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Asymptotic and Oscillatory Behaviour of Third Order Non-linear Differential Equations with Canonical Operator and Mixed Neutral Terms

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Abstract
This paper deals with the asymptotic and oscillatory behaviour of third-order non-linear differential equations with mixed non-linear neutral terms and a canonical operator. The results are obtained via utilising integral conditions as well as comparison theorems with the oscillatory properties of first-order advanced and/or delay differential equations. The proposed theorems improve, extend, and simplify existing ones in the literature. The results are illustrated by two numerical examples.

Keywords Non-linear differential equations · Oscillation · Asymptotic behavior · Canonical operator · Mixed neutral terms

Mathematical Subject Classification 34K11 · 34N05

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1 Introduction

In this work, we aim to investigate the asymptotic and oscillatory behaviour of all solutions of the non-linear third-order differential equations with mixed neutral terms of the form:

$$\big(a(\zeta)\big(y''(\zeta)\big)^{\alpha}\big)' + q(\zeta)x^{\gamma}(\tau(\zeta)) + p(\zeta)x^{\lambda}(\omega(\zeta)) = 0,\ \ \zeta \geq \zeta_0, \tag{1.1}$$

where $y(\zeta) = x(\zeta) + p_1(\zeta)x^{\nu}(\sigma(\zeta)) - p_2(\zeta)x^{\kappa}(\sigma(\zeta))$. Throughout the paper, we always assume that

(A1) $\alpha, \nu, \kappa, \gamma$ and λ are the ratios of positive odd integers with $\alpha \geq 1$,
(A2) a, p_1, p_2, p and $q \in C([\zeta_0, \infty), \mathbb{R}^+)$ with $a'(\zeta) \geq 0$ for $\zeta \geq \zeta_0$,
(A3) $\omega, \tau, \sigma \in C([\zeta_0, \infty), \mathbb{R})$ such that $\tau(\zeta), \sigma(\zeta) \leq \zeta$, $\omega(\zeta) \geq \zeta$ and $\tau(\zeta), \sigma(\zeta), \omega(\zeta) \to \infty$ as $\zeta \to \infty$,
(A4) $h(\zeta) = \sigma^{-1}(\tau(\zeta)) \leq \zeta$, $h^*(\zeta) = \sigma^{-1}(\omega(\zeta)) \geq \zeta$ with $h(\zeta) \to \infty$ as $\zeta \to \infty$,
(A5) $A(\zeta, \zeta_0) = \int_{\zeta_0}^{\zeta} \frac{1}{a^{1/\alpha}(s)} ds$ with $A(\zeta, \zeta_0) \to \infty$ as $\zeta \to \infty$.

A solution of Eq. (1.1) is a function $x(\zeta)$ which is continuous on $[T_x, \infty)$, $T_x \geq \zeta_0$ and satisfies Eq. (1.1) on $[T_x, \infty)$. The solutions which are vanishing identically in some neighborhood of infinity will be excluded from our consideration. Such a solution of Eq. (1.1) is said to be oscillatory if it has arbitrarily large zeros, and to be nonoscillatory otherwise. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

A variety of physical and technological issues raise the question of developing a mathematical model that describes a specific process or structure. It is known that most differential equations, such as those used to model real-life processes, may not have closed-form solutions. This led to a new branch of the theory of differential equations, namely, qualitative theory. In particular, we are especially interested in the study of the oscillatory behaviour of some classes of functional differential equations. Over past years, the oscillation theory of functional differential equations has received much attention since it has a great number of applications in engineering and natural sciences, see, e.g., [1–3, 7–9, 20, 30, 40, 42] and the references cited therein. For example, third-order differential equations appear in a variety of real-world problems, such as in the study of curved beam deflection, scattering cross-section, steam turbine regulation, control of a flying apparatus in cosmic space, entry-flow phenomenon, and so on; see, e.g., [22, 40]. Danziger and Elemergreen [22] discovered a class of third-order linear differential equations by observing the thyroid-pituitary interaction over time. The governing equations that describe the variation of thyroid hormone with time are as follows:

$$a_1 x'''(\zeta) + a_2 x''(\zeta) + a_3 x'(\zeta) + (1 + l)x = lc, \quad x < c,$$
$$a_1 x'''(\zeta) + a_2 x''(\zeta) + a_3 x'(\zeta) + x = 0, \quad x > c,$$

where $x(\zeta)$ is the concentration of thyroid hormone at time ζ and a_1, a_2, a_3, l and c are constants.

Apart from this, neutral delay differential equations arise when lossless transmission lines are employed to interconnect switching circuits in high-speed computers, see [30].

In the recent years, some authors considered the special cases of Eq. (1.1), that is, $p(\zeta) = 0$, or, $\alpha = 1$, or $\nu = \kappa = 1$, or $\lambda = 1$, or $\gamma = 1$, see, e.g., [11, 12, 14–18, 21, 23, 24, 26, 27, 29, 32, 38, 39, 43–45] and references cited therein. In particular, Grace and Jadlovska [25] established several oscillation theorems for the odd-order neutral delay differential equation

$$(x(\zeta) - p_1(\zeta)x(\sigma(\zeta)))^n + q(\zeta)x^\beta(\tau(\zeta)) = 0,$$

where $n \geq 3$ is an odd natural number, $0 \leq p_1(\zeta) < 1$ and the delay terms τ, σ are non-decresaing. This paper's contribution is that it employs the comparison technique to provide conditions that only ensure the oscillation of the aforementioned problem. In [14], Chatzarakis and Grace considered the coupled of third-order neutral differential equation

$$\left(a(\zeta)\left(y''(\zeta)\right)^\alpha\right)' + q(\zeta)x^\gamma(\tau(\zeta)) = 0,$$

where $y(\zeta) = x(\zeta) \pm p_1(\zeta)x^\beta(\sigma(\zeta))$, $0 \leq p_1(\zeta) < \infty$, $\alpha \geq 1$ and $\sigma(\zeta)$ is strictly increasing. By using the comparison method, their two main conclusions (Theorems 1 and 2) guarantee that every solution of the aforementioned equations either oscillates or converges to zero.

