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Local Topological Stability for Diffeomorphisms

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Abstract

Let $f : M \to M$ be a diffeomorphism of compact smooth Riemannian manifold M, an let $\Lambda \subset M$ be a closed f-invariant set. We obtain conditions for Λ to be topologically stable which is called Λ-topologically stable. Moreover, we prove that if f is C^1 robustly Λ-topologically stable then Λ satisfies star condition for f. Then in the above, if a closed f-invariant set Λ is chain transitive (or transitive) then it is hyperbolic for f.

Keywords Locally topologically stable · Local star · Chain transitive set · Transitive set · Hyperbolic

Mathematics Subject Classification 37C50 · 37D20

1 Introduction

In this paper, we assume that M is a compact smooth Riemannian manifold with $\dim M \geq 2$. Denote by $\text{Diff}(M)$ the space of diffeomorphisms of M endowed with the C^1-topology. Let d be the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. A closed f-invariant set $\Lambda \subset M$ is *hyperbolic* for f if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n_{|_{E^s_x}}\| \leq C\lambda^n \quad \text{and} \quad \|Df^{-n}_{|_{E^u_x}}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$ then we say that f is *Anosov*. For any homeomorphisms $f, g : M \to M$, we defined by the C^0 metric

$$d_0(f, g) = \sup\{x \in M : d(f(x), g(x)), d(f^{-1}(x), g^{-1}(x))\}.$$

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Walters [17, 18] studies topologically stable, that is, a diffeomorphism f is *topologically stable* if for any $\epsilon > 0$, there is $\delta > 0$ such that for any homeomorphism $g : M \to M$ with $d_0(f, g) < \delta$ there is a continuous map $h : M \to M$ such that $h \circ g = f \circ h$ and $d(h(x), x) < \epsilon$ for any $x \in M$, where d_0 is the C^0 metric.

We say that a diffeomorphism f is *expansive* if there is $\delta > 0$ such that for any $x, y \in M$, if $d(f^i(x), f^i(y)) \le \delta$ for all $i \in \mathbb{Z}$ then $x = y$. Given $d > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ of points in M is called a *d-pseudo orbit of f* if the following is satisfied

$$d(f(x_k), x_{k+1}) < d,$$

for all $k \in \mathbb{Z}$. We say that a diffeomorphism f has the *shadowing property* if for any $\epsilon > 0$ there is $d > 0$ such that any d-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ of points in M of f there is a point $y \in M$ such that

$$d(f^i(y), x_i) < \epsilon$$

for all $i \in \mathbb{Z}$.

Walters [18] proved that if an expansive diffeomorphism f has the shadowing property then f is topologically stable. Also, he [17] proved that if a diffeomorphism f is Anosov then f is topologically stable. Denote by $P(f)$ the set of all periodic points of f. We say that a diffeomorphism f satisfies *Axiom A* if the non-wandering set $\Omega(f)$ is hyperbolic and it is the closure of $P(f)$. For any diffeomorphisms $f, g : M \to M$, we defined by the C^1 metric

$$d_1(f, g) = d_0(f, g) + \max_{x\in M} \max(\|Df(x) - Dg(x)\|, \|Df^{-1}(x) - Dg^{-1}(x)\|).$$

A diffeomorphism f is *Ω-stable* if there is a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ there is a homeomorphism $h : \Omega(f) \to \Omega(g)$ such that $f \circ h = h \circ g$, where $\Omega(g)$ is the non-wandering set of g. Smale [16] and Palis [13] proved that a diffeomorphism f satisfies Axiom A and the no-cycle condition if and only if f is Ω-stable. The following notion was introduced by Gan and Wen [5]. We say that a diffeomorphism f is *star* if there is a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$, every $p \in P(g)$ is hyperbolic, where $P(g)$ is the set of all periodic point of g. Denote by $\mathcal{F}(M)$ the set of all star diffeomorphisms. By Aoki [1] and Hayashi [6], if a diffeomorphism $f \in \mathcal{F}(M)$ then f is Ω-stable.

