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Global Dynamics and Optimal Control of Multi-Age Structured Vector Disease Model with Vaccination, Relapse and General Incidence

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Abstract

Vaccine effectiveness, disease recovery and recurrence are important issues that must be faced in the prevention and control of vector-borne infectious diseases. We develop, in this paper, a dynamical model of vector disease with multi-age-structure to describe the transmission of parasites (or bacteria) between vectors and hosts, where vaccination, relapse and general incidence are introduced to study how these factors influence the spread and control of disease. First, the accurate formulation of the basic reproduction number is gained, which determines the existence and local asymptotic stability of the disease-free and endemic steady states. Further, by utilizing the fluctuation theorem and the method of Lyapunov function, we verify that the disease-free steady state is globally asymptotically stable if the basic reproduction number is less than one. In addition, we also prove that the endemic steady state of this model without relapse is globally asymptotically stable if the basic reproduction number is greater than one. Moreover, the optimal control problem for our model is formulated and analyzed. Finally, some numerical simulations are conducted to explain these analytical results.

Keywords Vector disease model with multi-age-structured · Vaccination and relapse · General incidence · Stability and optimal control · Fluctuation lemma

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1 Introduction

Malaria, Chagasdisease (Americantrypanosomiasis), Leishmaniasis, Tungiasis, African trypanosomiasis (sleeping sickness), etc.) are typical vector infectious diseases, which are caused by parasites or bacteria and transmitted by certain insects (such as mosquitoes, triatome bugs, sandflies, fleas, ticks, etc.) as vectors. Every year there are hundreds of thousands deaths due to these diseases around the world. Such as, approximately 500 million people worldwide are threatened by Malaria, and an estimated 10 million people develop clinical symptoms of Malaria each year, 90% of whom are on the African continent, where more than 2 million people die from the disease each year [1].

There is a very long history of using mathematical models to study the transmission patterns and prevention and control measures of vector-borne infectious diseases [2–4]. In particular, Niger et al. [5] proposed an ordinary differential equations model to evaluate the effect of reinfection on Malaria transmission. Chitnis et al. [6] developed a model to describe the spread of Malaria between mosquitoes and humans, and discussed the existence and stability of the equilibria which are determined by the basic reproduction number. In Ref. [7], Osman et al. introduced a model to study the different vector bias values of Malaria between low and high transmission areas, obtained the existence and global stability of equilibria, and predicted the course of this disease using a fit to actual data. Zheng et al. [8] developed a Malaria model with two strains to analyze the impacts on incubation period and diversity of Plasmodium on the transmission of this disease. Related studies are still continuous.

In recent years, clinical data have shown that the viral load in a host infected by a pathogen is not constant, but varies with the time of viral invasion. This leads to the fact that the transmission rate of pathogens from infected individuals to susceptible individuals is a function of infection time, not a constant. For example, the infection cycle for influenza is typically 2-10 days, with the rate of viral infection being almost zero on the first day, peaking and beginning to decline on the second day, and gradually converging to zero after four days. This also means that the rate of transmission of the virus is closely related to the time of being infected (also known as the age of infection). In addition, for chronic infectious diseases such as Malaria, HIV, Tuberculosis, etc., the situation is more complicated [9]. The age factors (such as, the age of infection, the age of vaccination, the age of relapse, etc.) not only influence the basic reproduction number, but also influence the peak value of cases and the duration of disease transmission. Recently, many epidemiological studies have attended to this crucial aspect by introducing the compartment age of infectious disease models and have yielded results which include the existence and stability of steady states [10–14]. Specifically, Wang et al. [15] presented a host-vector model with age-structure and non-linear incidence for the transmission of Malaria, obtained the global asymptotic stability of steady-states, and discussed the effects of the general incidence and age of infection. Liu et al. [16] established an SEIR model with age-dependent latency and recurrence periods, studied the asymptotic smoothness and uniform persistence of solutions, and achieved the local and global stability of steady states by constructing appropriate Volterra-type Lyapunov functions. Other studies on age-structured models of vector-borne infectious diseases can be found [17–20], to name just a few.

In addition, the study of optimal control for infectious diseases is an important part of the dynamics of epidemic models, which is also an important guide to the rational use of limited medical resources, effective control of disease transmission, and reduction of control costs [8, 21, 22]. However, there are also fewer research studies on the optimal control of epidemic models with age structure. This is because the optimal control of epidemic models with age-structure are usually coupling with ordinary differential equations (ODEs) and partial differential equations (PDEs), which is more complex than the optimal control of ODEs or PDEs. This will bring more computational challenges. Very recently, Mohammed-Awel et al. [23] proposed a vector-borne infectious disease model with age of vaccination, and obtained the adjoints equations, the existence and uniqueness of the optimal control for the optimal problem subject to their model by using the Gâteaux derivatives and the Ekeland's Principle. Until now, control measures for mosquito-borne infections have typically involved the use of medications to reduce the risk of recurrence, the use of bed nets and insecticides to reduce the likelihood of mosquito bites, extensive spraying to destroy mosquito breeding sites, and vaccination of susceptible hosts [24–27].

We present, in this paper, a model of vector-borne with multi-age structures (that is, age of vaccination, age of infection and age of relapse), where the general incidence is also introduced to describe the complexity of parasites/bacteria transmission from vectors to hosts. This article is organized as below. The model is presented in Sect. 2. The global dynamics of this model is discussed in Sect. 3, which includes the existence and uniqueness of global positive solutions, the basic reproduction number, the existence and local stability of the disease-free and endemic steady states, the global asymptotic stability of the disease-free steady state, and the global asymptotic stability of the endemic steady state under ignoring relapse. The adjoint equations are derived and the optimal control is described in Sect. 4. Numerical simulations and a brief discussions are presented in Sects. 5 and 6, respectively.

2 Model Formulation and Preliminaries

Following the process of parasites/bacteria transmission between hosts and vectors, the host population is divided into susceptible individuals S_h, vaccinated immune individuals V_h, infected individuals I_h, and recovered individuals R_h. Let $V_h(t, a)$ denotes the number of immunized individuals with the age of vaccination a at time t (here, a denotes the time-since-vaccination), then the total number of vaccinated humans at t is $\int_0^\infty V_h(t, a)\mathrm{d}a$. The age-dependent decay rate of the vaccine is denoted by $\omega_h(a)$, so the total number of vaccine decay for vaccinated individuals entering the susceptible hosts is $\int_0^\infty \omega_h(a)V_h(t, a)\mathrm{d}a$. Similarly, $I_h(t, b)$ is the number of infected hosts with the age of infection b at time t (here, b represents the time-since-infection), then the total number of infected humans at t is $\int_0^\infty I_h(t, b)\mathrm{d}b$. The age-related removal rate of infected humans is given by $k_h(b)$, so the number of infected humans entering the recovery term is $\int_0^\infty k_h(b)I_h(t, b)\mathrm{d}b$. Let $R_h(t, c)$ denotes the number of recovered individuals with the age of recover c at time t (here, c denotes the time-since-recover), then the total number at time t is $\int_0^\infty R_h(t, c)\mathrm{d}c$. The age-related relapse rate of recovered individuals is denoted by $r_h(c)$, so that the number of recovery individuals entering the infected compartment is $\int_0^\infty r_h(c)R_h(t, c)\mathrm{d}c$. The vector population is

Table 1 The means of model parameters

Param.	The biological meanings	Units
Λ_h	The total birth/recruitment rate of host population	day^{-1}
Λ_v	The total birth/recruitment rate of vector population	day^{-1}
μ_h, μ_v	Natural mortality rate of hosts and vectors, respectively	day^{-1}
ψ_h	Vaccination rate of susceptible individuals	day^{-1}
$k_h(b)$	Age-dependent recovery rate of infected individuals	day^{-1}
$\nu_h(b)$	Age-dependent death rate of infected individuals due to disease	day^{-1}
$\omega_h(a)$	Immunity loss rate of vaccinated individuals	day^{-1}
$r_h(c)$	From recovery class to infected class age-dependent relapse rate	day^{-1}

divided into susceptible S_v and infected I_v, where, assume that the infected hosts will carry the parasites/bacteria until it dies due to its short life cycle. Defined the infectivity from infected hosts to susceptible vectors, $\beta_v(b) = q_v\beta_{1v}(b)$, where q_v denotes the average biting rate of vectors, and $\beta_{1v}(b)$ denotes the probability of virus transmission from infected hosts with infectious b to vectors after a successful bite. Under the above hypothesis, a model with multi-age structure is formulated

$$\begin{cases} \dfrac{\mathrm{d}S_h(t)}{\mathrm{d}t} = \Lambda_h - \mu_h S_h(t) - f(S_h(t), I_v(t)) - \dfrac{1}{\alpha}\psi_h S_h(t) + \displaystyle\int_0^\infty \omega_h(a)V_h(t,a)\mathrm{d}a, \\ \left(\dfrac{\partial}{\partial t} + \alpha\dfrac{\partial}{\partial a}\right)V_h(t,a) = -(\mu_h + \omega_h(a))V_h(t,a),\ V_h(t,0) = \dfrac{1}{\alpha}\psi_h S_h(t), \\ \left(\dfrac{\partial}{\partial t} + \alpha\dfrac{\partial}{\partial b}\right)I_h(t,b) = -(\mu_h + k_h(b) + \nu_h(b))I_h(t,b), \\ I_h(t,0) = f(S_h(t), I_v(t)) + \displaystyle\int_0^\infty r_h(c)R_h(t,c)\mathrm{d}c, \\ \left(\dfrac{\partial}{\partial t} + \alpha\dfrac{\partial}{\partial c}\right)R_h(t,c) = -(\mu_h + r_h(c))R_h(t,c),\ R_h(t,0) = \displaystyle\int_0^\infty k_h(b)I_h(t,b)\mathrm{d}b, \\ \dfrac{\mathrm{d}S_v(t)}{\mathrm{d}t} = \Lambda_v - \displaystyle\int_0^\infty \beta_v(b)S_v(t)I_h(t,b)\mathrm{d}b - \mu_v S_v(t), \\ \dfrac{\mathrm{d}I_v(t)}{\mathrm{d}t} = \displaystyle\int_0^\infty \beta_v(b)S_v(t)I_h(t,b)\mathrm{d}b - \mu_v I_v(t), \end{cases} \tag{1}$$

for $t \geq 0$. Here, the initial conditions $S_h(0) = S_{h0}$, $V_h(0,a) = V_{h0}(a)$, $I_h(0,b) = I_{h0}(b)$, $R_h(0,c) = R_{h0}(c)$, $S_v(0) = S_{v0}$, $I_v(0) = I_{v0}$, for $a \geq 0$, $b \geq 0$, $c \geq 0$; $I_{v0} \in \mathbb{R}^+ := (0,\infty)$, and $V_{h0}(a)$, $I_{h0}(b)$, $R_{h0}(c) \in L^1_+(\mathbb{R}^+)$, $L^1_+(\mathbb{R}^+)$ is the positive cone of the function space $L^1(\mathbb{R})$ that are defined on $\mathbb{R} := (-\infty,\infty)$ and Lebesgue integrable. The means of model parameters can be found in Table 1.

Remark 1 Where, parameter α is introduced to balance the difference in units of age and time [23, 28]. For example, if the units of age and time are week and day, respectively, then $\alpha = \frac{1}{7}$. In this paper, in order to simplify the calculation, the units of age and time are both selected as day, so in this case, $\alpha = 1$.

In addition, the age-related parameters fulfill the following assumptions.

(H_1) Functions $\omega_h(a)$, $\beta_v(b)$, $k_h(b)$, $\nu_h(b)$, $r_h(c) \in L^1_+(\mathbb{R}^+)$ are Lipschitz continuous and $\beta_v(b) > 0$.

(H_2) There is a constant $\mu_0 \in (0, \mu_h]$ such that $\omega_h(\tau), k_h(\tau), \nu_h(\tau), r_h(\tau) \geq \mu_0$ for $\tau > 0$.

(H_3) $f(S_h, I_v)$ is locally Lipschitz continuous on S_h, I_v, that is, for every $K > 0$, there exists some $C_h(K) > 0$ and $C_v(K) > 0$ such that $\|f(S_h, I_v) - f(\tilde{S}_h, I_v)\| \leq C_h(K)|S_h - \tilde{S}_h|$, $\|f(S_h, I_v) - f(S_h, \tilde{I}_v)\| \leq C_v(K)|I_v - \tilde{I}_v|$, whenever $0 \leq S_h, \tilde{S}_h, I_v, \tilde{I}_v \leq K$.

(H_4) Function f satisfies $f(0, I_v) = f(S_h, 0) = 0$, and $f(S_h, I_v)$ is differentiable such that $\frac{\partial f(S_h, I_v)}{\partial I_v} > 0$, $\frac{\partial f(S_h, I_v)}{\partial S_h} > 0$ and $\frac{\partial f^2(S_h, I_v)}{\partial I_v^2} \leq 0$, for $S_h > 0$, $I_v > 0$.

Remark 2 There are many concrete nonlinear incidence functions satisfying (H_4). For example, $f(S_h, I_v) = \frac{\beta S_h I_v}{1+qS_h+pI_v}$ (Beddington-DeAngelis incidence [29, 30]), $f(S_h, I_v) = \frac{\beta S_h I_v}{(1+qS_h)(1+pI_v)}$ (Crowley-Martin incidence [31]) and $f(S_h, I_v) = \frac{\beta S_h I_v^p}{1+pI_v^l}$, $p, l \geq 1$ (nonmonotone incidence [32, 33]).

Now, let the function space $\mathbb{X} = \mathbb{R} \times L^1(\mathbb{R}^+) \times L^1(\mathbb{R}^+) \times L^1(\mathbb{R}^+) \times \mathbb{R} \times \mathbb{R}$ endowed with the norm

$$\|(x_1, \cdots, x_6)\|_{\mathbb{X}} = |x_1| + \int_0^\infty |x_2(a)|\mathrm{d}a + \int_0^\infty |x_3(b)|\mathrm{d}b + \int_0^\infty |x_4(c)|\mathrm{d}c + |x_5| + |x_6|,$$

for any $(x_1, \cdots, x_6) \in \mathbb{X}$, and its positive cone is denoted by $\mathbb{X}_+ = \mathbb{R}^+ \times L^1_+(\mathbb{R}^+) \times L^1_+(\mathbb{R}^+) \times L^1_+(\mathbb{R}^+) \times \mathbb{R}^+ \times \mathbb{R}^+$. The initial conditions of model (1) can be rewritten as $x_0 = (S_{h0}, V_{h0}(a), I_{h0}(b), R_{h0}(c), S_{v0}, I_{v0}) \in \mathbb{X}_+$. It's easy to get that

$$\begin{aligned} V_h(0,0) &= \psi_h S_{h0}, \\ I_h(0,0) &= f(S_{h0}, I_{v0}) + \int_0^\infty r_h(c) R_{h0}(c)\mathrm{d}c, \\ R_h(0,0) &= \int_0^\infty k_h(b) I_{h0}(b)\mathrm{d}b. \end{aligned} \tag{2}$$

From the basic theory of age-structured model in Ref. [34], model (1) has a unique non-negative solution. That is, model (1) generates a semiflow $\Phi(t, x_0) = (S_h(t), V_h(t, \cdot), R_h(t, \cdot), I_h(t, \cdot), S_v(t), I_v(t))$, $t \geq 0$, which is continuous and

$$\begin{aligned} \|\Phi(t, x_0)\|_{\mathbb{X}} = |S_h(t)| &+ \int_0^\infty |V_h(t,a)|\mathrm{d}a \\ &+ \int_0^\infty |I_h(t,b)|\mathrm{d}b \\ &+ \int_0^\infty |R_h(t,c)|\mathrm{d}c + |S_v(t)| + |I_v(t)|. \end{aligned} \tag{3}$$

The following substitution is introduced below, for $a, b, c \geq 0$,

$$
\begin{aligned}
&\varepsilon_1(a) = \omega_h(a) + \mu_h, && \varepsilon_2(b) = k_h(b) + \nu_h(b) + \mu_h, && \varepsilon_3(c) = r_h(c) + \mu_h,\\
&\Omega(a) = e^{-\int_0^a \varepsilon_1(s)\mathrm{d}s}, && \tau(b) = e^{-\int_0^b \varepsilon_2(s)\mathrm{d}s}, && \Delta(c) = e^{-\int_0^c \varepsilon_3(s)\mathrm{d}s},\\
&L = \int_0^\infty \omega_h(a)\Omega(a)\mathrm{d}a, && H = \int_0^\infty k_h(b)\tau(b)\mathrm{d}b, && N = \int_0^\infty r_h(c)\Delta(c)\mathrm{d}c.
\end{aligned}
$$

By the characteristic line method, solving $V_h(t, a)$ from the second equation of (1), one has

$$
V_h(t, a) = \begin{cases} V_h(t-a, 0)e^{-\int_0^a \varepsilon_1(s)\mathrm{d}s}, & t > a \geq 0,\\ V_{h0}(a-t)e^{-\int_{a-t}^a \varepsilon_1(s)\mathrm{d}s}, & a \geq t \geq 0. \end{cases} \tag{4}
$$

Similarly, we can obtain $I_h(t, b)$ and $R_h(t, c)$

$$
\begin{aligned}
I_h(t, b) &= \begin{cases} I_h(t-b, 0)e^{-\int_0^b \varepsilon_2(s)\mathrm{d}s}, t > b \geq 0,\\ I_{h0}(b-t)e^{-\int_{b-t}^b \varepsilon_2(s)\mathrm{d}s}, b \geq t \geq 0, \end{cases}\\
R_h(t, c) &= \begin{cases} R_h(t-c, 0)e^{-\int_0^c \varepsilon_3(s)\mathrm{d}s}, t > c \geq 0,\\ R_{h0}(c-t)e^{-\int_{c-t}^c \varepsilon_3(s)\mathrm{d}s}, c \geq t \geq 0. \end{cases}
\end{aligned} \tag{5}
$$

Next, define that the state space for model (1) as follow

$$
\begin{aligned}
\mathbb{D} := \Bigg\{ &(S_h, V_h(t, \cdot), I_h(t, \cdot), R_h(t, \cdot), S_v, I_v) \in \mathbb{X}_+\\
&: S_h + \int_0^\infty V_h(t, a)\mathrm{d}a + \int_0^\infty I_h(t, b)\mathrm{d}b\\
&+ \int_0^\infty R_h(t, c)\mathrm{d}c \leq \frac{\Lambda_h}{\mu_h},\ S_v + I_v \leq \frac{\Lambda_v}{\mu_v} \Bigg\}.
\end{aligned}
$$

Proposition 1 *For model (1), region $\mathbb{D}$ is positive invariant for Φ, that is, for any $x_0 \in \mathbb{D}$, $\Phi(t, x_0) \in \mathbb{D}$, $t \geq 0$. Further, Φ is ultimately bounded and $\mathbb{D}$ attracts every point in $\mathbb{X}_+$.*