Therefore, we aim here to initiate the study of the oscillation problem of (1.1) with either $\nu < \kappa \leq 1$ or $\nu < 1$ and $\kappa > 1$, via comparison with the known oscillatory behaviour of first order equations. The method we employ here in this work has naturally a partial resemblance of the works [14, 24, 28], however the results and most arguments are quite different due to more general nature of Eq. (1.1). The obtained results improve and correlate many of the known oscillation criteria existing in the literature, even for the case of Eq. (1.1) with $p_1(\zeta) = 0$, or $p_2(\zeta) = 0$, or $p_1(\zeta) = p_2(\zeta) = 0$.

To make it easier to read, we simplify our notations here: for $b \in C([\zeta_0, \infty), \mathbb{R}_+)$,

$$g_1(\zeta) := (1-\nu)\nu^{\frac{\nu}{1-\nu}} p_1^{\frac{1}{1-\nu}}(\zeta) b^{\frac{\nu}{\nu-1}}(\zeta),$$

$$g_2(\zeta) := (\kappa-1)\kappa^{\frac{\kappa}{1-\kappa}} p_2^{\frac{1}{1-\kappa}}(\zeta) b^{\frac{\kappa}{\kappa-1}}(\zeta),$$

$$P(\zeta) := \frac{p(\zeta)}{\left(p_2(h^*(\zeta))\right)^{\frac{\lambda}{\kappa}}}, \quad Q(\zeta) := \frac{q(\zeta)}{\left(p_2(h(\zeta))\right)^{\frac{\gamma}{\kappa}}},$$

and

$$\mathcal{P}(\zeta) := \left(\frac{\kappa-\nu}{\nu}\right)\left[\frac{\nu}{\kappa}p_1(\zeta)\right]^{\frac{\kappa}{\kappa-\nu}} \left(p_2(\zeta)\right)^{\frac{\nu}{\nu-\kappa}}.$$

2 Some Preliminaries Lemmas

In order to prove our results later, we have replicated some lemmas that are required.

Lemma 2.1 *Let $q:[\zeta_0,\infty)\to\mathbb{R}^+$, $g:[\zeta_0,\infty)\to\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ are continuous functions, f is non-decreasing and $xf(x)>0$ for $x\neq 0$ and $g(\zeta)\to\infty$ as $\zeta\to\infty$. If*

(I) the first order delay differential inequality (i.e., $g(\zeta)\le\zeta$)

$$y'(\zeta)+q(\zeta)f\big(y(g(\zeta))\big)\le 0$$

has an eventually positive solution, then so does the corresponding delay differential equation.

(II) the first order advanced differential inequality (i.e., $g(\zeta)\ge\zeta$)

$$y'(\zeta)-q(\zeta)f\big(y(g(\zeta))\big)\ge 0$$

has an eventually positive solution, then so does the corresponding advanced differential equation.

Proof This Lemma is an extension of known results in [10, Lemma 2.3] and [41, Corollary 1] and hence the proof is omitted. □

Lemma 2.2 [31] *If $\mathcal{X}$ and $\mathcal{Y}$ are non-negative, then*

$$\mathcal{X}^{\varphi}+(\varphi-1)\mathcal{Y}^{\varphi}-\varphi\mathcal{X}\mathcal{Y}^{\varphi-1}\ge 0 \quad \text{for} \quad \varphi>1 \tag{2.1}$$

and

$$\mathcal{X}^{\varphi}-(1-\varphi)\mathcal{Y}^{\varphi}-\varphi\mathcal{X}\mathcal{Y}^{\varphi-1}\le 0 \quad \text{for} \quad 0<\varphi<1, \tag{2.2}$$

where equalities hold if and only if $\mathcal{X}=\mathcal{Y}$.

Lemma 2.3 (Young's Inequality) [31] If $\mathcal{X}$, Y be nonnegative real numbers and if $m,n>1$ are real numbers such that $\frac{1}{n}+\frac{1}{m}=1$. Then

$$\mathcal{X}\mathcal{Y}\le\frac{1}{n}\mathcal{X}^n+\frac{1}{m}\mathcal{Y}^m. \tag{2.3}$$

Equality holds if and only if $\mathcal{X}^n$=$\mathcal{Y}^m$.

3 The Case When $\nu<1$ and $\kappa>1$

In this section, we present some oscillation criteria for Eq. (1.1) when $\nu<1$ and $\kappa>1$.

Theorem 3.1 *Let (A1) – (A5) hold with $\nu<1$ and $\kappa>1$. Furthermore, assume that*

(A6) there exists a function $b\in C\left([\zeta_0,\infty),\mathbb{R}_+\right)$ such that

$$\lim_{\zeta\to\infty}[g_1(\zeta)+g_2(\zeta)]=0 \tag{3.1}$$

and

(A7) there exist non-decreasing functions $\mu(\zeta),\ \pi(\zeta) \in C([\zeta_0, \infty), \mathbb{R})$ *such that* $\mu_1(\zeta) = \mu(\zeta) < \zeta$, $\mu_2(\zeta) = \mu\big(\mu(\zeta)\big)$ *with* $\rho(\zeta) = h^*\big(\mu_2(\zeta)\big) > \zeta$,

$$\tau(\zeta) \le \pi(\zeta) \le \zeta \ \ for\ \zeta \ge \zeta_0 \tag{3.2}$$

hold. If there exist numbers $\theta_1,\ \theta_2 \in (0, 1)$ *such that the delay differential equations*

$$W'(\zeta) + \theta_1 q(\zeta)\left(\int_{\zeta_0}^{\tau(\zeta)} A(s, \zeta_1)ds\right)^{\gamma} W^{\frac{\gamma}{\alpha}}(\tau(\zeta)) = 0, \tag{3.3}$$

$$Y'(\zeta) + \theta_2 \tau^{\gamma/\kappa}(\zeta) Q(\zeta) A^{\gamma/\kappa}\big(\pi(\zeta), \tau(\zeta)\big) Y^{\frac{\gamma}{\alpha\kappa}}\big(\pi(\zeta)\big) = 0 \tag{3.4}$$

and the advanced differential equation

$$\hat{y}'(\zeta) - \left[\int_{\mu(\zeta)}^{\zeta} \left(a^{\frac{-1}{\alpha}}(u)\right)\left(\int_{\mu(u)}^{u} P(s)ds\right)^{\frac{1}{\alpha}} du\right] \hat{y}^{\frac{\lambda}{\alpha\kappa}}\big(\rho(\zeta)\big) = 0 \tag{3.5}$$

are oscillatory, then every solution of Eq. (1.1) *is oscillatory or converges to zero.*