We say that a diffeomorphism f is *Ω-topologically stable* if for any $\epsilon > 0$ there is $\delta > 0$ such that for any homeomorphism $g : M \to M$ with $d_0(f, g) < \delta$ there is a continuous map $h : \Omega(g) \to \Omega(f)$ such that

(a) $f \circ h = h \circ g$ on $\Omega(g)$, and
(b) $d(h(x), x) < \epsilon$ for $x \in \Omega(g)$.

Moriyasu [10] proved that if a diffeomorphism f belongs to the C^1 interior of the set of all topologically stable then it is structurally stable. Moreover, he proved that if a diffeomorphism f belongs to the C^1 interior of the set of all Ω-topologically stable then it satisfies Axiom A and the no-cycle condition.

In this paper we generalize Moriyasu result. We say that $\Lambda \subseteq M$ is *locally maximal* with respect to some $f : M \to M$ if there is a compact neighborhood U (called *an isolating block*) such that

$$\Lambda = U_f = \bigcap_{n\in\mathbb{Z}} f^n(U).$$

Every locally maximal invariant set is clearly a compact invariant set of f.

Definition 1.1 Let $f : M \to M$ be a diffeomorphism and Λ be a locally maximal invariant set of f. We say that f is Λ-*topologically stable* if for any $\epsilon > 0$ there are an isolating block U of Λ and a C^0 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ there is a continuous map $h : U_g \to \Lambda$ such that $d(h(x), x) \leq \epsilon$ for every $x \in U_g$ and $f \circ h = h \circ g$ on U_g, where $U_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is the continuation of Λ.

Note that we introduce another definition of the topological stability of a set Λ which is corresponding to the notion of the C^0 lower semistability of the germ of Λ(see [11]).

Definition 1.2 Let $f : M \to M$ be a diffeomorphism and Λ be a locally maximal invariant set of f. We say that f is C^1 *robustly* Λ-*topologically stable* if there are an isolated block U of Λ and a C^1 neighborhood $\mathcal{U}$ of f such that g is U_g-topologically stable, for every $g \in \mathcal{U}$.

A closed f-invariant set $\Lambda \subset M$ satisfies a *local star condition* for f (or f is *local star* on Λ) if there are an isolated block U of Λ and a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$, every $p \in U_g \cap P(g)$ is hyperbolic, where $P(g)$ is the set of all periodic points of g. The notion is a generalization of star diffeomorpisms, because if $\Lambda = M$ then it satisfies star condition. For the notion, a remarkable result is Lee [8]. In [8], the author showed that if a transitive set Λ satisfies a local star condition for f then Λ is hyperbolic for f. From the local star condition, we will show the following Theorem.

Theorem A *Let $\Lambda \subset M$ be a closed f-invariant set for f. If a diffeomorphism f is C^1 robustly Λ-topologically stable, then Λ satisfies a local star condition for f.*

A closed f-invariant set $\mathcal{C}$ is called *chain transitive* if for any $\delta > 0$ and $x, y \in \mathcal{C}$, there is a δ-pseudo orbit $\{x_i\}_{i=0}^{n}(n \geq 1) \subset \mathcal{C}$ such that $x_0 = x$ and $x_n = y$. For any hyperbolic periodic point p, we denote index$(p) = \dim W^s(p)$, where $W^s(p) = \{x \in M : f^i(x) \to p \text{ for } i \to \infty\}$ which is called the stable manifold of x. A closed f-invariant set Λ with index$(p) = i(p \in \Lambda \cap P(f))$ is *robustly homogenous index* if there are an isolating block U of Λ and a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ every hyperbolic $q \in U_g \cap P(g)$ has index$(q) = i$. Lee [9] proved that if every periodic points in $\mathcal{C}$ is robustly homogenous index then the set is hyperbolic. In the paper, we consider the chain transitive set $\mathcal{C}$ under a type of a locally topological stability. We prove the following.

