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The Controllability for Second-Order Semilinear Impulsive Systems

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Abstract
This paper studies the controllability of the initial value problems of linear and semilinear second-order impulsive systems. Necessary and sufficient conditions of controllability for linear problems are obtained, and a new rank criterion is presented. We also show semilinear problems are controllable via Krasnoselskii's fixed point theorem. Finally, two examples are provided to verify the theoretically results.

Keywords Controllability · Second-order · Impulsive differential equations · Rank criterion · Semilinear

1 Introduction

Many evolution processes in science and technology, such as mechanics, population dynamics, pharmacokinetics, industrial robotics, biotechnology, economics and so

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on, may change their state rapidly, or the duration of the change is negligible. We describe these processes with impulsive effects by impulsive differential equations and the theory of impulsive differential equations is an important branch of differential equation theory; see [1] and the references therein.

Control theory is an important branch in applied mathematics and engineering and modern control theory was developed by Kalman. Roughly speaking, the object of control theory is to find a control function that can steer the state function to the desired result at the end (terminal). Numerous papers are devoted to the controllability of differential equations in Banach space [2–22], such as exact controllability, approximate controllability and null controllability, and the main techniques are based on fixed point theorems [3, 4, 14, 18, 23], variational methods [5, 24], semigroup theory [2, 8], and so on.

Second-order systems capture the dynamic behavior of many natural phenomena and have applications in many fields such as mathematical physics, electrical power systems, quantum mechanics, biology, long transmission lines and finance [25, 26]. Numerous papers focus on the controllability of second-order impulsive systems (see [2, 3, 6, 7, 11, 27] for cosine family theory and [2, 8, 10, 28] where the corresponding operators of the cosine family are compact). However, as noted by Travis and Webb [29], some of these results work only to finite-dimensional spaces. We refer the reader also to [3, 30] for other results on the controllability of second-order impulsive systems.

For the controllability of initial value problems for second-order differential equations

$$\begin{cases} x''(t) = Ax(t) + Bu(t), \quad t \in [0, b], \\ x(0) = x_0, \quad x'(0) = y_0, \end{cases} \tag{1.1}$$

many authors consider the controllability of the solution $x(t)$ i.e., one finds a control u which makes the state function $x(t)$ arrive at the value that we wish at the terminal. As mentioned in [7], it is unreasonable to regard the damped term $x'(t)$ in the controllability. Recently, the authors in [7, 10, 11, 27] consider the controllability of $x(t)$ and $x'(t)$.

In [7], Li et al. consider the approximate controllability of system (1.1). Let $J = [0, b]$, the state $x(\cdot)$ takes values in a Banach space X, $u(\cdot) \in L^2(J, U)$ is the control function where U is a Banach space, the definition of controllability defined as follows: Systems (1.1) are said to be approximately controllable on J if $\overline{D} = X \times X$, where $D = \{(x(b, x_0, y_0, u), y(b, x_0, y_0, u)) : u \in L^2(J, U)\}$, $y(\cdot, x_0, y_0, u) = x'(\cdot, x_0, y_0, u)$ and $x(\cdot, x_0, y_0, u)$ is a mild solution of (1.1).

Their aim is to pick a control function u which controls both $x(t)$ and $x'(t)$. In [10, 11, 27], the following two assumptions are used,

$(A1)$ The linear operator $G_1 : L^2(J, U) \to X$, defined by

$$G_1 u := \int_0^b S(b - s)Bu(s)ds,$$

has an invertible operator G_1^{-1} which takes the values in $L^2(J, U)/\ker G_1$ and there exists positive constant M_1 such that $\|G_1^{-1}\| \le M_1$.

$(A2)$ The linear operator $G_2 : L^2(J, U) \to X$, defined by

$$G_2 u := \int_0^b C(b-s)Bu(s)ds,$$

has an invertible operator G_2^{-1} which takes the values in $L^2(J, U)/\ker G_2$ and there exists positive constant M_2 such that $\|G_2^{-1}\| \le M_2$.

As pointed by Balachandran and Kim [31] the control function defined in [11, 27] can not steer the value of the state function to what we want at the terminal unless the condition

(H) $G_1 G_2^{-1} = G_2 G_1^{-1} = 0$ is satisfied.

For the second-order systems in finite dimensional space, $(A1)$ or $(A2)$ will lead to a contradiction with the definition of controllability. Since if we assume system (1.1) is controllable. Then for any $(x_1, y_1) \in X \times X$, there exists a control u_1 such that $x(b) = x_1$, and $x'(b) = y_1$ under the control u_1. For another point (x_1, y_2), since $y_1 \ne y_2$, there exists a control u_2 such that $x(b) = x_1$, and $x'(b) = y_2$ under the control u_2 as well. Then if $u_1 = u_2$, we have $y_1 = y_2$, a contradiction; if $u_1 \ne u_2$, since A is the infinitesimal generator of a strongly continuous cosine family $C(t)$ on X, hence, the Cauchy problem (1.1) is well posed. Then from the expression of the solution for (1.1),

$$x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)Bu(s)ds,$$

and we get

$$x_1 = C(b)x_0 + S(b)y_0 + \int_0^b S(b-s)Bu_1(s)ds,$$

and

$$x_1 = C(b)x_0 + S(b)y_0 + \int_0^b S(b-s)Bu_2(s)ds.$$

Combining these two equalities with conditions $(A1)$, we find

$$u_1 = G_1^{-1}(x_1 - C(b)x_0 - S(b)y_0) = u_2,$$

a contradiction to the assumption $u_1 \ne u_2$. Hence, if assumptions $(A1)$ or $(A2)$ hold, we cannot obtain the controllability result of system (1.1) under the definition of controllability defined in [7]. In view of this, we introduce a weaker definition of controllability in Sect. 2.

To the best of our knowledge, there are only a few articles on the controllability of second-order linear systems, and we note that, for finite-dimensional linear systems, all the concepts of controllability are equivalent (exact controllability, approximate controllability and null controllability). In this paper, we consider the controllability of the following initial value problems for second-order impulsive differential equations

$$\begin{cases} x''(t) = Ax(t) + Bu(t), \quad t \in J = [0, b], \quad t \neq t_i, \\ \Delta x(t_i) = B_1 x(t_i^-), \quad i = 1, 2, \ldots, m, \\ \Delta x'(t_i) = B_2 x'(t_i^-), \quad i = 1, 2, \ldots, m, \\ x(0) = x_0, \quad x'(0) = y_0, \end{cases} \tag{1.2}$$

and semilinear second-order impulsive differential equations

$$\begin{cases} x''(t) = Ax(t) + Bu(t) + f(t, x(t)), \quad t \in J' = J \backslash \{t_i\}, \quad i = 1, 2, \ldots, m, \\ x(t_i^+) = x(t_i^-) + B_1 x(t_i^-), \quad i = 1, 2, \ldots, m, \\ x'(t_i^+) = x'(t_i^-) + B_2 x'(t_i^-), \quad i = 1, 2, \ldots, m, \\ x(0) = x_0, \quad x'(0) = y_0, \end{cases} \tag{1.3}$$

where A, B_1 and B_2 are constant $n \times n$ matrices satisfying $AB_1 = B_1 A$, $AB_2 = B_2 A$, $B_1 B_2 = B_2 B_1$, $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = b$ are impulsive points, $u \in L^2(J, \mathbb{R}^n)$ is a control function, and $f \in C(J \times \mathbb{R}^n; \mathbb{R}^n)$.

The contributions of this paper are as follows:

(1) We introduce a weaker definition of controllability with respect to the state function $x(t)$ and the damped term $x'(t)$.
(2) We present a new algebraic method to obtain a rank criterion, and a rank criterion of controllability for second-order impulsive linear systems is given.
(3) Based on the controllability of the linear systems, we give a sufficient condition to guarantee the controllability of the semilinear second-order impulsive systems.

The paper is structured in the following way. In Sect. 2, we give a weaker definition of controllability and some associated notations and essential lemmas. In Sect. 3, instead of converting a second-order system into a first order system, we obtain a new rank criterion of controllability of system (1.2) by direct analysis of the second-order system itself. In Sect. 4, we give a sufficient condition of the controllability of the system (1.3). Finally, in Sect. 5, some examples are provided to illustrate the suitability of our results.

2 Preliminaries

In this section, we modify the definition of controllability and list some notations and properties needed to establish our main results.

Let$PC(J, \mathbb{R}^n)$ denote the Banach space of piecewise continuous functions on the interval J, that is $PC(J, \mathbb{R}^n) = \{v : J \to \mathbb{R}^n | u \in C((t_{k-1}, t_k], \mathbb{R}^n)$ for $k \in \{1, \ldots, m+1\}$ and there exists $v(t_k^-)$ and $v(t_k^+)$, $k \in \{1, \ldots, m\}$ with $v(t_k) = v(t_k^-)\}$ equipped with the Chebyshev PC-norm $||v||_{PC} := \sup\{||v(t)|| : t \in J\}$.

Let $PC^1(J, \mathbb{R}^n) := \{x \in PC(J, \mathbb{R}^n) : x' \in PC(J, \mathbb{R})\}$ equipped with the norm $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$. Obviously, $PC(I, \mathbb{R}^n)$ endowed with the norm $\|\cdot\|_{PC^1}$ is also a Banach space. We use the notation

$$A_1 = I + \frac{B_1 + B_2}{2}, \quad A_2 = \frac{B_1 - B_2}{2}.$$

Let $m = i(t, 0)$ denote the number of impulsive points on $(0, t)$, and assume $AB = BA$.

Definition 2.1 The system (1.2) is said to be exact controllability in $\mathbb{R}^n$, if for each pair $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^n$, there exists a pair of control functions $(u_1(\cdot), u_2(\cdot)) \in L^2([0, b], \mathbb{R}^n) \times L^2([0, b], \mathbb{R}^n)$ such that for any $(x_1, y_1) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$x(b) = x_1, \quad y'(b) = y_1,$$

here $x(\cdot)$ is the solution of (1.2) under the control u_1, $y(\cdot)$ is the solution of (1.2) under the control u_2, and $y'(t) = dy(t)/dt$.

Remark 2.2 In [4, 5, 9, 14, 18], the definition of controllability imply that one find a control function which steer the state function $x(\cdot)$ to the target value, and in [7, 10, 11, 27], which imply that one find a control function which steer both the state function $x(\cdot)$ and damped term $x'(\cdot)$ to the value we wanted. However, Definition 2.1 indicates that one pick a pair of control functions (u_0, u_1) such that u_0 control the state function $x(t)$ and u_1 control the damped term $y'(t)$. Notice that at this moment except for a constant difference, the antiderivative of damped term $y'(t)$ may be different with the state function $x(t)$.

The following Lemmas is crucial to our proof of main results.