Proof Suppose $x(\zeta)$ is a non-oscillatory solution of Eq. (1.1) with $x(\zeta) > 0$ and $\lim_{\zeta\to\infty} x(\zeta) \neq 0$ for $\zeta \ge \zeta_0$. Therefore, $x\big(\tau(\zeta)\big) > 0$, $x\big(\sigma(\zeta)\big) > 0$ and $x\big(\omega(\zeta)\big) > 0$ for $\zeta \ge \zeta_1$ for some $\zeta_1 > \zeta_0$. It follows from Eq. (1.1) that

$$\big(a(\zeta)\big(y''(\zeta)\big)^{\alpha}\big)' = -q(\zeta)x^{\gamma}\big(\tau(\zeta)\big) - p(\zeta)x^{\lambda}\big(\omega(\zeta)\big) < 0. \tag{3.6}$$

Hence, $a(\zeta)\big(y''(\zeta)\big)^{\alpha}$ is decreasing and of one sign, that is, there exists a $\zeta_2 > \zeta_1$ such that $y''(\zeta) > 0$ or $y''(\zeta) < 0$ for $\zeta \ge \zeta_2$. We shall distinguish the following four cases:

1. $y(\zeta) > 0,\ \ y''(\zeta) < 0$, 2. $y(\zeta) > 0,\ \ y''(\zeta) > 0$
3. $y(\zeta) < 0,\ \ y''(\zeta) > 0$, 4. $y(\zeta) < 0,\ \ y''(\zeta) < 0$.

Case 1: Since $y'''(\zeta) < 0$ and $y''(\zeta) < 0$, then a constant $K > 0$ exists such that $y''(\zeta) \le \frac{-K^{\frac{1}{\alpha}}}{a^{\frac{1}{\alpha}}(\zeta)} < 0$ for $\zeta \ge \zeta_3 > \zeta_2$, which on integration from ζ_3 to ζ gives

$$y'(\zeta) \le y'(\zeta_3) - K^{\frac{1}{\alpha}} \int_{\zeta_3}^{\zeta} \frac{1}{a^{\frac{1}{\alpha}}(s)} ds.$$

Letting $\zeta \to \infty$ and using (A_5), we get $\lim_{\zeta\to\infty} y'(\zeta) = -\infty$. Therefore, $y'(\zeta) < 0$. But conditions $y''(\zeta) < 0$ and $y'(\zeta) < 0$ imply that $y(\zeta) < 0$, which contradicts our assumption $y(\zeta) > 0$.

Case 2: For this case we have following two sub-cases:

Case 2_1**:** Let $y'(\zeta) < 0$ for $\zeta \geq \zeta_2$. This case is excluded because of the choice $\lim_{\zeta\to\infty} x(\zeta) \neq 0$.
Case 2_2**:** Let $y'(\zeta) > 0$ for $\zeta \geq \zeta_2$. From the associated function $y(\zeta)$, we have

$$y(\zeta) = x(\zeta) + \big(b(\zeta)x(\sigma(\zeta)) - p_2(\zeta)x^{\kappa}(\sigma(\zeta))\big) + \big(p_1(\zeta)x^{\nu}(\sigma(\zeta)) - b(\zeta)x(\sigma(\zeta))\big),$$

or,

$$x(\zeta) = y(\zeta) - \big(b(\zeta)x(\sigma(\zeta)) - p_2(\zeta)x^{\kappa}(\sigma(\zeta))\big) - \big(p_1(\zeta)x^{\nu}(\sigma(\zeta)) - b(\zeta)x(\sigma(\zeta))\big). \tag{3.7}$$

If we apply (2.1) to $[b(\zeta)x(\sigma(\zeta)) - p_2(\zeta)x^{\kappa}(\sigma(\zeta))]$ with $\varphi = \kappa > 1$, $\mathcal{X} = p_2^{\frac{1}{\kappa}}(\zeta)x(\sigma(\zeta))$ and $\mathcal{Y} = \left(\frac{1}{\kappa}b(\zeta)p_2^{\frac{-1}{\kappa}}(\zeta)\right)^{\frac{1}{\kappa-1}}$, we get

$$\big(b(\zeta)x(\sigma(\zeta)) - p_2(\zeta)x^{\kappa}(\sigma(\zeta))\big) \leq (\kappa-1)\kappa^{\frac{\kappa}{1-\kappa}} p_2^{\frac{1}{1-\kappa}}(\zeta) b^{\frac{\kappa}{\kappa-1}}(\zeta) := g_2(\zeta).$$

Similarly, if we apply (2.2) to $[p_1(\zeta)x^{\nu}(\sigma(\zeta)) - b(\zeta)x(\sigma(\zeta))]$ with $\varphi = \nu < 1$, $\mathcal{X} = p_1^{\frac{1}{\nu}}(\zeta)x(\sigma(\zeta))$ and $\mathcal{Y} = \left(\frac{1}{\nu}b(\zeta)p_1^{\frac{-1}{\nu}}(\zeta)\right)^{\frac{1}{\nu-1}}$, we get

$$(p_1(\zeta)x^{\nu}(\sigma(\zeta)) - b(\zeta)x(\sigma(\zeta))) \leq (1-\nu)\nu^{\frac{\nu}{1-\nu}} p_1^{\frac{1}{1-\nu}}(\zeta) b^{\frac{\nu}{\nu-1}}(\zeta) := g_1(\zeta).$$

Thus, from (3.7), we see that

$$x(\zeta) \geq \left[1 - \frac{g_1(\zeta) + g_2(\zeta)}{y(\zeta)}\right] y(\zeta). \tag{3.8}$$

Since $y(\zeta) > 0$ is increasing, a constant $\mathcal{C} > 0$ for $\zeta_3 > \zeta_2$ exists such that $y(\zeta) \geq \mathcal{C}$ for $\zeta \geq \zeta_3$ and so, we have

$$x(\zeta) \geq \left[1 - \frac{g_1(\zeta) + g_2(\zeta)}{\mathcal{C}}\right] y(\zeta) \quad \text{for } \zeta \geq \zeta_3. \tag{3.9}$$

Now, in view of (3.1), a constant $c \in (0, 1)$ exists such that

$$x(\zeta) \geq c y(\zeta). \tag{3.10}$$

Thus, we have

$$\big(a(\zeta)(y''(\zeta))^{\alpha}\big)' \leq -c^{\gamma} q(\zeta) y^{\gamma}\big(\tau(\zeta)\big) - c^{\lambda} p(\zeta) y^{\lambda}\big(\omega(\zeta)\big). \tag{3.11}$$

