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Nehari-type Oscillation Theorems for Second Order Functional Dynamic Equations

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Abstract
This paper is devoted to studying the half-linear functional dynamic equations of second-order on an unbounded above time scale $\mathbb{T}$. We present some Nehari-type oscillation criteria for a class of second-order dynamic equations. The obtained results show that there is a substantial improvement in the literature on second-order dynamic equations. We include some examples illustrating the significance of our results.

Keywords Oscillation behavior · Second order · Functional dynamic equations · Time scales

1 Introduction

In order to combine continuous and discrete analysis, Stefan Hilger [25] has proposed the theory of dynamic equations on time scales. In many applications, different types of time scales can be applied. The theory of dynamic equations includes the classical theories for the differential equations and difference equations cases, and other cases in between these classical cases. That is, we are worthy of considering the $q-$difference equations when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^\lambda : \lambda \in \mathbb{N}_0 \text{ for } q > 1\}$ which has important applications in quantum theory (see [27]), and various types of time scales such as $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$, and $\mathbb{T} = \mathbb{T}_n$, where $\mathbb{T}_n$ is the set of the harmonic numbers, can also be considered. See [1, 9, 10] for more details of time scales calculus.

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Oscillation phenomena take part in different models from real world applications; we refer to the papers [23, 30] for models from mathematical biology where oscillation and/or delay actions may be formulated by means of cross-diffusion terms. The study of half-linear dynamic equations is dealt with in this paper because these equations arise in various real-world problems such as non-Newtonian fluid theory, the turbulent flow of a polytrophic gas in a porous medium, and in the study of $p-$Laplace equations; see, e.g., the papers [4, 7, 8, 12, 29, 37] for more details. Therefore, we are concerned with the behavior of the oscillatory solutions to the half-linear functional dynamic equation of second-order

$$\left[a(t)\phi_\beta\left(y^\Delta(t)\right)\right]^\Delta + b(t)\phi_\beta\left(y(k(t))\right) = 0 \tag{1}$$

on an arbitrary unbounded above time scale $\mathbb{T}$, where $t \in [t_0, \infty)_\mathbb{T} := [t_0, \infty) \cap \mathbb{T}$, $t_0 \geq 0$, $t_0 \in \mathbb{T}$, $\phi_\beta(u) := |u|^{\beta-1} u$, $\beta > 0$, b is a positive rd-continuous function on $\mathbb{T}$, $k : \mathbb{T} \to \mathbb{T}$ is a rd-continuous function satisfying $\lim_{t\to\infty} k(t) = \infty$, and a is a positive rd-continuous function on $\mathbb{T}$ such that $a^\Delta \geq 0$ such that $\int_{t_0}^{\infty} a^{-\frac{1}{\beta}}(\tau)\Delta\tau = \infty$.

By a solution of Eq. (1) we mean a nontrivial real–valued function $y \in \mathrm{C}^1_{\mathrm{rd}}[t_y, \infty)_\mathbb{T}$ for some $t_y \geq t_0$ with $t_0 \in \mathbb{T}$ such that $y^\Delta, a(t)\phi_\beta\left(y^\Delta(t)\right) \in \mathrm{C}^1_{\mathrm{rd}}[t_y, \infty)_\mathbb{T}$ and $y(t)$ satisfies Eq. (1) on $[t_y, \infty)_\mathbb{T}$, where C_{rd} is the space of right-dense continuous functions. It may be noted that in a particular case when $\mathbb{T} = \mathbb{R}$ then

$$\mu(t) = 0,\ \eta^\Delta(t) = \eta'(t),\ \int_a^b \eta(t)\Delta t = \int_a^b \eta(t)dt,$$

and the equation (1) becomes the half-linear differential equation

$$\left[a(t)\phi_\beta\left(y'(t)\right)\right]' + b(t)\phi_\beta\left(y(k(t))\right) = 0. \tag{2}$$

The oscillation properties of special cases of equation (2) are investigated by Nehari [32] as follows: every solution of the linear differential equation

$$y''(t) + b(t)y(t) = 0, \tag{3}$$

is oscillatory if

$$\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \tau^2 b(\tau)\mathrm{d}\tau > \frac{1}{4}, \tag{4}$$

We will show that our results not only extend some of the known oscillation results for differential equations, but we can also perform these results on other cases in which the oscillatory behaviour of solutions to these equations on various types of time scales is not known. Notice that, if $\mathbb{T} = \mathbb{Z}$, then

$$\mu(t) = 1,\ \eta^\Delta(t) = \Delta\eta(t),\ \int_a^b \eta(t)\Delta t = \sum_{t=a}^{b-1} \eta(t),$$

and (1) becomes the half-linear difference equation

$$\Delta\left[a(t)\phi_\beta\left(\Delta y(t)\right)\right]+b(t)\phi_\beta\left(y(k(t))\right)=0. \tag{5}$$