Proof Let $\Phi(t, x_0) = Z_1(t, x_{10}) + Z_2(t, x_{20})$, where $Z_1(t, x_{10}) = (S_h, V_h(t, \cdot), I_h(t, \cdot), R_h(t, \cdot), 0, 0)$, $Z_2(t, x_{20}) = (0, 0, 0, 0, S_v, I_v)$, $x_{10} = (S_{h0}, V_{h0}(a), I_{h0}(b), R_{h0}(c), 0, 0)$ and $x_{20} = (0, 0, 0, 0, S_{v0}, I_{v0})$. Then for any $x_{10} \in \mathbb{X}_+$, we have

$$
\begin{aligned}
\frac{\mathrm{d}\|Z_1(t, x_{10})\|_{\mathbb{X}}}{\mathrm{d}t} \leq \Lambda_h - \mu_h &\\
&\left(S_h + \int_0^\infty V_h(t, a)\mathrm{d}a + \int_0^\infty I_h(t, b)\mathrm{d}b + \int_0^\infty R_h(t, c)\mathrm{d}c \right).
\end{aligned}
$$

Hence, it yields that

$$\|Z_1(t, x_{10})\|_{\mathbb{X}} \le \frac{\Lambda_h}{\mu_h} - e^{-\mu_h t}\left(\frac{\Lambda_h}{\mu_h} - \|x_{10}\|_{\mathbb{X}}\right). \tag{6}$$

Similarly, we can get that

$$\|Z_2(t, x_{20})\|_{\mathbb{X}} = \frac{\Lambda_v}{\mu_v} - e^{-\mu_v t}\left(\frac{\Lambda_v}{\mu_v} - \|x_{20}\|_{\mathbb{X}}\right). \tag{7}$$

Therefore, for any $x_0 = x_{10} + x_{20} \in \mathbb{D}$, it follows that $\Phi(t, x_0) \in \mathbb{D}$ for $t \ge 0$. This means that $\mathbb{D}$ is a positive invariant for model (1). Further, it is also observe from inequalities (6) and (7) that $\Phi(t, x_0)$ is ultimately bounded and attracts every point in $\mathbb{X}_+$. This completes the proof. □

Proposition 2 *For some* $\mathcal{A} \ge \frac{\Lambda_h}{\mu_h}$, $\mathcal{B} \ge \frac{\Lambda_v}{\mu_v}$, *if* $x_0 \in \mathbb{X}$ *and* $\|x_0\|_{\mathbb{X}} \le \min\{\mathcal{A}, \mathcal{B}\}$, *then* $0 \le S_h(t)$, $\int_0^\infty V_h(t,a)\mathrm{d}a$, $\int_0^\infty I_h(t,b)\mathrm{d}b$, $\int_0^\infty R_h(t,c)\mathrm{d}c \le \mathcal{A}$, $0 \le S_v,\ I_v \le \mathcal{B}$ *for all* $t \ge 0$.

Proof By Proposition 1, it concludes that $\mathbb{D}$ is positive invariant and Φ attracts every point in $\mathbb{D}$. Therefore, there exists $\mathcal{A} \ge \frac{\Lambda_h}{\mu_h}$ and $\mathcal{B} \ge \frac{\Lambda_v}{\mu_v}$ such that $0 \le Z_1(t) \le \mathcal{A}$, $0 \le Z_2(t) \le \mathcal{B}$ for all $t > 0$. This means that the conclusion of Proposition 2 is valid. The proof is completed. □

The following two lemmas are useful to prove the global stability of the steady state.

Lemma 1 (Fluctuation Lemma, Ref. [35]) *Suppose that* $M : \mathbb{R}^+ \to \mathbb{R}$ *be a bounded and continuously differentiable function, then, there exists sequences* $\{s_n\}$ *and* $\{t_n\}$ *such that* $s_n \to \infty$, $t_n \to \infty$, $M(s_n) \to M_\infty$, $M'(s_n) \to 0$, $M(t_n) \to M^\infty$, *and* $M'(t_n) \to 0$ *as* $n \to \infty$, *where* $M_\infty = \liminf_{t\to\infty} M(t)$, $M^\infty = \limsup_{t\to\infty} M(t)$.

Lemma 2 (Ref. [36]) *Suppose that* $f : \mathbb{R}^+ \to \mathbb{R}$ *is a bounded function and* $k \in L^1(\mathbb{R}^+)$, *then,* $\limsup_{t\to\infty} \int_0^t k(\theta) f(t-\theta)\mathrm{d}\theta \le f^\infty \|k\|_1$.

3 Global Behavior of the Model

In this section we focus on the local and global asymptotiical stability of the disease-free and endemic steady states of model (1).

3.1 Existence and Local Stability of Steady States

Model (1) always has the disease-free steady state $\mathcal{E}^0(S_h^0, V_h^0(a), 0, 0, S_v^0, 0)$, where

$$S_h^0 = \frac{\Lambda_h}{\mu_h + \psi_h(1-L)}, \quad V_h^0(a) = \psi_h S_h^0 \Omega(a), \quad S_v^0 = \frac{\Lambda_v}{\mu_v}.$$

The basic reproduction number $\mathcal{R}_0$ is defined as

$$\mathcal{R}_0 = \frac{\partial f(S_h^0, 0)}{\partial I_v} \frac{\eta \Lambda_v}{\mu_v^2} \frac{1}{1-NH}, \qquad \eta = \int_0^\infty \beta_v(b)\tau(b)\mathrm{d}b. \tag{8}$$

Next, we discuss the existence and uniqueness of the endemic steady state. Assume that $\mathcal{E}^*(S_h^*, V_h^*(\cdot), I_h^*(\cdot), R_h^*(\cdot), S_v^*, I_v^*)$ is the endemic steady state of model (1), then

$$\begin{cases} \Lambda_h - \mu_h S_h^* - f(S_h^*, I_v^*) - \psi_h S_h^* + \int_0^\infty \omega_h(a) V_h^*(a)\mathrm{d}a = 0, \\ \frac{\mathrm{d}}{\mathrm{d}a} V_h^*(a) = -(\mu_h + \omega_h(a)) V_h^*(a), \quad V_h^*(0) = \psi_h S_h^*, \\ \frac{\mathrm{d}}{\mathrm{d}b} I_h^*(b) = -(\mu_h + k_h(b) + v(b)) I_h^*(b), \quad I_h^*(0) = f(S_h^*, I_v^*) + \int_0^\infty r_h(c) R_h^*(c)\mathrm{d}c, \\ \frac{\mathrm{d}}{\mathrm{d}c} R_h^*(c) = -\left(\mu_h + r_h(c)\right) R_h^*(c), \quad R_h^*(0) = \int_0^\infty k_h(b) I_h^*(b)\mathrm{d}b, \\ \Lambda_v - \int_0^\infty \beta_v(b) S_v^* I_h^*(b)\mathrm{d}b - \mu_v S_v^* = 0, \\ \int_0^\infty \beta_v(b) S_v^* I_h^*(b)\mathrm{d}b - \mu_v I_v^* = 0. \end{cases} \tag{9}$$

Solving the second, third and fourth equations of (9), one can get

$$\begin{aligned} V_h^*(a) &= \psi_h S_h^* \Omega(a), \\ I_h^*(b) &= \left(f(S_h^*, I_v^*) + HN I_h^*(0)\Delta(c)\right)\tau(b), \\ R_h^*(c) &= H I_h^*(0)\Delta(c). \end{aligned} \tag{10}$$

Further, substituting (10) into the fifth and sixth equations of (9), we have

$$S_v^* = \frac{\Lambda_v - \mu_v I_v^*}{\mu_v}, \quad I_h^*(0) = \frac{\mu_v^2 I_v^*}{\eta(\Lambda_v - \mu_v I_v^*)}. \tag{11}$$

Combining $I_h^*(0) = f(S_h^*, I_v^*) + \int_0^\infty r_h(c) R_h^*(c)\mathrm{d}c$ and substituting the above formulations of S_v^*, $I_h^*(0)$ and $V_h^*(0)$ into the first Eq. of (9) and rearrange yields

$$S_h^* = \frac{\Lambda_h - I_h^*(0)(1-NH)}{\mu_h + \psi_h(1-L)} = \frac{\eta \Lambda_h(\Lambda_v - \mu_v I_v^*) - \mu_v^2(1-NH) I_v^*}{\eta(\mu_h + \psi_h(1-L))(\Lambda_v - \mu_v I_v^*)}. \tag{12}$$

Inserting (10)–(12) into the last Eq. of (9), it follows that

$$0 = \eta S_v^* I_h^*(0) - \mu_v I_v^* = \eta \frac{\Lambda_v - \mu_v I_v^*}{\mu_v} \frac{f(S_h^*, I_v^*)}{1-NH} - \mu_v I_v^*.$$

In the light of the above discussion, model (1) has a positive steady state (S_h^*, $V_h^*(a)$, $I_h^*(b)$, $R_h^*(c)$, S_v^*, I_v^*) if and only if I_v^* is a positive solution of function F, where

$$F(y) = \eta \frac{\Lambda_v - \mu_v y}{\mu_v} f\left(\frac{\Lambda_h - \frac{(1-NH)\mu_v^2 y}{\eta(\Lambda_v - \mu_v y)}}{\mu_h + \psi_h(1-L)}, y \right) \frac{1}{1-NH} - \mu_v y.$$

If $\mathcal{R}_0 \leq 1$, then, by using the monotonicity and the concavity of f in (H$_4$), one has

$$\begin{aligned} F(y) &\leq \eta \frac{\Lambda_v}{\mu_v} f\left(\frac{\Lambda_h}{\mu_h + \psi_h(1-L)}, y \right) \frac{1}{1-NH} - \mu_v y \\ &\leq \eta \frac{\Lambda_v}{\mu_v} \frac{1}{1-NH} \frac{\partial f(S_h^0, 0)}{\partial y} y - \mu_v y = \mu_v(\mathcal{R}_0 - 1) y, \end{aligned}$$

for $y > 0$. This indicates that model (1) has no positive steady state for $\mathcal{R}_0 \leq 1$.

Now, we turn to the case $\mathcal{R}_0 > 1$. Note that $F(0) = 0$,

$$\begin{aligned} &F\left(\frac{\Lambda_h \Lambda_v \eta}{\mu_v \Lambda_h + (1-NH)\mu_v^2} \right) \\ &= \frac{(1-\eta)\Lambda_h \Lambda_v + \Lambda_v(1-NH)\mu_v}{\mu_v(\Lambda_h + (1-NH)\mu_v)} f\left(0, \frac{\Lambda_h \Lambda_v \eta}{\mu_v \Lambda_h + (1-NH)\mu_v^2} \right) \\ &\quad - \frac{\Lambda_h \Lambda_v \eta}{\Lambda_h + (1-NH)\mu_v} = - \frac{\Lambda_h \Lambda_v \eta}{\Lambda_h + (1-NH)\mu_v} < 0, \end{aligned}$$

and

$$F'(0) = \frac{\eta \Lambda_v}{\mu_v(1-NH)} \frac{\partial f(S_h^0, 0)}{\partial y} - \mu_v = \mu_v(\mathcal{R}_0 - 1) > 0.$$

From the intermediate value theorem of continuous function, there is at least one positive root of function F, so model (1) has at least one positive steady state. Next, we focus on the uniqueness of positive steady state. Using the contradictory method, suppose that there is another positive steady state ($\bar{S}_h$, $\bar{V}_h$, $\bar{I}_h$, $\bar{R}_h$, $\bar{S}_v$, $\bar{I}_v$). Without loss of generality, assume that $\bar{I}_v > I_v^* > 0$. Let $l = \frac{\bar{I}_v}{I_v^*}$, by (H$_4$), we obtain

$$I_h^*(0) = \frac{f(S_h^*, I_v^*)}{1-NH} = \frac{1}{1-NH} f\left(S_h^*, \frac{\bar{I}_v}{l} \right) \geq \frac{1}{l} \frac{1}{1-NH} f(\bar{S}_h, \bar{I}_v) = \frac{1}{l} \bar{I}_h(0),$$

where $\bar{S}_h < S_h^*$. From the sixth Eq. of (9), it follows that

$$I_v^* = \frac{\frac{\Lambda_v}{\mu_v} \eta I_h^*(0)}{\mu_v + \eta I_h^*(0)} > \frac{\frac{\Lambda_v}{\mu_v} \eta \frac{1}{l} \bar{I}_h(0)}{\mu_v + \eta \frac{1}{l} \bar{I}_h(0)} = \frac{1}{l} \frac{\frac{\Lambda_v}{\mu_v} \eta \bar{I}_h(0)}{\mu_v + \eta \frac{1}{l} \bar{I}_h(0)} > \frac{1}{l} \frac{\frac{\Lambda_v}{\mu_v} \eta \bar{I}_h(0)}{\mu_v + \eta \bar{I}_h(0)} = \frac{1}{l} \bar{I}_v,$$

which is a contradiction.

Summing up the above discussion, we can come to the following conclusion.

Theorem 1 *If $\mathcal{R}_0 \leq 1$, then model (1) has the disease-free steady state $\mathcal{E}^0(S_h^0, V_h0(a), 0, 0, S_v^0, 0)$; if $\mathcal{R}_0 > 1$, then model (1) has a unique endemic steady state $\mathcal{E}^*(S_h^*, V_h^*(a), I_h^*(b), R_h^*(c), S_v^*, I_v^*)$ in addition to the $\mathcal{E}_0$, where I_v^* is a positive root of $F(I_v) = 0$.*

Next, we proceed to investigate the local stability of the steady state using the linearization method. Let $\bar{\mathcal{E}}(\bar{S}_h, \bar{V}_h(\cdot), \bar{I}_h(\cdot), \bar{R}_h(\cdot), \bar{S}_v, \bar{I}_v)$ be a steady state of (1), linearization at $\bar{\mathcal{E}}$ yields that

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -\left(\mu_h + \psi_h + \dfrac{\partial f(\bar{S}_h, \bar{I}_v)}{\partial S_h}\right) x_1(t) - \dfrac{\partial f(\bar{S}_h, \bar{I}_v)}{\partial I_v} x_6(t) \\ \qquad\qquad + \displaystyle\int_0^\infty \omega_h(a) x_2(t, a)\mathrm{d}a, \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) x_2(t, a) = -(\mu_h + \omega_h(a)) x_2(t, a), \quad x_2(t, 0) = \psi_h x_1(t), \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b}\right) x_3(t, b) = -(\mu_h + k_h(b) + \nu_h(b)) x_3(t, b), \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial c}\right) x_4(t, c) = -(\mu_h + r_h(c)) x_4(t, c), \quad x_4(t, 0) = \displaystyle\int_0^\infty k_h(b) x_3(t, b)\mathrm{d}b, \\ \dfrac{\mathrm{d}x_5(t)}{\mathrm{d}t} = -\displaystyle\int_0^\infty \beta_v(b)\bar{S}_v x_3(t, b)\mathrm{d}b - \int_0^\infty \beta_v(b)\bar{I}_h(b) x_5(t)\mathrm{d}b - \mu_v x_5(t), \\ \dfrac{\mathrm{d}x_6(t)}{\mathrm{d}t} = \displaystyle\int_0^\infty \beta_v(b)\bar{S}_v x_3(t, b)\mathrm{d}b + \int_0^\infty \beta_v(b)\bar{I}_h(b) x_5(t)\mathrm{d}b - \mu_v x_6(t), \end{cases} \tag{13}$$

and $x_3(t, 0) = \frac{\partial f(\bar{S}_h, \bar{I}_v)}{\partial S_h} x_1(t) + \frac{\partial f(\bar{S}_h, \bar{I}_v)}{\partial I_v} x_6(t) + \int_0^\infty r_h(c) x_4(t, c)\mathrm{d}c$.

Now, we look for the solution of system (13) for the form $x_1(t) = \bar{x}_1 e^{\lambda t}$, $x_2(t, a) = \bar{x}_2(a)e^{\lambda t}$, $x_3(t, b) = \bar{x}_3(b)e^{\lambda t}$, $x_4(t, c) = \bar{x}_4(c)e^{\lambda t}$, $x_5(t) = \bar{x}_5 e^{\lambda t}$, $x_6(t) = \bar{x}_6 e^{\lambda t}$, where $\bar{x}_i > 0$, $(i = 1, 5, 6)$, $\bar{x}_2(a)$, $\bar{x}_3(b)$ and $\bar{x}_4(c)$ will be determined later in the calculation. Then,

$$\begin{aligned} &\left(\lambda + \mu_h + \psi_h + \frac{\partial f(\bar{S}_h, \bar{I}_v)}{\partial S_h}\right) \bar{x}_1 + \frac{\partial f(\bar{S}_h, \bar{I}_v)}{\partial S_h} \bar{x}_6 \\ &\qquad\qquad - \int_0^\infty \omega_h(a)\bar{x}_2(0)\Omega(a)e^{-\lambda a}\mathrm{d}a = 0, \\ &\bar{x}_2(0) - \psi_h \bar{x}_1 = 0, \\ &\bar{x}_3(0) - \frac{\partial f(\bar{S}_h, \bar{I}_v)}{\partial S_h}\bar{x}_1 - \frac{\partial f(\bar{S}_h, \bar{I}_v)}{\partial I_v}\bar{x}_6 - \int_0^\infty r_h(c)\bar{x}_4(0)\Delta(c)e^{-\lambda c}\mathrm{d}c = 0, \\ &\bar{x}_4(0) - \int_0^\infty k_h(b)\bar{x}_3(0)\tau(b)e^{-\lambda b}\mathrm{d}b = 0, \end{aligned}$$

$$(\lambda+\mu_v)\bar{x}_5+\int_0^\infty \beta_v(b)\bar{S}_v\bar{x}_3(0)\tau(b)e^{-\lambda b}\mathrm{d}b+\int_0^\infty \beta_v(b)\bar{I}_h(b)\bar{x}_5\mathrm{d}b=0,$$
$$(\lambda+\mu_v)\bar{x}_6-\int_0^\infty \beta_v(b)\bar{S}_v\bar{x}_3(0)\tau(b)e^{-\lambda b}\mathrm{d}b-\int_0^\infty \beta_v(b)\bar{I}_h(b)\bar{x}_5\mathrm{d}b=0.$$

Theorem 2 *The disease-free steady state $\mathcal{E}^0$ of model (1) is locally asymptotically stable for $\mathcal{R}_0<1$and is unstable for $\mathcal{R}_0>1$.*

Proof Notice that $\frac{\partial f(S_h^0,0)}{\partial S_h}=0$ for $S_h\in\mathbb{R}^+$, we can acquire the characteristic equation at $\mathcal{E}^0$ as

$$(\lambda+\mu_v)C_1(\lambda)C_2(\lambda)=0, \tag{14}$$

where $C_1(\lambda)=\lambda+\mu_h+\psi_h[1-\hat{K}_1(\lambda)]$,

$$C_2(\lambda)=(\lambda+\mu_v)\left(1-\hat{K}_2(\lambda)\hat{K}_3(\lambda)\right)-S_v^0\frac{\partial f(S_h^0,0)}{\partial I_v}\hat{K}_4(\lambda),$$
$$\hat{K}_1(\lambda)=\int_0^\infty \omega_h(a)\Omega(a)e^{-\lambda a}\mathrm{d}a,$$
$$\hat{K}_2(\lambda)=\int_0^\infty k_h(b)\tau(b)e^{-\lambda b}\mathrm{d}b,$$
$$\hat{K}_3(\lambda)=\int_0^\infty r_h(c)\Delta(c)e^{-\lambda c}\mathrm{d}c,$$
$$\hat{K}_4(\lambda)=\int_0^\infty \beta_v(b)\tau(b)e^{-\lambda b}\mathrm{d}b.$$

We claim that all roots of $C_1(\lambda)=0$ have negative real parts. As a matter of fact, if λ_0 with nonnegative real part, then $\psi_h<|\lambda_0+\mu_h+\psi_h|=\psi_h|\hat{K}_1(\lambda)|\le\psi_h$. This is a contradiction. Thus, the claim is true.