Since $y'(\zeta) > 0$, then the last inequality can be written as

$$\big(a(\zeta)(y''(\zeta))^{\alpha}\big)' \leq -c^{\gamma} q(\zeta) y^{\gamma}(\tau(\zeta)). \tag{3.12}$$

Because $y''(\zeta) > 0$ and $y'(\zeta) > 0$ for $\zeta \geq \zeta_3$, then following [1, Lemma 2.2.3], a constant $k \in (0, 1)$ exists such that

$$y'(\zeta) \geq kA(\zeta, \zeta_1)a^{\frac{1}{\alpha}}(\zeta)y''(\zeta).$$

Integrating this inequality from ζ_3 to ζ, we get

$$y(\zeta) \geq \left(k \int_{\zeta_3}^{\zeta} A(s, \zeta_1)ds\right) a^{\frac{1}{\alpha}}(\zeta)y''(\zeta).$$

Using this inequality in (3.12) and setting $W(\zeta) = a(\zeta)(y''(\zeta))^{\alpha}$, we have

$$W'(\zeta) + \theta_1 q(\zeta)\left(\int_{\zeta_3}^{\tau(\zeta)} A(s, \zeta_1)ds\right)^{\gamma} W^{\frac{\gamma}{\alpha}}(\tau(\zeta)) \leq 0, \tag{3.13}$$

where $\theta_1 = (ck)^{\gamma} \in (0, 1)$. It follows from Lemma 2.1 (I) that the corresponding differential Eq. (3.3) also has a positive solution, which is a contradiction.

Case 3: For $y(\zeta) < 0$, we consider

$$\hat{y}(\zeta) = -y(\zeta) = -x(\zeta) - p_1(\zeta)x^{\nu}(\sigma(\zeta)) + p_2(\zeta)x^{\kappa}(\sigma(\zeta)) \leq p_2(\zeta)x^{\kappa}(\sigma(\zeta)),$$

or,

$$x(\sigma(\zeta)) \geq \left(\frac{\hat{y}(\zeta)}{p_2(\zeta)}\right)^{\frac{1}{\kappa}},$$

or,

$$x(\zeta) \geq \left(\frac{\hat{y}(\sigma^{-1}(\zeta))}{p_2(\sigma^{-1}(\zeta))}\right)^{\frac{1}{\kappa}}$$

and so,

$$\begin{aligned}\left(a(\zeta)(\hat{y}''(\zeta))^{\alpha}\right)' &\geq q(\zeta)x^{\gamma}(\tau(\zeta)) \\ &\geq \frac{q(\zeta)}{\left(p_2(\sigma^{-1}(\tau(\zeta)))\right)^{\frac{\gamma}{\kappa}}}\hat{y}^{\frac{\gamma}{\kappa}}\left(\sigma^{-1}(\tau(\zeta))\right) := Q(\zeta)\hat{y}^{\frac{\gamma}{\kappa}}\left(h(\zeta)\right).\end{aligned} \tag{3.14}$$

From $\hat{y}''(\zeta) < 0$ and for $\zeta_2 \leq u \leq v$, it follows that

$$\begin{aligned}\hat{y}'(u) \geq \hat{y}'(u) - \hat{y}'(v) &= -\int_u^v a^{\frac{-1}{\alpha}}(s)\left(a(s)(\hat{y}''(s)^{\alpha})\right)^{\frac{1}{\alpha}}ds \\ &\geq A(v, u)\left(-a^{\frac{1}{\alpha}}(v)\right)\hat{y}''(v).\end{aligned}$$

In the above inequality, we let $u = \tau(\zeta)$ and $v = \pi(\zeta)$, then

$$\hat{y}'\big(\tau(\zeta)\big) \geq A(\pi(\zeta), \tau(\zeta))\big(-a^{\frac{1}{\alpha}}(\pi(\zeta))\big)\hat{y}''\big(\pi(\zeta)\big). \tag{3.15}$$

According to [1, Lemma 2.2.3], a constant $\theta_3 \in (0, 1)$ exists such that

$$\hat{y}(\tau(\zeta)) \geq \theta_3 \tau(\zeta)\hat{y}'\big(\tau(\zeta)\big). \tag{3.16}$$

Combining (3.16) in (3.15), we have

$$\hat{y}(\tau(\zeta)) \geq \theta_3 \tau(\zeta) A\big(\pi(\zeta), \tau(\zeta)\big)\big(-a^{\frac{1}{\alpha}}(\pi(\zeta))\big)\hat{y}''\big(\pi(\zeta)\big). \tag{3.17}$$

Using (3.17) in (3.14), we get

$$Y'(\zeta) + \theta_2 \tau^{\gamma/\kappa}(\zeta) Q(\zeta) A^{\gamma/\kappa}\big(\pi(\zeta), \tau(\zeta)\big) Y^{\frac{\gamma}{\alpha\kappa}}\big(\pi(\zeta)\big) \leq 0, \tag{3.18}$$

where $Y(\zeta) := -a(\zeta)\big(\hat{y}''(\zeta)\big)^{\alpha}$ and $\theta_2 = (\theta_3)^{\gamma/\kappa}$. As a result of Lemma 2.1(I), the differential Eq. (3.4) also has a positive solution, which is a contradiction.

Case 4: Clearly, we see that $\hat{y}''(\zeta) > 0$. In this case we have $\hat{y}'(\zeta) > 0$. From Case 3, it follows that

$$\big(a(\zeta)(\hat{y}''(\zeta))^{\alpha}\big)' \geq P(\zeta)\hat{y}^{\frac{\lambda}{\kappa}}\big(h^*(\zeta)\big). \tag{3.19}$$

Integrating (3.19) from $\mu(\zeta)$ to ζ, we have

$$\begin{aligned} a(\zeta)\big(\hat{y}''(\zeta)\big)^{\alpha} - a\big(\mu(\zeta)\big)\big(\hat{y}''(\mu(\zeta))\big)^{\alpha} &\geq \int_{\mu(\zeta)}^{\zeta} P(s)\hat{y}^{\frac{\lambda}{\kappa}}\big(h^*(s)\big)ds \\ &\geq \hat{y}^{\frac{\lambda}{\kappa}}\big(h^*(\mu(\zeta))\big)\int_{\mu(\zeta)}^{\zeta} P(s)ds. \end{aligned}$$