Theorem B *Let $f : M \to M$ be a diffeomorphism and $\mathcal{C} \subset M$ be a locally maximal chain transitive set. If f is C^1 robustly $\mathcal{C}$-topologically stable, then $\mathcal{C}$ is a hyperbolic set of f.*

2 Proof of Theorem A

Let $f : M \to M$ be a diffeomorphism. A closed f-invariant set Λ is called *transitive* if there is a point $x \in \Lambda$ such that $\omega(x) = \Lambda$. It is known that according to C^1 closing lemma, if Λ is transitive and locally maximal, then for every isolating block U there is g C^1 close to f such that g has a periodic point in $U_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$. Then, we have the following lemma.

Lemma 2.1 *Let Λ be a locally maximal invariant set of f. If a diffeomorphism f is Λ-topologically stable, then $\Lambda \cap P(f) \neq \emptyset$.*

Proof Let $\epsilon > 0$ be given. For this ϵ take the isolating block U of Λ and a C^0 neighborhood $\mathcal{U}$ of f from the Λ-topological stability of f. Take any $x \in \Lambda$. We have $\omega(x) \subset \Lambda$. By C^0 closing lemma, we can find $g \in \mathcal{U}$ and a periodic orbit $Orb(p_g)$ such that the orbit of $Orb(p_g)$ close to $\omega(x)$ (Hausdorff arbitrarily close). This implies $p_g \in U_g$. Now let $h : U_g \to \Lambda$ be the continuous map given in the definition of Λ-topological stability. It follows that $h(p_g)$ is well defined and belongs to Λ. Since $h \circ g = f \circ h$ on U_g,

$$h(g^n(p_g)) = f^n(h(p_g))$$

for all $n \in \mathbb{Z}$. So, if k is the period of p_g with respect to g,

$$f^k(h(p_g)) = h(g^k(p_g)) = h(p_g),$$

proving that $h(p_g)$ is the periodic point of f in Λ. □

The following is called Franks' lemma [4] which is a very useful lemma for the C^1 perturbation property.

Lemma 2.2 *Let $\mathcal{U}(f)$ be any given C^1 neighborhood of f. Then there exist $\epsilon > 0$ and a C^1 neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that for given $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, \cdots, x_N\}$, a neighborhood W of $\{x_1, x_2, \cdots, x_N\}$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \le \epsilon$ for all $1 \le i \le N$, there exists $\widehat{g} \in \mathcal{U}(f)$ such that $\widehat{g}(x) = g(x)$ if $x \in \{x_1, x_2, \cdots, x_N\} \cup (M \setminus W)$ and $D_{x_i}\widehat{g} = L_i$ for all $1 \le i \le N$.*

According to Lemma 2.1, if f is Λ-topologically stable then there is a periodic point p which contained in Λ. From this we can see the following fact.

Proposition 2.3 *Let $f : M \to M$ be a diffeomorphism and Λ be a locally maximal invariant set of f. Suppose that f is C^1 robustly Λ-topologically stable. Then, there are an isolating block U of Λ and a C^1 neighborhood $\mathcal{U}$ of f such that every $p \in U_g \cap P(g)$ is hyperbolic, for any $g \in \mathcal{U}$.*

Proof Let U be the isolating block of Λ and $\mathcal{U}$ be the C^1 neighborhood of f given in the definition of C^1 robustly Λ-topological stability. Put $\mathcal{U}(f) = \mathcal{U}$ in Lemma 2.2 to get ϵ and the neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}$. Suppose that there are $g \in \mathcal{U}_0(f)$ and

$p \in P(g) \cap U_g$ such that p is not hyperbolic for g. Since $p \in P(g) \cap U_g$ is not hyperbolic, there is an eigenvalue λ of $D_p g^{\pi(p)}$ such that $|\lambda| = 1$, where $\pi(p)$ is the period of p. For simplicity, we may assume that $\pi(p) = 1$. Since p is not hyperbolic for g, we assume that $T_p M = E^c_p \oplus E^s_p \oplus E^u_p$ is the $D_p g$-invariant splitting of $T_p M$, where E^σ_p, $\sigma = c, s, u$, are subspaces $T_p M$ corresponding to eigenvalues λ of $D_p g$ for $|\lambda| = 1$, $|\lambda| < 1$ and $|\lambda| > 1$, respectively.