Lemma 2.3 (see [32]) *Let $|\cdot|$ be a norm on $\mathbb{R}^n$ and B be an $n \times n$ matrix. Then for any $\varepsilon > 0$ there exist $T_{B,\varepsilon} \geq 1$ such that $||B^k|| \leq T_{B,\varepsilon}(\rho(B) + \varepsilon)^k$, where $\rho(B)$ is the spectral radius of B.*

Lemma 2.4 (Krasnoselskii's fixed point theorem) *Let B be a bounded closed and convex subset of a Banach space X and let F_1, F_2 be maps of B into X such that $F_1 x + F_2 y \in B$ for every $x, y \in B$. If F_1 is a contraction and F_2 is compact and continuous, then the equation $F_1 x + F_2 x = x$ has a solution on B.*

Lemma 2.5 (PC-type Ascoli–Arzela theorem, see [33]) *Let $Q \subset PC(\Omega, X)$ where X is a Banach space. Then Q is a relatively compact subset of $PC(\Omega, X)$ if, (a) Q is uniformly bounded subset of $PC(\Omega, X)$; (b) Q is equicontinuous in (t_i, t_{i+1}), $i = 0, 1, \ldots, k$; and (c) $Q(t) = \{v(t) | v \in Q, t \in \Omega \backslash \{t_i\}, i = 0, 1, \ldots, k\}$, $Q(t_i^+) = \{v(t_i^+) | v \in Q\}$ and $Q(t_i^-) = \{v(t_i^-) | v \in Q\}$ are relatively compact subsets of X.*

Lemma 2.6 (see [34]) *For $t \in (t_m, t_{m+1}]$, $m = 0, 1, \ldots, k$, the solution of* (1.2) *is given by*

$$x(t) = W(A, t, x_0, y_0) + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} W_i(A, t, s)Bu(s)ds + A^{-\frac{1}{2}} \int_{t_m}^{t} \sinh A^{\frac{1}{2}}(t-s)Bu(s)ds,$$

where $W(A, t, x_0, y_0)$ is the solution of the homogeneous initial value problem of (1.2)*, and*

$$\begin{aligned} &W_i(A, t, s) \\ &= A_1^{m-i} A^{-\frac{1}{2}} \sinh A^{\frac{1}{2}}(t-s) - A_1^{m-i-1} A_2 A^{-\frac{1}{2}} \sum_{i+1 \le i_{11} \le m} \sinh A^{\frac{1}{2}}(t - 2t_{i_{11}} + s) \\ &+ A_1^{m-i-2} A_2^2 A^{-\frac{1}{2}} \sum_{i+1 \le i_{21} < i_{22} \le m} \sinh A^{\frac{1}{2}}(t - 2t_{i_{22}} + 2t_{i_{21}} - s) \\ &+ \cdots + (-1)^{m-i-1} A_1 A_2^{m-i-1} A^{-\frac{1}{2}} \cdot \\ &\sum_{i+1 \le i_{m-i-1,1} < i_{m-i-1,2} < \cdots < i_{m-i-1,m-i-1} \le m} \sinh A^{\frac{1}{2}}(t - 2t_{i_{m-i-1,m-i-1}} \\ &2t_{i_{m-i-1,m-i-2}} - \cdots \pm 2t_{i_{m-i-1,1}} \mp s) + (-1)^{m-i} A_2^{m-i} A^{-\frac{1}{2}} \sinh A^{\frac{1}{2}} \\ &(t - 2t_m + 2t_{m-1} - \cdots \pm 2t_{i+1} \mp s), \quad i = 0, 1, \ldots, m-1. \end{aligned}$$

Consider the notation

$$Q_m(t, s) = \begin{cases} W_0(A, t, s), & t_0 \le s \le t_1, \\ W_1(A, t, s), & t_1 < s \le t_2, \\ \cdots \\ W_{m-1}(A, t, s), & t_{m-1} < s \le t_m, \\ A^{-\frac{1}{2}} \sinh A^{\frac{1}{2}}(t-s), & t_m < s \le t, \end{cases}$$

then the solution of (1.2) can be expressed by

$$x(t) = W(A, t, x_0, y_0) + \int_0^t Q_m(t, s)Bu(s)ds. \tag{2.1}$$

Lemma 2.7 *For any $t_m < \tau_1 \le \tau_2 \le b$, and $t_m < t \le t_{m+1} \le b$, we have*

$$\left\| \int_0^t \big(Q_m(\tau_2, s) - Q_m(\tau_1, s)\big)ds \right\| \le \theta_1 |\tau_2 - \tau_1|,$$

and

$$\Big\| \int_0^t \big(Q'_m(\tau_2, s) - Q'_m(\tau_1, s)\big)ds \Big\| \le \theta_2 |\tau_2 - \tau_1|,$$

where θ_1 and θ_2 are positive constants, and $Q'_m(t, s)$ denotes the function that takes derivative with respect to t.

Proof Since

$$A^{\frac{1}{2}} \sinh A^{\frac{1}{2}} t = A \sum_{n=0}^{\infty} \frac{A^n t^{2n+1}}{(2n+1)!},$$
$$\cosh A^{\frac{1}{2}} t = \sum_{n=0}^{\infty} \frac{A^n t^{2n}}{(2n)!},$$

combining this with the Lemma 2.3, we have

$$\|A^{\frac{1}{2}} \sinh A^{\frac{1}{2}} t\| \le T_{A,\varepsilon} \sqrt{\rho(A) + \varepsilon} e^{\sqrt{\rho(A)+\varepsilon} t}, \tag{2.2}$$

and

$$\| \cosh A^{\frac{1}{2}} t\| \le T_{A,\varepsilon} e^{\sqrt{\rho(A)+\varepsilon} t}. \tag{2.3}$$

According to the definition of Q_m, inequality (2.3), and the mean value theorem, we find

$$\begin{aligned}
&\Big\| \int_0^t \big(Q_m(\tau_2, s) - Q_m(\tau_1, s)\big)ds \Big\| \\
&\le \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \|W_i(A, \tau_2, s) - W_i(A, \tau_1, s)\| ds \\
&\le \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \|A_1^{m-i} \cosh A^{\frac{1}{2}} \varsigma_0\| + \|A_1^{m-i-1} A_2 \sum_{j=1}^{C_{m-i}^1} \cosh A^{\frac{1}{2}} \varsigma_{1,j}\| \\
&\quad + \cdots + \|A_1 A_2^{m-i-1} \sum_{j=1}^{C_{m-i}^{m-i-1}} \cosh A^{\frac{1}{2}} \varsigma_{m-i-1,j}\| \\
&\quad + \|A_2^{m-i} \cosh A^{\frac{1}{2}} \varsigma_{m-i}\| ds \cdot |\tau_2 - \tau_1| \\
&\le T_{A,\varepsilon} e^{\sqrt{\rho(A)+\varepsilon} b} \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} T_\varepsilon^2 (\rho(A_1) + \rho(A_2) + 2\varepsilon)^{m-i} ds \\
&\quad \cdot |\tau_2 - \tau_1|
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=0}^{m} \big(\rho(A_1)+\rho(A_2)+2\varepsilon\big)^{m-i} b T_\varepsilon^3 e^{\sqrt{\rho(A)+\varepsilon}b} |\tau_2-\tau_1| \\
&=: \theta_1|\tau_2-\tau_1|,
\end{aligned}
$$

where $\varsigma_0, \varsigma_{1,j}, \ldots, \varsigma_{m-i-1,j}, \varsigma_{m-i}$ are selected by the mean value theorem located in $[-b, b]$, $T_\varepsilon = \max\{T_{A,\varepsilon}, T_{A_1,\varepsilon}, T_{A_2,\varepsilon}\}$. Similarly, by virtue of the definition of Q_m, inequality (2.2), and the mean value theorem, we have

$$
\begin{aligned}
&\Big\| \int_0^t \big(Q_m'(\tau_2,s)-Q_m'(\tau_1,s)\big)ds \Big\| \\
&\le \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \|W_i'(A,\tau_2,s)-W_i'(A,\tau_1,s)\|ds \\
&\le \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \|A_1^{m-i}A^{\frac12}\sinh A^{\frac12}\xi_0\| + \|A_1^{m-i-1}A_2A^{\frac12}\sum_{j=1}^{C_{m-i}^1}\sinh A^{\frac12}\xi_{1,j}\| \\
&\quad + \cdots + \|A_1A_2^{m-i-1}A^{\frac12}\sum_{j=1}^{C_{m-i}^{m-i-1}}\sinh A^{\frac12}\xi_{m-i-1,j}\| \\
&\quad + \|A_2^{m-i}A^{\frac12}\sinh A^{\frac12}\xi_{m-i}\|ds\cdot|\tau_2-\tau_1| \\
&\le T_{A,\varepsilon}\sqrt{\rho(A)+\varepsilon}e^{\sqrt{\rho(A)+\varepsilon}b}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}} T_\varepsilon^2(\rho(A_1)+\rho(A_2)+2\varepsilon)^{m-i}ds \\
&\quad \cdot|\tau_2-\tau_1| \\
&\le \sum_{i=0}^{m} \big(\rho(A_1)+\rho(A_2)+2\varepsilon\big)^{m-i} bT_\varepsilon^3\sqrt{\rho(A)+\varepsilon}e^{\sqrt{\rho(A)+\varepsilon}b}|\tau_2-\tau_1| \\
&=: \theta_2|\tau_2-\tau_1|,
\end{aligned}
$$

where $\xi_0, \xi_{1,j}, \ldots, \xi_{m-i-1,j}, \xi_{m-i}$ are selected by the mean value theorem located in $[-b, b]$. □

3 The controllability of linear systems

In this section, we present some controllability criteria for systems (1.2) by using an algebraic method.

Theorem 3.1 *The following statements are equivalent:* 1°*The system* (1.2) *is exact controllability;* 2° *The matrix* $\Gamma_0^b = \int_0^b Q_k(b,s)BB^*Q_k^*(b,s)ds$ *and* $\Lambda_0^b = \int_0^b Q_k'(b,s)BB^*Q_k'^*(b,s)ds$ *are nonsingular;* 3° *There at least exists a pair of integers* $0 \le i \le k$, $0 \le j \le k$ *such that both* $\int_{t_i}^{t_{i+1}} W_i(b,s)BB^*W_i^*(b,s)ds$ *and* $\int_{t_j}^{t_{j+1}} W_j'(b,s)BB^*W_j'^*(b,s)ds$ *are nonsingular.*

Proof First, we show the equivalence of 1° and 2°. Assume the systems are exact controllability. We show that the matrix Γ_0^b and Λ_0^b both are nonsingular. If the result is not true, then at least one of matrices Γ_0^b and Λ_0^b is singular. Suppose Γ_0^b is singular. Then there exists a nonzero vector $\overline{x}_0 \in \mathbb{R}^n$ such that

$$\int_0^b \overline{x}_0^T Q_k(b,s)BB^* Q_k^*(b,s)\overline{x}_0 ds = 0.$$

Hence we have

$$\int_0^b \|B^* Q_k^*(b,s)\overline{x}_0\|^2 ds = 0,$$

that is

$$B^* Q_k^*(b,s)\overline{x}_0 = \mathbf{0}, \quad \forall s \in (0,b]. \tag{3.1}$$

On the other hand, since the systems are exact controllability, then because of the definition of exactly controllability, there exists a pair of control functions (u_1, u_2) such that for $\overline{x}_0 + W(A, b, x_0, y_0) \in \mathbb{R}^n$, the solution $x(\cdot)$ of systems (1.2) under the control $u_1(\cdot)$ arrives at $\overline{x}_0 + W(A, b, x_0, y_0) \in \mathbb{R}^n$ at the terminal b, i.e.

$$\overline{x}_0 + W(A, b, x_0, y_0) = W(A, b, x_0, y_0) + \int_0^b Q_k(b,s)Bu_1 ds. \tag{3.2}$$

Now (3.1) with (3.2) allows us to affirm that

$$\|\overline{x}_0\|^2 = \overline{x}_0^T \overline{x}_0 = \int_0^b u_1^T B^* Q_k^*(b,s)\overline{x}_0 ds = 0,$$

which implies $\overline{x}_0 = \mathbf{0}$ and this contradicts the hypothesis. Hence Γ_0^b is nonsingular. In a similar way, we obtain that Λ_0^b is nonsingular,

If both the matrices Γ_0^b and Λ_0^b are nonsingular, we prove that systems (1.2) are exactly controllability, that is for any fixed $(x_1, y_1) \in \mathbb{R}^n \times \mathbb{R}^n$, we show that there exists a pair of control functions $(u_1(\cdot), u_2(\cdot)) \in L^2([0,b], \mathbb{R}^n) \times L^2([0,b], \mathbb{R}^n)$ such that the solution $x(t)$ of systems (1.2) satisfies $x(b) = x_1$ under the control $u_1(\cdot)$ and $y'(b) = y_1$ under the control $u_2(\cdot)$. We choose the control functions by

$$u_1(t) = B^* Q_k^*(b,t)(\Gamma_0^b)^{-1}(x_1 - W(A, b, x_0, y_0)), \tag{3.3}$$

and

$$u_2(t) = B^* Q_k'^*(b,t)(\Lambda_0^b)^{-1}(y_1 - W'(A, b, x_0, y_0)). \tag{3.4}$$

Then we have

$$x(t) = W(A, t, x_0, y_0) + \int_0^t Q_k(t, s)BB^* Q_k^*(b, s)(\Gamma_0^b)^{-1}(x_1 - W(A, b, x_0, y_0))ds.$$

Obviously, $x(b) = x_1$. Similarly, under the control function $u_2(t)$, $y'(t)$ satisfies

$$y'(t) = W'(A, t, x_0, y_0) + \int_0^t Q_k'(t, s)BB^* Q_k'^*(b, s)(\Lambda_0^b)^{-1}(y_1 - W'(A, b, x_0, y_0))ds,$$

and we have $y'(b) = y_1$. Hence the systems (1.2) are exact controllability.