If $\mathbb{T}=h\mathbb{Z}$, $h>0$, thus

$$\mu(t)=h,\ \eta^\Delta(t)=\Delta_h\eta(t)=\frac{\eta(t+h)-\eta(t)}{h},$$
$$\int_a^b\eta(t)\Delta t=\sum_{k=0}^{\frac{b-a-h}{h}}\eta(a+kh)h,$$

and (1) gets the half-linear difference equation

$$\Delta_h\left[a(t)\phi_\beta\left(\Delta_h y(t)\right)\right]+b(t)\phi_\beta\left(y(k(t))\right)=0. \tag{6}$$

If

$$\mathbb{T}=q^{\mathbb{N}_0}=\{t:t=q^k,\ k\in\mathbb{N}_0,\ q>1\},$$

then

$$\mu(t)=(q-1)t,\ \eta^\Delta(t)=\Delta_q\eta(t)=\frac{y(q\,t)-y(t)}{(q-1)\,t},$$
$$\int_{t_0}^\infty\eta(t)\Delta t=\sum_{k=n_0}^\infty\eta(q^k)\mu(q^k),$$

where $t_0=q^{n_0}$, and (1) becomes the half-linear $q-$difference equation

$$\Delta_q\left[a(t)\phi_\beta\left(\Delta_q y(t)\right)\right]+b(t)\phi_\beta\left(y(k(t))\right)=0. \tag{7}$$

If

$$\mathbb{T}=\mathbb{N}_0^2:=\{n^2:n\in\mathbb{N}_0\},$$

then

$$\mu(t)=1+2\sqrt{t},\ \Delta_q\eta(t)=\frac{\eta((\sqrt{t}+1)^2)-\eta(t)}{1+2\sqrt{t}},$$

and (1) converts to the half-linear difference equation

$$\Delta_N\left[a(t)\phi_\beta\left(\Delta_N y(t)\right)\right]+b(t)\phi_\beta\left(y(k(t))\right)=0. \tag{8}$$

If $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$ where H_n is the harmonic numbers defined by

$$H_0 = 0,\ H_n = \sum_{k=1}^{n} \frac{1}{k},\ n \in \mathbb{N},$$

then

$$\mu(H_n) = \frac{1}{n+1},\ \eta^{\Delta}(t) = \Delta_{H_n}\eta(H_n) = (n+1)\Delta\eta(H_n),$$

and (1) becomes the half-linear harmonic difference equation

$$\Delta_{H_n}\left[a(H_n)\phi_\beta\left(\Delta_{H_n}y(H_n)\right)\right] + b(H_n)\phi_\beta\left(y(k(H_n))\right) = 0. \tag{9}$$

For Nehari-type oscillation criteria of second-order dynamic equations, Erbe et al. [20] examined the nonlinear dynamic equation

$$\left(\left(y^{\Delta}(t)\right)^{\beta}\right)^{\Delta} + b(t)y^{\beta}(k(t)) = 0, \tag{10}$$

where $\beta \geq 1$ is a quotient of odd positive integers and $k(t) \leq t$ for $t \in \mathbb{T}$ and showed that every solution of (10) is oscillatory, if

$$\int_{t_0}^{\infty} k^{\beta}(\tau)b(\tau)\Delta\tau = \infty \tag{11}$$

and

$$\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \tau^{\beta+1}\left(\frac{k(\tau)}{\sigma(\tau)}\right)^{\beta} b(\tau)\Delta\tau + \liminf_{t\to\infty} t^{\beta}\int_{\sigma(t)}^{\infty}\left(\frac{k(\tau)}{\sigma(\tau)}\right)^{\beta} b(\tau)\Delta\tau > \frac{1}{l^{\beta(\beta+1)}}, \tag{12}$$

where $l := \liminf_{t\to\infty} t/\sigma(t) > 0$. Erbe et al. [21] investigated Nehari-type oscillation criterion for the half-linear dynamic equation

$$\left(a(t)\left(y^{\Delta}(t)\right)^{\beta}\right)^{\Delta} + b(t)y^{\beta}(k(t)) = 0, \tag{13}$$

where $0 < \beta \leq 1$ is a quotient of odd positive integers, $a^{\Delta} \geq 0$, and $k(t) \leq t$ for $t \in \mathbb{T}$ and proved that every solution of (13) is oscillatory, if (11) holds,

$$\int_{t_0}^{\infty} a^{-\frac{1}{\beta}}(\tau)\Delta\tau = \infty, \tag{14}$$

and

$$\liminf_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\frac{\tau^{\beta+1}}{a(\tau)}\left(\frac{k(\tau)}{\sigma(\tau)}\right)^{\beta}b(\tau)\Delta\tau+\liminf_{t\to\infty}\frac{t^{\beta}}{a(t)}\int_{\sigma(t)}^{\infty}\left(\frac{k(\tau)}{\sigma(\tau)}\right)^{\beta}b(\tau)\Delta\tau>\frac{1}{l^{\beta(\beta+1)}}, \tag{15}$$

where $l := \liminf_{t\to\infty} t/\sigma(t) > 0$.