If $\lambda \in \mathbb{R}$ then we consider $\lambda = 1$. In Lemma 2.2 we put $N = 1, x_1 = p, W = B_\alpha(p)$ in a way that $W \subseteq U$ and $L = L_1 : T_p M \to T_p M$ such that $L|_{E^*_p} = D_p g|_{E^*_p} + \epsilon I|_{E^*_p}$ for $* = s, u$ and $L|_{E^c_p} = I$. Then, $\|L - D_p g\| \leq \epsilon$ and so, by Lemma 2.2, we can choose $\varphi = \widehat{g} \in \mathcal{U}(f)$ such that $\varphi(p) = g(p)$, $\varphi(x) = g(x)$ for $x \in M \setminus W$ and $D_p \varphi = L$. We can assume that the exponential map $exp_p : T_p M \to W$ is a well defined diffeomorphisms.

Take a non-zero vector $u \in E^c_p$ such that $\|u\| \leq \alpha/2$. Then we have

$$\varphi(\exp_p(u)) = \exp_p \circ L \circ \exp_p^{-1}(\exp_p(u)) = \exp_p(u).$$

Denoting by $E^c_p(\alpha/4)$ the ball of radius $\alpha/4$, centered in $\overrightarrow{O}_p$ and inside E^c_p, we have an invariant small arc $\mathfrak{I}_p \subset B_\alpha(p) \cap \exp_p(E^c_p(\alpha/4))$ with center at p which satisfies the following:

(1) $\mathfrak{I}_p \subset U_\varphi = \bigcap_{n \in \mathbb{Z}} \varphi^n(U)$;
(2) $\varphi(\mathfrak{I}_p) = \mathfrak{I}_p$;
(3) $\varphi|_{\mathfrak{I}_p} : \mathfrak{I}_p \to \mathfrak{I}_p$ is the identity map;
(4) $\mathfrak{I}_p$ is a normally hyperbolic set of φ (see proof of Proposition A p. 730 in [19]).

Since f is C^1 robustly Λ-topologically stable, φ is U_φ-topologically stable. However, we shall prove that φ is not U_φ-topologically stable as follows.

Let $diam(\mathfrak{I}_p)$ be the diameter of $\mathfrak{I}_p$. By Item (4) above we can choose $0 < \rho < diam(\mathfrak{I}_p)$ so that

$$\mathfrak{I}_p = \bigcap_{n \in \mathbb{Z}} \varphi^n(O) \tag{1}$$

where O is the ρ-ball centered at $\mathfrak{I}_p$. Choose δ from the U_φ-topological stability of φ for $\rho/4$. Now take ϕ C^1 δ-close to φ so that $\phi(\mathfrak{I}_p) = \mathfrak{I}_p$ and the dynamics of $\phi|_{\mathfrak{I}_p}$ is Pole North-South one namely we identify $\mathfrak{I}_p = [0, 1]$ with $\phi(0) = 0$, $\phi(1) = 1$ and $\phi^n(y) \to 0$ or 1 as $n \to \infty$ respectively for $y \in]0, 1[$.

It follows that there is a continuous map $h : U_\phi \to U_\varphi$ so that $d(h(y), y) < \rho/4$ and $\varphi \circ h = h \circ \phi$. Since $\phi(\mathfrak{I}_p) = \mathfrak{I}_p$, we have $\mathfrak{I}_p \subseteq U_\phi$. Therefore h is defined on $\mathfrak{I}_p$.

Note if $y \in \mathfrak{I}_p$ then $\varphi(h(y)) = h(\phi(y)) \in h(\mathfrak{I}_p)$ proving that $h(\mathfrak{I}_p)$ is φ-invariant. By (1) we get $h(\mathfrak{I}_p) \subseteq \mathfrak{I}_p$.

