w), w_j\right) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} \\
&\quad + \lim_{n\to\infty} \frac{|\mathbb{J}_x^c \cap \{0, 1, \dots, n-1\}|}{n} \\
&\geq \frac{1}{2}
\end{aligned}$$

Hence $\lim_{n\to\infty} \frac{1}{n} \mathcal{A}_n(\{w_i\}, w, f, \mathcal{E}) \neq 0$, which is a contradiction. Therefore $\overline{d}(\mathbb{J}_x) > 0$. Now, let $\mathbb{J}_m(y) = \{i \in \mathbb{J}_y : w_i = y_{-m}\}$, for $m \geq 0$. Since the density of the disjoint union of sets with zero density is zero, by claim, there is an integer $m_0 \geq 0$ such that $\overline{d}(\mathbb{J}_{m_0}(y)) > 0$.

Choose $i_0 > 0$ and $0 \leq k_0 \leq 2^{i_0-1} - 1$ such that $f^{i_0}(w) \in D[x_{-k_0}]$. For any $j \in \mathbb{J}_{m_0}(y)$ with $j \geq i_0 + k_0$, we have $f^j(w) \in D[y_{-m_0}]$. Since f is topological Lyapunov stable, we have

$$f^{i_0+k_0}(w) \in E[x] \ \textit{and} \ f^{j+m_0}(w) \in E[y].$$

Let $n_j = (j + m_0) - (i_0 + k_0)$. Hence $f^{n_j}(E[x]) \cap E[y] \neq \emptyset$. So, $f^{n_j}(U) \cap V \neq \emptyset$. Thus $\overline{d}(N(U, V)) \geq \overline{d}(\mathbb{J}_{m_0}(y)) > 0$. This implies that f is topologically ergodic. □

Remark Theorem 1.2 states that a dynamical system with topological average shadowing property and a dense set of minimal points is topologically totally strongly ergodic. Note that Lyapunov stability and density of minimal points are mutually exclusive. For example the doubling map on the circle has a dense set of minimal points and no points of the circle is stable. On the other hand, the map $f : S^1 \to S^1$ defined by $f(x) = x - x(x - \frac{1}{2})(x - 1)$ has no minimal points and it is topologically Lyapunov stable.

The follwing example shows that the stability condition is necessary in the previous theorem.

Example Let $\mathbb{P} = \mathbb{R} \backslash \mathbb{Q}$. For $r > 0$ let

$$U_r = \{(x, y) \in \mathbb{R}^2 : x = y \in \mathbb{P} \text{ or } |x - y| < r\}.$$

Then

$$\mathscr{U} = \{U \subseteq \mathbb{R}^2 : U \supseteq U_r \text{ for some } r > 0\},$$

is a uniformity on $\mathbb{R}$. Most of the conditions are very easy to check. For (4), let $U \in \mathscr{U}$. Denote by τ the topology generated by $\mathscr{U}$ on $\mathbb{R}$. For each $U \in \mathcal{U}$ and $x \in \mathbb{R}$ let

$$U[x] = \{y \in \mathbb{R} : \langle x, y \rangle \in U\}.$$

Suppose first that x is irrational. Then $U_r[x] = \{x\}$ for each $r > 0$ is τ-open and therefore each irrational is an isolated point of $(\mathbb{R}, \tau)$. Now suppose that x is rational. Then

$$U_r[x] = \{y \in \mathbb{R} : |x - y| < r\} = (x - r, x + r)$$

for each $r > 0$ and $\mathscr{U}$, $\{U_r[x] : r > 0\}$ is a local base for the topology τ at the point x, so $\{(x - r, x + r) : r > 0\}$ is a local base at each rational. Consider this uniformity on S^1 (just taking the projection modulo 1). Then the topology generated by this uniformity on S^1 is the Michael line topology $\tau_M =$

$\{U \cup F : U$ is open in usual topology and $F \subset \mathbb{P}\}$ which is neither compact nor metrizable [15, b-13]. Consider the map $f : S^1 \to S^1$ by

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{3}, \\ \frac{1}{2}x + \frac{1}{2} & \text{if } \frac{1}{3} \leq x \leq 1 \end{cases}.$$

It is easy to see that f is uniformly continuous, in fact, $x \in \mathbb{P}$ iff $f^{-1}(x) \in \mathbb{P}$ and for any $U_a \in \mathcal{B}$,

$$f^{-1}(U_a) = \bigcup_{i \in \mathbb{Z}} \Big\{ \Big\{ \Big(f^{-1}(a_i), f^{-1}(a_i) \Big) \Big\} \cup \Big(f^{-1}(a_i), f^{-1}(a_{i+1}) \Big) \\ \times \Big(f^{-1}(a_i), f^{-1}(a_{i+1}) \Big) \Big\} \in \mathcal{B}.$$