Next, we show the equivalence of 2° and 3°. Assume the matrix Γ_0^b is singular. Then there exists a nonzero vector $\overline{x}_0 \in \mathbb{R}^n$ such that

$$\int_0^b \overline{x}_0^T Q_k(b, s)BB^* Q_k^*(b, s)\overline{x}_0 ds = 0,$$

that is

$$\begin{aligned}
&\int_0^{t_1} \overline{x}_0^T W_0(b, s)BB^* W_0^*(b, s)\overline{x}_0 ds + \int_{t_1}^{t_2} \overline{x}_0^T W_1(b, s)BB^* W_1^*(b, s)\overline{x}_0 ds \\
&\quad + \cdots + \int_{t_{k-1}}^{t_k} \overline{x}_0^T W_{k-1}(b, s)BB^* W_{k-1}^*(b, s)\overline{x}_0 ds \\
&\quad + \int_{t_k}^b \overline{x}_0^T A^{-\frac{1}{2}} \sinh A^{\frac{1}{2}}(b - s)BB^*(\sinh A^{\frac{1}{2}}(b - s))^*(A^{-\frac{1}{2}})^*\overline{x}_0 ds = 0,
\end{aligned}$$

which is equivalent to

$$\int_{t_i}^{t_{i+1}} \overline{x}_0^T W_i(b, s)BB^* W_i^*(b, s)\overline{x}_0 ds = 0, \quad \forall 0 \le i \le k,$$

that is $\int_{t_i}^{t_{i+1}} W_i(b, s)BB^* W_i^*(b, s)ds$ is singular for all $0 \le i \le k$. Hence Γ_0^b is nonsingular iff there at least exists a constant $0 \le i \le k$ such that $\int_{t_i}^{t_{i+1}} W_i(b, s)BB^* W_i^*(b, s)ds$ is nonsingular.

By the same argument, we also can show that Λ_0^b is nonsingular iff there at least exists a constant $0 \le j \le k$ such that the matrix $\int_{t_j}^{t_{j+1}} W_j'(b, s)BB^* W_j'^*(b, s)ds$ is nonsingular. □

Remark 3.2 Theorem 3.1 shows that initial value problems of second-order linear impulsive systems (1.2) are controllable iff there exist constants $\lambda > 0$ and $\gamma > 0$ such that for all $x \in \mathbb{R}^n$,

$$(\Gamma_0^b x, x) \ge \gamma \|x\|^2,$$

and

$$(\Lambda_0^b x, x) \geq \lambda \|x\|^2.$$

Then

$$\|(\Gamma_0^b)^{-1}\| \leq \frac{1}{\gamma}, \quad \|(\Lambda_0^b)^{-1}\| \leq \frac{1}{\lambda}. \tag{3.5}$$

Since the conditions which guarantee the controllability in Theorem 3.1 are formal and are hard to verify. In what follows, we give a new rank criterion of controllability of systems (1.2). For convenience in writing, in what follows, we use the notation

$$\Psi(t) = A^{-\frac{1}{2}} \sinh A^{\frac{1}{2}} t.$$

Theorem 3.3 *Systems* (1.2) *are exact controllability iff there exists a pair of integers* $l_1, l_2 \in \{0, 1, 2, \ldots, k\}$ *such that*

$$Rank \left(A_1^{l_1} B \cdots A_1^{l_1 - i} A_2^i B \cdots A_2^{l_1} B \right) = n,$$

and

$$Rank \left(A_1^{l_2} B \cdots A_1^{l_2 - i} A_2^i B \cdots A_2^{l_2} B \right) = n.$$

Proof Theorem 3.1 shows that systems (1.2) are exact controllability iff there is a pair of integers $0 \leq i \leq k$, $0 \leq j \leq k$ such that both $\int_{t_i}^{t_{i+1}} W_i(b, s) B B^* W_i^*(b, s) ds$ and $\int_{t_j}^{t_{j+1}} W_j'(b, s) B B^* W_j'^*(b, s) ds$ are nonsingular. We subdivide the proof into several cases.

Case 1 If $i = j = k$, that is both $\int_{t_k}^b W_k(b, s) B B^* W_k^*(b, s) ds$ and $\int_{t_k}^b W_k'(b, s) B B^* W_k'^*(b, s) ds$ are nonsingular. Now $\int_{t_k}^b W_k(b, s) B B^* W_k^*(b, s) ds$ is singular iff there exists a nonzero vector $x_0 \in \mathbb{R}^n$ such that $x_0^T W_k(b, s) B = \mathbf{0}$ for all $t_k \leq s < b$. Since A is a nonsingular matrix, we have Rank $W_k(b, s) = n$, hence Rank $B < n$. Likewise, we can show that $\int_{t_k}^b W_k'(b, s) B B^* W_k'^*(b, s) ds$ is singular iff Rank $B < n$. Hence both $\int_{t_k}^b W_k(b, s) B B^* W_k^*(b, s) ds$ and $\int_{t_k}^b W_k'(b, s) B B^* W_k'^*(b, s) ds$ are nonsingular iff Rank $B = n$.

Case 2 If $i = j = k - 1$, we show that both $\int_{t_{k-1}}^{t_k} W_{k-1}(b, s) B B^* W_{k-1}^*(b, s) ds$ and $\int_{t_{k-1}}^{t_k} W_{k-1}'(b, s) B B^* W_{k-1}'^*(b, s) ds$ are nonsingular iff

$$\text{Rank} \left(A_1 B \ \ A_2 B \right) = n.$$

Assume $\int_{t_{k-1}}^{t_k} W_{k-1}(b, s) B B^* W_{k-1}^*(b, s) ds$ is nonsingular. Then we show that

$$\text{Rank} \left(A_1 B \ \ A_2 B \right) = n.$$

If this is not true, that is Rank $\left(A_1B \ A_2B\right) < n$, then there exists a nonzero vector $x_0 \in \mathbb{R}^n$ such that

$$x_0^T \left(A_1B \ A_2B\right) = \mathbf{0}_{1\times 2n},$$

i.e.,

$$x_0^T A_1 B = \mathbf{0}, \quad x_0^T A_2 B = \mathbf{0}.$$

Hence, for all $t_{k-1} < s \le t_k$,

$$\begin{aligned} x_0^T W_{k-1}(b,s)B &= x_0^T [A_1 A^{-\frac{1}{2}} \sinh A^{\frac{1}{2}}(b-s)B - A_2 A^{-\frac{1}{2}} \sinh A^{\frac{1}{2}}(b-2t_k+s)B] \\ &= x_0^T [A_1 B A^{-\frac{1}{2}} \sinh A^{\frac{1}{2}}(b-s) - A_2 B A^{-\frac{1}{2}} \sinh A^{\frac{1}{2}}(b-2t_k+s)] \\ &= \mathbf{0}, \end{aligned}$$

which implies $\int_{t_{k-1}}^{t_k} W_{k-1}(b,s)BB^*W_{k-1}^*(b,s)ds$ is singular. This contradicts the hypothesis. Therefore, Rank $\left(A_1B \ A_2B\right) = n$.

Assume Rank $\left(A_1B \ A_2B\right) = n$. We will show that $\int_{t_{k-1}}^{t_k} W_{k-1}(b,s)BB^*W_{k-1}^*(b,s)$ ds is nonsingular. Assume $\int_{t_{k-1}}^{t_k} W_{k-1}(b,s)BB^*W_{k-1}^*(b,s)ds$ is singular. First, we prove that there exists a number sequence $(\lambda_1, \lambda_2) \in (t_{k-1}, t_k]^2$, where $\lambda_1 \neq \lambda_2$, such that the matrix

$$\begin{pmatrix} \Psi(b-\lambda_1) & \Psi(b-\lambda_2) \\ \Psi(b-2t_k+\lambda_1) & \Psi(b-2t_k+\lambda_2) \end{pmatrix} \tag{3.6}$$

is nonsingular. Suppose for every $(\lambda_1, \lambda_2) \in (t_{k-1}, t_k]^2$, we have

$$\begin{vmatrix} \Psi(b-\lambda_1) & \Psi(b-\lambda_2) \\ \Psi(b-2t_k+\lambda_1) & \Psi(b-2t_k+\lambda_2) \end{vmatrix} = 0. \tag{3.7}$$

Take $\lambda_2 = t_k$ in (3.7), since $|\Psi(b-\lambda_2)| \neq 0$, we find

$$|\Psi(b-2t_k+\lambda_1) - \Psi(b-\lambda_1)| = 0, \tag{3.8}$$

however, by the Jordan decomposition, we find zero is not an eigenvalue of $\Psi(b-2t_k+\lambda_1) - \Psi(b-\lambda_1)$, hence, (3.8) is not valid, that is there exists a number sequence $(\lambda_1, \lambda_2) \in (t_{k-1}, t_k]^2$ such that (3.6) is nonsingular.

Suppose $\int_{t_{k-1}}^{t_k} W_{k-1}(b,s)BB^*W_{k-1}^*(b,s)ds$ is singular. Then there exists a nonzero vector $x_0 \in \mathbb{R}^n$ such that

$$\int_{t_{k-1}}^{t_k} x_0^T W_{k-1}(b,s)BB^*W_{k-1}^*(b,s)x_0 ds = 0,$$

that is

$$\int_{t_{k-1}}^{t_k} \|x_0^T W_{k-1}(b,s)B\|^2 ds = 0,$$

which implies

$$\begin{aligned} & x_0^T \left[A_1 A^{-\frac{1}{2}} \sinh A^{\frac{1}{2}}(b-s)B - A_2 A^{-\frac{1}{2}} \sinh A^{\frac{1}{2}}(b-2t_k+s)B \right] \\ & = \mathbf{0}, \quad \forall t_{k-1} < s \le t_k. \end{aligned} \tag{3.9}$$

Take $(\lambda_1, \lambda_2) \in (t_{k-1}, t_k]^2$ such that (3.6) is nonsingular, and then by (3.9), we find

$$x_0^T \begin{pmatrix} A_1 B & -A_2 B \end{pmatrix} \begin{pmatrix} \Psi(b-\lambda_1) & \Psi(b-\lambda_2) \\ \Psi(b-2t_k+\lambda_1) & \Psi(b-2t_k+\lambda_2) \end{pmatrix} = \mathbf{0}_{1\times 2n},$$

therefore,

$$x_0^T \begin{pmatrix} A_1 B & -A_2 B \end{pmatrix} = \mathbf{0}_{1\times 2n},$$

that is

$$\text{Rank} \begin{pmatrix} A_1 B & -A_2 B \end{pmatrix} < n,$$

which contradict the hypothesis.

Thus $\int_{t_{k-1}}^{t_k} W_{k-1}(b,s)BB^* W_{k-1}^*(b,s)ds$ is nonsingular iff

$$\text{Rank} \begin{pmatrix} A_1 B & A_2 B \end{pmatrix} = n.$$

Using the same argument we can establish that $\int_{t_{k-1}}^{t_k} W'_{k-1}(b,s)BB^* W'^*_{k-1}(b,s)ds$ is nonsingular iff

$$\text{Rank} \begin{pmatrix} A_1 B & A_2 B \end{pmatrix} = n.$$

Case 3 If $i = j = k-2$, we show that both $\int_{t_{k-2}}^{t_{k-1}} W_{k-2}(b,s)BB^* W_{k-2}^*(b,s)ds$ and $\int_{t_{k-2}}^{t_{k-1}} W'_{k-2}(b,s)BB^* W'^*_{k-2}(b,s)ds$ are nonsingular iff

$$\text{Rank} \begin{pmatrix} A_1^2 B & A_1 A_2 B & A_2^2 B \end{pmatrix} = n.$$

To do this, we first show an auxiliary result. For $s_1, s_2, s_3 \in (t_{k-2}, t_{k-1}]$, let

$$\Sigma_3 = \begin{pmatrix} \Psi(b-s_1) & \Psi(b-s_2) & \Psi(b-s_3) \\ \sum\limits_{k-1 \le i_{11} \le k} \Psi(b-2t_{i_{11}}+s_1) & \sum\limits_{k-1 \le i_{11} \le k} \Psi(b-2t_{i_{11}}+s_2) & \sum\limits_{k-1 \le i_{11} \le k} \Psi(b-2t_{i_{11}}+s_3) \\ \Psi(b-2t_k+2t_{k-1}-s_1) & \Psi(b-2t_k+2t_{k-1}-s_2) & \Psi(b-2t_k+2t_{k-1}-s_3) \end{pmatrix},$$

and

$$\Sigma_3' = \begin{pmatrix} \Psi'(b-s_1) & \Psi'(b-s_2) & \Psi'(b-s_3) \\ \sum\limits_{k-1\le i_{11}\le k} \Psi'(b-2t_{i_{11}}+s_1) & \sum\limits_{k-1\le i_{11}\le k} \Psi'(b-2t_{i_{11}}+s_2) & \sum\limits_{k-1\le i_{11}\le k} \Psi'(b-2t_{i_{11}}+s_3) \\ \Psi'(b-2t_k+2t_{k-1}-s_1) & \Psi'(b-2t_k+2t_{k-1}-s_2) & \Psi'(b-2t_k+2t_{k-1}-s_3) \end{pmatrix}.$$