If $\mathcal{R}_0>1$, then $C_2(0)=\mu_v(1-\mathcal{R}_0)<0$. By incorporating $\lim_{\lambda\to 0}C_2(\lambda)=+\infty$ and the intermediate value theorem of continuous function, we know that $C_2(\lambda)=0$ has a positive root. Hence, $\mathcal{E}^0$ is unstable if $\mathcal{R}_0>1$.

Now, we claim that all roots of $C_2(\lambda)=0$ have negative real parts for $\mathcal{R}_0<1$. Otherwise, let λ_0 be a root of $C(\lambda)=0$ with $Re(\lambda_0)\ge 0$, then $\mu_v(1-NH)\le|(\lambda_0+\mu_v)(1-\hat{K}_2(\lambda_0)\hat{K}_3(\lambda_0))|$ and

$$\left|S_v^0\hat{K}_4(\lambda)\frac{\partial f(S_h^0,0)}{\partial I_v}\right|\le\eta S_v^0\frac{\partial f(S_h^0,0)}{\partial I_v}=\mu_v(1-NH)\mathcal{R}_0<\mu_v(1-NH).$$

This is a contradiction. Therefore, all roots of (14) have negative real parts. This implies $\mathcal{E}^0$ is locally asymptotically stable for $\mathcal{R}_0<1$. The proof is finished. □

3.2 Uniform Persistence

To obtain the global dynamics of the steady state of model (1), we introduce the asymptotic smoothness of solution semiflow. Denote operators $\Phi_1(t, x_0)$, $\Phi_2(t, x_0)$: $\mathbb{R}^+ \times \mathbb{X}_+ \to \mathbb{X}_+$ as follows $\Phi_1(t, x_0) := (0, \tilde{\varphi}_V(t, \cdot), \tilde{\varphi}_I(t, \cdot), \tilde{\varphi}_R(t, \cdot), 0, 0)$, $\Phi_2(t, x_0) := (S_h(t), \tilde{V}_h(t, \cdot), \tilde{I}_h(t, \cdot), \tilde{R}_h(t, \cdot), S_v(t), I_v(t))$, where,

$$\tilde{\psi}_V(t,a) = \begin{cases} 0, & t > a \geq 0, \\ V_h(t,a), & a \geq t \geq 0, \end{cases} \quad \text{and}$$
$$\tilde{\psi}_I(t,b) = \begin{cases} 0, & t > b \geq 0, \\ I_h(t,b), & b \geq t \geq 0, \end{cases} \tag{15}$$

$$\tilde{\psi}_R(t,c) = \begin{cases} 0, & t > c \geq 0, \\ R_h(t,c), & c \geq t \geq 0, \end{cases} \quad \text{and}$$
$$\tilde{V}_h(t,a) = \begin{cases} V_h(t,a), & t > a \geq 0, \\ 0, & a \geq t \geq 0, \end{cases} \tag{16}$$

$$\tilde{I}_h(t,b) = \begin{cases} I_h(t,b), & t > b \geq 0, \\ 0, & b \geq t \geq 0, \end{cases} \quad \text{and}$$
$$\tilde{R}_h(t,b) = \begin{cases} R_h(t,c), & t > c \geq 0, \\ 0, & c \geq t \geq 0. \end{cases} \tag{17}$$

It's obvious that $\Phi(t, x_0) = \Phi_1(t, x_0) + \Phi_2(t, x_0)$ for all $t \geq 0$.

Proposition 3 *For any $r_1 > 0$, if $x_0 \in \mathbb{X}_+$ and $\|x_0\|_{\mathbb{X}} \leq r_1$, then $\|\Phi_1(t, x_0)\|_{\mathbb{X}} \leq e^{-\mu_h t} r_1$ for $t \geq 0$.*

Proof Based on (4), (5), (15) and (16), one can get

$$\tilde{\psi}_V(t,a) = \begin{cases} 0, & t > a \geq 0, \\ V_{h0}(a-t)\dfrac{\Omega(a)}{\Omega(a-t)}, & a \geq t \geq 0, \end{cases}$$
$$\tilde{\psi}_I(t,b) = \begin{cases} 0, & t > b \geq 0, \\ I_{h0}(b-t)\dfrac{\tau(b)}{\tau(b-t)}, & b \geq t \geq 0, \end{cases}$$
$$\tilde{\psi}_R(t,c) = \begin{cases} 0, & t > c \geq 0, \\ R_{h0}(c-t)\dfrac{\Delta(c)}{\Delta(c-t)}, & c \geq t \geq 0. \end{cases}$$

For $x_0 \in \mathbb{X}_+$ and $\|x_0\|_{\mathbb{X}} \leq r_1$, we have

$$
\begin{aligned}
&\|\Phi_1(t,x_0)\|_{\mathbb{X}}\\
&=\int_0^\infty \left|V_{h0}(\sigma)\frac{\Omega(\sigma+t)}{\Omega(\sigma)}\right| \mathrm{d}\sigma+\int_0^\infty \left|I_{h0}(\sigma)\frac{\tau(\sigma+t)}{\tau(\sigma)}\right| \mathrm{d}\sigma\\
&\quad+\int_0^\infty \left|R_{h0}(\sigma)\frac{\Delta(\sigma+t)}{\Delta(\sigma)}\right| \mathrm{d}\sigma\\
&=\int_0^\infty \left|V_{h0}(\sigma)e^{-\int_\sigma^{t+\sigma}\varepsilon_1(s)\mathrm{d}s}\right| \mathrm{d}\sigma\\
&\quad+\int_0^\infty \left|I_{h0}(\sigma)e^{-\int_\sigma^{t+\sigma}\varepsilon_2(s)\mathrm{d}s}\right| \mathrm{d}\sigma+\int_0^\infty \left|R_{h0}(\sigma)e^{-\int_\sigma^{t+\sigma}\varepsilon_3(s)\mathrm{d}s}\right| \mathrm{d}\sigma.
\end{aligned}
$$

It follows from $\varepsilon_1(a)$, $\varepsilon_2(b)$ and $\varepsilon_3(c)\geq \mu_h+\mu_0$, for $a,b,c\in\mathbb{R}^+$ that one has

$$
\begin{aligned}
\|\Phi_1(t,x_0)\|_{\mathbb{X}} &\leq e^{-(\mu_h+\mu_0)t}\left(\int_0^\infty |V_{h0}(\sigma)|\mathrm{d}\sigma\int_0^\infty |I_{h0}(\sigma)|\mathrm{d}\sigma+\int_0^\infty |R_{h0}(\sigma)|\mathrm{d}\sigma\right)\\
&\leq e^{-(\mu_h+\mu_0)t}\|x_0\|_{\mathbb{X}}\leq r_1 e^{-(\mu_h+\mu_0)t},
\end{aligned}
$$

for $t\in\mathbb{R}^+$. The proof is completed. □

Due to, $L^1_+(\mathbb{R}^+)$ is the infinite dimensional Banach space, and boundness does not imply compactness in $L^1_+(\mathbb{R}^+)$, we introduce the following proposition to guarantee tightness $L^1_+(\mathbb{R}^+)$.

Proposition 4 *Semiflow* $\{\Phi(t,x_0)\}_{t\geq 0}$ *is completely continuous.*

Proof By the Proposition 2, $S_h(t)$ remains in the compact set $[0,\mathcal{A}]$, $S_v(t)$ and $I_v(t)$ remain in the compact set $[0,\mathcal{B}]$. Therefore, we just need to prove that $\tilde{V}_h(t,\cdot)$, $\tilde{I}_h(t,\cdot)$ and $\tilde{R}_h(t,\cdot)$ belong to the compact set of $L^1_+(\mathbb{R}^+)$, without dependence on $x_0\in\mathbb{X}_+$. Since $\tilde{V}_h(t,\cdot)$, $\tilde{I}_h(t,\cdot)$ and $\tilde{R}_h(t,\cdot)$ are bounded, combined with the Fréchet-Kolmogorov Theorem (see Ref. [43]), it need to prove that the condition $\lim_{l\to 0^+}\int_0^\infty |\tilde{y}_i(t,a+l)-\tilde{y}_i(t,a)|\,\mathrm{d}a=0$ uniformly with respect to $x_0\in\mathbb{X}_+$.

For $l\in(0,t)$, by (15) and (H$_3$), one has

$$
\begin{aligned}
&\int_0^\infty \left|\tilde{V}_h(t,a+l)-\tilde{V}_h(t,a)\right| \mathrm{d}a\\
&\quad=\int_0^{t-l} |V_h(t,a+l)-V_h(t,a)|\,\mathrm{d}a\\
&\qquad+\int_{t-l}^{t} |0-V_h(t,a)|\,\mathrm{d}a\leq \psi_h\mathcal{A}l\\
&\qquad+\int_0^{t-l} V_h(t-a-l,0)\,|\Omega(a+l)-\Omega(a)|\,\mathrm{d}a\\
&\qquad+\int_0^{t-l} |V_h(t-a-l,0)-V_h(t-a,0)|\,\Omega(a)\mathrm{d}a\\
&\quad=\psi_h\mathcal{A}l+\int_0^{t-l} V_h(t-a-l,0)\,|\Omega(a+l)-\Omega(a)|\,\mathrm{d}a
\end{aligned}
$$

$$+\int_0^{t-l} \psi_h \left|S_h(t-a-l) - S_h(t-a)\right| \Omega(a)\mathrm{d}a.$$

Note that $0 \leq \Omega(a) \leq e^{-(\mu_0+\mu_h)a} \leq 1$, we obtain

$$\int_0^{t-l} |\Omega(a+l) - \Omega(a)|\, \mathrm{d}a = \int_0^{l} \Omega(a)\mathrm{d}a - \int_{t-l}^{l} \Omega(a)\mathrm{d}a \leq l$$

and

$$\left|\frac{\mathrm{d}S_h(t)}{\mathrm{d}t}\right| \leq \Lambda_h + (\mu_h + \psi_h + \bar{\omega})\mathcal{A} + f(\mathcal{A}, \mathcal{B}) := L_S.$$

Therefore,

$$\int_0^{\infty} \left|\tilde{V}_h(t, a+l) - \tilde{V}_h(t, a)\right| \mathrm{d}a \leq 2\psi_h \mathcal{A} l + \frac{L_S \psi_h l}{\mu_h},$$

which indicates $\lim_{l\to 0^+} \int_0^{\infty} \left|\tilde{V}_h(t, a+l) - \tilde{V}_h(t, a)\right| \mathrm{d}a = 0$. Similarly, it also has

$$\lim_{l\to 0^+} \int_0^{\infty} \left|\tilde{I}_h(t, b+l) - \tilde{I}_h(t, b)\right| \mathrm{d}b = 0, \ \lim_{l\to 0^+} \int_0^{\infty} \left|\tilde{R}_h(t, c+l) - \tilde{R}_h(t, c)\right| \mathrm{d}c = 0.$$

This completes to verify that $\tilde{V}_h(t, a)$, $\tilde{I}_h(t, b)$ and $\tilde{R}_h(t, c)$ satisfy the above condition. □

Now, we turn to the uniform persistence of (1). Define $\rho : \mathbb{X}_+ \to \mathbb{R}^+$ as

$$\rho(S_h, V_h, I_h, R_h, S_v, I_v) = \int_0^{\infty} \beta_v(b) I_h(\cdot, b)\mathrm{d}b, \qquad (S_h, V_h, I_h, R_h, S_v, I_v) \in \mathbb{X}_+,$$

and

$$\mathbb{X}_0 = \{(S_{h0}, V_{h0}, I_{h0}, R_{h0}, S_{v0}, I_{v0}) \in \mathbb{D} : \rho(\Phi(t, (S_{h0}, V_{h0}, I_{h0}, R_{h0}, S_{v0}, I_{v0}))) > 0, \ t \in \mathbb{R}^+\},$$

and $\partial\mathbb{X}_0 = \mathbb{X}_+ \backslash \mathbb{X}_0$. Before proving uniform persistence, we give the following Proposition 5.

Proposition 5 *The $\mathcal{E}^0$ of model (1) is globally attractive for the $\{\Phi(t)\}_{t\geq 0}$ restricted to $\partial\mathbb{X}_0$.*

Proof Similar to the analysis of Ref. [17], $\Phi(t, \mathbb{X}_0) \subset \mathbb{X}_0$ and $\Phi(t, \partial\mathbb{X}_0) \subset \partial\mathbb{X}_0$, that is, $\mathbb{X}_0$ and $\partial\mathbb{X}_0$ is the positive invariant set for semiflow $\Phi(t)$ of model (1). Based on the positive invariance of $\partial\mathbb{X}_0$ for $(S_{h0}, \cdots, I_{v0}) \in \partial\mathbb{X}_0$, it yields that $\Phi(t, (S_{h0}, \cdots, I_{v0})) \in \partial\mathbb{X}_0$. Therefore, $\rho(\Phi(t, (S_{h0}, \cdots, I_{v0}))) = 0$, i.e., $\int_0^{\infty} \beta_v(b) I_h(\cdot, b)\mathrm{d}b = 0$. This implies that $I_h(t,b) \equiv 0$ for $t,\ b \geq 0$. Substituting it into the fifth and seventh equations of model (1), one can derives $\lim_{t\to\infty} R_h(t,b) = 0$ and $\lim_{t\to\infty} I_v(t) = 0$. In conclusion, from the above results and the first, second and sixth equations of model (1), one has $\lim_{t\to\infty} S_h(t) = S_h^0$, $\lim_{t\to\infty} V_h(t,a) = V_h^0(a)$ and $\lim_{t\to\infty} S_v(t) = S_v^0$. This means $\lim_{t\to\infty} \Phi(t,(S_{h0},\cdots,I_{v0})) = \mathcal{E}^0$. This completes the proof. □

Theorem 3 *If* $\mathcal{R}_0 > 1$, *then semiflow* $\{\Phi(t,x_0)\}_{t\geq 0}$ *uniformly weakly* ρ*-persistent, i.e., for* $(S_{h0}, V_{h0}, I_{h0}, R_{h0}, S_{v0}, I_{v0}) \in \mathbb{X}_0$, *there is a positive constant* ε, *independent of the initial conditions, such that* $\limsup_{t\to\infty} \rho(\Phi(t_0,(S_{h0}, V_{h0}, I_{h0}, R_{h0}, S_{v0}, I_{v0}))) > \varepsilon$.

Proof For contradiction, for any $\varepsilon > 0$, there exists a positive $x^{\varepsilon} = (S_{h0}, V_{h0}, I_{h0}, R_{h0}, S_{v0}, I_{v0}) \in \mathbb{D}\setminus\mathbb{X}_0$ such that $\limsup_{t\to\infty}\int_0^\infty \beta_v(b)I_h(t,b)\mathrm{d}b \leq \varepsilon$. Since $\mathcal{R}_0 > 1$, we can select $l > 0$ such that

$$\frac{\hat{\eta}(\lambda)}{1-\hat{N}(\lambda)\hat{H}(\lambda)}\frac{\partial f(\omega_2(l),\omega_1(l))}{\partial I_v}\frac{N_v-\omega_1(l)}{\mu_v} > 1 \quad \text{and} \quad \omega_2(l) := \frac{\Lambda_h - C_v\omega_1(l)}{\mu_h+\psi_h}, \tag{18}$$

where $\omega_1(l) = \frac{N_v l}{\mu_v + l}$. By the hypothesis, there exists $x^{\frac{l}{2}} \in \mathbb{D}\setminus\mathbb{X}_0$ (for convenience, the remaining part of the proof is denoted by x) such that $\limsup_{t\to\infty}\int_0^\infty \beta_v(b)I_h(t,b)\mathrm{d}b \leq \frac{l}{2}$. Then there exists $t_0 \in \mathbb{R}^+$ such that $\int_0^\infty \beta_v(b)I_h(t,b)\mathrm{d}b \leq l$ for all $t \geq t_0$. Without loss of generality, suppose that $t_0 = 0$, because we can substitute $\Phi(t_0,x)$ for x. From the seventh Eq. of (1), one has

$$\frac{\mathrm{d}I_v(t)}{\mathrm{d}t} = (N_v - I_v(t))\int_0^\infty \beta_v(b)I_h(t,b)\mathrm{d}b - \mu_v I_v(t) \leq N_v l - (\mu+l)I_v(t),$$

that is, $\limsup_{t\to\infty} I_v(t) \leq \frac{N_v l}{\mu_v+l}$. Again, without loss of generality, assume that

$$I_v(t) \leq \omega_1(l) \qquad \text{for } t \in \mathbb{R}_+. \tag{19}$$

Combining (H_3) and the first equation of (1), we have

$$\begin{aligned}\frac{\mathrm{d}S_h(t)}{\mathrm{d}t} &\geq \Lambda_h - \mu_h S_h(t) - f(S_h(t),I_v(t)) - \psi_h S_h(t)\\ &\geq \Lambda_h - C_v(K)\omega_1(l) - (\mu_h+\psi_h)S_h(t),\end{aligned}$$

it follows that $\liminf_{t\to\infty} S_h(t) \geq \frac{\Lambda_h - C_v(K)\omega_1(l)}{\mu_h+\psi_h}$. Again, without loss of generality, suppose that

$$S_h(t) \geq \omega_2(l) \qquad \text{for } t \in \mathbb{R}^+. \tag{20}$$

According to the seventh Eq. of (1), we have

$$\frac{\mathrm{d}I_v(t)}{\mathrm{d}t} \geq \int_0^\infty \beta_v(b)(N_v - \omega_1(l))I_h(t,b)\mathrm{d}b - \mu_v I_v(t),$$

which implies that $\liminf_{t\to\infty} I_v(t) \geq \frac{1}{\mu_v}\int_0^\infty \beta_v(b)(N_v-\omega_1(l))I_h(t,b)\mathrm{d}b$. Again, without loss of generality, assume that

$$I_v(t) \geq \frac{\int_0^\infty \beta_v(b)(N_v-\omega_1(l))I_h(t,b)\mathrm{d}b}{\mu_v} \quad \text{for } t\in\mathbb{R}^+.$$