For any two disjoint irrational points x and y we can consider two open sets $U = \{x\}$ and $V = \{y\}$. Note that for any integer n we obtain $f^n(x) \in \mathbb{P}$. Hence $f^n(U) \cap V = \emptyset$ for all $n \in \mathbb{Z}$. Therefore f is not topologically transitive. This implies that f is not topologically ergodic. It is easy to check that f is chain transitive. If $x, y \in (0, 1)$ and $r > 0$, then $f^n(x) \to 1$ and $f^{-n}(y) \to 1$ (note that $1 \in \mathbb{Q}$). Hence there exists $N \in \mathbb{N}$ such that $(f^N(x), 1) \in U_{r/2}$ and $(f^{-N}(x), 1) \in U_{r/2}$. Therefore the sequence $x, f(x), \ldots, f^{N-1}(x), f^{-N}(y), f^{-N+1}(y), \ldots, f^{-1}(y), y$ is an U_r-chain from x to y.

Also we can show taht f doesn't have the topological shadowing. For $E = U_{0.1}$ and any entourage D, choose $r > 0$ such that $U_r \subset D$. Since $f^n(\frac{1}{2}) \to 1$ and $f^{-n}(\frac{1}{2}) \to 0$, there exists $k > 0$ such that $f^{-k+1}(\frac{1}{2}) \in (0, r)$ and $f^{k-1}(\frac{1}{2}) \in (1 - r, 1)$. Then the cyclic sequence

$$\xi = \left\{ \overline{0, f^{-k+1}\left(\frac{1}{2}\right), f^{-k+2}\left(\frac{1}{2}\right), f^{-k+3}\left(\frac{1}{2}\right), \ldots, f^{k-1}\left(\frac{1}{2}\right)} \right\}$$

is a D-pseudo orbt. Suppose that ξ is E-shadowed by some point $x \in (0, 1)$. Since $f^n(x) \to 1$, there exists $n > 0$ such that $f^{(2n+1)k}(x) \in (0.9, 1)$, but $x_{(2n+1)k} = \frac{1}{2}$ which implies that $(f^{(2n+1)k}(x), x_{(2n+1)k}) \notin E$, a contadiction. In [20, Example 3] we showed that f has the topological average shadowing property.

The fixed point $x = 0$ is not a stable point. Since if $E = U_{\frac{1}{4}}$ for any $D \in \mathscr{U}$ there exists $r > 0$ such that $U_r \subset D$. Let $y \in U_r[0] \setminus \{0\}$. Then there exists a positive n such that $f^n(y) > \frac{1}{4}$. Hence $f^n(D[0]) \not\subset E[0] = E[f^n(0)]$. Therefore f is not Lyapunov stable.

Theorem 3.4 *Let $(X, \mathscr{U})$ be a uniform space. Let f be a continuous map from X onto itself. Let f has topological ergodic shadowing property, then f has the the topological average shadowing property.*

Proof Let $\mathscr{E} \in \Sigma_{\mathscr{J}}$. Suppose that $D_0 = E_0$ and for each $i \geq 1$ the entourage D_i is E_i-modulus of the ergodic shadowing property. Without loose of generality we can assume that $\mathcal{D} = \{D_i\} \in \Sigma_{\mathscr{U}}$. If $\xi = \{x_i\}$ is an average $\mathcal{D}$-pseudo orbit, then by

Lemma 3.0.1 there exists $\mathbb{J} \subset \mathbb{N}$ and $N \in \mathbb{N}$ such that $(f(x_{j-1}), x_j) \in D_k$ for all $j \in \mathbb{J}^c \cap \{N, N+1, N+2, \cdots\}$. Therefore $d(\Lambda^c(\xi, f, D_k)) = 0$. That is, $\xi = \{x_i\}$ is an ergodic D_k-pseudo-orbit and can be E_k-shadowed by some point $z \in X$. Therefore $d(\Lambda^c(\xi, z, f, E_k)) = 0$. Since $(x_j, f^j(z)) \in E_k$ for each $j \in \Lambda(\xi, z, f, E_k)$, again by Lemma 3.0.1 we obtain that

$$\lim_{n\to\infty} \frac{1}{n} \inf \left\{ \sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad \left(f^j(z), x_j\right) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} = 0.$$

Hence f has the topological average shadowing property. □

Corollary 3.5 *Let $(X, \mathscr{U})$ be a compact uniform space. Let f be a uniformly continuous surjective map with topological shadowing property. Then for the dynamical system (X, f), the following prperties are equivalent:*

1. *Totally topological chain transitive*
2. *Totally transitive.*
3. *Topological weak mixing.*
4. *Topological mixing.*
5. *Topological chain mixing.*
6. *Topological weak specification.*
7. *Topological pseudo orbital specification property.*
8. *Topological $\underline{d}$-shadowing property.*
9. *Topological ergodic shadowing property.*
10. *Topological average shadowing property.*

Proof By Theorems 3.4 and 1.1, we have 9 $\Rightarrow$ 10 and 10 $\Rightarrow$ 5. Then [8, Main Theorem] completes the proof. □