We claim that

$$\Sigma_3 x_0 = \mathbf{0}_{3n\times n}, \quad \forall (s_1, s_2, s_3) \in (t_{k-2}, t_{k-1})^3, \tag{3.10}$$

or

$$\Sigma_3' x_0 = \mathbf{0}_{3n\times n}, \quad \forall (s_1, s_2, s_3) \in (t_{k-2}, t_{k-1})^3, \tag{3.11}$$

where $x_0 = (\lambda_1 I, \lambda_2 I, \lambda_3 I)^T \in \mathbb{R}^{3n\times n}$, implies $x_0 = \mathbf{0}_{3n\times n}$. We only show that (3.10) implies that $x_0 = \mathbf{0}_{3n\times n}$, and the other case can be treated similarly. For any $s \in (t_{k-2}, t_{k-1})$ and $\varepsilon_1, \varepsilon_2 > 0$ small enough, let

$$s_1 = s, \quad s_2 = (1+\varepsilon_1)s, \quad s_3 = (1+\varepsilon_2)s.$$

By the first row of equality (3.10), we have

$$\lambda_1 \Psi(b-s) + \lambda_2 \Psi(b-(1+\varepsilon_1)s) + \lambda_3 \Psi(b-(1+\varepsilon_2)s) = \mathbf{0}_{n\times n}, \tag{3.12}$$

take the second and fourth derivatives with respect to s in (3.12) respectively and we have

$$\begin{aligned} &\lambda_1 A\Psi(b-s) + \lambda_2 A(1+\varepsilon_1)^2 \Psi(b-(1+\varepsilon_1)s) + \lambda_3 A(1+\varepsilon_2)^2 \Psi(b-(1+\varepsilon_2)s) \\ &\quad = \mathbf{0}_{n\times n}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\lambda_1 A^2\Psi(b-s) + \lambda_2 A^2(1+\varepsilon_1)^4 \Psi(b-(1+\varepsilon_1)s) + \lambda_3 A^2(1+\varepsilon_2)^4 \Psi(b-(1+\varepsilon_2)s) \\ &\quad = \mathbf{0}_{n\times n}. \end{aligned} \tag{3.14}$$

Combine (3.12), (3.13) and (3.14) to obtain

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 A & \lambda_2 A(1+\varepsilon_1)^2 & \lambda_3 A(1+\varepsilon_2)^2 \\ \lambda_1 A^2 & \lambda_2 A^2(1+\varepsilon_1)^4 & \lambda_3 A^2(1+\varepsilon_2)^4 \end{pmatrix} \begin{pmatrix} \Psi(b-s) \\ \Psi(b-(1+\varepsilon_1)s) \\ \Psi(b-(1+\varepsilon_2)s) \end{pmatrix} = \mathbf{0}_{3n\times n},$$

and since $(\Psi(b-s), \Psi(b-(1+\varepsilon_1)s), \Psi(b-(1+\varepsilon_2)s))^T$ is a nonzero vector therefore

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 A & \lambda_2 A(1+\varepsilon_1)^2 & \lambda_3 A(1+\varepsilon_2)^2 \\ \lambda_1 A^2 & \lambda_2 A^2(1+\varepsilon_1)^4 & \lambda_3 A^2(1+\varepsilon_2)^4 \end{vmatrix} = 0,$$

which implies that at least one of λ_1, λ_2, and λ_3 is zero.

Let $\lambda_1 = 0$, then by the second row of (3.10), we find

$$\lambda_2 \sum_{k-1\le i_{11}\le k} \Psi(b-2t_{i_{11}}+(1+\varepsilon_1)s) + \lambda_3 \sum_{k-1\le i_{11}\le k} \Psi(b-2t_{i_{11}}+(1+\varepsilon_2)s) = \mathbf{0}_{n\times n}. \tag{3.15}$$

Take the second derivative with respect to s in (3.15) to get

$$\begin{aligned}\lambda_2 A(1+\varepsilon_1)^2 &\sum_{k-1\le i_{11}\le k} \Psi(b-2t_{i_{11}}+(1+\varepsilon_1)s)\\ &+\lambda_3 A(1+\varepsilon_2)^2 \sum_{k-1\le i_{11}\le k} \Psi(b-2t_{i_{11}}+(1+\varepsilon_2)s) = \mathbf{0}_{n\times n}.\end{aligned} \tag{3.16}$$

Combine (3.15) with (3.16) and we have

$$\begin{pmatrix} \lambda_2 & \lambda_3 \\ \lambda_2 A(1+\varepsilon_1)^2 & \lambda_3 A(1+\varepsilon_2)^2 \end{pmatrix} \begin{pmatrix} \sum_{k-1\le i_{11}\le k} \Psi(b-2t_{i_{11}}+(1+\varepsilon_1)s) \\ \sum_{k-1\le i_{11}\le k} \Psi(b-2t_{i_{11}}+(1+\varepsilon_2)s) \end{pmatrix} = \mathbf{0}_{2n\times n},$$

since

$$\begin{pmatrix} \sum_{k-1\le i_{11}\le k} \Psi(b-2t_{i_{11}}+(1+\varepsilon_1)s) \\ \sum_{k-1\le i_{11}\le k} \Psi(b-2t_{i_{11}}+(1+\varepsilon_2)s) \end{pmatrix} \ne \mathbf{0}_{n\times n},$$

hence,

$$\begin{vmatrix} \lambda_2 & \lambda_3 \\ \lambda_2 A(1+\varepsilon_1)^2 & \lambda_3 A(1+\varepsilon_2)^2 \end{vmatrix} = 0,$$

which implies that at least one of λ_2 and λ_3 is zero.

Let $\lambda_2 = 0$, then by the third row of (3.10), we have

$$\lambda_3 \Psi(b-2t_k+2t_{k-1}-(1+\varepsilon_2)s) = \mathbf{0}_{n\times n}, \tag{3.17}$$

obviously, (3.17) implies $\lambda_3 = 0$. Thus, $x_0 = \mathbf{0}_{3n\times n}$.

Suppose $\int_{t_{k-2}}^{t_{k-1}} W_{k-2}(b,s)BB^* W_{k-2}^*(b,s)ds$ is nonsingular and assume Rank $\left(A_1^2 B \;\; A_1 A_2 B \;\; A_2^2 B\right) < n$. Then there exists a nonzero vector $x_0 \in \mathbb{R}^n$ such that

$$x_0^T \left(A_1^2 B \;\; A_1 A_2 B \;\; A_2^2 B\right) = \mathbf{0},$$

that is, for all $s \in (t_{k-2}, t_{k-1}]$,

$$
\begin{aligned}
x_0^T W_{k-2}(b, s)B =& x_0^T \Big(A_1^2 B\Psi(b-s) - A_1A_2B\big(\Psi(b-2t_k+s) \\
&+ A^{-\frac{1}{2}}\Psi(b-2t_{k-1}+s)\big) + A_2^2BA^{-\frac{1}{2}}\Psi(b-2t_k+2t_{k-1}-s)\Big) \\
=& \mathbf{0},
\end{aligned}
$$

which contradicts the fact that $\int_{t_{k-2}}^{t_{k-1}} W_{k-2}(b, s)BB^* W_{k-2}^*(b, s)ds$ is nonsingular.

Suppose Rank $\left(A_1^2B \ \ A_1A_2B \ \ A_2^2B\right) = n$ and assume

$$
\int_{t_{k-2}}^{t_{k-1}} W_{k-2}(b, s)BB^* W_{k-2}^*(b, s)ds
$$

is singular. There exists a nonzero vector $x_0 \in \mathbb{R}^n$ such that

$$
\int_{t_{k-2}}^{t_{k-1}} x_0^T W_{k-2}(b, s)BB^* W_{k-2}^*(b, s)x_0 ds = 0.
$$

which implies

$$
x_0^T W_{k-2}(b, s)B = \mathbf{0}, \quad \forall s \in (t_{k-2}, t_{k-1}]. \tag{3.18}
$$

For s_1, s_2, s_3 selected in the auxiliary result, by equation (3.18), we have

$$
\begin{aligned}
& x_0^T \Big(A_1^2BA^{-\frac{1}{2}}\Psi(b-s_1) - A_1A_2B\big(A^{-\frac{1}{2}}\Psi(b-2t_k+s_1) \\
&\qquad + A^{-\frac{1}{2}}\Psi(b-2t_{k-1}+s_1)\big) + A_2^2BA^{-\frac{1}{2}}\Psi(b-2t_k+2t_{k-1}-s_1)\Big) = \mathbf{0}, \\
& x_0^T \Big(A_1^2BA^{-\frac{1}{2}}\Psi(b-s_2) - A_1A_2B\big(A^{-\frac{1}{2}}\Psi(b-2t_k+s_2) \\
&\qquad + A^{-\frac{1}{2}}\Psi(b-2t_{k-1}+s_2)\big) + A_2^2BA^{-\frac{1}{2}}\Psi(b-2t_k+2t_{k-1}-s_2)\Big) = \mathbf{0}, \\
& x_0^T \Big(A_1^2BA^{-\frac{1}{2}}\Psi(b-s_3) - A_1A_2B\big(A^{-\frac{1}{2}}\Psi(b-2t_k+s_3) \\
&\qquad + A^{-\frac{1}{2}}\Psi(b-2t_{k-1}+s_3)\big) + A_2^2BA^{-\frac{1}{2}}\Psi(b-2t_k+2t_{k-1}-s_3)\Big) = \mathbf{0},
\end{aligned}
$$

that is

$$
x_0^T \left(A_1^2B \ \ -A_1A_2B \ \ A_2^2B\right) \Sigma_3 = \mathbf{0},
$$

then by the auxiliary result, we find

$$
x_0^T \left(A_1^2B \ \ -A_1A_2B \ \ A_2^2B\right) = \mathbf{0},
$$

which implies

$$\text{Rank}\left(A_1^2B \ -A_1A_2B \ A_2^2B\right) < n,$$

which contradict the hypothesis.

Thus $\int_{t_{k-2}}^{t_{k-1}} W_{k-2}(b,s)BB^*W_{k-2}^*(b,s)ds$ is nonsingular iff

$$\text{Rank}\left(A_1^2B \ A_1A_2B \ A_2^2B\right) = n.$$

In similar way, we obtain $\int_{t_{k-2}}^{t_{k-1}} W'_{k-2}(b,s)BB^*W'^*_{k-2}(b,s)ds$ is nonsingular iff

$$\text{Rank}\left(A_1^2B \ A_1A_2B \ A_2^2B\right) = n.$$

Thus, we have case 3.

Similarly, taking the proof method of auxiliary result in case 3, for any $0 \le m \le k-2$, we can show an auxiliary result for $i = j = m$, i.e.

$$\Sigma_{k-m+1}x_0 = \mathbf{0}_{(k-m+1)n\times n},$$

or

$$\Sigma'_{k-m+1}x_0 = \mathbf{0}_{(k-m+1)n\times n}$$

implies $x_0 = \mathbf{0}_{(k-m+1)n\times n}$, where Σ_{k-m+1} is constructed the same way as Σ_3. Making use of this auxiliary result and proceeding as the technique in case 3, we can show that for $i = j = 0$, both $\int_{t_0}^{t_1} W_0(b,s)BB^*W_0^*(b,s)ds$ and $\int_{t_0}^{t_1} W'_0(b,s)BB^*W'^*_0(b,s)ds$ are nonsingular iff

$$\text{Rank}\left(A_1^kB \ \cdots \ A_1^{k-i}A_2^iB \ \cdots \ A_2^kB\right) = n.$$

By Theorem 3.1, obviously, if there exists an integer $l \in \{0, 1, 2, \ldots, k\}$ such that

$$\text{Rank}\left(A_1^lB \ \cdots \ A_1^{l-i}A_2^iB \ \cdots \ A_2^lB\right) = n,$$

then system (1.2) is controllable.