If $y \in]0, 1[$ then $\phi^n(y) \to 1$ as $n \to \infty$ thus $h(y) = \varphi^n(h(y)) = h(\phi^n(y)) \to h(1)$ proving $h(y) = 1$ for every $y \in]0, 1[$. Then, $h(y) = h(1)$ for every y by continuity. It follows that

$$d(y, w) \leq d(h(y), y) + d(h(y), h(w)) + d(h(w), w) < \rho/2 < diam(\mathfrak{I}_p), \quad \forall y, w \in \mathfrak{I}_p.$$

This is a contradiction proving the result.

If the eigenvalue $\lambda \in \mathbb{C}$, then to avoid notational complexity, we consider only the case $g(p) = p$. As in the above case, there are $\alpha > 0$ and g_1 C^1 close to g $(h \in \mathcal{U})$ such that $g_1(p) = g(p) = p$ and

$$g_1(x) = \exp_p \circ D_p g \circ \exp_p^{-1}(x) \text{ if } x \in B_\alpha(p).$$

Then there is $k > 0$ such that $D_p g_1^k(v) = v$ for any non-zero vector $v \in E_p^c$. As in the above argument, we can get a contradiction. □

Denote by $\mathcal{F}(\Lambda)$ the set of all diffeomorphisms satisfying the local star condition on Λ. To prove Theorem A, it is enough to show that a diffeomorphism $f \in \mathcal{F}(\Lambda)$.

Proof of Theorem A Since a diffeomorphism f is Λ-topologically stable, according to Lemma 2.1, $P(f) \cap \Lambda \neq \emptyset$. Since f is C^1 robustly Λ-topologically stable, by Proposition 2.3, $f \in \mathcal{F}(\Lambda)$. This ends proof of Theorem A. □ .

3 Proof of Theorem B

Let M be as before and let $f \in \mathrm{Diff}(M)$.

Lemma 3.1 *Let $f : M \to M$ be a diffeomorphism and let $\Lambda \subset M$ be a closed f-invariant set. Let $k > 0$ be an integer and $\epsilon > 0, \eta > 0$ be given. Then for any sequence $\{x_0, x_1, \dots, x_k\} \subset \Lambda$ with $d(f(x_i), x_{i+1}) < \epsilon$ for $i \in \{0, 1, \dots, k-1\}$, there exists a sequence $\{y_0, y_1, \dots, y_k\} \subset \Lambda$ such that*

(i) $d(x_i, y_i) < \eta$, *for* $i \in \{0, 1, \dots, k\}$,
(ii) $d(f(y_i), y_{i+1}) < 2\epsilon$, *for* $i \in \{0, 1, \dots, k-1\}$, *and*
(iii) $y_i \neq y_j (i \neq j)$, *for* $0 \leq i, j \leq k$.

Proof The proof is similar to [2, Lemma 2.4.10]. □

For any $\delta > 0$, a sequence $\{x_i\}_{i \in \mathbb{Z}}$ is said to be δ *pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. We say that a diffeomorphism f has the *shadowing property* on Λ if for any $\epsilon > 0$ there is $\delta > 0$ such that for any δ pseudo orbit $\{x_i\}_{i \in \mathbb{Z}} \subset \Lambda$ we can take a point $z \in M$ satisfying $d(f^i(z), x_i) < \epsilon$ for all $i \in \mathbb{Z}$.