Combining Eqs. (19) and (20), we have

$$I_h(t,0) \geq \frac{\partial f(\omega_2(l),\omega_1(l))}{\partial I_v} I_v(t) + \int_0^t r_h(c)\Delta(c) \int_0^{t-c} k_h(b) I_h(t-b-c,0)\tau(b)\mathrm{d}b\mathrm{d}c,$$

for $t\in\mathbb{R}^+$. Applying the Laplace transform to each side of the above inequality, one can get

$$\begin{aligned}
\mathcal{L}[\hat{I}_h(t,0)] &\geq \int_0^\infty e^{-\lambda s}\frac{\partial f(\omega_2(l),\omega_1(l))}{\partial I_v} I_v(s)\mathrm{d}s + \mathcal{L}[\hat{I}_h(t,0)] \\
&\quad \int_0^\infty r_h(c)\Delta(c)e^{-\lambda c}\mathrm{d}c \int_0^\infty k_h(b)\tau(b)e^{-\lambda b}\mathrm{d}b \\
&\geq \frac{1}{1-\hat{N}(\lambda)\hat{H}(\lambda)}\frac{\partial f(\omega_2(l),\omega_1(l))}{\partial I_v}\int_0^\infty e^{-\lambda t} I_v(t)\mathrm{d}t \\
&\geq \frac{1}{1-\hat{N}(\lambda)\hat{H}(\lambda)}\frac{\partial f(\omega_2(l),\omega_1(l))}{\partial I_v}\frac{N_v-\omega_1(l)}{\mu_v}\int_0^\infty \beta_v(b)\tau(b)e^{-\lambda b}\mathrm{d}b \\
&\quad \int_0^\infty e^{-\lambda t} I_h(t,0)\mathrm{d}t.
\end{aligned}$$

Here, $\hat{H}(\lambda) = \int_0^\infty k_h(b)\tau(b)e^{-\lambda b}\mathrm{d}b$, $\hat{N}(\lambda) = \int_0^\infty r_h(c)\Delta(c)e^{-\lambda c}\mathrm{d}b$, $\hat{\eta}(\lambda) = \int_0^\infty \beta_v(b)\tau(b)e^{-\lambda b}\mathrm{d}b$ and $\mathcal{L}[\hat{I}_h(t,0)]$ denotes the Laplace transform of $I_h(t,0)$, which is strictly positive. Dividing both sides of the above inequality by $\mathcal{L}[\hat{I}_h(t,0)]$, we have

$$1 \geq \frac{(N_v-\omega_1(l))\hat{\eta}(\lambda)}{\mu_v(1-\hat{N}(\lambda)\hat{H}(\lambda))}\frac{\partial f(\omega_2(l),\omega_1(l))}{\partial I_v},$$

which contradicts with Eq. (18). This completes the proof. □

Theorem 4 *Semiflow* $\{\Phi(t,x_0)\}_{t\geq 0}$ *is uniformly strongly* ρ*-persistent for* $\mathcal{R}_0 > 1$. *That is, there exists a positive constant* ε *such that* $\liminf_{t\to\infty}\rho(\Phi(t_0,(S_{h0},V_{h0},I_{h0},R_{h0},S_{v0},I_{v0}))) > \varepsilon$.

The proof method is consistent with that in Ref. [37] and the proof is omitted here.

Theorem 5 *If* $\mathcal{R}_0 > 1$, *then model* (1) *is uniformly persistent, i.e., there exists a constant* $\epsilon > 0$ *such that for every* $x_0 \in \mathbb{X}_0$, $\liminf_{t\to\infty}(S_h(t), \|V_h(t,\cdot)\|_1, \|I_h(t,\cdot)\|_1, \|R_h(t,\cdot)\|_1, S_v(t), I_v(t)) \geq \epsilon$, *where* $\|f(t,\cdot)\|_1 = \int_0^\infty f(t,x)\mathrm{d}x$, $f(t,\cdot) = V_h(t,\cdot)$, $I_h(t,\cdot)$ *and* $R_h(t,\cdot)$, *respectively.*

Proof It follows from (H_3) and the first Eq. of (1) that

$$\frac{\mathrm{d}S_h(t)}{\mathrm{d}t} \geq \Lambda_h - (\mu_h + C_h(K) + \psi_h)S_h(t), \quad \frac{\mathrm{d}S_v(t)}{\mathrm{d}t} \geq \Lambda_v - \mu_v S_v(t),$$

which indicates that $\liminf_{t\to\infty} S_h(t) \geq \frac{\Lambda_h}{(\mu_h+\psi_h+C_h(K))} := \epsilon_1$, $\liminf_{t\to\infty} S_v(t) \geq \frac{\Lambda_v}{\mu_v} := \epsilon_3$. By Theorem 4, there exists a $\epsilon_2 > 0$ such that $\liminf_{t\to\infty} \int_0^\infty \beta_v(b)I_h(t,b)\mathrm{d}b \geq \epsilon_2$. Then, for $l_1 \in (0, \epsilon_2)$, there is a constant $T_1 > 0$ such that $\int_0^\infty \beta_v(b)I_h(t,b)\mathrm{d}b \geq \epsilon_2 - l_1$ for $t \geq T_1$. It follows from

$$\frac{\mathrm{d}I_v(t)}{\mathrm{d}t} \geq N_v(\epsilon_2 - l_1) - (\mu_v + \epsilon_2 - l_1)I_v(t) \quad \text{for all } t > T_1,$$

that $\liminf_{t\to\infty} I_v(t) \geq \frac{N_v(\epsilon_2-l_1)}{(\mu_v+\epsilon_2-l_1)}$. Since l_1 is arbitrary, we can get $\liminf_{t\to\infty} I_v(t) \geq \frac{N_v\epsilon_2}{(\mu_v+\epsilon_2)} := \epsilon_4$. For any $l_2 \in (0, \min\{\epsilon_1, \epsilon_4\})$, there exists $T_2 > 0$ such that $S_h(t) \geq \epsilon_1 - l_2$, $I_v(t) \geq \epsilon_4 - l_2$ for all $t \geq T_2$. Then, for all $t \geq T_2$,

$$\begin{aligned}\|I_h(t,b)\|_1 &\geq \int_0^{t-T_2} \tau(b) f(S_h(t-b), I_v(t-b))\mathrm{d}b \\ &\geq \int_0^{t-T_2} \tau(b) f((\epsilon_1 - l_2), (\epsilon_4 - l_2))\mathrm{d}b,\end{aligned}$$

which imply that $\liminf_{t\to\infty} \|I_h(t,\cdot)\| \geq \|\tau\|_1 f((\epsilon_1 - l_1), (\epsilon_4 - l_2))$. Similarly, $\liminf_{t\to\infty} \|R_h(t,\cdot)\|_1 \geq \|\Delta k_h \tau\|_1 f(\epsilon_1, \epsilon_4)$, and $\liminf_{t\to\infty} \|V_h(t,\cdot)\|_1 \geq \psi_h\epsilon_1\|\Omega(a)\|_1$. Letting $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ immediately completes the proof. □

3.3 Global Stability

We first show the global stability of the disease-free steady state $\mathcal{E}^0$ by using Fluctuation Lemma, that is Lemma 1.

Theorem 6 *Assume that $\mathcal{R}_0 < 1$, then $\mathcal{E}^0$ of model (1) is globally asymptotically stable.*

Proof Combining the positive invariance and attractive properties of $\mathbb{D}$ with the Theorem 2, we just need to prove that $\{\mathcal{E}^0\}$ attracts to $\mathbb{D}$. Suppose $(S_h(t), V_h(t,a), I_h(t,b), R_h(t,c), S_v(t), I_v(t))$ be the solution of model (1) with $(S_{h0}, V_{h0}(\cdot), I_{h0}(\cdot), R_{h0}(\cdot), S_{v0}, I_{v0}) \in \mathbb{D}$. From the Eq. (5), one has

$$I_h(t-b, 0) \leq \frac{\partial f(S_h^0, 0)}{\partial I_v} I_v + \int_0^\infty r_h(c)R_h(t-b, c)\mathrm{d}c. \tag{21}$$

Combining the fifth Eq. of (1) and (H_3), we can get

$$\int_0^\infty r_h(c)R_h(t,c)\mathrm{d}c = \int_0^t r_h(c)R_h(t-c, 0)\Delta(c)\mathrm{d}c$$

$$
\begin{aligned}
&+\int_t^{\infty} r_h(c)\frac{\Delta(c)}{\Delta(c-t)}R_{h0}(c-t)\mathrm{d}c\\
&\le \int_0^t r_h(c)R_h(t-c,0)\Delta(c)\mathrm{d}c + e^{-\mu_h t}\|r_h\|_\infty\|R_{h0}\|_1\\
&= \int_0^t r_h(c)\Delta(c)\bigg(\int_0^{t-c} k_h(b)I_h(t-c-b,0)\tau(b)\mathrm{d}b\\
&\quad+\int_{t-c}^{\infty} k_h(b)\frac{\tau(b)}{\tau(b-t+c)}I_{h0}(b-t+c)\mathrm{d}b\bigg)\mathrm{d}c\\
&\quad+e^{-\mu_h t}\|r_h\|_\infty\|R_{h0}\|_1\\
&\le \int_0^t r_h(c)\Delta(c)\bigg(\int_0^{t-c} k_h(b)I_h(t-c-b,0)\tau(b)\mathrm{d}b\\
&\quad+e^{-\mu_h(t-c)}\|k_h\|_\infty\|I_{h0}\|_1\bigg)\mathrm{d}c + e^{-\mu_h t}\|r_h\|_\infty\|R_{h0}\|_1.
\end{aligned}
$$

Applying Lemma 1 and Lemma 2,

$$
\left(\int_0^{\infty} r_h(c)R_h(\cdot,c)\mathrm{d}c\right)^{\infty} \le N\left(\int_0^{\infty} k_h(b)I_h(\cdot,0)\tau(b)\mathrm{d}b\right)^{\infty}. \tag{22}
$$

According to the seventh equation of the model (1), we have

$$
\begin{aligned}
I_v(t) &\le e^{-\mu_v t}I_v(0) + \frac{\Lambda_v}{\mu_v}\int_0^t e^{-\mu_v(t-s)}\int_0^{\infty}\beta_v(b)I_h(s,b)\mathrm{d}b\mathrm{d}s\\
&= e^{-\mu_v t}I_v(0) + \frac{\Lambda_v}{\mu_v}\int_0^t e^{-\mu_v s}\int_0^{\infty}\beta_v(b)I_h(t-s,b)\mathrm{d}b\mathrm{d}s.
\end{aligned}
$$

It yields from Lemma 1 that $(I_v)^{\infty} \le \frac{\Lambda_v}{\mu_v^2}\left(\int_0^{\infty}\beta_v(b)I_h(\cdot,b)\mathrm{d}b\right)^{\infty}$. Further,

$$
\begin{aligned}
\int_0^{\infty}\beta_v(b)I_h(t,b)\mathrm{d}b &= \int_0^t \beta_v(b)I_h(t-b,0)\tau(b)\mathrm{d}b\\
&\quad+\int_t^{\infty}\beta_v(b)\frac{\tau(b)}{\tau(b-t)}I_{h0}(b-t)\mathrm{d}b\\
&\le \int_0^t \beta_v(b)I_h(t-b,0)\tau(b)\mathrm{d}b + e^{-\mu_v t}\|\beta_v\|_\infty\|I_{h0}\|_1.
\end{aligned} \tag{23}
$$

Combined with (21), (22) and (23), it follows that

$$
(I_h(\cdot,0))^{\infty} \le \frac{1}{1-NH}\frac{\eta\Lambda_v}{\mu_v^2}\frac{\partial f(S_h^0,0)}{\partial I_v}(I_h(\cdot,0))^{\infty} = \mathcal{R}_0(I_h(\cdot,0))^{\infty}. \tag{24}
$$

To summarize, we obtained $(I_h(\cdot,0))^{\infty} \le \mathcal{R}_0(I_h(\cdot,0))^{\infty}$. Because $\mathcal{R}_0 < 1$, we can get $(I_h(\cdot,0))^{\infty} = 0$. Further, one can get $\lim_{t\to\infty} I_h(t,b) = 0$, $\lim_{t\to\infty} R_h(t,c) = 0$ and $\lim_{t\to\infty} I_v(t) = 0$. Finally, we come to the proof that $\lim_{t\to\infty} S_h(t) = S_h^0$,

$\lim_{t\to\infty} V_h(t,a) = V_h^0(a)$ and $\lim_{t\to\infty} S_v(t) = S_v^0$. Due to $(S_h)^\infty \le S_h^0$, $(S_v)^\infty \le S_v^0$, it is only necessary to prove that $(S_h)_\infty \ge S_h^0$, $(S_v)_\infty \ge S_v^0$. From Lemma 1, there exists a sequence $\{\tilde{t}_n\}$ such that $\tilde{t}_n \to \infty$, $S_h(\tilde{t}_n) \to (S_h)_\infty$, $S_v(\tilde{t}_n) \to (S_v)_\infty$ and $\frac{\mathrm{d}S_h(\tilde{t}_n)}{\mathrm{d}t} \to 0$, $\frac{\mathrm{d}S_v(\tilde{t}_n)}{\mathrm{d}t} \to 0$ as $n \to \infty$. And then, from the first and sixth equations of model (1), it follows that

$$\begin{cases} \dfrac{\mathrm{d}S_h(\tilde{t}_n)}{\mathrm{d}t} = \Lambda_h - (\mu_h + \psi_h)S_h(\tilde{t}_n) - f(S_h(\tilde{t}_n), I_v(\tilde{t}_n)) + \displaystyle\int_0^\infty \omega_h(a)V_h(\tilde{t}_n, a)\mathrm{d}a, \\ \dfrac{\mathrm{d}S_v(\tilde{t}_n)}{\mathrm{d}t} = \Lambda_v - \displaystyle\int_0^\infty \beta_v(b)S_v(\tilde{t}_n)I_h(\tilde{t}_n, b)\mathrm{d}b - \mu_v S_v(\tilde{t}_n). \end{cases}$$

When $n \to \infty$, one has $0 = \Lambda_h - (\mu_h + \psi_h)(S_h)_\infty + (S_h)_\infty \int_0^\infty \psi_h\omega_h(a)\Omega(a)\mathrm{d}a$, $0 = \Lambda_v - \mu_v(S_v)_\infty$. Then

$$\lim_{t\to\infty} S_h(t) = \frac{\Lambda_h}{\mu_h + \psi_h(1-L)}, \qquad \lim_{t\to\infty} S_v(t) = \frac{\Lambda_v}{\mu_v}.$$

According to (4), it is easy to derive $\lim_{t\to\infty} V_h(t,a) = \frac{\psi_h\Lambda_h\Omega(a)}{\mu_h+\psi_h(1-L)}$. Thus, $\lim_{t\to\infty}(S_h(t), V_h(t,\cdot), I_h(t,\cdot), R_h(t,\cdot), S_v(t), I_v(t)) = \mathcal{E}^0$. The proof is completed. □

According to Proposition 1, when t tends to ∞, it is obtained that that $N_v(t)$ tends to a constant N_v and $S_v(t) = N_v - I_v(t)$ for $t \ge 0$. Therefore, to simplify theoretical calculations, we only discuss the global asymptotical stability of the endemic steady state $\mathcal{E}^{**}(S_h^*, V_h^*(\cdot), I_h^*(\cdot), R_h^*(\cdot), I_v^*)$ of the limiting system (25) of model (1)

$$\begin{cases} \dfrac{\mathrm{d}S_h(t)}{\mathrm{d}t} = \Lambda_h - \mu_h S_h(t) - f(S_h(t), I_v(t)) - \psi_h S_h(t) + \displaystyle\int_0^\infty \omega_h(a)V_h(t,a)\mathrm{d}a, \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) V_h(t,a) = -(\mu_h + \omega_h(a))V_h(t,a),\ V_h(t,0) = \psi_h S_h(t), \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b}\right) I_h(t,b) = -(\mu_h + k_h(b) + \nu_h(b))I_h(t,b), \\ I_h(t,0) = f(S_h(t), I_v(t)) + \displaystyle\int_0^\infty r_h(c)R_h(t,c)\mathrm{d}c, \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial c}\right) R_h(t,c) = -(\mu_h + r_h(c))R_h(t,c),\ R_h(t,0) = \displaystyle\int_0^\infty k_h(b)I_h(t,b)\mathrm{d}b, \\ \dfrac{\mathrm{d}I_v(t)}{\mathrm{d}t} = \displaystyle\int_0^\infty \beta_v(b)(N_v - I_v(t))I_h(t,b)\mathrm{d}b - \mu_v I_v(t). \end{cases} \tag{25}$$

Define that function $g : \mathbb{R}^+ \to \mathbb{R}^+$ by $g(x) = x - 1 - \ln x$, $L(a) = \int_a^\infty \omega_h(\theta)e^{-\int_a^\theta \varepsilon_1(s)\mathrm{d}s}\mathrm{d}\theta$, and $\eta(b) = \int_b^\infty \beta_v(b)e^{-\int_b^\theta \varepsilon_2(s)\mathrm{d}s}\mathrm{d}\theta$. The following assumption on the nonlinear incidences is necessary.