On the other hand, if system (1.2) is controllable. Then, by Theorem 3.1, there at least exists a pair of integers $0 \le \mathrm{i} \le k$, $0 \le \mathrm{j} \le k$ such that both $\int_{t_\mathrm{i}}^{t_{\mathrm{i}+1}} W_\mathrm{i}(b,s)BB^*W_\mathrm{i}^*(b,s)ds$ and $\int_{t_\mathrm{j}}^{t_{\mathrm{j}+1}} W'_\mathrm{j}(b,s)BB^*W'^*_\mathrm{j}(b,s)ds$ are nonsingular, that is

$$\text{Rank}\left(A_1^{k-\mathrm{i}}B \ \cdots \ A_1^{k-\mathrm{i}-i}A_2^iB \ \cdots \ A_2^{k-\mathrm{i}}B\right) = n,$$

and

$$\text{Rank}\left(A_1^{k-\mathrm{j}}B \ \cdots \ A_1^{k-\mathrm{j}-i}A_2^iB \ \cdots \ A_2^{k-\mathrm{j}}B\right) = n. \qquad \square$$

4 The controllability of semilinear systems

In this section, we consider the controllability of the initial value problems of second-order semilinear systems (1.3).

For convenience in writing, let us introduce the notation $\|B\| = K$, and the following assumptions.

(H_1) $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and there exist a positive constant L such that

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|,$$

for every $x, y \in \mathbb{R}^n$, and $N = \max_{t\in[0,b]} \|f(t, 0)\|$

(H_2) The linear systems (1.2) are exactly controllable.

(H_3) Let

$$\rho(A_1) + \rho(A_2) < 1.$$

Theorem 4.1 *Let $x_0, y_0 \in \mathbb{R}^n$ and assume the condition (H_1)-(H_3) are satisfied. Then the initial value problems of semilinear second-order impulsive systems (1.3) are exactly controllable provided that*

$$L\left(K^2 \frac{T_\varepsilon^9}{2\gamma\rho(A)^{5/2}} e^{3\sqrt{\rho(A)}b} + \frac{T_\varepsilon^3}{\sqrt{\rho(A)}} e^{\sqrt{\rho(A)}b}\right) < 1 \tag{4.1}$$

and

$$L\left(K^2 T_\varepsilon^9 \frac{1}{2\lambda} \mu e^{3\sqrt{\rho(A)}b} + T_\varepsilon^3 e^{\sqrt{\rho(A)}b}\right) < \min\{\rho(A)^{1/2}, 1\}, \tag{4.2}$$

where $\mu = \max\{\frac{1}{\rho(A)^{1/2}}, \frac{1}{\rho(A)^{3/2}}\}$.

Proof From Lemma 2.6, for $t \in (t_k, b]$, (1.3) are equivalent to the integral equation

$$\begin{aligned} x(t) =\ & W(A, t, x_0, y_0) + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} W_i(A, t, s)(Bu(s) + f(s, x(s)))ds \\ & + A^{-\frac{1}{2}} \int_{t_k}^{t} \sinh A^{\frac{1}{2}}(t - s)(Bu(s) + f(s, x(s)))ds \\ :=\ & W(A, t, x_0, y_0) + \int_0^t Q_k(t, s)(Bu(s) + f(s, x(s)))ds. \end{aligned}$$

In light of (H_3), we choose $\varepsilon > 0$ small enough such that

$$\rho(A_1) + \rho(A_2) + 2\varepsilon < 1.$$

Combine with Lemma 2.3 and it follows that for $t \in (t_k, b]$,

$$\begin{aligned} \|W_i(A,t,s)\| &\le \frac{T_\varepsilon^3}{\sqrt{\rho(A)+\varepsilon}}(\rho(A_1)+\rho(A_2)+2\varepsilon)^{k-i}e^{\sqrt{\rho(A)+\varepsilon}(t-s)} \\ &\le \frac{T_\varepsilon^3}{\sqrt{\rho(A)+\varepsilon}}e^{\sqrt{\rho(A)+\varepsilon}(t-s)}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \|W_i'(A,t,s)\| &\le T_{A,\varepsilon}^3(\rho(A_1)+\rho(A_2)+2\varepsilon)^{k-i}e^{\sqrt{\rho(A)+\varepsilon}(t-s)} \\ &\le T_{A,\varepsilon}^3 e^{\sqrt{\rho(A)+\varepsilon}(t-s)}, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \|W(A,t,x_0,y_0)\| &\le \Delta_1 e^{\sqrt{\rho(A)+\varepsilon}b}(\rho(A_1)+\rho(A_2)+2\varepsilon)^k \\ &\le \Delta_1 e^{\sqrt{\rho(A)+\varepsilon}b}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \|W'(A,t,x_0,y_0)\| &\le \Delta_1 e^{\sqrt{\rho(A)+\varepsilon}b}(\rho(A_1)+\rho(A_2)+2\varepsilon)^k \\ &\le \Delta_1 e^{\sqrt{\rho(A)+\varepsilon}b}, \end{aligned} \tag{4.6}$$

where

$$\Delta_1 = 2\max\{\frac{1}{\sqrt{\rho(A)+\varepsilon}}, \sqrt{\rho(A)+\varepsilon}\}T_\varepsilon^3,$$

$T_\varepsilon = \max\{T_{A,\varepsilon}, T_{A_1,\varepsilon}, T_{A_2,\varepsilon}\}$ and $\varepsilon > 0$ small enough.

We show the controllability of the solutions of (1.3). Define the feedback control function

$$u_{1x}(t) = B^* Q_k^*(b,t)(\Gamma_0^b)^{-1}\left(x_1 - W(A,b,x_0,x_1) - \int_0^b Q_k(b,s)f(s,x(s))ds\right). \tag{4.7}$$

and the operator $\mathcal{F} : PC(J,\mathbb{R}^n) \to PC(J,\mathbb{R}^n)$ as follows,

$$(\mathcal{F}x)(t) = W(A,t,x_0,y_0) + \int_0^t Q_k(t,s)\big(Bu_{1x}(s) + f(s,x(s))\big)ds, \quad t \in (t_k, b].$$

Let

$$B_r = \{v \in PC(J,\mathbb{R}^n)\big|\|v\|_{PC} \le r\},$$

and we use the notation

$$(F_1x)(t) = W(A,t,x_0,y_0) + \int_0^t Q_k(t,s)Bu_{1x}(s)ds, \quad t \in (t_k, b],$$

$$(F_2x)(t) = \int_0^t Q_k(t,s)f(s,x(s))ds, \quad t \in (t_k, b].$$

Now $\mathcal{F} = F_1 + F_2$.

We subdivide the proof into several steps.

Step 1 We show that for every $x, y \in B_r$, $F_1x + F_2y \in B_r$. In fact, for every $x, y \in B_r$, from (4.3), (4.5), (4.7) and (H_1), we have

$$\begin{aligned}
&\|F_1x + F_2y\|_{PC} \\
&\quad \le \|W(A,t,x_0,y_0)\| + \left\| \int_0^t Q_k(t,s)Bu_{1x}(s)ds \right\| + \left\| \int_0^t Q_k(t,s)f(s,y(s))ds \right\| \\
&\quad \le \|W(A,t,x_0,y_0)\| + K^2 \frac{1}{\gamma} \int_0^b \frac{T_\varepsilon^6}{\rho(A)+\varepsilon} e^{2\sqrt{\rho(A)+\varepsilon}(b-s)} ds \\
&\qquad \cdot \left(\|x_1\| + \|W(A,b,x_0,y_0)\| + (N+Lr) \int_0^b \frac{T_\varepsilon^3}{\sqrt{\rho(A)+\varepsilon}} e^{\sqrt{\rho(A)+\varepsilon}(b-s)} ds \right) \\
&\qquad + (N+Lr) \int_0^b \frac{T_\varepsilon^3}{\sqrt{\rho(A)+\varepsilon}} e^{\sqrt{\rho(A)+\varepsilon}(b-s)} ds \\
&\quad \le \|W(A,t,x_0,y_0)\| + K^2 \frac{1}{\gamma} \frac{T_\varepsilon^6}{2(\rho(A)+\varepsilon)^{3/2}} e^{2\sqrt{\rho(A)+\varepsilon}b} \\
&\qquad \cdot \left(\|x_1\| + \|W(A,b,x_0,y_0)\| + (N+Lr) \frac{T_\varepsilon^3}{\rho(A)+\varepsilon} e^{\sqrt{\rho(A)+\varepsilon}b} \right) \\
&\qquad + (N+Lr) \frac{T_\varepsilon^3}{\rho(A)+\varepsilon} e^{\sqrt{\rho(A)+\varepsilon}b} \\
&\quad \le \Delta_1 e^{\sqrt{\rho(A)+\varepsilon}b} (\|x_0\| + \|y_0\|) + K^2 \frac{1}{\gamma} \frac{T_\varepsilon^6}{2(\rho(A)+\varepsilon)^{3/2}} e^{2\sqrt{\rho(A)+\varepsilon}b} \\
&\qquad \cdot \left(\|x_1\| + \Delta_1 e^{\sqrt{\rho(A)+\varepsilon}b} (\|x_0\| + \|y_0\|) + (N+Lr) \frac{T_\varepsilon^3}{\rho(A)+\varepsilon} e^{\sqrt{\rho(A)+\varepsilon}b} \right) \\
&\qquad + (N+Lr) \frac{T_\varepsilon^3}{\rho(A)+\varepsilon} e^{\sqrt{\rho(A)+\varepsilon}b} \\
&\quad \le \Delta + L(K^2 \frac{1}{\gamma} \frac{T_\varepsilon^9}{2\rho(A)^{5/2}} e^{3\sqrt{\rho(A)+\varepsilon}b} + \frac{T_\varepsilon^3}{\rho(A)} e^{\sqrt{\rho(A)+\varepsilon}b})r,
\end{aligned}$$

where

$$\begin{aligned}
\Delta &= \Delta_1 e^{\sqrt{\rho(A)+\varepsilon}b} (\|x_0\| + \|y_0\|) + N \left(\frac{T_\varepsilon^3}{\rho(A)} e^{\sqrt{\rho(A)+\varepsilon}b} + K^2 \frac{1}{\gamma} \frac{T_\varepsilon^9}{2\rho(A)^{5/2}} e^{3\sqrt{\rho(A)+\varepsilon}b} \right) \\
&\quad + K^2 \frac{1}{\gamma} \frac{T_\varepsilon^6}{2\rho(A)^{3/2}} e^{2\sqrt{\rho(A)+\varepsilon}b} \big(\|x_1\| + \Delta_1 e^{\sqrt{\rho(A)+\varepsilon}b} (\|x_0\| + \|y_0\|) \big).
\end{aligned}$$

By inequality (4.1), we can pick

$$r \geq \frac{\Delta}{1 - L(K^2 \frac{1}{\gamma} \frac{T_\varepsilon^9}{2\rho(A)^{5/2}} e^{3\sqrt{\rho(A)+\varepsilon b}} + \frac{T_\varepsilon^3}{\rho(A)} e^{\sqrt{\rho(A)+\varepsilon b}})}, \tag{4.8}$$

Then, we have

$$\|F_1 x + F_2 y\|_{PC} \leq r,$$

that is

$$F_1 x + F_2 y \in B_r.$$

Step 2 We claim that $F_1 : B_r \to PC(J, \mathbb{R}^n)$ is a contraction mapping. For every $x, y \in B_r$, by (3.5), (4.1), (4.3), and (H_1), we have

$$\begin{aligned}
\|F_1 x - F_1 y\|_{PC} &= \left\| \int_0^t Q_k(t,s) B(u_{1x}(s) - u_{1y}(s)) ds \right\| \\
&\leq \frac{T_\varepsilon^6}{\rho(A)+\varepsilon} K^2 \frac{1}{\gamma} \int_0^t e^{2\sqrt{\rho(A)+\varepsilon}(b-s)} ds \\
&\quad \cdot \left\| \int_0^b Q_k(b,s) \big(f(s, x(s)) - f(s, y(s)) \big) ds \right\| \\
&\leq \frac{T_\varepsilon^9 K^2 L}{2(\rho(A)+\varepsilon)^{5/2}} \frac{1}{\gamma} e^{3\sqrt{\rho(A)+\varepsilon b}} \|x - y\|_{PC},
\end{aligned}$$

so F_1 is a contraction mapping.