Theorem 3.2 *Let $\Lambda \subset M$ be a closed f-invariant set. If a diffeomorphism f is Λ-topological stable then f has the shadowing property on Λ.*

Proof For any $\epsilon > 0$, let U be a locally maximal neighborhood of Λ and let $\delta > 0$ be as corresponding to the definition of Λ-topologically stable. We assume that $\{x_0, x_1, \dots, x_k\} \subset \Lambda$ be choosen such that $d(f(x_i), x_{i+1}) < \delta/4\pi$ for $i = \{0, 1, \dots, k-1\}$. According to Lemma 3.1, there is $\{y_0, y_1, \dots, y_k\} \subset \Lambda$ such that

(i) $d(x_i, y_i) < \epsilon$, for $i \in \{0, 1, \dots, k\}$,
(ii) $d(f(y_i), y_{i+1}) < \delta/2\pi$ for $i \in \{0, 1, \dots, k-1\}$,
(iii) $y_i \neq y_j (i \neq j)$ for $0 \leq i, j \leq k$, and
(iv) $f(y_i) \neq f(y_j)(i \neq j)$ for $0 \leq i, j \leq k$.

According to [12, Lemma 13], there is a homeomorphism $\zeta : M \to M$ such that $d(\zeta(x), x) < \delta$ for $x \in M$ and $\zeta \circ f(y_i) = y_{i+1}$ for $i \in \{0, 1, \ldots, k-1\}$. Let $g = \zeta \circ f$. Then we have $d(g(x), f(x)) < \delta$ for $x \in M$ and $g(y_i) = y_{i+1}$ for $i \in \{0, 1, \ldots, k-1\}$. Since f is Λ-topologically stable, there are a closed invariant set $U_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$ and a continuous map $h : U_g \to \Lambda$ such that $d(h(x), x) < \epsilon$ for $x \in U_g$ and $h \circ g = f \circ h$ on U_g. Then we have

$$\begin{aligned} d(f^i \circ h(y_0), x_i) &= d(h \circ g^i(y_0), x_i) \\ &= d(h(y_i), x_i) \le d(h(y_i), y_i) + d(y_i, x_i) \\ &< \epsilon + \epsilon = 2\epsilon, \end{aligned}$$

for $i \in \{0, 1, \ldots, k\}$. Thus for each $\{x_i\}_{i=0}^k \subset \Lambda$ with $d(f(x_i), x_{i+k}) < \delta/4\pi$ for $i \in \{0, 1, \ldots, k-1\}$, there is $y \in \Lambda$ such that $d(f^i(y), x_i) < 2\epsilon$, for $i \in \{0, 1, \ldots, k-1\}$. Since $\Lambda \subset M$ is a closed and invariant set for f, by [14, Lemma 1.1.1] f has the shadowing property on Λ. □

We say that a diffeomorpism f has the C^1 robustly shadowing property on Λ if there are a C^1 neighborhood $\mathcal{U}(f)$ of f and an isolating block U of Λ such that for any $g \in \mathcal{U}(f)$, g has the shadowing property on U_g.

Proof of Theorem B Since f is C^1 robustly $\mathcal{C}$-topologically stable, there are a C^1 neighborhood $\mathcal{U}(f)$ of f and an isolated block U of $\mathcal{C}$ such that for any $g \in \mathcal{U}$, g is U_g-topologically stable, where $U_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is the continuation of Λ. Since f is $\mathcal{C}$-topologically stable, f has the shadowing property on $\mathcal{C}$. Thus if f has the C^1 robustly $\mathcal{C}$-topologically stable then it exactly is the notion of the C^1 robustly shadowing property on $\mathcal{C}$. Thus as in the result of Sakai [15], $\mathcal{C}$ is hyperbolic. □

We know that Diff(M) is a Baire space in the C^1 topology. A residual subset of Diff(M) is a countable intersection of open dense subsets. According to the Baire category theorem, a residual set is dense. We say that a property holds for the C^1 generic diffeomorphism f if it holds on a residual subset of Diff(M).

Theorem 3.3 *There is a residual set $\mathcal{G} \subset$ Diff(M) such that for any $f \in \mathcal{G}$ and a chain transitive set $\mathcal{C}$ for f, if f is $\mathcal{C}$-topologically stable then $\mathcal{C}$ is hyperbolic for f.*

Proof By Lemma 3.2, the diffeomorphism f is a C^1 generic f having the shadowing property on a locally maximal chain transitive set $\mathcal{C}$. From the result of Lee and Wen [7], we have $\mathcal{C}$ is hyperbolic for f. □

For a chain transitive set $\mathcal{C}$, it is easily show that if a diffeomorphism f has the shadowing property on $\mathcal{C}$ then $\mathcal{C}$ is transitive. According to the above theorems, we have the following results.