(H$_5$) For $S_h > 0$,

$$\begin{cases} \dfrac{I_v}{I_v^*} \le \dfrac{S_h^* f(S_h, I_v)}{S_h f(S_h^*, I_v^*)} \le 1,\ 0 < I_v \le I_v^*, \\ 1 \le \dfrac{S_h^* f(S_h, I_v)}{S_h f(S_h^*, I_v^*)} \le \dfrac{I_v}{I_v^*},\ 0 < I_v^* \le I_v. \end{cases} \tag{26}$$

Theorem 7 *Assume that $r_h(c) = 0$ and* (H$_5$) *holds. If $\mathcal{R}_0 > 1$, then the endemic steady state $\mathcal{E}^{**}$ of system* (25) *is globally asymptomatically stable in $\mathbb{X}_0$.*

Proof By Theorem 5, we know that for any $\varepsilon > 0$, there exists $T > 0$ such that, for $t > T$, $(S_h(t), \|V_h(t,\cdot)\|_1, \|I_h(t,\cdot)\|_1, \|R_h(t,\cdot)\|_1, S_v(t), I_v(t)) \ge \epsilon$. Without loss of generality, assume that $(S_h(t), \|V_h(t,\cdot)\|_1, \|I_h(t,\cdot)\|_1, \|R_h(t,\cdot)\|_1, S_v(t), I_v(t)) > \varepsilon$ for $t > 0$. Define a Lyapunov function as $L_{EE}(t) = W_1(t) + W_2(t) + W_3(t) + W_4(t)$, where

$$W_1(t) = \eta S_h^* g\left(\frac{S_h(t)}{S_h^*}\right), \qquad W_2(t) = \eta \int_0^\infty L(a) V_h^*(a) g\left(\frac{V_h(t,a)}{V_h^*(a)}\right) \mathrm{d}a,$$

$$W_3(t) = \int_0^\infty \eta(b) I_h^*(b) g\left(\frac{I_h(t,b)}{I_h^*(b)}\right) \mathrm{d}b, \quad W_4(t) = \frac{\eta f(S_h^*, I_v^*)}{\mu_v} g\left(\frac{I_v(t)}{I_v^*}\right).$$

Calculate the derivative of function $W_1(t)$ along the solution of model (1), it follow that

$$\begin{aligned} \frac{\mathrm{d}W_1(t)}{\mathrm{d}t} &= \eta(\mu_h + \psi_h) S_h^* \left(2 - \frac{S_h^*}{S_h(t)} - \frac{S_h(t)}{S_h^*}\right) \\ &\quad + \eta f(S_h^*, I_v^*) \left(1 - \frac{f(S_h(t), I_v(t))}{f(S_h^*, I_v^*)} - \frac{S_h^*}{S_h(t)} + \frac{S_h^* f(S_h(t), I_v(t))}{S_h(t) f(S_h^*, I_v^*)}\right) \\ &\quad + \eta \int_0^\infty \omega_h(a) V_h^*(a) \left(\frac{V_h(t,a)}{V_h^*(a)} - 1 - \frac{S_h^* V_h(t,a)}{S_h(t) V_h^*(a)} + \frac{S_h^*}{S_h(t)}\right) \mathrm{d}a, \end{aligned} \tag{27}$$

where the $\Lambda_h = (\mu_h + \psi_h) S_h^* + f(S_h^*, I_v^*) - \int_0^\infty \omega_h(a) V_h^*(a) \mathrm{d}a$ is used. Derivation of $W_2(t)$ yields

$$\frac{\mathrm{d}W_2(t)}{\mathrm{d}t} = \eta \int_0^\infty L(a) \left(1 - \frac{V_h(t,a)}{V_h^*(a)}\right) \left[-\frac{\partial V_h(t,a)}{\partial a} - \varepsilon_1(a) V_h(t,a)\right] \mathrm{d}a.$$

Since

$$V_h^*(a) \frac{\partial}{\partial a} g\left(\frac{V_h(t,a)}{V_h^*(a)}\right) = V_h^*(a) \left(1 - \frac{V_h^*(a)}{V_h(t,a)}\right) \left[\varepsilon_1(a) V_h(t,a) + \frac{\partial V_h(t,a)}{\partial a}\right],$$

we can get

$$\frac{\mathrm{d}W_2(t)}{\mathrm{d}t} = -\eta L(a) V_h^*(a) g\left(\frac{V_h(t,a)}{V_h^*(a)}\right)\Bigg|_0^\infty + \eta \int_0^\infty \left[L'(a) V_h^*(a) - L(a) \varepsilon_1(a) V_h^*(a)\right] \mathrm{d}a$$

$$
\begin{aligned}
&= -\eta L(a)V_h^*(a)g\left(\frac{V_h(t,a)}{V_h^*(a)}\right)\Big|_{a=\infty} + \eta L\psi_h S_h^* g\left(\frac{S_h(t)}{S_h^*}\right) \\
&\quad - \eta\int_0^\infty \omega_h(a)V_h^*(a)g\left(\frac{V_h(t,a)}{V_h^*(a)}\right)\mathrm{d}a. \qquad (28)
\end{aligned}
$$

Similarly, for $W_3(t)$, one has

$$
\begin{aligned}
\frac{\mathrm{d}W_3(t)}{\mathrm{d}t} &= -\eta(b)I_h^*(b)g\left(\frac{I_h(t,b)}{I_h^*(0)}\right)\Big|_{b=\infty} \\
&\quad + \eta I_h(0)g\left(\frac{I_h(t,0)}{I_h^*(0)}\right) - \int_0^\infty \beta_v(b)I_h^*(b)g\left(\frac{I_h(t,b)}{I_h^*(b)}\right)\mathrm{d}b. \qquad (29)
\end{aligned}
$$

Calculating the derivative of $W_4(t)$ along with the solutions of model (1) yields

$$
\begin{aligned}
\frac{\mathrm{d}W_4(t)}{\mathrm{d}t} &= \frac{\eta f(S_h^*, I_v^*)}{\mu_v I_v^*}\left(1-\frac{I_v^*}{I_v(t)}\right)\left(\int_0^\infty \beta_v(b)(N_v - I_v^*)I_h(t,b)\mathrm{d}b\right) \\
&\quad - \frac{\eta f(S_h^*, I_v^*)}{\mu_v I_v^*}\mu_v I_v(t)\left(1-\frac{I_v^*}{I_v(t)}\right) \\
&\quad - \frac{\eta f(S_h^*, I_v^*)(I_v^* - I_v(t))^2}{\mu_v I_v(t) I_v^*}\int_0^\infty \beta_v(b)I_h(t,b)\mathrm{d}b. \qquad (30)
\end{aligned}
$$

To sum up (27)–(30), from (H_4), we have

$$
\begin{aligned}
\frac{\mathrm{d}L_{EE}(t)}{\mathrm{d}t} &= \eta(\mu_h + \psi_h)S_h^*\left(2 - \frac{S_h^*}{S_h(t)} - \frac{S_h(t)}{S_h^*}\right) + \eta f(S_h^*, I_v^*)\left(1 - \frac{f(S_h(t), I_v(t))}{f(S_h^*, I_v^*)} - \frac{S_h^*}{S_h(t)}\right. \\
&\quad \left. + \frac{S_h^* f(S_h(t), I_v(t))}{S_h(t) f(S_h^*, I_v^*)}\right) + \eta\int_0^\infty \omega_h(a)V_h^*(a)\left(\frac{V_h(t,a)}{V_h^*(a)} - 1 - \frac{S_h^* V_h(t,a)}{S_h(t)V_h^*(a)}\right. \\
&\quad \left. + \frac{S_h^*}{S_h(t)}\right)\mathrm{d}a - \eta L(a)V_h^*(a)g\left(\frac{V_h(t,a)}{V_h^*(a)}\right)\Big|_{a=\infty} + \eta L\psi_h S_h^* g\left(\frac{S_h(t)}{S_h^*}\right) \\
&\quad - \eta\int_0^\infty \omega_h(a)V_h^*(a)g\left(\frac{V_h(t,a)}{V_h^*(a)}\right)\mathrm{d}a - \eta(b)I_h^*(b)g\left(\frac{I_h(t,b)}{I_h^*(0)}\right)\Big|_{b=\infty} + \eta I_h(0) \\
&\quad \times g\left(\frac{I_h(t,0)}{I_h^*(0)}\right) - \int_0^\infty \beta_v(b)I_h^*(b)g\left(\frac{I_h(t,b)}{I_h^*(b)}\right)\mathrm{d}b + \frac{\eta f(S_h^*, I_v^*)}{\mu_v I_v^*}\left(1 - \frac{I_v^*}{I_v(t)}\right) \\
&\quad \times \int_0^\infty \beta_v(b)(N_v - I_v^*)I_h(t,b)\mathrm{d}b - \frac{\eta f(S_h^*, I_v^*)}{\mu_v I_v^*}\mu_v I_v(t)\left(1 - \frac{I_v^*}{I_v(t)}\right) \\
&\quad - \frac{\eta f(S_h^*, I_v^*)(I_v^* - I_v(t))^2}{\mu_v I_v(t) I_v^*}\int_0^\infty \beta_v(b)I_h(t,b)\mathrm{d}b \\
&= -\eta\left(\mu_h + (1-L)\psi_h\right)\left(g\left(\frac{S_h^*}{S_h(t)}\right) + g\left(\frac{S_h(t)}{S_h^*}\right)\right) - \eta f(S_h^*, I_v^*)g\left(\frac{S_h^*}{S_h(t)}\right) \\
&\quad - \eta\int_0^\infty \omega_h(a)V_h^*(a)g\left(\frac{S_h^* V_h(t,a)}{S_h(t)V_h^*(a)}\right)\mathrm{d}a - \frac{\eta f(S_h^*, I_v^*)(I_v^* - I_v(t))^2}{\mu_v I_v(t) I_v^*} \\
&\quad \times \int_0^\infty \beta_v(b)I_h(t,b)\mathrm{d}b - \eta L(a)V_h^*(a)g\left(\frac{V_h(t,a)}{V_h^*(a)}\right)\Big|_{a=\infty} - \eta(b)I_h^*(b)
\end{aligned}
$$

$$\times\, g\left(\frac{I_h(t,b)}{I_h^*(0)}\right)\bigg|_{b=\infty} - \int_0^\infty \beta_v(b) I_h^*(b) g\left(\frac{I_v^* I_h(t,b)}{I_v(t) I_h^*(b)}\right) \mathrm{d}b.$$

It follows that $\frac{\mathrm{d}L_{EE}(t)}{\mathrm{d}t} \leq 0$ if $\mathcal{R}_0 > 1$. In addition, the strict equality holds only if $S_h^* = S_h(t)$, $S_h^* V_h(t,a) = S_h(t) V_h^*(a)$, $I_v^* I_h(t,b) = I_v(t) I_h^*(b)$. This is easy to see $\mathcal{E}^{**}$ is the largest invariant subset of $\{(S_h, \cdots, I_v) \in \mathbb{X}_0 : \frac{\mathrm{d}L_{EE}(t)}{\mathrm{d}t} = 0\}$. By the Lyapunov-LaSalle's invariance principle, $\mathcal{E}^{**}$ is globally asymptomatically stable when all conditions of Theorem 7 are hold. The proof is completed. □

Remark 3 By the limit theory of dynamical systems [38, 39] and Theorem 7, we have the following result: assume that $r_h(c) = 0$ and (H_5) holds, if $\mathcal{R}_0 > 1$, then the endemic steady state $\mathcal{E}^*$ of model (1) is globally asymptotically stable.

Remark 4 It should be noted that the condition (H_5) is a technical condition that we attach to prove global asymptotic stability of the endemic steady state. It is also a condition often used by epidemic models with general incidence in proving the global asymptotic stability of the endemic steady state. As in the Refs. [15, 44] and the references therein. In fact, if the incidence $f(S_h, I_v)$ degenerates into some special incidence such as $f(S_h, I_v) = \beta S_h I_v$, $f(S_h, I_v) = \frac{\beta S_h I_v}{1+\alpha I_v}$ and $f(S_h, I_v) = \frac{\beta S_h I_v}{1+q S_h + p I_v}$, then (H_5) is clearly valid (i.e. it is not needed in some special cases).

4 Optimal Control Problem

In this section, two control functions $u_1(t)$ and $u_2(t)$ are introduced to understand the impact of controlling the transmission of mosquito-borne infectious diseases, where, $u_1(t)$ represents the additional vaccination rate at time t, $u_2(t)$ indicates the level of larvicides and adulticides to be used in mosquito concentration sites in order to minimize the size of mosquitoes. Hence, the control model is given by

$$\begin{aligned}
\frac{\mathrm{d}S_h}{\mathrm{d}t} &= \Lambda_h - \mu_h S_h - f(S_h, I_v) - (\psi_h + u_1) S_h + \int_0^\infty \omega_h(a) V_h(t,a) \mathrm{d}a,\\
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) V_h(t,a) &= -(\mu_h + \omega_h(a)) V_h(t,a),\ V_h(t,0) = (\psi_h + u_1) S_h,\\
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial b}\right) I_h(t,b) &= -\,(\mu_h + k_h(b) + \nu_h(b)) I_h(t,b),\\
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial c}\right) R_h(t,c) &= -(\mu_h + r_h(c)) R_h(t,c),\ R_h(t,0) = \int_0^\infty k_h(b) I_h(t,b) \mathrm{d}b,\\
\frac{\mathrm{d}S_v}{\mathrm{d}t} &= \Lambda_v - \int_0^\infty \beta_v(b) S_v I_h(t,b) \mathrm{d}b - (\mu_v + u_2) S_v,\\
\frac{\mathrm{d}I_v}{\mathrm{d}t} &= \int_0^\infty \beta_v(b) S_v I_h(t,b) \mathrm{d}b - (\mu_v + u_2) I_v,
\end{aligned} \tag{31}$$

with $I_h(t,0) = f(S_h, I_v) + \int_0^\infty r_h(c) R_h(t,c) \mathrm{d}c$. Here, the control set $U = \{(u_1, u_2) : u_i \in L^1([0,T]),\ 0 \leq u_i(t) < 1,\ i = 1, 2\}$. Hence, our optimal control problem is to minimize the objective function

$$\mathcal{J}(u_1, u_2) = \int_0^T \left(A_1 I_h + A_2 S_v + A_3 I_v + \frac{1}{2}\left(B_1 u_1^2 + B_2 u_2^2\right)\right) \mathrm{d}t,$$

where A_i and B_j are positive constants to balance the differences between state variables and control variables ($i = 1, 2, 3;\ \ j = 1, 2$).

Remark 5 The introduction of the control variable u_2 is derived from Ross [2] proposal that "$\cdots$ *in order to counteract malaria anywhere we need not banish Anopheles there entirely — we need only to reduce their numbers below a certain figure.*" Therefore, th objective function aims to simultaneously minimize the number of infected humans and mosquitoes at the end of the control period and the accumulated cost of control strategies.

To obtain the optimal control system, it is needed to differentiate the objective function about the control functions. The Gâteaux derivative rule in Ref. [40] is applied to derive the derivative about $u_1(t)$ and $u_2(t)$. Given a control u_1 and u_2, select an additional control $u_1^\epsilon = u_1 + \epsilon l_1$ and $u_2^\epsilon(t) = u_2 + \epsilon l_2$, where l_i ($i = 1, 2$) is a variation function and $\epsilon \in (0, 1)$. Assume that $S_h = S_h(u_i)$, $V_h = V_h(u_i)$, $I_h = I_h(u_i)$, $R_h = R(u_i)$, $S_v = S_v(u_i)$, $I_v = I_v(u_i)$ and $S_h^\epsilon = S_h(u_i^\epsilon)$, $V_h^\epsilon = V_h(u_i^\epsilon)$, $I_h^\epsilon = I_h(u_i^\epsilon)$, $R_h^\epsilon = R_h(u_i^\epsilon)$, $S_v^\epsilon = S_v(u_i^\epsilon)$, $I_v^\epsilon = I_v(u_i^\epsilon)$. Then the state equations corresponding to controls $u_i^\epsilon (i = 1, 2)$ are given as

$$\begin{aligned}
\frac{\mathrm{d}S_h^\epsilon}{\mathrm{d}t} &= \Lambda_h - \mu_h S_h^\epsilon - f(S_h^\epsilon, I_v^\epsilon) - (\psi_h + u_1^\epsilon) S_h^\epsilon + \int_0^\infty \omega_h(a) V_h^\epsilon(t, a)\mathrm{d}a,\\
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) V_h^\epsilon(t, a) &= -(\mu_h + \omega_h(a)) V_h^\epsilon(t, a),\ V_h^\epsilon(t, 0) = (\psi_h + u_1^\epsilon) S_h^\epsilon,\\
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial b}\right) I_h^\epsilon(t, b) &= -(\mu_h + k_h(b) + \nu_h(b)) I_h^\epsilon(t, b),\\
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial b}\right) R_h^\epsilon(t, b) &= -(\mu_h + r(b)) R_h^\epsilon(t, b),\ R_h^\epsilon(t, 0) = \int_0^\infty k_h(b) I_h^\epsilon(t, b)\mathrm{d}b,\\
\frac{\mathrm{d}S_v^\epsilon}{\mathrm{d}t} &= \Lambda_v - \int_0^\infty \beta_v(b) S_v^\epsilon I_h^\epsilon(t, b) - (\mu_v + u_2^\epsilon) S_v^\epsilon,\\
\frac{\mathrm{d}I_v^\epsilon}{\mathrm{d}t} &= \int_0^\infty \beta_v(b) S_v^\epsilon I_h^\epsilon(t, b) - (\mu_v + u_2^\epsilon) I_v^\epsilon,
\end{aligned} \tag{32}$$

with $I_h^\epsilon(t, 0) = f(S_h^\epsilon, I_v^\epsilon) + \int_0^\infty r(b) R_h^\epsilon(t, b)\mathrm{d}b$. Define $V_h^\epsilon = \int_0^\infty V_h^\epsilon(t, a)\mathrm{d}a$, $I_h^\epsilon = \int_0^\infty I_h^\epsilon(t, b)\mathrm{d}b$ and $R_h^\epsilon = \int_0^\infty R_h^\epsilon(t, c)\mathrm{d}c$, we find the following difference quotient

$$\begin{aligned}
&\frac{S_h^\epsilon - S_h}{\epsilon} \to S_h, \quad \frac{V_h^\epsilon(t, a) - V_h(t, a)}{\epsilon} \to V_h(t, a), \quad \frac{I_h^\epsilon(t, b) - I_h(t, b)}{\epsilon} \to I_h(t, b),\\
&\frac{R_h^\epsilon(t, c) - R_h(t, c)}{\epsilon} \to R_h(t, c), \quad \frac{S_v^\epsilon - S_v}{\epsilon} \to S_v, \quad \frac{I_v^\epsilon - I_v}{\epsilon} \to I_v,
\end{aligned}$$

as $\epsilon \to 0$, where $(\bar{S}_h, \bar{V}_h(t,a), \bar{I}_h(t,b), \bar{R}_h(t,c), \bar{S}_v, \bar{I}_v)$ complies with the following system