Step 3 We show that F_2 is compact and continuous. For any $x, y \in B_r$, by the inequality (4.3), we have

$$\begin{aligned}
\|F_2 x - F_2 y\|_{PC} &\leq \int_0^t \|Q_k(t,s)\| \cdot \|f(s, x(s)) - f(s, y(s))\| ds \\
&\leq \frac{bLT_\varepsilon^3}{\rho(A)+\varepsilon} e^{\sqrt{\rho(A)+\varepsilon b}} \|x - y\|_{PC},
\end{aligned}$$

therefore, $F_2 : B_r \to PC(J, \mathbb{R})$ is continuous. To check the compactness of F_2, we prove that F_2 is uniformly bounded and equicontinuous. In fact, for any $x \in B_r$, by the inequality (4.3), we have

$$\begin{aligned}
\|F_2 x\|_{PC} &= \left\| \int_0^t Q_k(t,s) f(s, x(s)) ds \right\| \\
&\leq \frac{T_\varepsilon^3}{\rho(A)+\varepsilon} e^{\sqrt{\rho(A)+\varepsilon b}} (N + Lr),
\end{aligned}$$

that is $F_2B_r = \{F_2x \big| x \in B_r\}$ is uniformly bounded. Next, we show that F_2 is equicontinuous. For any $t_k < \tau_1 \le \tau_2 \le b$, by Lemma 2.7, and (4.3), we have

$$\begin{aligned}\|(F_2x)(\tau_2) - (F_2x)(\tau_1)\| \le& \Big\| \int_0^{\tau_2} Q_k(\tau_2, s) f(s, x(s))ds - \int_0^{\tau_1} Q_k(\tau_1, s) f(s, x(s))ds \Big\| \\ \le& \Big\| \int_0^{\tau_1} \big(Q_k(\tau_2, s) - Q_k(\tau_1, s)\big) f(s, x(s))ds \Big\| \\ &+ \Big\| \int_{\tau_1}^{\tau_2} Q_k(\tau_2, s) f(s, x(s))ds \Big\| \\ \le& \theta_1(N + Lr)|\tau_2 - \tau_1| + \frac{T_\varepsilon^3}{\sqrt{\rho(A) + \varepsilon}} e^{\sqrt{\rho(A)+\varepsilon}b}(N + Lr) \\ &\cdot |\tau_2 - \tau_1|,\end{aligned}$$

therefore, F_2B_r is the equicontinuous family of functions in $PC(J, \mathbb{R}^n)$. From Lemma 2.5, F_2B_r is relatively compact in $PC(J, \mathbb{R}^n)$.

From Krasnoselskii's fixed point theorem, we obtain that F has a fixed point x in B_r, which is the solution of (1.3) and satisfies $x(b) = x_1$.

In what follows, we show the controllability of the derivative of solutions for systems (1.3). Define the feedback control function

$$u_{2x}(t) = B^* Q_k'^*(b, t)(\Lambda_0^b)^{-1}\big(y_1 - W'(A, b, x_0, y_0) - \int_0^b Q_k'(b, s) f(s, x(s))ds\big).$$

and the operator $\mathcal{H} : PC^1(J, \mathbb{R}^n) \to PC^1(J, \mathbb{R}^n)$ as follows,

$$(\mathcal{H}x)(t) = W(A, t, x_0, y_0) + \int_0^t Q_k(t, s) \big(Bu_{2x}(s) + f(s, x(s))\big)ds, \quad t \in (t_k, b].$$

Let

$$\begin{aligned}(H_1x)(t) &= W(A, t, x_0, y_0) + \int_0^t Q_k(t, s)Bu_{2x}(s)ds, \\ (H_2x)(t) &= \int_0^t Q_k(t, s) f(s, x(s))ds,\end{aligned}$$

and $\mathcal{H} = H_1 + H_2$. Let

$$D_\ell = \{x \in PC^1(J, \mathbb{R}^n) \big| \|x\|_{PC^1} \le \ell\}.$$

We show that $\mathcal{H} : D_\ell \to PC^1(J, \mathbb{R}^n)$ has a fixed point. Proceeding as before, we subdivide the proof into several steps.

Step 1 We show that $H_1x + H_2y \in D_\ell$, for any $x, y \in D_\ell$.

In fact, for any $x, y \in D_\ell$, proceeding as in the proof for the operator $\mathcal{F}$, and by inequalities (3.5), (4.3)–(4.6) and condition (H_1), we have

$$
\begin{aligned}
&\|H_1x + H_2y\|_{PC} \\
&\quad \le \|W(A,t,x_0,y_0)\| + \left\| \int_0^t Q_k(t,s)Bu_{2x}(s)ds \right\| + \left\| \int_0^t Q_k(t,s)f(s,y(s))ds \right\| \\
&\quad \le \|W(A,t,x_0,y_0)\| + K^2 \frac{1}{\lambda} \int_0^b \frac{T_\varepsilon^6}{\sqrt{\rho(A)+\varepsilon}} e^{2\sqrt{\rho(A)+\varepsilon}(b-s)} ds \\
&\qquad \cdot \left(\|y_1\| + \|W'(A,b,x_0,y_0)\| + (N+Lr)\int_0^b T_\varepsilon^3 e^{\sqrt{\rho(A)+\varepsilon}(b-s)} ds \right) \\
&\qquad + (N+Lr)\int_0^b \frac{T_\varepsilon^3}{\sqrt{\rho(A)+\varepsilon}} e^{\sqrt{\rho(A)+\varepsilon}(b-s)} ds \\
&\quad \le \|W(A,t,x_0,y_0)\| + K^2 \frac{1}{\lambda} \frac{T_\varepsilon^6}{2(\rho(A)+\varepsilon)} e^{2\sqrt{\rho(A)+\varepsilon}b} \\
&\qquad \cdot \left(\|y_1\| + \|W'(A,b,x_0,y_0)\| + (N+Lr)\frac{T_\varepsilon^3}{\sqrt{\rho(A)+\varepsilon}} e^{\sqrt{\rho(A)+\varepsilon}b} \right) \\
&\qquad + (N+Lr)\frac{T_\varepsilon^3}{\rho(A)+\varepsilon} e^{\sqrt{\rho(A)+\varepsilon}b} \\
&\quad \le \Delta_1 e^{\sqrt{\rho(A)+\varepsilon}b}(\|x_0\|+\|y_0\|) + K^2 \frac{1}{\lambda} \frac{T_\varepsilon^6}{2(\rho(A)+\varepsilon)} e^{2\sqrt{\rho(A)+\varepsilon}b} \\
&\qquad \cdot \left(\|y_1\| + \Delta_1 e^{\sqrt{\rho(A)+\varepsilon}b}(\|x_0\|+\|y_0\|) + (N+Lr)\frac{T_\varepsilon^3}{\sqrt{\rho(A)+\varepsilon}} e^{\sqrt{\rho(A)+\varepsilon}b} \right) \\
&\qquad + (N+Lr)\frac{T_\varepsilon^3}{\rho(A)+\varepsilon} e^{\sqrt{\rho(A)+\varepsilon}b} \\
&\quad \le \overline{\Delta} + L\left(K^2 \frac{1}{\lambda} \frac{T_\varepsilon^9}{2\rho(A)^{3/2}} e^{3\sqrt{\rho(A)+\varepsilon}b} + \frac{T_\varepsilon^3}{\rho(A)} e^{\sqrt{\rho(A)+\varepsilon}b} \right) r,
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{\Delta} &= \Delta_1 e^{\sqrt{\rho(A)+\varepsilon}b}(\|x_0\|+\|y_0\|) + N\left(\frac{T_\varepsilon^3}{\rho(A)} e^{\sqrt{\rho(A)+\varepsilon}b} + K^2 \frac{1}{\lambda} \frac{T_\varepsilon^9}{2\rho(A)^{3/2}} e^{3\sqrt{\rho(A)+\varepsilon}b} \right) \\
&\quad + K^2 \frac{1}{\lambda} \frac{T_\varepsilon^6}{2\rho(A)^{3/2}} e^{2\sqrt{\rho(A)+\varepsilon}b} \big(\|y_1\| + \Delta_1 e^{\sqrt{\rho(A)+\varepsilon}b}(\|x_0\|+\|y_0\|) \big).
\end{aligned}
$$

By inequality (4.2), we can pick

$$
r \ge \frac{\overline{\Delta}}{1 - L\left(K^2 \frac{1}{\lambda} \frac{T_\varepsilon^9}{2\rho(A)^{3/2}} e^{3\sqrt{\rho(A)+\varepsilon}b} + \frac{T_\varepsilon^3}{\rho(A)} e^{\sqrt{\rho(A)+\varepsilon}b} \right)}, \tag{4.9}
$$

then, we have

$$
\|H_1x + H_2y\|_{PC} \le r.
$$

It follows that for any $x, y \in D_r$, $H_1x + H_2y \in D_r$.

Similarly, making use of (3.5), (4.4), (4.6), (4.7) and (H_1), we get

$$
\begin{aligned}
&\|(H_1x)' + (H_2y)'\|_{PC} \\
&\le \|W'(A, t, x_0, y_0)\| + \left\| \int_0^t Q_k'(t, s)Bu_{2x}(s)ds \right\| + \left\| \int_0^t Q_k'(t, s)f(s, y(s))ds \right\| \\
&\le \|W'(A, t, x_0, y_0)\| + K^2 T_\varepsilon^6 \frac{1}{\lambda} \int_0^b e^{2\sqrt{\rho(A)+\varepsilon}(b-s)} ds \\
&\quad \cdot \left(\|y_1\| + \|W'(A, b, x_0, y_0)\| + (N + Lr) \int_0^b T_\varepsilon^3 e^{\sqrt{\rho(A)+\varepsilon}(b-s)} ds \right) \\
&\quad + (N + Lr) \int_0^b T_\varepsilon^3 e^{\sqrt{\rho(A)+\varepsilon}(b-s)} ds \\
&\le \Delta_1 e^{\sqrt{\rho(A)+\varepsilon} b} (\|x_0\| + \|y_0\|) + \frac{K^2 T_\varepsilon^6}{2\sqrt{\rho(A)+\varepsilon}} \frac{1}{\lambda} e^{2\sqrt{\rho(A)+\varepsilon} b} \\
&\quad \cdot \left(\|y_1\| + \Delta_1 e^{\sqrt{\rho(A)+\varepsilon} b} (\|x_0\| + \|y_0\|) + \frac{T_\varepsilon^3}{\sqrt{\rho(A)+\varepsilon}} (N + Lr) e^{\sqrt{\rho(A)+\varepsilon} b} \right) \\
&\quad + (N + Lr) \frac{T_\varepsilon^3}{\sqrt{\rho(A)+\varepsilon}} e^{\sqrt{\rho(A)+\varepsilon} b} \\
&\le \Gamma + L \left(K^2 \frac{1}{\lambda} \frac{T_\varepsilon^9}{2\rho(A)} e^{3\sqrt{\rho(A)+\varepsilon} b} + \frac{T_\varepsilon^3}{\sqrt{\rho(A)}} e^{\sqrt{\rho(A)+\varepsilon} b} \right) r,
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma &= K^2 \frac{1}{\lambda} \frac{T_\varepsilon^6}{2\sqrt{\rho(A)}} e^{2\sqrt{\rho(A)+\varepsilon} b} \left(\|y_1\| + \Delta_1 e^{\sqrt{\rho(A)+\varepsilon} b} (\|x_0\| + \|y_0\|) \right) \\
&\quad + \Delta_1 e^{\sqrt{\rho(A)+\varepsilon} b} (\|x_0\| + \|y_0\|) + N \left(\frac{T_\varepsilon^3}{\sqrt{\rho(A)}} e^{\sqrt{\rho(A)+\varepsilon} b} + K^2 \frac{T_\varepsilon^9}{2\rho(A)} \frac{1}{\lambda} e^{3\sqrt{\rho(A)+\varepsilon} b} \right).
\end{aligned}
$$

By inequality (4.2), we can pick

$$
r \ge \frac{\Gamma}{1 - L(K^2 \frac{1}{\lambda} \frac{T_\varepsilon^9}{2\rho(A)^2} e^{3\sqrt{\rho(A)+\varepsilon} b} + \frac{T_\varepsilon^3}{\sqrt{\rho(A)}} e^{\sqrt{\rho(A)+\varepsilon} b})}, \tag{4.10}
$$

Then, we have

$$
\|(H_1x)' + (H_2y)'\|_{PC} \le r.
$$

Take r be the maximum of the right hand of (4.8) and (4.9), then we obtain

$$
\|H_1x + H_2y\|_{PC^1} \le r,
$$

that is

$$
H_1x + H_2y \in B_r.
$$

Step 2 We state that H_1 is a contraction.