Therefore,

$$\hat{y}''(\zeta) \geq \hat{y}^{\frac{\lambda}{\alpha\kappa}}\big(h^*(\mu(\zeta))\big)\big(a^{\frac{-1}{\alpha}}(\zeta)\big)\left(\int_{\mu(\zeta)}^{\zeta} P(s)ds\right)^{\frac{1}{\alpha}}.$$

An integration from $\mu(\zeta)$ to ζ yields

$$\hat{y}'(\zeta) \geq \hat{y}^{\frac{\lambda}{\alpha\kappa}}\big(\rho(\zeta)\big)\int_{\mu(\zeta)}^{\zeta}\big(a^{\frac{-1}{\alpha}}(u)\big)\left(\int_{\mu(u)}^{u} P(s)ds\right)^{\frac{1}{\alpha}} du.$$

Consequently, $\hat{y}(\zeta)$ is a positive solution of the advanced differential inequality

$$\hat{y}'(\zeta) - \left[\int_{\mu(\zeta)}^{\zeta}\left(a^{\frac{-1}{\alpha}}(u)\right)\left(\int_{\mu(u)}^{u} P(s)ds\right)^{\frac{1}{\alpha}} du\right]\hat{y}^{\frac{\lambda}{\alpha\kappa}}\big(\rho(\zeta)\big) \geq 0. \tag{3.20}$$

As a result of Lemma 2.1(II), the differential Eq. (3.5) also has a positive solution, which is a contradiction. This completes the proof. □

Next, we have the following corollary that follows immediately from Theorem 3.1.

Corollary 3.1 *Let* $(A1) - (A5)$ *hold with* $\nu < 1$ *and* $\kappa > 1$. *Furthermore, assume that there exists a function* $b \in C([\zeta_0, \infty), \mathbb{R}_+)$ *such that condition* (3.1), *and non-decreasing functions* $\mu(\zeta), \eta(\zeta) \in C([\zeta_0, \infty), \mathbb{R})$ *such that condition* (3.2) *are satisfied. If*

$$\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta} q(s)\left(\int_{\zeta_0}^{\tau(s)} A(u,\zeta_1)du\right)^{\gamma} ds = \infty \ for\ \gamma < \alpha, \tag{3.21}$$

$$\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta} Q(s)\tau^{\gamma/\kappa}(s)A^{\gamma/\kappa}\big(\pi(s),\tau(s)\big)ds = \infty \ \ for\ \gamma < \alpha\kappa, \tag{3.22}$$

and

$$\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}\left[\int_{\mu(l)}^{l}\left(a^{\frac{-1}{\alpha}}(u)\right)\left(\int_{\mu(u)}^{u} P(s)ds\right)^{\frac{1}{\alpha}} du\right]dl = \infty \ \ for\ \lambda > \alpha\kappa, \tag{3.23}$$

then every solution of Eq. (1.1) *is oscillatory or converges to zero.*

Proof Suppose $x(\zeta)$ is a non-oscillatory solution of Eq. (1.1) with $x(\zeta) > 0$ and $\lim_{\zeta\to\infty} x(\zeta) \neq 0$ for $\zeta \geq \zeta_0$. Therefore, $x\big(\tau(\zeta)\big) > 0, x\big(\sigma(\zeta)\big) > 0$ and $x\big(\omega(\zeta)\big) > 0$ for $\zeta \geq \zeta_1$ for some $\zeta_1 > \zeta_0$. Proceeding as in the proof of Theorem 3.1, we arrive at (3.13) for $\zeta \geq \zeta_3$, (3.18) for $\zeta > \zeta_2$, and (3.20) for $\zeta > \zeta_2$ respectively. Upon using the fact that $\tau(\zeta) \leq \zeta$, and $W(\zeta) = a(\zeta)(y''(\zeta))^{\alpha}$ is positive and decreasing, we have $W(\tau(\zeta)) \geq W(\zeta)$ and hence inequality (3.13) can be written as

$$W'(\zeta) + \theta_1 q(\zeta)\left(\int_{\zeta_3}^{\tau(\zeta)} A(s,\zeta_1)ds\right)^{\gamma} W^{\frac{\gamma}{\alpha}}(\zeta) \leq 0,$$

that is,

$$\frac{W'(\zeta)}{W^{\frac{\gamma}{\alpha}}(\zeta)} + \theta_1 q(\zeta)\left(\int_{\zeta_3}^{\tau(\zeta)} A(s,\zeta_1)ds\right)^{\gamma} \leq 0,$$

which on integration from ζ_4 to ζ gives

$$\int_{\zeta_4}^{\zeta} q(s)\left(\int_{\zeta_3}^{\tau(s)} A(u,\zeta_1)du\right)^{\gamma} ds \leq \frac{1}{\theta_1}\left(\frac{W^{1-\frac{\gamma}{\alpha}}(\zeta_4)}{1-\frac{\gamma}{\alpha}}\right)$$

and letting $\zeta \to \infty$, we get a contradiction to (3.21). The reminder of proof follows from the inequalities (3.18) and (3.20), and noting that $\pi(\zeta) \leq \zeta$ and $\rho(\zeta) \geq \zeta$, respectively. Hence, we omit the details. □

The following example illustrate the applicability of Corollary 3.1.

Example 3.1 Consider

$$\left(\left(\zeta\left(x(\zeta)+\frac{1}{\zeta}x^{\frac{3}{7}}\left(\frac{\zeta}{2}\right)-\zeta x^{\frac{9}{7}}\left(\frac{\zeta}{2}\right)\right)''\right)\right)' \\ +x^{\frac{5}{7}}\left(\frac{\zeta}{8}\right)+\frac{1}{\zeta}x^{\lambda}(2\zeta)=0,\ \zeta>\zeta_0=1, \tag{3.24}$$