$$
\begin{aligned}
\frac{\mathrm{d}\bar{S}_h}{\mathrm{d}t} &= -\left(\mu_h + \psi_h + u_1 + \frac{\partial f(0, I_v)}{\partial S_h}\right)\bar{S}_h + \int_0^\infty \omega_h(a)\bar{V}_h(t,a)\mathrm{d}a - \frac{\partial f(S_h, 0)}{\partial I_v}\bar{I}_v - l_1 S_h,\\
&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)\bar{V}_h(t,a) = -(\mu_h + \omega_h(a))\bar{V}_h(t,a),\ \bar{V}_h(t,0) = (\psi_h + u_1)\bar{S}_h + l_1 S_h,\\
&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial b}\right)\bar{I}_h(t,b) = -(\mu_h + k_h(b) + \nu_h(b))\bar{I}_h(t,b),\\
&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial c}\right)\bar{R}_h(t,c) = -(\mu_h + r_h(c))\bar{R}_h(t,c),\ \bar{R}_h(t,0) = \int_0^\infty k_h(b)\bar{I}_h(t,b)\mathrm{d}b,\\
\frac{\mathrm{d}\bar{S}_v}{\mathrm{d}t} &= -\int_0^\infty \beta_v(b)\left(\bar{S}_v I_h(t,b) + S_v\bar{I}_h(t,b)\right)\mathrm{d}b - (\mu_v + u_2)\bar{S}_v - l_2 S_v,\\
\frac{\mathrm{d}\bar{I}_v}{\mathrm{d}t} &= \int_0^\infty \beta_v(b)\left(\bar{S}_v I_h(t,b) + S_v\bar{I}_h(t,b)\right)\mathrm{d}b - (\mu_v + u_2)\bar{I}_v - l_2 I_v,
\end{aligned}
\tag{33}
$$

and $\bar{I}_h(t,0) = \frac{\partial f(S_h,0)}{\partial I_v}\bar{I}_v + \frac{\partial f(0,I_v)}{\partial S_h}\bar{S}_h + \int_0^\infty r_h(c)\bar{R}_h(t,c)\mathrm{d}c$. For the purpose of solving the equation, we can write the first equation of system (33) in the following form

$$
\begin{aligned}
0 &= \Big\langle \frac{\mathrm{d}S_h}{\mathrm{d}t} + \left(\mu_h + \psi_h + u_1 + \frac{\partial f(0, I_v)}{\partial S_h}\right)\bar{S}_h \\
&\quad - \int_0^\infty \omega_h(a)\bar{V}_h(t,a)\mathrm{d}a + \frac{\partial f(S_h, 0)}{\partial I_v}\bar{I}_h + l_1 S_h, \lambda_1 \Big\rangle \\
&= \Big\langle \bar{S}_h, -\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} + \left(\mu_h + \psi_h + u_1 + \frac{\partial f(0, I_v)}{\partial S_h}\right)\lambda_1 \Big\rangle - \int_0^T \int_0^\infty \lambda_1 \omega_h(a)\bar{V}_h(t,a)\mathrm{d}a\mathrm{d}t \\
&\quad + \int_0^T \lambda_1 \frac{\partial f(S_h, 0)}{\partial I_v}\bar{I}_v\mathrm{d}t + \int_0^\infty \lambda_1 l_1 S_h \mathrm{d}t
\end{aligned}
\tag{34}
$$

under the initial conditions $\lambda_1(T) = 0$, $\bar{S}_h(0) = 0$ and $\langle f, g\rangle = \int_0^T fg\mathrm{d}t$. Now the second equation of system (33) can be reformulated as

$$
\begin{aligned}
0 &= \Big\langle\Big\langle \frac{\partial \bar{V}_h(t,a)}{\partial t} + \frac{\partial \bar{V}_h(t,a)}{\partial a} + (\mu_h + \omega_h(a))\bar{V}_h(t,a), \lambda_2(t,a) \Big\rangle\Big\rangle \\
&= \Big\langle\Big\langle \bar{V}_h(t,a), -\frac{\partial \lambda_2(t,a)}{\partial t} - \frac{\partial \lambda_2(t,a)}{\partial a} + (\mu_h + \omega_h(a))\lambda_2(t,a) \Big\rangle\Big\rangle \\
&\quad - \int_0^T (\psi_h + u_1)\lambda_2(t,0)\bar{S}_h\mathrm{d}t - \int_0^T l_1 S_h \lambda_2(t,0)\mathrm{d}t.
\end{aligned}
\tag{35}
$$

and the initial conditions $\lambda_2(T,a) = 0$, $\bar{V}(t,\infty) = 0$, $\bar{V}_h(0,a) = 0$ and $\langle\langle f, g\rangle\rangle = \int_0^T \int_0^\infty fg\mathrm{d}a\mathrm{d}t$. third-sixth equation of system (33) can be expressed as

$$
0 = \Big\langle\Big\langle \frac{\partial \bar{I}_h(t,b)}{\partial t} + \frac{\partial \bar{I}_h(t,b)}{\partial a} + (\mu_h + k_h(b) + \nu_h(b))\bar{I}_h(t,b), \lambda_3(t,b) \Big\rangle\Big\rangle
$$

$$= \left\langle\left\langle \bar{I}_h(t,b), -\frac{\partial \lambda_3(t,b)}{\partial t} - \frac{\partial \lambda_3(t,b)}{\partial b} + (\mu_h + k_h(b) + \nu_h(b))\lambda_3(t,b) \right\rangle\right\rangle$$
$$- \int_0^T \lambda_3(t,0) \left(\frac{\partial f(S_h,0)}{\partial I_v} \bar{I}_v + \frac{\partial f(0,I_v)}{\partial S_h} \bar{S}_h + \int_0^\infty r_h(c)\bar{R}_h(t,c)\mathrm{d}c \right) \mathrm{d}t \tag{36}$$

under the initial conditions $\lambda_3(T,b) = 0$, $\bar{I}(t,\infty) = 0$, $\bar{I}_h(0,b) = 0$.

$$0 = \left\langle\left\langle \frac{\partial \bar{R}_h(t,c)}{\partial t} + \frac{\partial \bar{R}_h(t,c)}{\partial c} + (\mu_h + r_h(c))\bar{R}_h(t,c), \lambda_4(t,c) \right\rangle\right\rangle$$
$$= \left\langle\left\langle \bar{R}_h(t,c), -\frac{\partial \lambda_4(t,c)}{\partial t} - \frac{\partial \lambda_4(t,c)}{\partial c} + (\mu_h + r_h(c))\lambda_4(t,c) \right\rangle\right\rangle$$
$$- \int_0^T \int_0^\infty \lambda_4(t,0)k_h(b)\bar{I}_h(t,b)\mathrm{d}b\mathrm{d}t, \tag{37}$$

and the initial conditions $\lambda_4(T,c) = 0$, $\bar{R}_h(t,\infty) = 0$, $\bar{R}_h(0,c) = 0$.

$$0 = \left\langle \frac{\mathrm{d}\bar{S}_v}{\mathrm{d}t} + \int_0^\infty \beta_v(b) \left(\bar{S}_v I_h(t,b) + S_v \bar{I}_h(t,b) \right) \mathrm{d}b + (\mu_v + u_2(t))S_v + l_2 S_v, \lambda_5 \right\rangle$$
$$= \left\langle \bar{S}_v, -\frac{\mathrm{d}\lambda_5}{\mathrm{d}t} + \left(\mu_v + u_2 + \int_0^\infty \beta_v(b) I_h(t,b)\mathrm{d}b \right) \lambda_5 \right\rangle$$
$$+ \int_0^T \int_0^\infty \beta_v(b) S_v \bar{I}_h(t,b)\lambda_5 \mathrm{d}b\mathrm{d}t + \int_0^T \lambda_5 l_2 S_v \mathrm{d}t, \tag{38}$$

under the initial conditions $\bar{S}_v(0) = 0$ and $\lambda_5(T) = 0$.

$$0 = \left\langle \frac{\mathrm{d}\bar{I}_v}{\mathrm{d}t} - \int_0^\infty \beta_v(b) \left(\bar{S}_v I_h(t,b) + S_v \bar{I}_h(t,b) \right) \mathrm{d}b + (\mu_v + u_2)\bar{I}_v + l_2 I_v, \lambda_6 \right\rangle$$
$$= \left\langle \bar{I}_v, -\frac{\mathrm{d}\lambda_6}{\mathrm{d}t} + (\mu_v + u_2)\lambda_6 \right\rangle$$
$$- \int_0^T \int_0^\infty \lambda_6 \beta_v(b)(\bar{S}_v I_h(t,b) + S_v \bar{I}_h(t,b))\mathrm{d}b\mathrm{d}t + \int_0^T \lambda_6 l_2 I_v \mathrm{d}t, \tag{39}$$

and the initial conditions $\bar{I}_v(0) = 0$ and $\lambda_6(T) = 0$.

Furthermore, by defining the Lagrangian function $\mathcal{L}$, the adjoint equations can be obtained according to the objective function and (33). As a result, the Lagrangian $\mathcal{L}$ is defined as

$$\mathcal{L}(\bar{S}_h, \bar{V}_h, \bar{I}_h, \bar{R}_h, \bar{S}_v, \bar{I}_v)$$
$$= \int_0^T \left(A_1 \bar{I}_h + A_2 \bar{S}_v + A_3 \bar{I}_v + \frac{1}{2} \left(B_1 u_1^2 + B_2 u_2^2 \right) \right) \mathrm{d}t - \lambda_1 \int_0^T \Big[\frac{\mathrm{d}\bar{S}_h}{\mathrm{d}t}$$
$$+ \left(\mu_h + \psi_h + u_1 + \frac{\partial f(0,I_v)}{\partial S_h} \right) \bar{S}_h - \int_0^\infty \omega_h(a)\bar{V}_h(t,a)\mathrm{d}a + \frac{\partial f(S_h,0)}{\partial I_v} \bar{I}_v - l_1 S_h \Big] \mathrm{d}t$$

$$
\begin{aligned}
& -\lambda_2(t,a)\int_0^T\int_0^\infty\left[\frac{\partial \bar{V}_h(t,a)}{\partial t}+\frac{\partial \bar{V}_h(t,a)}{\partial a}+(\mu_h+\omega_h(a))\bar{V}_h(t,a)\right]\mathrm{d}a\mathrm{d}t \\
& -\lambda_3(t,b)\int_0^T\left[\frac{\partial \bar{I}_h(t,b)}{\partial t}+\frac{\partial \bar{I}_h(t,b)}{\partial b}+(\mu_h+k_h+\nu_h)\bar{I}_h(t,b)\right]\mathrm{d}b\mathrm{d}t \\
& -\lambda_4(t,c)\int_0^T\int_0^\infty\left[\frac{\partial \bar{R}_h(t,c)}{\partial t}+\frac{\partial \bar{R}_h(t,c)}{\partial c}+(\mu_h+r_h(c))\bar{R}_h(t,c)\right]\mathrm{d}c\mathrm{d}t \\
& -\lambda_5\int_0^T\left[\frac{\mathrm{d}\bar{S}_v}{\mathrm{d}t}+\int_0^\infty\beta_v(b)\left(\bar{S}_v I_h(t,b)+S_v\bar{I}_h(t,b)\right)\mathrm{d}b+(\mu_v+u_2)\bar{S}_v+l_2 S_v\right]\mathrm{d}t \\
& -\lambda_6\int_0^T\left[\frac{\mathrm{d}\bar{I}_v}{\mathrm{d}t}-\int_0^\infty\beta_v(b)\left(\bar{S}_v I_h(t,b)+S_v\bar{I}_h(t,b)\right)\mathrm{d}b+(\mu_v+u_2)\bar{I}_v+l_2 I_v\right]\mathrm{d}t.
\end{aligned}
$$

By solving $\frac{\partial\mathcal{L}}{\partial\bar{S}_h}=0$, $\frac{\partial\mathcal{L}}{\partial\bar{V}_h}=0$, $\frac{\partial\mathcal{L}}{\partial\bar{I}_h}=0$, $\frac{\partial\mathcal{L}}{\partial\bar{R}_h}=0$, $\frac{\partial\mathcal{L}}{\partial\bar{S}_v}=0$, $\frac{\partial\mathcal{L}}{\partial\bar{I}_v}=0$, combined with the Eqs. (34)–(39), we can obtain the adjoint equations

$$
\begin{aligned}
\frac{\mathrm{d}\lambda_1(t)}{\mathrm{d}t} &= \left(\mu_h+\psi_h+u_1(t)+\frac{\partial f(0,I_v)}{\partial S_h}\right)\lambda_1 \\
&\quad -(\psi_h+u_1)\lambda_2(t,0)-\lambda_3(t,0)\frac{\partial f(0,I_v)}{\partial S_h}, \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\lambda_2(t,a) &= (\mu_h+\omega_h(a))\lambda_2(t,a)-\omega_h(a)\lambda_1, \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial b}\right)\lambda_3(t,b) &= (\mu_h+k_h(b)+\nu_h(b))\lambda_3(t,b) \\
&\quad +\beta_v(b)S_v(\lambda_5-\lambda_6)-k_h(b)\lambda_4(t,0)-A_1, \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial c}\right)\lambda_4(t,c) &= (\mu_h+r_h(c))\lambda_4(t,c)-\lambda_3(t,0)r_h(c), \\
\frac{\mathrm{d}\lambda_5}{\mathrm{d}t} &= \left(\mu_v+u_2+\int_0^\infty\beta_v(b)I_h(t,b)\mathrm{d}b\right)\lambda_5 \\
&\quad -\lambda_6\int_0^\infty\beta_v(b)I_h(t,b)\mathrm{d}b-A_2, \\
\frac{\mathrm{d}\lambda_6}{\mathrm{d}t} &= (\mu_v+u_2)\lambda_6+\lambda_1\frac{\partial f(S_h,0)}{\partial I_v}-\lambda_3(t,0)\frac{\partial f(S_h,0)}{\partial I_v}-A_3.
\end{aligned} \tag{40}
$$

Theorem 8 *If u_1^*, u_2^* in U is an optimal control that minimizes $\mathcal{J}$ and $(S_h^*, V_h^*(t,a), I_h^*(t,b), R_h^*(t,c), S_v^*, I_v^*)$ and $(\lambda_1,\lambda_2(t,a),\lambda_3(t,b),\lambda_4(t,c),\lambda_5,\lambda_6)$ are the corresponding state variables as well as the adjoint variables variables, respectively, and then $u_1^*=\min\{\max\{0,\bar{u}_1\},u_{1\max}\}$, $u_2^*=\min\{\max\{0,\bar{u}_2\},u_{2\max}\}$.*

Proof From the object function $\mathcal{J}(u_1,u_2)$, one has

$$
0\le\mathcal{J}'(u_1,u_2)=\int_0^T\left(A_1\bar{I}_h+A_2\bar{S}_v+A_3\bar{I}_v+B_1u_1l_1+B_2u_2l_2\right)\mathrm{d}t
$$

$$
\begin{aligned}
&= \int_0^T \bar{S}_h \Big[-\frac{d\lambda_1}{dt} + \Big(\mu_h + \psi_h + u_1 + \frac{\partial f(0, I_v)}{\partial S_h} \Big) \lambda_1 \\
&- (\psi_h + u_1)\lambda_2(t,0) - \lambda_3(t,0)\frac{\partial f(0, I_v)}{\partial S_h} \Big] dt \\
&+ \int_0^T \int_0^\infty \bar{V}_h(t,a) \Big[-\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \Big) \lambda_2(t,a) + (\mu_h + \omega_h(a))\lambda_2(t,a) \\
&- \omega_h(a)\lambda_1 \Big] da dt + \int_0^T \bar{I}_h(t,b) \Big[-\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial b} \Big) \lambda_3(t,b) + (\mu_h + k_h(b) + \nu_b)\lambda_3(t,b) \\
&+ \beta_v(b) S_v(\lambda_5 - \lambda_6) - k_h(b)\lambda_4(t,0) \Big] dt + \int_0^T \int_0^\infty \bar{R}_h(t,c) \Big[-\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial c} \Big) \lambda_4(t,c) \\
&+ (\mu_h + r_h(c))\lambda_4(t,c) - \lambda_3(t,0) r_h(c) \Big] dc dt + \int_0^T \bar{S}_v \Big[-\frac{d\lambda_5}{dt} + \Big(\mu_v + u_2 \\
&+ \int_0^\infty \beta_v(b) I_h(t,b) db \Big) \lambda_5 - \lambda_6 \int_0^\infty \beta_v(b) I_h(t,b) db \Big] dt + \int_0^\infty \bar{I}_v \Big[-\frac{d\lambda_6}{dt} \\
&+ (\mu_v + u_2)\lambda_6 + \lambda_1 \frac{\partial f(S_h, 0)}{\partial I_v} - \lambda_3(t,0)\frac{\partial f(S_h, 0)}{\partial I_v} \Big] dt + \int_0^T (B_1 u_1 l_1 + B_2 u_2 l_2) dt \\
&= \int_0^T \Big[S_h(\lambda_2(t,0) - \lambda_1) + B_1 u_1 \Big] l_1 dt + \int_0^\infty \Big[B_2 u_2 - I_v \lambda_6 - S_v \lambda_5 \Big] l_2 dt,
\end{aligned}
$$

where $\mathcal{J}'(u_1, u_2)$ denotes differentiation of the objective function $\mathcal{J}(u_1, u_2)$ with respect to ϵ and set $\epsilon = 0$. When $l_1, l_2 \neq 0$, the rest of the integrand function must be equivalent to zero. Thus,

$$
\bar{u}_1 = \frac{S_h(\lambda_2(t,0) - \lambda_1)}{B_1}, \quad \bar{u}_2 = \frac{I_v \lambda_6 + S_v \lambda_5}{B_2}.
$$

Hence, $u_1^* = \min\{\max\{0, \bar{u}_1\}, u_{1\max}\}$, $u_2^* = \min\{\max\{0, \bar{u}_2\}, u_{2\max}\}$, in which $u_{1\max}$ and $u_{2\max}$ are the upper bounds of the two control functions. □

Now, we apply Ekeland's principle (see Ref. [40]) to acquire the sequence of minima of the approximation function, and then according to Ref. [41], there exists a set of objective function sequences of the following form

$$
\mathcal{J}_\varepsilon(u_1, u_2) = \mathcal{J}(u_1, u_2) + \sqrt{\varepsilon}(\|u_1^\varepsilon - u_1\|_{L^1(0,T)} + \|u_2^\varepsilon - u_2\|_{L^1(0,T)}).
$$

Theorem 9 *If $(u_1^\varepsilon, u_2^\varepsilon)$ is a pair of minimizers for $\mathcal{J}_\varepsilon(u_1, u_2)$, then*

$$
u_1^\varepsilon = \min\left\{ \max\left\{ 0, \frac{S_h^\epsilon(\lambda_2^\epsilon(t,0) - \lambda_1^\epsilon)}{B_1} \right\}, u_{1\max} \right\},
$$
$$
u_2^\varepsilon = \min\left\{ \max\left\{ 0, \frac{I_v^\epsilon \lambda_6^\epsilon + S_v^\epsilon \lambda_5^\epsilon}{B_2} \right\}, u_{2\max} \right\},
$$

where the functions $(\theta_1^\varepsilon, \theta_2^\varepsilon)$ belong to $L^\infty(E)$ such that $|\theta_i^\varepsilon| \leq 1 (i = 1, 2)$ for $t \in E := (0, T)$.