Example Consider the set of integers $\mathbb{Z}$. The p-adic structure on $\mathbb{Z}$, for a given prime p, is the uniformity $\mathscr{U}_p$ generated by subsets $D_{[n]}$ of $\mathbb{Z}\times\mathbb{Z}$, $(n = 1, 2, 3, \cdots)$, where $s, t \in D_{[n]}$ if and only if $s \equiv t (\mathrm{mod}\ p^n)$. This structure is very important in the theory of numbers [6]. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a polynomial with integer coefficients. Then we show that f has the topological shadowing property. Suppose that $E \in \mathscr{U}_p$, then there exists $n \in \mathbb{N}$ such that $D_{[n]} \subset E$. We claim that any $D_{[n]}$-pseudo-orbit can be E-shadowed by some point in $\mathbb{Z}$. If $\xi = \{x_i\}$ is a $D_{[n]}$-pseudo-orbit then $(f(x_j, x_{j+1}) \in D_{[n]}$ for $j = 0, 1, 2, \cdots$. This implies that $f^j(x_0) \equiv x_j (\mathrm{mod}\ p^n)$. Therefore $(x_j, f^j(x_0)) \in D_{[n]} \subset E$ for $j = 0, 1, 2, \cdots$. This means that ξ is E-shadowed by x_0.
Let $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(x) = ax$ where a is co-prime with p. Then g has the topological shadowing property. We show that g does not have the topological average shadowing property. Suppose that $E_0 = \mathbb{Z} \times \mathbb{Z}$ and $E_i = D_{[i]}$ for $i = 1, 2, 3, \cdots$ and $\mathcal{E} = \{E_i\}$. Define the sequence $\xi = \{x_i\}$ as follows:

$$x_0 = 0, \quad x_1 = ax_0;$$
$$x_2 = a^2x_0 + 1, \quad x_3 = ax_2;$$
$$x_4 = a^2x_0 + 1, \quad x_5 = ax_4, \quad x_6 = a^2x_4, \quad x_7 = a^3x_4;$$

$$x_8 = a^4 x_4 + 1, \quad x_9 = a x_8, \quad \cdots x_{15} = a^7 x_8;$$
$$\vdots$$
$$x_{2^{j+1}} = a^{2^j} x_{2^j} + 1 \quad \cdots x_{2^{j+2}-1} = a^{2^{j+1}-1} x_{2^{j+1}};$$
$$\vdots$$

It is easy to see that ξ is an $\mathcal{D}$-average pseudo-orbit for any $\mathcal{D} \in \Sigma_{\mathscr{U}_p}$. No point in $\mathbb{Z}$ can $\{E_i\}$-shadows ξ. Indeed if $z \in \mathbb{Z}$ and $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\xi, z, g, \mathcal{E}) = 0$. Then for any k there exists $0 \leq l_k < 2^k$ such that $(x_{2^k+l_k}, g^{2^k+l_k}(z)) \in E_1 = D_{[1]}$. Hence

$$a^{(2^k+l_k)} z \overset{p}{\equiv} x_{(2^k+l_k)} = a^{l_k} x_{2^k} \tag{1}$$

and

$$a^{(2^{k+1}+l_{k+1})} z \overset{p}{\equiv} x_{(2^{k+1}+l_{k+1})} = a^{l_{k+1}} x_{2^{k+1}} = a^{l_{k+1}} (a^{2^k} x_{2^k} + 1) = a^{2^k+l_{k+1}} x_{2^k} + a^{l_{k+1}} \tag{2}$$

Suppose that $r = (2^{k+1} + l_{k+1}) - (2^k + l_k)$. By multiplying 1 by r, we obtain

$$a^{(2^{k+1}+l_{k+1})} z \overset{p}{\equiv} a^{r+l_k} x_{2^k} = a^{(l_{k+1}+2^{k+1}-2^k)} x_{2^k} = a^{(l_{k+1}+2^k)} x_{2^k}. \tag{3}$$

Therefore by 2 and 3 we obtain

$$a^{(2^k+l_{k+1})} x_{2^k} + a^{l_{k+1}} \overset{p}{\equiv} a^{(2^k+l_{k+1})} x_{2^k},$$

which implies that $p \mid a^{l_{k+1}}$, a contradiction.

Let $k \in \mathbb{N}$ be arbitrary. Then by the Euler theorem we have $a^{\phi(p^k)} \overset{p^k}{\equiv} 1$. This implies that for any $x \in \mathbb{Z}$ the sequence $x_0 = x, x_1 = ax, \cdots x_{\phi(p^k)} = a^{\phi(p^k)} x$ is a $D_{[k]}$-chain of length $\phi(p^k)$ from x to itself. Therefore g is topologically chain recurrent. Furthermore g is not topologically chain transitive, for example, there is no $D_{[1]}$-chain from 0 to 1 (If $x_0 = 0, x_1, \cdots, x_n = 1$ is a $D_{[1]}$-chain from 0 to 1, then we obtain $p \mid a^n$, a contradiction).

Acknowledgements The authors are grateful to anonymous reviewers for careful reading and valuable suggestions.

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