where $\alpha = 1$, $\nu = \frac{3}{7}$, $\kappa = \frac{9}{7}$, $\gamma = \frac{5}{7}$, $\lambda > \frac{9}{7}$, $a(\zeta) = \zeta$, $p_1(\zeta) = p(\zeta) = \frac{1}{\zeta}$, $p_2(\zeta) = \zeta$, $q(\zeta) = 1$, $\sigma(\zeta) = \frac{\zeta}{2}$, $\tau(\zeta) = \frac{\zeta}{8}$ and $\omega(\zeta) = 2\zeta$. Also, $h(\zeta) = \frac{\zeta}{4}$, $h(\zeta) = 4\zeta$ $Q(\zeta) = \frac{1}{\left(\frac{\zeta}{4}\right)^{5/9}}$ and $P(\zeta) = \frac{(2\zeta)^{\frac{5}{9}}}{\zeta}$. Letting $b = 1$, it is not difficult to see that (3.1) holds. We let $\mu(\zeta) = \frac{3}{4}\zeta$, then $\rho(\zeta) = \frac{9}{4}\zeta$. Since $A(\zeta, \zeta_0) = \int_1^{\zeta} \frac{ds}{s} = \ln \zeta$, then all conditions of Corollary 3.1 are met. Indeed, from (3.21), (3.22) and (3.23), we have

$$\lim_{\zeta\to\infty}\int_1^{\zeta}\left(\int_1^{s/8}\ln u\,du\right)^{\frac{5}{7}}ds=\infty,$$

$$\lim_{\zeta\to\infty}\int_1^{\zeta}\frac{1}{\left(\frac{s}{4}\right)^{5/9}}\left(\frac{s}{4}\right)^{5/9}\left(\frac{s}{8}\left(\ln\frac{s}{2}-1\right)\right)^{5/9}ds=\infty,$$

and

$$\lim_{\zeta\to\infty}\int_{\zeta_0}^{\zeta}\left[\int_{\frac{3l}{4}}^{l}\left(\frac{1}{u}\right)\left(\int_{\frac{3u}{4}}^{u}\frac{(2s)^{\frac{5}{9}}}{s}ds\right)du\right]dl=\infty$$

respectively. Thus, every solution to (3.24) is oscillatory or else converges to zero.

Remark 3.1 We may note that [24, Theorem 2.1] is not applicable to (3.24) due to the restriction that $p_1(\zeta) = 0 = p(\zeta)$ and $0 < \kappa \le 1$. Apart from this, suppose that $p_1(\zeta) = 0 = p(\zeta)$ in (3.24), then it is not difficult to see that Theorem 3.1 generalised/improved the results reported in [24]. A similar observation can be made for the papers [11, 13, 23, 25, 39].

Now, we shall present some special cases of Theorem 3.1. First we consider the case when $p_1(\zeta) = p_2(\zeta) = 0$, i.e., for the non-neutral equation

$$\left(a(\zeta)(x''(\zeta))^{\alpha}\right)'+q(\zeta)x^{\gamma}\big(\tau(\zeta)\big)+p(\zeta)x^{\lambda}\big(\omega(\zeta)\big)=0. \tag{3.25}$$

Accordingly, Theorem 3.1 can be expressed in the following form:

Corollary 3.2 *Let assumptions* $(A1)-(A5)$ *hold. If there exists a number* $\theta_1 \in (0, 1)$ *such that the first order delay differential Eq.* (3.3) *is oscillatory, then every solution of Eq.* (3.25) *is oscillatory or converges to zero.*

Following that, we consider the case when $p_2(\zeta) = 0$, i.e., the neutral equation

$$\left(a(\zeta)\left(\left(x(\zeta) + p_1(\zeta)x^{\nu}(\sigma(\zeta))\right)''\right)^{\alpha}\right)' + q(\zeta)x^{\gamma}\left(\tau(\zeta)\right) + p(\zeta)x^{\lambda}\left(\omega(\zeta)\right) = 0. \tag{3.26}$$

For the Eq. (3.26) with $0 < \nu \le 1$, we have the following new result:

Corollary 3.3 *In addition to the hypotheses of Corollary 3.2, assume that* $\lim_{\zeta\to\infty} p_1(\zeta) = 0$. *Then every solution of Eq.* (3.26) *is oscillatory or converges to zero.*

For complete oscillation criteria of Eq. (3.25), we have the following result.

Theorem 3.2 *Let conditions* $(A1) - (A5)$ *hold. Assume that there exists a non-decreasing function* $\eta(\zeta) \in C([\zeta_0, \infty), \mathbb{R})$ *such that*

$$\eta_1(\zeta) = \eta(\zeta) > \zeta,\ \eta_2(\zeta) = \eta_1(\eta(\zeta))\ \ with\ \ \varphi(\zeta) = \eta_2(\tau(\zeta)) < \zeta. \tag{3.27}$$

If there exist numbers $\theta_1 \in (0, 1)$ *such that the first order delay differential Eq.* (3.3), *and*

$$X'(\zeta) + q(\zeta)\left(\int_{\tau(\zeta)}^{\eta(\tau(\zeta))} A(\eta(s), s)ds\right)^{\gamma} X^{\frac{\gamma}{\alpha}}(\varphi(\zeta)) = 0 \tag{3.28}$$

are oscillatory, then Eq. (3.25) *is oscillatory.*

Proof Let $x(\zeta)$ be a non-oscillatory solution of Eq. (3.25), say $x(\zeta) > 0, x(\tau(\zeta)) > 0$ and $x(\omega(\zeta)) > 0$ for $\zeta \ge \zeta_1$ for some $\zeta_1 > \zeta_0$. Hence, $a(\zeta)\left(x''(\zeta)\right)^{\alpha}$ is of one sign, that is, there exists a $\zeta_2 \ge \zeta_1$ such that $x''(\zeta) > 0$ or $x''(\zeta) < 0$ for $\zeta \ge \zeta_2$. We shall distinguish the following two cases:

$$1.\ x(\zeta) > 0,\ x''(\zeta) < 0, \quad 2.\ x(\zeta) > 0,\ x''(\zeta) > 0.$$

Case 1: Since $x''(\zeta)$ is non-increasing and negative, a constant $\mathcal{C} > 0$ exists for $\zeta \ge \zeta_3 > \zeta_2$ such that

$$a(\zeta)\left(x''(\zeta)\right)^{\alpha} \le -\mathcal{C} < 0.$$

Integrating the last inequality from ζ_3 to ζ, we get

$$x'(\zeta) \le x'(\zeta_3) - \mathcal{C}^{1/\alpha}\int_{\zeta_3}^{\zeta} a^{-1/\alpha}(s)ds.$$

Letting $\zeta \to \infty$ and then using $(A5)$, we get $x'(\zeta) \to -\infty$. Therefore, $x'(\zeta) < 0$ together with $x''(\zeta) < 0$ implies that $x(\zeta) < 0$, a contradiction.
Case 2: For this case, we have following two subcases:

Case 2_1 $(x'(\zeta) > 0)$**:** This case can be follows from the proof of Case 2_2 of Theorem 3.1 and hence we omit the details.
Case 2_2 $(x'(\zeta) < 0)$**:** One can easily see that $x(\zeta)$ satisfies

$$(-1)^i x^{(i)}(\zeta) \geq 0, \quad i = 1, 2, 3.$$

Therefore,

$$-x'(\zeta) \geq x'(\eta(\zeta)) - x'(\zeta) = \int_{\zeta}^{\eta(\zeta)} a^{\frac{-1}{\alpha}}(s)ds \left(a^{\frac{1}{\alpha}}(\eta(\zeta))x''(\eta(\zeta))\right),$$

which implies that

$$-x'(\zeta) \geq A(\eta(\zeta), \zeta) \left(a^{\frac{1}{\alpha}}(\eta(\zeta))x''(\eta(\zeta))\right).$$

Integrate this inequality from ζ to $\eta(\zeta)$ yields

$$x(\zeta) \geq \left(a^{\frac{1}{\alpha}}(\eta_2(\zeta))x''(\eta_2(\zeta))\right) \left(\int_{\zeta}^{\eta(\zeta)} A(\eta(s), s)ds\right),$$

that is,

$$x(\tau(\zeta)) \geq \left(a^{\frac{1}{\alpha}}(\eta_2(\tau(\zeta)))x''(\eta_2(\tau(\zeta)))\right) \left(\int_{\tau(\zeta)}^{\eta(\tau(\zeta))} A(\eta(s), s)ds\right).$$

Using this inequality in (3.25), we have

$$X'(\zeta) + q(\zeta) \left(\int_{\tau(\zeta)}^{\eta(\tau(\zeta))} A(\eta(s), s)ds\right)^{\gamma} X^{\frac{\gamma}{\alpha}}(\varphi(\zeta)) \leq 0,$$

where $X(\zeta) = a(\zeta)\big(x''(\zeta)\big)^{\alpha}$. It follows that the rest of the proof is similar to those mentioned above, so it is omitted. This completes the proof. □

Finally, we consider the case when $p_1(\zeta) = 0$, i.e., the neutral equation

$$\Big(a(\zeta)\big(\big(x(\zeta) - p_2(\zeta)x^{\kappa}(\sigma(\zeta))\big)''\big)^{\alpha}\Big)' + q(\zeta)x^{\gamma}\big(\tau(\zeta)\big) + p(\zeta)x^{\lambda}\big(\omega(\zeta)\big) = 0. \tag{3.29}$$

Now, we have the following oscillation result for Eq. (3.29).

Theorem 3.3 *Let the hypotheses of Theorem 3.1 hold with* $p_1(\zeta) = 0$*. Then every solution of Eq.* (3.29) *is oscillatory or converges to zero.*

Proof Suppose $x(\zeta)$ is a non-oscillatory solution of (3.29) with $x(\zeta) > 0$ and $\lim_{\zeta\to\infty} x(\zeta) \neq 0$ for $\zeta \geq \zeta_0$. Therefore, $x\big(\tau(\zeta)\big) > 0, x\big(\sigma(\zeta)\big) > 0$ and $x\big(\omega(\zeta)\big) > 0$ for $\zeta \geq \zeta_1$ for some $\zeta_1 > \zeta_0$. Following the same procedure used for the proof of Theorem 3.1, we obtain Cases 1 through 4.

If $y(\zeta) = x(\zeta) - p_2(\zeta)x^{\kappa}\big(\sigma(\zeta)\big)$ is positive, then $x(\zeta) \geq y(\zeta)$ and so, Eq. (3.29) becomes

$$\big(a(\zeta)\big(y''(\zeta)\big)^{\alpha}\big)' \leq -q(\zeta)y^{\gamma}\big(\tau(\zeta)\big) - p(\zeta)y^{\lambda}\big(\omega(\zeta)\big)$$

and we may apply Corollary 3.2. For the case when $y(\zeta) < 0$, we apply the Theorem 3.1 when the two cases, Case 3 and Case 4 hold. Therefore, we omit the details. □

Remark 3.2 Theorem 3.3 improved or generalised the results reportted in [11, 23–25].

4 The Case When $\nu < \kappa \leq 1$

In this section, we present some oscillation criteria for Eq. (1.1) when $\nu < \kappa \leq 1$.

Theorem 4.1 *Let* $(A1) - (A5)$ *hold with* $\nu < \kappa \leq 1$. *Assume that all the hypotheses of the Theorem 3.1 hold, and the condition* (3.1) *is replaced by*

$$\lim_{\zeta\to\infty} \mathcal{P}(\zeta) = 0. \tag{4.1}$$

Then the conclusion of Theorem 3.1 holds.

Proof Suppose $x(\zeta)$ is a non-oscillatory solution of (1.1) with $x(\zeta) > 0$ and $\lim_{\zeta\to\infty} x(\zeta) \neq 0$ for $\zeta \geq \zeta_0$. Therefore, $x\big(\tau(\zeta)\big) > 0, x\big(\sigma(\zeta)\big) > 0$ and $x\big(\omega(\zeta)\big) > 0$ for $\zeta \geq \zeta_1$ for some $\zeta_1 > \zeta_0$. Following the same procedure used for the proof of Theorem 3.1, we obtain Cases 1 through 4.

First, we consider Case 1 and 2. Clearly, we see that $y'(\zeta) > 0$ for $\zeta \geq \zeta_2$. It is not difficult to see that

$$[p_1(\zeta)x^{\nu}\big(\sigma(\zeta)\big) - p_2(\zeta)x^{\kappa}\big(\sigma(\zeta)\big)] = \frac{\kappa}{\nu}p_2(\zeta)\left[x^{\nu}\big(\sigma(\zeta)\big)\frac{\nu}{\kappa}\frac{p_1(\zeta)}{p_2(\zeta)} - \frac{\nu}{\kappa}\big(x^{\nu}(\sigma(\zeta))\big)^{\frac{\kappa}{\nu}}\right].$$