Theorem 10 *There exists a unique pair of optimal controls* (u_1^*, u_2^*) *minimizing* $\mathcal{J}(u_1, u_2)$ *if* T/B_1 *and* T/B_2 *are sufficiently small.*

Proof Define two functions by

$$\mathcal{F}_1(u_1) = \min\left\{\max\left\{0, \frac{S_h^\epsilon(\lambda_2^\epsilon(t,0) - \lambda_1^\epsilon) - \sqrt{\varepsilon}\theta_1^\varepsilon}{B_1}\right\}, u_{1\max}\right\},$$
$$\mathcal{F}_2(u_2) = \min\left\{\max\left\{0, \frac{I_v^\epsilon\lambda_6^\epsilon + S_v^\epsilon\lambda_5^\epsilon - \sqrt{\varepsilon}\theta_2^\varepsilon}{B_2}\right\}, u_{2\max}\right\}.$$

For two pairs of controls (u_1, u_2) and $(\hat{u}_1, \hat{u}_2)$, according to the Lipschitz properties of the states and adjoints and Ref. [41], we have

$$\|\mathcal{F}_1(u_1) - \mathcal{F}_1(\hat{u}_1)\| \leq \frac{K_T}{B_1}\|u_1 - \hat{u}_1\|_{L^\infty}, \quad \|\mathcal{F}_2(u_2) - \mathcal{F}_2(\hat{u}_2)\| \leq \frac{K_T}{B_2}\|u_2 - \hat{u}_2\|_{L^\infty},$$

where K_T depends on the L^∞ bounds on the state and adjoint solutions and the Lipschitz constants.

If K_T/B_1 small enough, it yields that

$$\|u_1 - u_1^\varepsilon\| \leq \frac{\sqrt{\varepsilon}}{B_1 - K_T}, \quad \|u_2 - u_2^\varepsilon\| \leq \frac{\sqrt{\varepsilon}}{B_2 - K_T}.$$

This imply that $(u_{1\varepsilon}, u_{2\varepsilon})$ converges to (u_1^*, u_2^*). Following Ekland's principle, one can get that $\mathcal{J}(u_1^*, u_2^*) \leq \inf \mathcal{J}_{(u_1,u_2)\in U}(u_1, u_2)$ as $\varepsilon \to 0^+$. □

5 Numerical Simulations

According to the extensive Refs. [15, 18, 46, 47] on vector-borne infectious diseases models, the basic model parameters are chosen as $\Lambda_h = 25$, $\mu_h = \frac{1}{79\times 365}$, $\mu_v = 0.04$, $\psi_h = 0.008$, $\alpha = 1$, and $q = 0.0001$. Further, select that $f(S_h(t), I_v(t)) = \frac{\beta_1 S_h(t) I_v(t)}{1+qI_v(t)}$ and $\omega_h(a) = x_m e^{-2.5a}a^2$, $k_h(b) = y_m e^{-2.5b}b^2$, $r_h(c) = u_m e^{-2.5c}c^2$, $\beta_v(b) = n_m e^{-2b}b^2$, $\nu_h(b) = 0.0001 \times (4 + 2e^{-0.9b})^{-1}$. By the direct calculation, it can be get

$$\mathcal{R}_0 = \frac{\Lambda_v \Lambda_h \beta_1 \int_0^\infty \beta_v(b) e^{-\int_0^b \varepsilon_2(s)\mathrm{d}s}\mathrm{d}b}{\mu_v^2\left(\mu_h + \psi_h\left(1 - \int_0^\infty \omega_h(a) e^{-\int_0^a \varepsilon_1(s)\mathrm{d}s}\mathrm{d}a\right)\right)} \times \frac{1}{1 - \left(\int_0^\infty r_h(c) e^{-\int_0^c \varepsilon_3(s)\mathrm{d}s}\mathrm{d}c \int_0^\infty k_h(b) e^{-\int_0^b \varepsilon_2(s)\mathrm{d}s}\mathrm{d}b\right)}.$$

We choose, firstly, $\Lambda_v = 11400$, $\beta_1 = 9.5\times 10^{-7}$ and take $x_m = 0.15$, $y_m = 0.08$, $u_m = 0.04$, $n_m = 0.1$. In the case, the basic reproduction number is $\mathcal{R}_0 \approx 0.4457 < 1$. From Theorem 6, the disease-free steady state $\mathcal{E}^0$ is globally asymptotically stable which shows in the Fig. 1a–c. That is, trajectories with different initial values converge to the disease-free steady state. These imply that the disease is eradicated, the infected classes are vanished.

Now, let $\Lambda_v = 22800, \beta_1 = 1.2\times10^{-5}, u_m = 0.1$ and $r_h(c) = 0$, no change in other parameters in Fig. 1. Then, the basic reproduction number as $\mathcal{R}_0 \approx 10.2868 > 1$ in this case. According to Theorem 7, the endemic steady state $\mathcal{E}^*$ is globally asymptotically stable, in agreement with the plots in Fig. 2a–f. However, if we only change $r_h(c) = 0.04e^{-2.5c}c^2$ and other parameters remain unchanged. A direct calculation shows that $\mathcal{R}_0 \approx 10.2870$ and the plots in Fig. 3a–d imply that the endemic steady state $\mathcal{E}^*$ is also globally asymptotically stable.

Next, we concerned about the effects of immune loss rate $\omega_h(a)$ and vaccination rate ψ_h on the distribution of this disease. The plots in Fig. 4a–c show that the distributions of $\int_0^\infty V_h(t,a)\mathrm{d}a$, $S_v(t)$ and $I_v(t)$ are a slight fluctuations when $\omega_h(a)$ gradually increases from $0.02e^{-2.5a}a^2, 0.30e^{-2.5a}a^2, 0.67e^{-2.5a}a^2$ to $0.95e^{-2.5a}a^2$. Numerical simulations show that as the rate of immune loss increases, the number of vaccinated individuals decreases and the number of infected vectors increases. Further, by Fig. 4d–f, this can also be found that when the vaccination rate ψ_h increases, from 0.0012, 0.004, 0.008 to 0.02, the quantity of infected vectors declined significantly. This suggests that both vaccine timeliness and vaccination rates can have important effects on the distribution of infected individuals. Relatively speaking, the vaccination rate has a greater impact. Therefore, increasing the vaccination rate will have a good effect on disease control without guaranteeing the perfect effectiveness of the vaccine. Of course, good vaccines and high vaccination rates are one of the best strategies for disease control.

Further, we consider the influences of age-related parameters $k_h(b)$, $\beta_v(b)$ and $r_h(c)$ on the distribution of disease transmission. According to the Fig. 5a–c, as the value of the recovery rate $k_h(b)$ is increased form $0.08e^{-2.5b}b^2, 0.20e^{-2.5b}b^2, 0.45e^{-2.5b}b^2$ to $0.9e^{-2.5b}b^2$, the number of $\int_0^\infty I_h(t,b)\mathrm{d}b$ peaks almost simultaneously at time $t = 100$ and as time increases the amount of $\int_0^\infty I_h(t,b)\mathrm{d}b$ and $I_v(t)$ also changes. The plots in Fig. 5d–f show that, when the relapse rate $r_h(c)$ increases from $0.01e^{-2.5c}c^2$, $0.08e^{-2.5c}c^2$, $0.20e^{-2.5c}c^2$ to $0.92e^{-2.5c}c^2$, the numbers of $\int_0^\infty I_h(t,b)\mathrm{d}b$ and $I_v(t)$ also change and the total number of infected individuals $\int_0^\infty I_h(t,b)\mathrm{d}b$ reach a peak at the same moment. The plots in Fig. 5g–i show that the number of $\int_0^\infty I_h(t,b)\mathrm{d}b$ and $I_v(t)$ changes dramatically and achieves a steady state at the same time when the values of transmission rate $\beta_v(b)$ to be taken $0.02e^{-2b}b^2, 0.05e^{-2b}b^2, 0.08e^{-2b}b^2$ and $0.11e^{-2b}b^2$, respectively. Therefore, while all of the age-dependent model parameters have an effect on disease transmission, however, the rate of viral transmission has an important effect on the magnitude of disease transmission. Therefore, avoiding vector bites on host populations is one of the most prove methods for vector-borne diseases. Additionally, it is easy to see from the expression of the basic reproduction number that the age-dependent parameters $k_h(b)$, $r_h(c)$ and $\beta_v(b)$ are positively correlated with $\mathcal{R}_0$, which implies that ignoring the age factor will overestimate or underestimate the basic reproduction number and produce erroneous judgments about disease control.

To further elaborate the effects of the key parameters ψ_h and μ_v, β_1 and ψ_h of model on the basic reproduction number, other parameters are fixed as shown above.

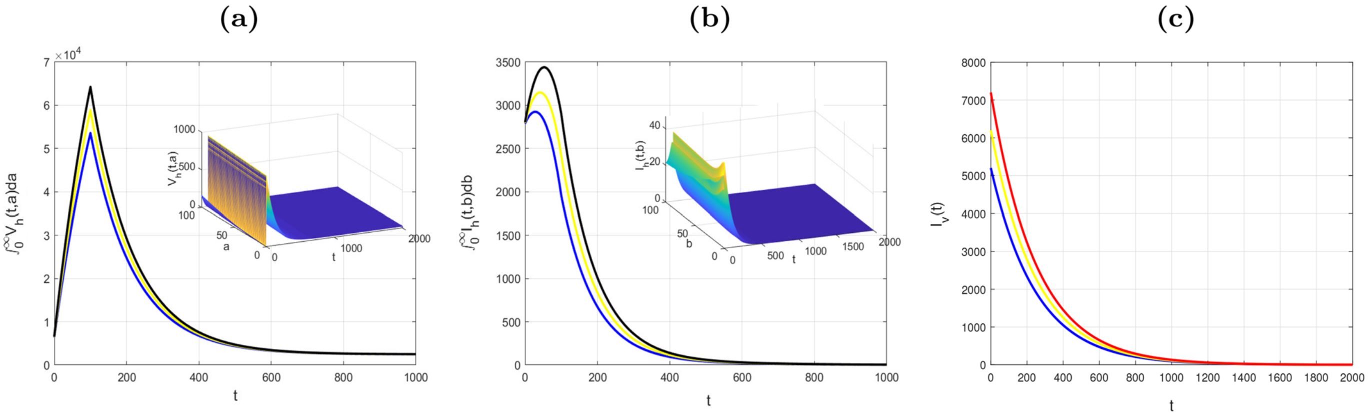

Fig. 1 The global asymptotical stability of $\mathcal{E}^0$ of model (1) with $\mathcal{R}_0 \approx 0.5648 < 1$: **a** $V_h(t,a)$ and $\int_0^\infty V_h(t,a)\mathrm{d}a$; **b** $I_h(t,a)$ and $\int_0^\infty I_h(t,b)\mathrm{d}b$; **c** $I_v(t)$

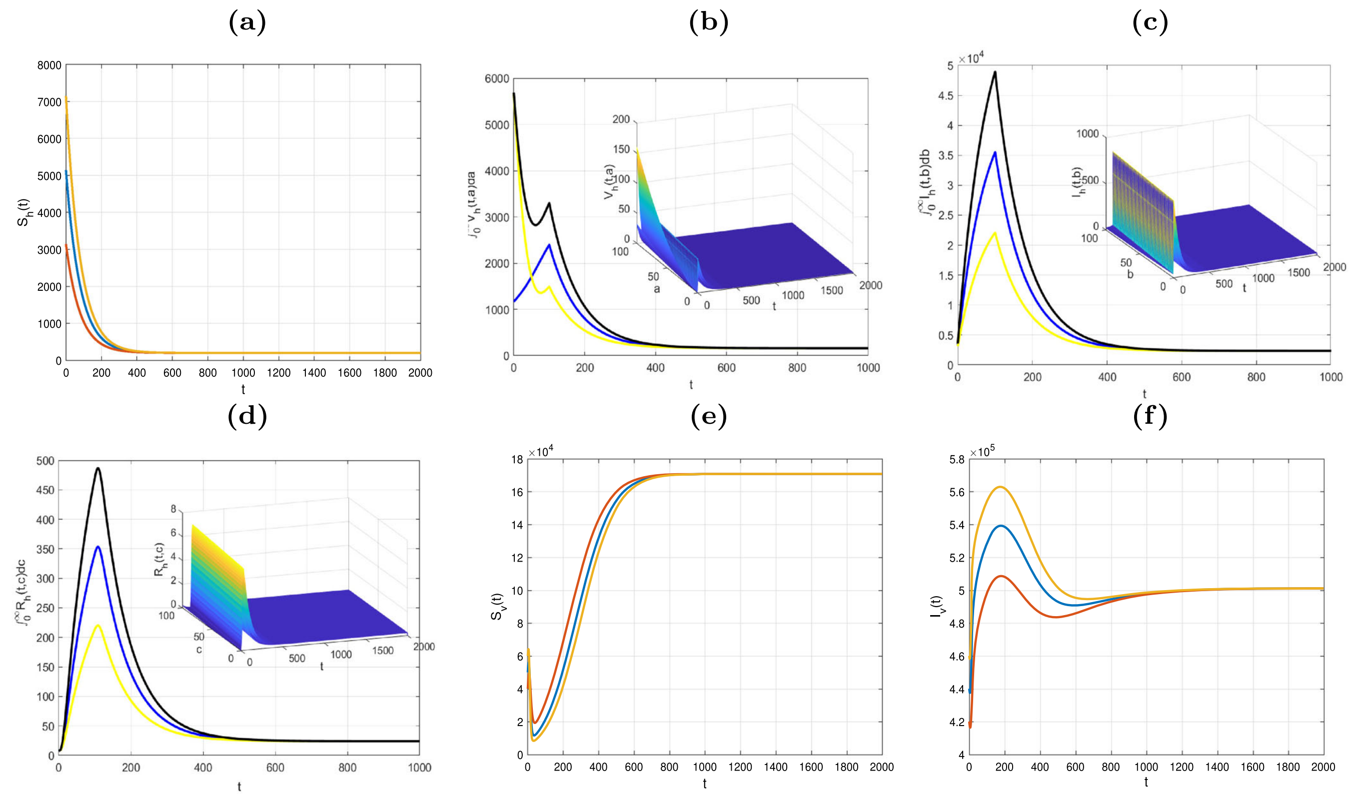

Fig. 2 The global asymptotic stability of $\mathcal{E}^{**}$ of model (1) with $r_h(c) = 0$ and $\mathcal{R}_0 \approx 10.2868 > 1$: **a** $S_h(t)$; **b** $V_h(t, a)$ and $\int_0^\infty V_h(t, a)\mathrm{d}a$; **c** $I_h(t, b)$ and $\int_0^\infty I_h(t, b)\mathrm{d}b$; **d** $R_h(t, c)$ and $\int_0^\infty R_h(t, c)\mathrm{d}c$; **e** $S_v(t)$; **f** $I_v(t)$

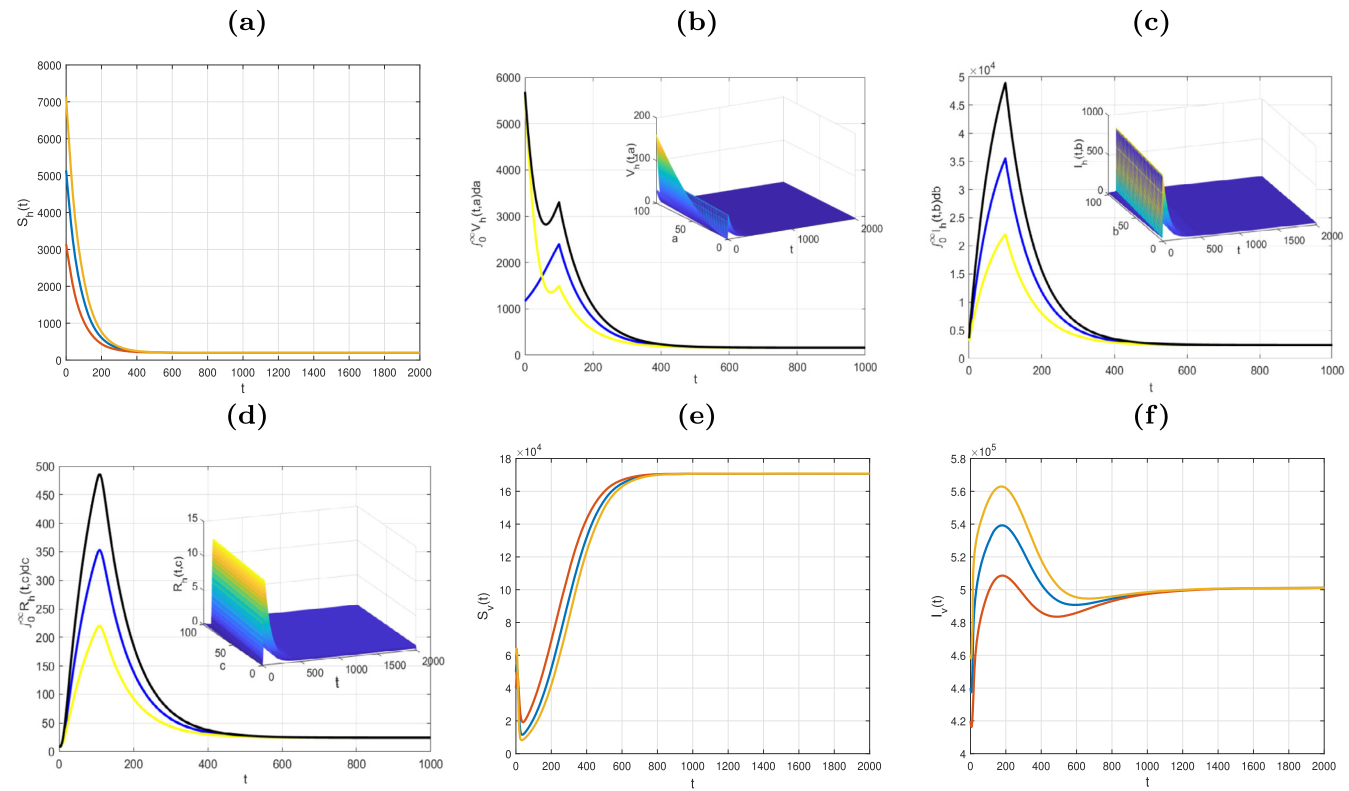

Fig. 3 The global asymptotic stability of $\mathcal{E}^*$ for model (1), where $\mathcal{R}_0 \approx 10.2870 > 1$, $r_h(c) = 0.04e^{-3.5c}c^2 \neq 0$: **a** $S_h(t)$; **b** $V_h(t,a)$ and $\int_0^\infty V_h(t,a)\mathrm{d}a$; **c** $I_h(t,b)$ and $\int_0^\infty I_h(t,b)\mathrm{d}b$; **d** $R_h(t,c)$ and $\int_0^\infty R_h(t,c)\mathrm{d}c$; **e** $S_v(t)$; **f** $I_v(t)$