Corollary 3.4 *Let Λ be a locally maximal transitive set of f. If a diffeomorphism f is C^1 robustly Λ-topologically stable then Λ is hyperbolic.*

Proof By Theorem A, the transitive set Λ satisfies a local star condition for f. By Lee [8], Λ is hyperbolic for f. □

According to Crovisier [3], a chain transitive set $\mathcal{C}$ is a transitive set. Thus by Theorem 3.3, we have the following.

Corollary 3.5 *There is a residual set $\mathcal{G} \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}$ and Λ is transitive set for f, if f is Λ-topologically stable then Λ is hyperbolic for f.*

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Author Contributions The author reviewed the manuscript.

Declarations

Conflict of interest The authors declare no competing interests.

References

1. Aoki, N.: The set of Axiom A diffeomorphisms with no cycles. Bol. Soc. Bras. Mat. **23**, 21–65 (1992)
2. Aoki, N., Hiraide, K.: Topological theory of dynamical systems, North-Holland, (1994)
3. Crobvisier, S.: Periodic orbits and chain-transitive sets of C^1 diffeomorphisms. Publ. Math. Inst. Hautes Etudes Sci. **104**, 87–141 (2006)
4. Franks, J.: Necessary conditions for stability of diffeomorphisms. Trans. Amer. Math. Soc. **158**, 301–308 (1971)
5. Gan, S., Wen, L.: Nonsingular star flows satisfy Axiom A and the no-cycle condition. Invent. Math. **164**, 279–315 (2006)
6. Hayashi, S.: Diffeomorphisms in $\mathcal{F}^1(M)$ satisfy Axiom A. Ergodic Thoery Dyn. Syst. **12**, 233–253 (1992)
7. Lee, K., Wen, X.: Shadowable chain transitive sets of C^1-generic diffeomorphisms. Bull. Korean Math. Soc. **49**, 263–270 (2012)
8. Lee, M. : Inverse pseudo orbit tracing property for robust diffeomorphisms. Chaos Soliton. Fract. **160**, 112150 (2022)
9. Lee, M.: Robustly chain transitive diffeomorphisms. J. Inequal. Appl. **2015**(230), 1–6 (2015)
10. Moriyasu, K.: The topological stability of diffeomorphisms. Nagoya Math. J. **123**, 91–102 (1991)
11. Nitecki, Z.: On semistability for diffeomorphisms. Invent. Math. **14**, 83–122 (1971)
12. Nitecki, Z., Shub, M.: Filtarations, decompositions and exposions. Amer. J. Math. **97**, 1029–1047 (1976)
13. Palis, J.: On the C^1-stability conjecture. Publ. Math. l'I.H.É.S. **66**, 211–215 (1988)
14. Pilyugin, S.Y.: The Space of Dynamical Systems with the C^0-Topology. Lecture Notes in Mathematics, vol. 1571. Springer-Verlag, Berlin (1994)
15. Sakai, K.: Shadowable chain transitive sets. J. Differ. Equ. Appl. **19**, 1601–1618 (2013)
16. Smale, S.: The Ω-stability thereom, in Global Analysis Proc. Symp. Pure Math. Amer. Math. Soc. **14**, 289–297 (1970)
17. Walters, P.: Anosov diffeomorphisms are topologically stable. Topology **9**, 71–78 (1970)
18. Walters, P.: On the pseudo orbit tracing property and its relationship to stability, Dynamical Systems-Warwick 1974A Manning (Ed.), Springer Lecture Notes No. 468, Springer Verlag, (1975), 231–244
19. Yang, D., Gan, S.: Expansive homoclinic classes. Nonlinearity **22**, 729–733 (2009)