Setting $n = \frac{\kappa}{\nu} > 1$, $\mathcal{X} = x^{\nu}\big(\sigma(\zeta)\big)$, $\mathcal{Y} = \frac{\nu}{\kappa}\left(\frac{p_1(\zeta)}{p_2(\zeta)}\right)$ and $m = \frac{\kappa}{\kappa-\nu}$, in $[p_1(\zeta)x^{\nu}\big(\sigma(\zeta)\big) - p_2(\zeta)x^{\kappa}\big(\sigma(\zeta)\big)]$, we have

$$[p_1(\zeta)x^{\nu}\big(\sigma(\zeta)\big) - p_2(\zeta)x^{\kappa}\big(\sigma(\zeta)\big)] = \frac{\kappa}{\nu}p_2(\zeta)\left[\mathcal{X}\mathcal{Y} - \frac{1}{n}\mathcal{X}^n\right].$$

Applying (2.3) to $[p_1(\zeta)x^{\nu}(\sigma(\zeta)) - p_2(\zeta)x^{\kappa}(\sigma(\zeta))]$, we obtain

$$\begin{aligned}[p_1(\zeta)x^{\nu}(\sigma(\zeta)) - p_2(\zeta)x^{\kappa}(\sigma(\zeta))] &\leq \frac{\kappa}{\nu}p_2(\zeta)\left(\frac{1}{m}\mathcal{Y}^m\right)\\ &= \frac{\kappa-\nu}{\nu}\left[\frac{\nu}{\kappa}p_1(\zeta)\right]^{\frac{\kappa}{\kappa-\nu}}(p_2(\zeta))^{\frac{\nu}{\nu-\kappa}} := \mathcal{P}(\zeta).\end{aligned}$$

Thus, using the last inequality to $x(\zeta) = y(\zeta) - p_1(\zeta)x^{\nu}(\sigma(\zeta)) + p_2(\zeta)x^{\kappa}(\sigma(\zeta))$, we see that

$$x(\zeta) \geq \left[1 - \frac{\mathcal{P}(\zeta)}{y(\zeta)}\right]y(\zeta). \tag{4.2}$$

Due to non-decreasing of $y(\zeta) > 0$, we can find a constant $\mathcal{C} > 0$ such that $y(\zeta) \geq \mathcal{C}$, therefore, we have

$$x(\zeta) \geq \left[1 - \frac{\mathcal{P}(\zeta)}{\mathcal{C}}\right]y(\zeta). \tag{4.3}$$

Now, in view of (4.1), we can find $\epsilon \in (0, 1)$ such that

$$x(\zeta) \geq \epsilon y(\zeta). \tag{4.4}$$

It follows that the remainder of the proof is similar to Theorem 3.1. This completes the proof. □

Remark 4.1 We may note that the results similar to Corollary 3.1-Corollary 3.2 can also be extracted from Theorem 4.1. The details are left to the reader.

The following example illustrate the applicability of Theorem 4.1.

Example 4.1 Consider

$$\begin{aligned}&\left(\left(\frac{1}{\zeta}\left(x(\zeta) + \frac{1}{\zeta}x^{\frac{3}{7}}\left(\frac{\zeta}{2}\right) - x^{\frac{5}{7}}\left(\frac{\zeta}{2}\right)\right)''\right)\right)'\\ &\quad + \frac{1}{\zeta^2}x^{\frac{5}{7}}\left(\frac{\zeta}{4}\right) + \frac{1}{\zeta^2}x^{\lambda}(2\zeta) = 0,\ \zeta > \zeta_0 = 1,\end{aligned} \tag{4.5}$$

where $\alpha = 1, \nu = \frac{3}{7}, \kappa = \frac{5}{7}, \gamma = \frac{5}{7}, \lambda > 1, a(\zeta) = \frac{1}{\zeta}, p_1(\zeta) = \frac{1}{\zeta}, q(\zeta) = \frac{1}{\zeta^2} = p(\zeta)$, $p_2(\zeta) = 1, \sigma(\zeta) = \frac{\zeta}{2}, \tau(\zeta) = \frac{\zeta}{4}$ and $\omega(\zeta) = 2\zeta$. It is not difficult to see that (4.1) holds. We let $\mu(\zeta) = \frac{3}{4}\zeta$, then $\rho(\zeta) = \frac{9}{8}\zeta$. Since $A(\zeta, \zeta_0) = \int_1^{\zeta} s\, ds \simeq \frac{\zeta^2}{2}$, then all conditions of Theorem 4.1 are met, and thus every solution of (4.5) is either oscillatory or converges to zero.

Remark 4.2 We may note that [24, Theorem 2.1] is not applicable to (3.24) due to the restriction that $p_1(\zeta) = 0$ and $p(\zeta) = 0$. Apart from this, suppose that $p_1(\zeta) = 0 = p(\zeta)$ in (3.24), then it is not difficult to see that Theorem 4.1 generalised the results reported in [24]. A similar observation can be made for the papers [11, 13, 23, 25, 39].

5 Concluding Remark

In this paper, with the help of a novel comparison technique with the behaviour of first order delay and/or advanced differential equations as well as an integral criterion, several results for the oscillation and asymptotic behaviour of solutions of Eq. (1.1) are presented. As an application of the main results, Corollary 3.1, as well as some examples, are then presented. Articles [13, 16, 17, 24, 27, 32, 33, 43–45] are concerned with the asymptotic behaviour and oscillation of solutions to third/odd order neutral differential equations, which is a topic very close to our investigations but does not compliment our findings. We present our findings in a way that is essentially new and has high generality. Our findings are also easily applicable to higher-order equations of the form

$$\big(a(\zeta)\big(y^{(n-1)}(\zeta)\big)^{\alpha}\big)' + q(\zeta)x^{\gamma}\big(\tau(\zeta)\big) + p(\zeta)x^{\lambda}\big(\omega(\zeta)\big) = 0, \tag{5.1}$$

where $n \in \mathbb{N}$ and $y(\zeta) = x(\zeta) + p_1(\zeta)x^{\nu}\big(\sigma(\zeta)\big) - p_2(\zeta)x^{\kappa}\big(\sigma(\zeta)\big)$. The details are left to the reader.

Secondly, in this work, we have considered third-order non-linear differential equations with mixed neutral terms in the sense of non-linearity of function, that is, sublinear and superlinear neutral terms. Therefore, following the work [39], we raise the question of whether it would be interesting to extend this work to third-order non-linear differential equations with mixed neutral terms, that is, the neutral term contains both retarded and advanced arguments. The details are left to the reader.

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Declarations

Conflict of interest The authors declare that they have no conflicts of interests.

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