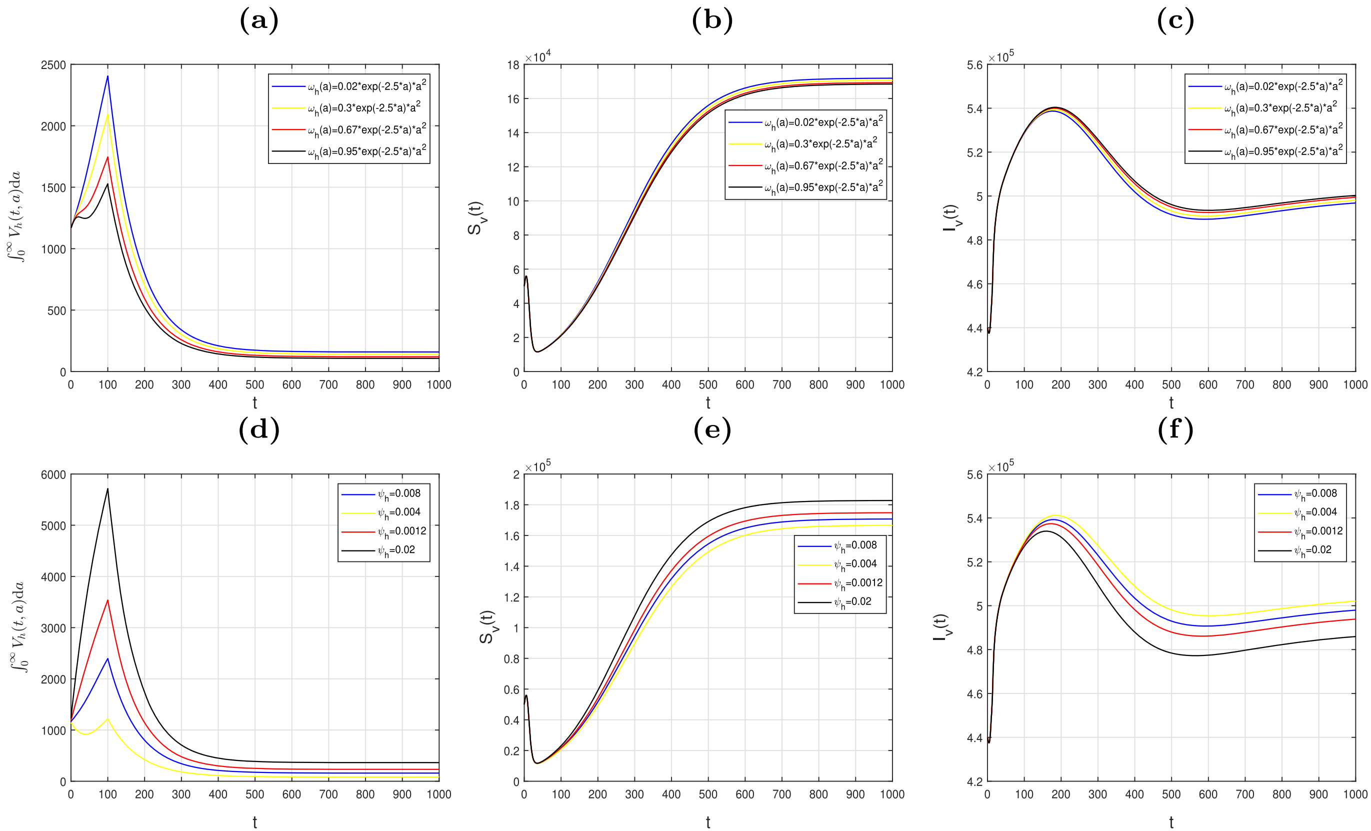

Fig. 4 The influences of the immune loss rate $\omega_h(a)$ and vaccination rate ψ_h on the spread of disease, where, **a–c** the effect of $\omega_h(a)$; **d–f** the effect of ψ_h

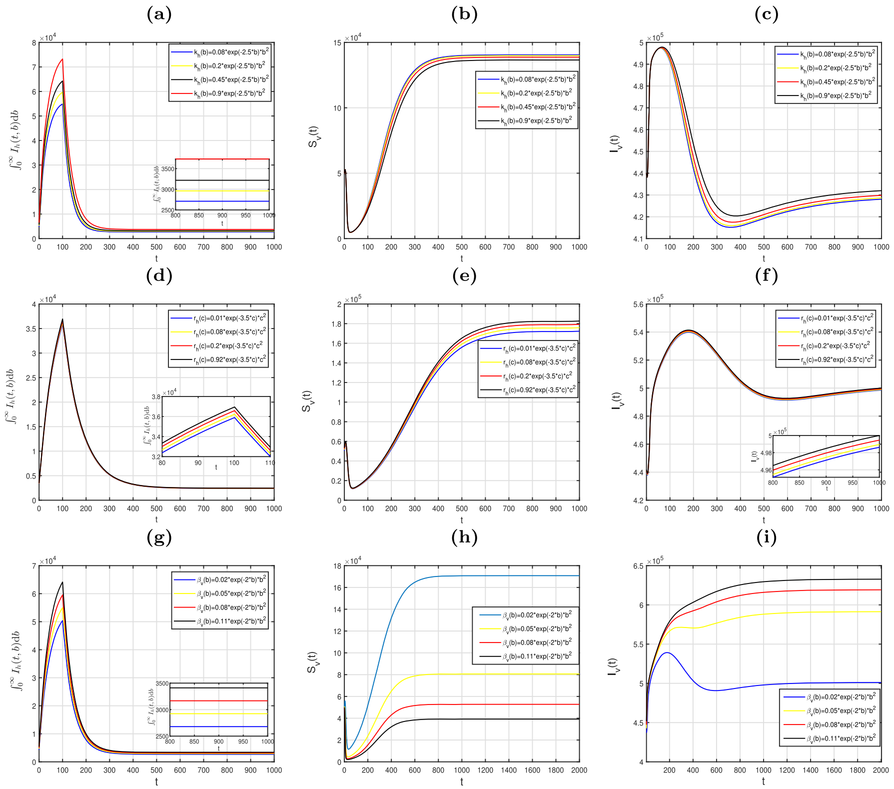

Fig. 5 The effects of age-dependent parameter $k_h(b)$, $r_h(c)$ and $\beta_v(b)$ on the distribution of disease, **a–c** the effect of $k_h(b)$; **d–f** the effect of $r_h(c)$; **g–i** the effect of $\beta_v(b)$

According to the expression of $\mathcal{R}_0$, this is obvious that ψ_h and μ_v are negatively correlated with $\mathcal{R}_0$, which can also be seen from Fig. 6a–b that $\mathcal{R}_0$ decreases as ψ_h and μ_v increase. As shown in Fig. 6, when the transmission rate of the virus β_1 (or the vaccination rate ψ_h) is unchangeable, we can only reduce the basic reproduction number $\mathcal{R}_0$ by increasing the vaccination rate ψ_h (or increasing the mortality rate μ_v of mosquitoes) in order to control the disease.

Finally, some numerical simulations are performed to explain the optimal control problem which is introduced in Sect. 4. Here, we extend the forward and backward sweep method of the ODE model in Ref. [42] that is used for the state equations and the accompanying system in the discrete age structure PDE model. Choose that weight constants $A_1 = 1$, $A_2 = 1$, $A_3 = 1$ and $B_1 = 20$, $B_2 = 900$ to balance these states, and the same parameters in Fig. 3 are used. Further, assume that the effectiveness of the control strategy cannot be 100%, the upper bound of each control function are lowed in order to better visualize the impact on control strategy, where the upper bounds for the control functions $u_1(t)$ and $u_2(t)$ are chosen to be 0.05 and 0.65, respectively. The distributions of infected and recovered individuals, and the time series of susceptible and infected vectors are shown in Fig. 7. More specially, the red curve indicates the

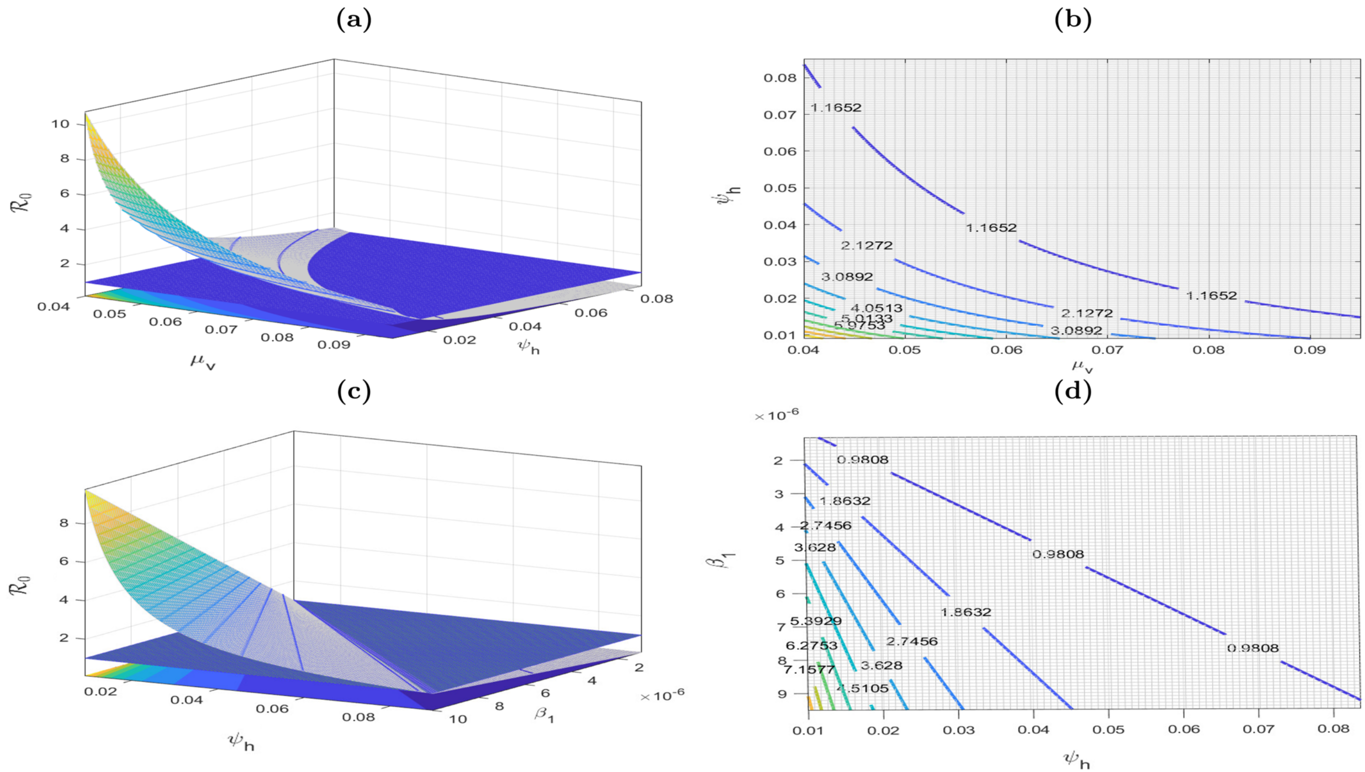

Fig. 6 Sensitivity of β_1 and ψ_h, ψ_h and μ_v on the basic reproduction number $\mathcal{R}_0$, where, **a, b** the effect of ψ_h and μ_v; **d, d** the effect of β_1 and ψ_h

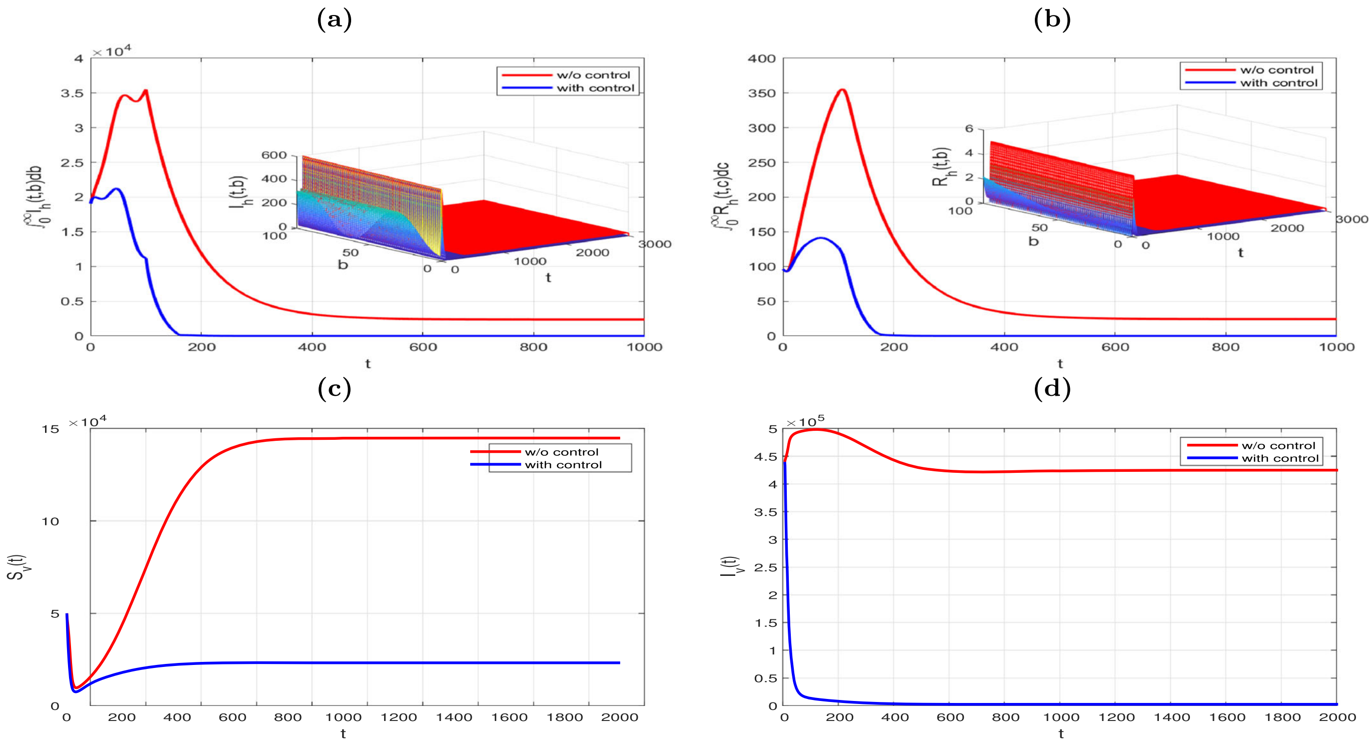

Fig. 7 Time series of the distributions of $\int_0^\infty I_h(t,b)\mathrm{d}b$, $\int_0^\infty R_h(t,c)\mathrm{d}c$, $S_v(t)$ and $I_v(t)$ of model (1) without control function (red curve) or with control function (blue curve): **a** $\int_0^\infty I_h(t,b)\mathrm{d}b$; **b** $\int_0^\infty R_h(t,c)\mathrm{d}c$; **c** $S_v(t)$; **d** $I_v(t)$

case of $u_1(t) = u_2(t) = 0$, while the blue curve indicates the situation with optimal control functions $u_1(t)$ and $u_2(t)$. Numerical simulations indicate that the numbers of infected individuals and and vectors are significantly reduced with optimal control measures. At same time, it can be seen from Fig. 7a–d that the quantities of susceptible and infected vector individuals are dramatically reduced under optimal control. This suggests that for vector-borne diseases, it is more effective to minimize the threat from disease transmission by increasing immunization rates as much as possible for the host and controlling mosquito populations for the vector.

6 Conclusion and Discussion

During the spread of some vector-borne infectious diseases, such as, Chagasdisease (Americantrypanosomiasis), Leishmaniasis, Tungiasis, African trypanosomiasis (sleeping sickness), etc, relapse is a common phenomenon. Due to the intervention in drugs, the concentration of the parasites or bacteria in host is suppressed at a very low level or in a pseudo-dead state. However, with the prolongation of the disease period and the emergence of drug resistance, a relapse of 'recover' can occur once the host's immune system declines. In addition, it is also noted that in the process of disease transmission, the transmission rate of parasites/bacteria from infected vectors to susceptible hosts is not constant, infected individuals have different infectivity at different age of infection. At the same time, the efficacy of the vaccine and the recurrence rate for vector-borne infectious diseases have similar characteristics. We develop, in this paper, a vector-borne disease model with multi-age structure, where the nonlinear incidence is also introduced to portray the complexity of parasites/bacteria transmission between hosts and vectors. The existence and uniqueness of disease-free and endemic steady states, asymptotic smoothness of solution semiflow, uniform persistence of system, and global stability of steady states are analyzed in detailed. In particular, the accurate expression of the basic reproduction number is inferred, which can be used as a threshold value for adjudicating the extinction and persistence of the disease. It is also easy to see from the expression for $\mathcal{R}_0$ that the age-dependent model parameters $\omega_h(a)$, $\beta_v(b)$, $k_h(b)$ and $r_h(c)$ have an important influences on the transmission of diseases. Treating these parameters simply as constants can underestimate or overestimate the risk of disease outbreaks.

Seeking the optimal control strategy for infectious diseases is one of the purposes of dynamical modeling. Therefore, we also discuss the optimal control problem induced by our model. Here, we are controlling this disease by increasing vaccination rates and reducing the mosquito population. The problem of optimal control of infectious disease models with class-age structure is a difficult and hot problem in the field of mathematical biology. By using the Gâteaux derivative rule, the Ekeland's principle, and full combination of the methods from the Refs. [18, 23], the existence of optimal control is obtained. Furthermore, it should be noted that the global asymptotic stability of the epidemic steady state is proved only for the case $r_h(c) = 0$. In the case of $r_h(c) \neq 0$, although we verified the global asymptotical stability of the endemic steady state by numerical simulation, the rigorous theoretical analysis continues to a challenge. This is an open problem that we will continue to focus on and study in the future.

Finally, it should be noted that the main purpose of this paper is to discuss the effects of age of vaccine, infection and relapse on the transmission of vector-borne diseases. Therefore, in order to highlight the research purpose and carry out the necessary theoretical analysis, we assume that the population size is relatively fixed and the host population and vector population have a stable recruitment rate, and ignore the intra-regional or inter-regional population flow and the physiological age of the population. It is also a subject worthy of further study to discuss the vector-borne models with population physiological age, inter-regional or intra-regional diffusion.

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Declarations

Conflict of interest The authors declare there is no conflict of interest.

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