For any $x, y \in D_r$, by inequalities (3.5), (4.2), (4.4), and (H_1), we have

$$\begin{aligned}\|H_1x - H_1y\|_{PC} &= \left\| \int_0^t Q_k(t,s)B(u_{2x}(s) - u_{2y}(s))ds \right\| \\ &\leq LK^2 \frac{T_\varepsilon^6}{\sqrt{\rho(A)+\varepsilon}} \frac{1}{\lambda} \int_0^t e^{2\sqrt{\rho(A)+\varepsilon}(b-s)} ds \cdot \int_0^b \|Q_k'(b,s)\| \\ &\quad \cdot \|x-y\|_{PC} ds \\ &\leq LK^2 T_\varepsilon^9 \frac{1}{\lambda} \frac{1}{2\rho(A)^{3/2}} e^{3\sqrt{\rho(A)+\varepsilon}b} \|x-y\|_{PC},\end{aligned}$$

and

$$\begin{aligned}\|(H_1x)' - (H_1y)'\|_{PC} &= \left\| \int_0^t Q_k'(t,s)B(u_{2x}(s) - u_{2y}(s))ds \right\| \\ &\leq LT_\varepsilon^6 K^2 \frac{1}{\lambda} \int_0^t e^{2\sqrt{\rho(A)+\varepsilon}(b-s)} ds \\ &\quad \cdot \int_0^b \|Q_k'(b,s)\| \cdot \|x-y\|_{PC} ds \\ &\leq \frac{T_\varepsilon^9 K^2 L}{2(\rho(A)+\varepsilon)} \frac{1}{\lambda} e^{3\sqrt{\rho(A)+\varepsilon}b} \|x-y\|_{PC},\end{aligned}$$

hence according to (4.2), there exists $\varepsilon > 0$ small enough such that

$$LK^2 T_\varepsilon^9 \frac{1}{\lambda} \frac{1}{2\rho(A)^{3/2}} e^{3\sqrt{\rho(A)+\varepsilon}b} < 1,$$

and

$$\frac{T_\varepsilon^9 K^2 L}{2(\rho(A)+\varepsilon)} \frac{1}{\lambda} e^{3\sqrt{\rho(A)+\varepsilon}b} < 1.$$

Therefore, H_1 is a contraction mapping.

Step 3 We show that H_2 is compact and continuous. Since, for every $x, y \in D_\ell$, by (4.3) and (4.4), we have

$$\begin{aligned}\|H_2x - H_2y\|_{PC} &\leq \int_0^t \|Q_k(t,s)\| \cdot \|f(s,x(s)) - f(s,y(s))\| ds \\ &\leq \frac{bLT_\varepsilon^3}{\rho(A)+\varepsilon} e^{\sqrt{\rho(A)+\varepsilon}b} \|x-y\|_{PC^1},\end{aligned}$$

and

$$\begin{aligned}\|(H_2x)'-(H_2y)'\|_{PC}\le&\int_0^t\|Q_k'(t,s)\|\cdot\|f(s,x(s))-f(s,y(s))\|ds\\\le&\frac{bLT_\varepsilon^3}{\sqrt{\rho(A)+\varepsilon}}e^{\sqrt{\rho(A)+\varepsilon}b}\|x-y\|_{PC^1},\end{aligned}$$

therefore, $H_2 : D_\ell \to PC^1(J,\mathbb{R})$ is continuous. To check the compactness of H_2, we consider the mapping

$$(H_2x)'(t)=\int_0^t Q_k'(t,s)f(s,x(s))ds.$$

For every $x\in D_\ell$, by inequality (4.4) and (H_1), we obtain

$$\begin{aligned}\|(H_2x)'\|_{PC}\le&\int_0^t\|Q_k'(t,s)\|\cdot\|f(s,x(s))\|ds\\\le&\int_0^t T_\varepsilon^3e^{\sqrt{\rho(A)+\varepsilon}(t-s)}(N+Lr)ds\\\le&\,bT_\varepsilon^3(N+Lr)e^{\sqrt{\rho(A)+\varepsilon}b},\end{aligned}$$

which implies $(H_2D_\ell)'=\{(H_2x)'\big|x\in D_\ell\}$ is uniformly bounded in $PC(J,\mathbb{R})$. We prove that for any $x\in D_\ell$, $(H_2x)'$ is equicontinuous. In fact, for any $t_k<\tau_1<\tau_2\le b$, in term of inequality (4.4), (H_1) and Lemma 2.7, we have

$$\begin{aligned}&\|(H_2x)'(\tau_2)-(H_2x)'(\tau_1)\|\\&=\Big\|\int_0^{\tau_2}Q_k'(\tau_2,s)f(s,x(s))ds-\int_0^{\tau_1}Q_k'(\tau_1,s)f(s,x(s))ds\Big\|\\&\le\int_0^{\tau_1}\|Q_k'(\tau_2,s)-Q_k'(\tau_1,s)\|\cdot\|f(s,x(s))\|ds\\&\quad+\int_{\tau_1}^{\tau_2}\|Q_k'(\tau_2,s)f(s,x(s))\|ds\\&\le\theta(N+Lr)|\tau_2-\tau_1|+T_\varepsilon^3e^{\sqrt{\rho(A)+\varepsilon}b}(N+Lr)\cdot|\tau_2-\tau_1|,\end{aligned}$$

therefore, $(H_2D_\ell)'$ is the equicontinuous family of functions in $PC(J,\mathbb{R}^n)$. From Lemma 2.5, $(H_2D_\ell)'$ is relatively compact in $PC(J,\mathbb{R}^n)$. Hence, for any sequence $\{x_n\}\subset D_\ell$, there exists a subsequence of $\{x_n\}$, again denoted by $\{x_n\}$, such that

$$(H_2x_n)'\to\phi\quad\text{in}\quad PC(J,\mathbb{R}^n)\quad\text{as}\quad n\to\infty.\tag{4.11}$$

Obviously,

$$\|x\|_{PC}\le b\|x'\|_{PC},$$

for any $x \in PC^1(J, \mathbb{R}^n)$. Let $\overline{\phi}$ be the antiderivative of ϕ, combining this inequality with (4.11), we have

$$\begin{aligned}\|H_2x_n - \overline{\phi}\|_{PC^1} &= \max\{\|H_2x_n - \overline{\phi}\|_{PC}, \|(H_2x_n)' - \phi\|_{PC}\}\\ &\leq \max\{b, 1\}\|(H_2x_n)' - \phi\|_{PC}\\ &< \varepsilon,\end{aligned}$$

as n is large enough, which implies that for any $\{H_2x_n\} \subset H_2D_\ell$, there exists a subsequence $\{H_2x_{n_k}\}$ which is convergence in $PC^1(J, \mathbb{R})$. Thus, $H_2 : D_\ell \to PC^1(J, \mathbb{R}^n)$ is a compact and continuous operator.

Hence, by the Krasnoselskii's fixed point theorem, we obtain that $\mathcal{H}$ has a fixed point x in D_ℓ which is the solution of (1.3) and satisfies $x'(b) = y_1$.

In conclusion, second-order impulsive systems (1.3) are exactly controllable.

□

5 Examples

In this section, we give some examples to illustrate the effectiveness of our results.

Example 5.1 For the simplicity of calculation, we consider the controllability of systems (1.2) with

$$A = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}, \quad B_1 = B_2 = \begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix}, \quad B = \begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}, \quad x_0 = \begin{pmatrix}0\\ 1\end{pmatrix}, \quad y_0 = \begin{pmatrix}1\\ 0\end{pmatrix},$$

and $0 = t_0 < 1 = t_1 < 2 = t_2 = b$. Obviously, B is nonsingular, then Theorem 3.1 holds for $l = 0$, that is system (1.2) is controllable. For the sake of convenience in calculating, we consider $x_1 = (30\ 40)^T$, then we show that we can choose a control function $u_1(t)$ such that, under $u_1(t)$, $x(2) = (30\ 40)^{\mathrm{T}}$. By Theorem 3.1 in [34], we obtain that, for $t \in (1, 2]$, the solution $W(A, t, x_0, y_0)$ of the homogeneous initial value problems of (1.2) is expressed as follows,

$$\begin{aligned}W(A, t, x_0, y_0) &= \begin{pmatrix}2\cosh t & 2\cosh\sqrt{2}t\\ 0 & 4\cosh\sqrt{2}t\end{pmatrix} x_0 + \begin{pmatrix}2\sinh t & \sqrt{2}\sinh\sqrt{2}t\\ 0 & 2\sqrt{2}\sinh\sqrt{2}t\end{pmatrix} y_0\\ &= \begin{pmatrix}2\sinh t + 2\cosh\sqrt{2}t\\ 4\cosh\sqrt{2}t\end{pmatrix}.\end{aligned} \tag{5.1}$$

By the calculation, we find

$$Q_1(t, s) = \begin{cases} W_0(A, t, s), & 0 \leq s \leq 1,\\ W_1(A, t, s), & 1 < s \leq t,\end{cases}$$

here

$$W_0(A,t,s)=\begin{pmatrix}2\sinh(t-s) & \sqrt{2}\sinh\sqrt{2}(t-s)\\ 0 & 2\sqrt{2}\sinh\sqrt{2}(t-s)\end{pmatrix},$$

$$W_1(A,t,s)=\begin{pmatrix}\sinh(t-s) & 0\\ 0 & \frac{1}{\sqrt{2}}\sinh\sqrt{2}(t-s)\end{pmatrix}.$$

and

$$\Gamma_0^2=\begin{pmatrix}-3\sinh 2+4\sinh 4-\frac{9}{2\sqrt{2}}\sinh 2\sqrt{2}+\frac{9}{2\sqrt{2}}\sinh 4\sqrt{2}-19 & -\frac{9}{\sqrt{2}}\sinh 2\sqrt{2}+\frac{9}{\sqrt{2}}\sinh 4\sqrt{2}-18\\ -\frac{9}{\sqrt{2}}\sinh 2\sqrt{2}+\frac{9}{\sqrt{2}}\sinh 4\sqrt{2}-18 & -\frac{18}{\sqrt{2}}\sinh 2\sqrt{2}+\frac{18}{\sqrt{2}}\sinh 4\sqrt{2}-36\end{pmatrix}_{2\times 2} \tag{5.2}$$

Hence, by (3.3), we can define the control function $u_1(t)$ by a piecewise function,piecewise function,

$$u_1(t)=\begin{cases}\begin{pmatrix}4\sinh(2-t) & 0\\ 3\sqrt{2}\sinh\sqrt{2}(2-t) & 6\sqrt{2}\sinh\sqrt{2}(2-t)\end{pmatrix}(\Gamma_0^2)^{-1}\big(x_1-W(A,2,x_0,y_0)\big), & t\in(0,1],\\ \begin{pmatrix}2\sinh(2-t) & 0\\ 0 & \frac{3}{\sqrt{2}}\sinh\sqrt{2}(2-t)\end{pmatrix}(\Gamma_0^2)^{-1}\big(x_1-W(A,2,x_0,y_0)\big), & t\in(1,2],\end{cases}$$

here $W(A,t,x_0,y_0)$ is expressed by (5.1), Γ_0^2 is expressed by (5.2), x_1 is the state we want to arrive. Therefore, under the control u_1, we have $x(2)=x_1$, see Fig. 1. Similarly, take $y_1=(0\ 0)^{\mathrm{T}}$, then we can take the control u_2 as follows,

$$u_2(t)=\begin{cases}\begin{pmatrix}4\cosh(2-t) & 0\\ 6\cosh\sqrt{2}(2-t) & 12\cosh\sqrt{2}(2-t)\end{pmatrix}(\Lambda_0^2)^{-1}\big(y_1-W'(A,2,x_0,y_0)\big), & t\in(0,1],\\ \begin{pmatrix}2\cosh(2-t) & 0\\ 0 & 3\cosh\sqrt{2}(2-t)\end{pmatrix}(\Lambda_0^2)^{-1}\big(y_1-W'(A,2,x_0,y_0)\big), & t\in(1,2],\end{cases}$$

and under this control, we can steer the derivative of the solution of systems (1.2) to $(0\ 0)^T$ at terminal, see Fig. 2.

Example 5.2 Consider the systems (1.3) with

$$A=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix},\quad B_1=B_2=\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\\ 0 & -\frac{1}{3}\end{pmatrix},\quad B=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix},$$

$0=t_0<\frac{1}{2}=t_1<1=t_2=b$, and $f(t,x(t))=\frac{1}{37}\sin x(t)$. Since B is nonsingular, by Theorem 3.3, we find condition (H_2) is satisfied. Obviously, (H_1) is satisfied with

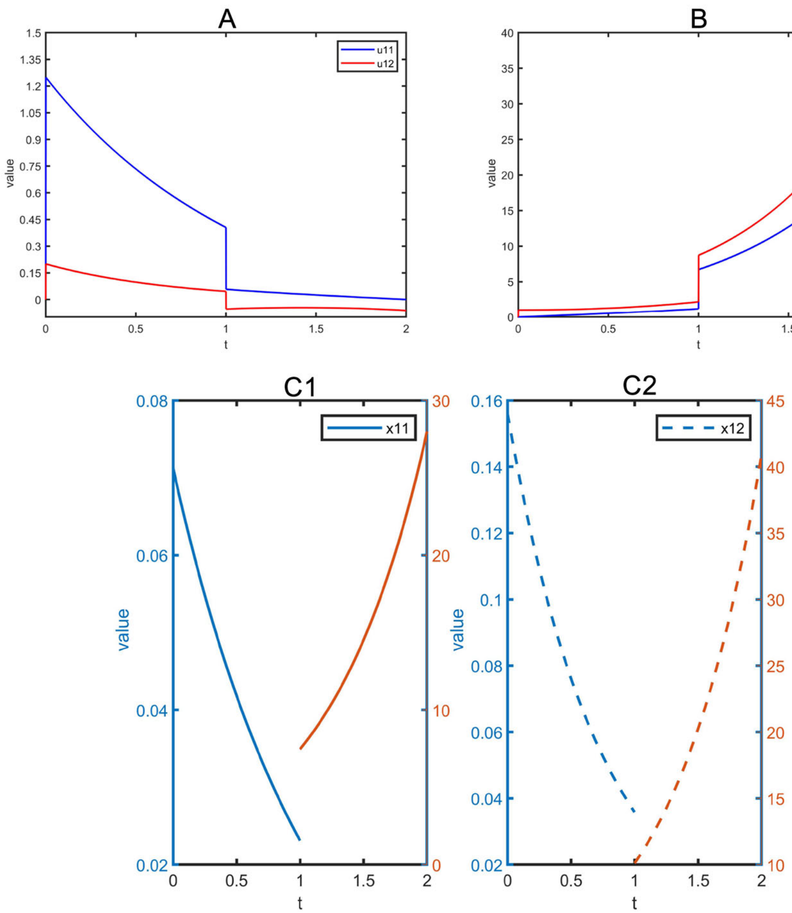

Fig. 1 **A** The control function $u_1(t)$, where blue line denote the first component of $u_1(t)$, red line denote the second component of $u_1(t)$. **B** The state function $X(t)$ of (1.2) without any control, similarly, where blue line denote the first component of $X(t)$, red line denote the second component of $X(t)$. **C** Figure C1 denote the first component of state function $x_1(t)$ under the control function $u_1(t)$, and Figure C2 denote the second component of state function $x_1(t)$ under the control function $u_1(t)$. It should be noted that, in figure C1 and C2, on the left side of impulsive point $t = 1$, we refer to the left scale, and on the right side of impulsive point $t = 1$, we refer to the right scale (color figure online)

$L = \frac{1}{37}$. Then we show that (4.1) and (4.2) hold. Define a new matrix norm $\|\cdot\|'$ by

$$\|x\|' = \|Qx\|, \quad \forall x \in \mathbb{R}^n,$$

where

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 2\varepsilon \end{pmatrix},$$

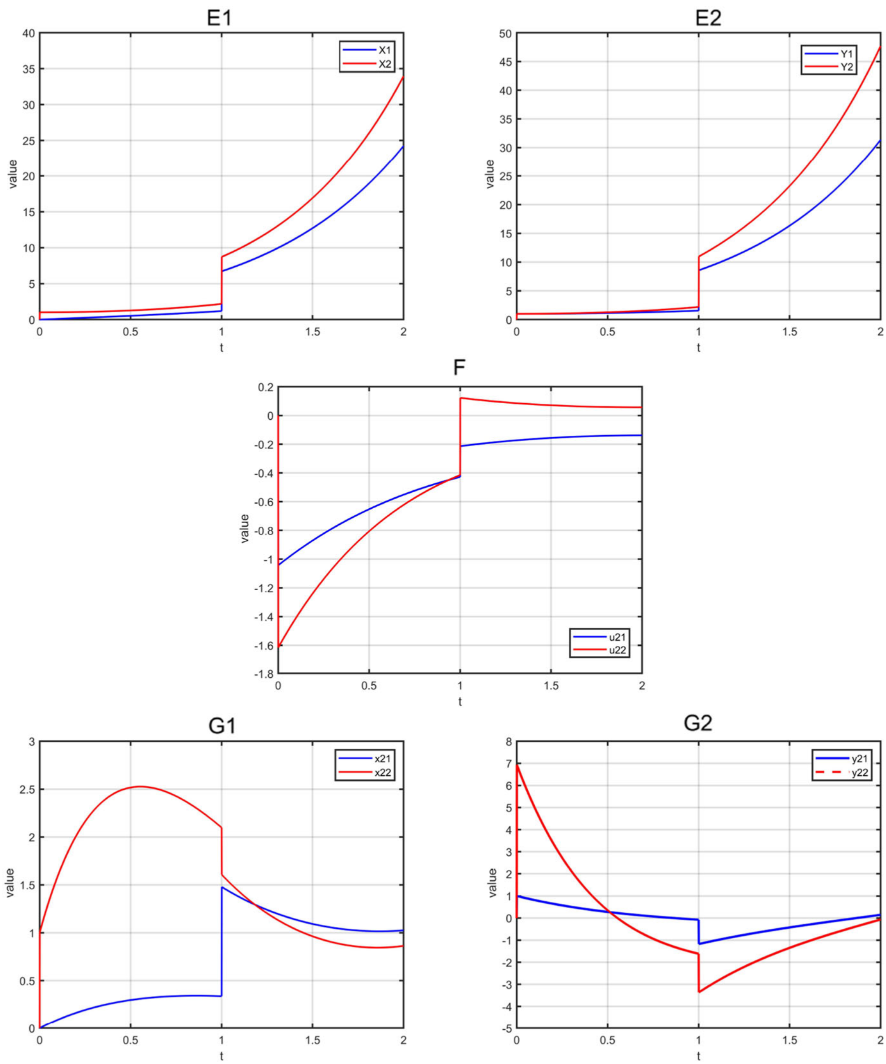

Fig. 2 **E1** The state function $X(t)$ of (1.2) without any control function. The blue line denote the first component of $X(t)$, and the red line denote the second component of $X(t)$. **E2** The derivative of state function $X(t)$ without any control function. Similarly, the blue line denote the first component, and the red line denote the second component. **F** The control function u_2 we picked to control the derivative function, the blue line denote the first component of u_2, and the red line denote the second component u_2. **G1** The state function $x_2(t)$ under the control u_2. The blue line denote the first component of $x_2(t)$, and the red line denote the second component $x_2(t)$. **G2** The derivative of state function $x_2(t)$. The blue line denote the first component, and the red line denote the second component (color figure online)

and $\|A\| = \max\limits_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$. According the Theorem 2.2.8 of [35], we find for $A_1 = I + \frac{B_1+B_2}{2}$,

$$\|A_1\|' = \|Q^{-1}A_1Q\| \leq \rho(A_1) + \varepsilon. \tag{5.3}$$

Let $\varepsilon = \frac{49}{100}$, then we have, for all $x \in \mathbb{R}^n$,

$$\frac{98}{100}\|x\| \leq \|x\|' \leq \|x\|,$$

hence,

$$\|A_1\| = \sup_{x\neq 0}\left\{\frac{\|A_1x\|}{\|x\|}\right\} \leq \sup_{x\neq 0}\left\{\frac{100}{98}\frac{\|A_1x\|'}{\|x\|'}\right\} = \frac{100}{98}\|A_1\|',$$

combining this with (5.3), we have

$$\|A_1\| \leq \frac{100}{98}(\rho(A) + \varepsilon),$$

that is $T_{A_1,\frac{49}{100}} = \frac{100}{98}$. Obviously, $T_{A,\varepsilon} = 1$, therefore, $T_{\frac{49}{100}} = \frac{100}{98}$. With a simple calculation, we find

$$\Gamma_0^1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

here

$$\begin{aligned}
a_{11} &= \frac{\sinh\sqrt{2}\cosh\sqrt{2} + \frac{1}{2} - \sinh\frac{\sqrt{2}}{2}\cosh\frac{\sqrt{2}}{2}}{32\sqrt{2}} + \frac{\frac{1}{2} + \sinh\frac{\sqrt{2}}{2}\cosh\frac{\sqrt{2}}{2}}{16\sqrt{2}}, \\
a_{12} &= \frac{\sinh\sqrt{2}\cosh\sqrt{2} + \frac{1}{2} - \sinh\frac{\sqrt{2}}{2}\cosh\frac{\sqrt{2}}{2}}{48\sqrt{2}}, \\
a_{21} &= \frac{\sinh\sqrt{2}\cosh\sqrt{2} + \frac{1}{2} - \sinh\frac{\sqrt{2}}{2}\cosh\frac{\sqrt{2}}{2}}{48\sqrt{2}}, \\
a_{22} &= \frac{\sinh\sqrt{2}\cosh\sqrt{2} + \frac{1}{2} - \sinh\frac{\sqrt{2}}{2}\cosh\frac{\sqrt{2}}{2}}{36\sqrt{2}} + \frac{\frac{1}{2} + \sinh\frac{\sqrt{2}}{2}\cosh\frac{\sqrt{2}}{2}}{16\sqrt{2}},
\end{aligned}$$

hence, for all $x \in \mathbb{R}^n$,

$$(\Gamma_0^1 x, x) > \frac{1}{10}\|x\|^2,$$

that is we can pick $\gamma = \frac{1}{10}$. Hence we obtain

$$L\left(K^2\frac{T_\varepsilon^9}{2\gamma\rho(A)^{5/2}}e^{3\sqrt{\rho(A)}b} + \frac{T_\varepsilon^3}{\sqrt{\rho(A)}}e^{\sqrt{\rho(A)}b}\right) = \frac{1}{37}\left(\frac{(\frac{100}{98})^9 e^{3\sqrt{2}}}{3.2\sqrt{2}} + \frac{(\frac{100}{98})^2 e^{\sqrt{2}}}{\sqrt{2}}\right) < 1. \tag{5.4}$$

Similarly, we can get

$$\Lambda_0^1 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

here

$$b_{11} = \frac{\sinh\sqrt{2}\cosh\sqrt{2} - \frac{1}{2} - \sinh\frac{\sqrt{2}}{2}\cosh\frac{\sqrt{2}}{2}}{16\sqrt{2}} + \frac{\sinh\frac{\sqrt{2}}{2}\cosh\frac{\sqrt{2}}{2} - \frac{1}{2}}{8\sqrt{2}},$$
$$b_{12} = \frac{\sinh\sqrt{2}\cosh\sqrt{2} - \frac{1}{2} - \sinh\frac{\sqrt{2}}{2}\cosh\frac{\sqrt{2}}{2}}{24\sqrt{2}},$$
$$b_{21} = \frac{\sinh\sqrt{2}\cosh\sqrt{2} - \frac{1}{2} - \sinh\frac{\sqrt{2}}{2}\cosh\frac{\sqrt{2}}{2}}{24\sqrt{2}},$$
$$b_{22} = \frac{\sinh\sqrt{2}\cosh\sqrt{2} - \frac{1}{2} - \sinh\frac{\sqrt{2}}{2}\cosh\frac{\sqrt{2}}{2}}{18\sqrt{2}} + \frac{\sinh\frac{\sqrt{2}}{2}\cosh\frac{\sqrt{2}}{2} - \frac{1}{2}}{8\sqrt{2}},$$

hence, for all $x \in \mathbb{R}^n$,

$$(\Lambda_0^1 x, x) > \frac{1}{10}\|x\|^2,$$

i.e., we can pick $\lambda = \frac{1}{10}$. Put these constants into (4.2), we obtain

$$L\left(K^2 T_\varepsilon^9 \frac{1}{2\lambda}\mu e^{3\sqrt{\rho(A)}b} + \frac{T_\varepsilon^3}{\sqrt{\rho(A)}}e^{\sqrt{\rho(A)}b}\right) < 1. \tag{5.5}$$

Therefore, all the assumptions of Theorem 4.1 are satisfied, that is the systems (1.3) are exactly controllable.

6 Conclusion

In this paper, we introduce a new definition of controllability, and obtain some sufficient and necessary conditions of second-order linear impulsive systems, the rank criterion of impulsive systems is obtained as well. Then, we present some sufficient conditions of nonlinear impulsive systems provided the linear systems are controllable. Finally, some examples are presented to illustrate our results.

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