We refer the reader to associated results [2, 5–8, 11, 14, 16, 19, 22, 24, 31, 33–35, 38] and the references cited therein. It may be noted that the contributions of Nehari [32] strongly motivated research in this paper. The objective of this paper is to conclude some Nehari-type oscillation criteria for Eq. (1) in the cases where $k(t) \leq t$ and $k(t) \geq t$. Besides, we reference that, contrary to [20, 21], a restrictive condition (11) is not needed in our oscillation theorems, and also, our results can function for any positive real numbers β. All functional inequalities deemed in the sequel are tacitly supposed to hold eventually. That is, they are satisfied for all sufficiently large t.

2 Main Results

We start this section with the following introductory lemmas.

Lemma 1 ([12, Lemma 2.1]) *Suppose that* (14) *holds. If y is a positive solution of Eq.* (1) *on* $[t_0,\infty)_{\mathbb{T}}$, *then*

$$y^{\Delta}(t)>0 \quad and \quad \left[a(t)\phi_{\beta}\left(y^{\Delta}(t)\right)\right]^{\Delta}<0$$

eventually.

Lemma 2 ([12, Lemma 2.2]) *If*

$$y(t)>0, \qquad y^{\Delta}(t)>0, \qquad \left[a(t)\phi_{\beta}\left(y^{\Delta}(t)\right)\right]^{\Delta}\leq 0 \qquad on\ [t_0,\infty)_{\mathbb{T}},$$

then $\dfrac{y(t)}{t-t_0}$ *is strictly decreasing on* $(t_0,\infty)_{\mathbb{T}}$.

In the sequel we will use the following notations $l := \liminf_{t\to\infty}\dfrac{t}{\sigma(t)}$ and

$$\varphi(t):=\begin{cases}1, & k(t)\geq t,\\ \left[\dfrac{k(t)}{t}\right]^{\beta}, & k(t)\leq t.\end{cases}$$

Theorem 1 *Suppose that* (14) *holds. If $l > 0$ and*

$$\liminf_{t\to\infty}\frac{1}{t}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau>\frac{1}{l^{\beta(\beta+1)+1}}\left(1-\frac{l^{\beta+1}}{\beta+1}\right), \tag{16}$$

for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$, then all solutions of Eq. (1) are oscillatory.

Proof Assume y is a nonoscillatory solution of Eq. (1) on $[t_0, \infty)_{\mathbb{T}}$. Then, let $y(t) > 0$ and $y(k(t)) > 0$ on $[t_0, \infty)_{\mathbb{T}}$, without loss of generality. From Lemma 1, we see that

$$\left[a(t)\phi_\beta\left(y^\Delta(t)\right)\right]^\Delta < 0 \text{ and } y^\Delta(t) > 0 \quad \text{for } t \geq t_0.$$

Define

$$x(t) := \frac{a(t)\phi_\beta\left(y^\Delta(t)\right)}{y^\beta(t)}. \tag{17}$$

By the rules of product and quotient, we get

$$\begin{aligned} x^\Delta(t) &= \left(\frac{a(t)\phi_\beta\left(y^\Delta(t)\right)}{y^\beta(t)}\right)^\Delta \\ &= \frac{1}{y^\beta(t)}\left[a(t)\phi_\beta\left(y^\Delta(t)\right)\right]^\Delta \\ &\quad + \left(\frac{1}{y^\beta(t)}\right)^\Delta \left[a(t)\phi_\beta\left(y^\Delta(t)\right)\right]^\sigma \\ &= \frac{\left[a(t)\phi_\beta\left(y^\Delta(t)\right)\right]^\Delta}{y^\beta(t)} - \frac{(y^\beta(t))^\Delta}{y^\beta(t)y^\beta(\sigma(t))}\left[a(t)\phi_\beta\left(y^\Delta(t)\right)\right]^\sigma. \end{aligned} \tag{18}$$

From (1) and the definition of $x(t)$, we have

$$x^\Delta(t) = -b(t)\left(\frac{y\left(k(t)\right)}{y(t)}\right)^\beta - \frac{(y^\beta(t))^\Delta}{y^\beta(t)}x\left(\sigma(t)\right).$$

Let $t \in [t_0, \infty)_{\mathbb{T}}$ be fixed. When $k(t) \leq t$, in view of Lemma 2, $\left(\frac{y(t)}{t-t_0}\right)^\Delta < 0$ on $(t_0, \infty)_{\mathbb{T}}$, we obtain

$$\frac{y(k(t))}{y(t)} \geq \frac{k(t)-t_0}{t} \text{ for } t \geq k(t) > t_0.$$

Then there exists $t_\lambda \in [t_0, \infty)_{\mathbb{T}}$, for each $0 < \lambda < 1$, such that

$$y(k(t)) \geq \lambda\frac{k(t)}{t}y(t) \text{ for } t \geq t_\lambda.$$

If $k(t) \geq t$, then $y(k(t)) \geq y(t) > \lambda y(t)$ for $t \geq t_\lambda$. In both cases, from the definition of $\varphi(t)$ we have that

$$y^\beta(k(t)) \geq \lambda^\beta\varphi(t)y^\beta(t) \text{ for } t \geq t_\lambda. \tag{19}$$

Therefore

$$x^{\Delta}(t) \leq -\lambda^{\beta}\varphi(t)b(t) - \frac{(y^{\beta}(t))^{\Delta}}{y^{\beta}(t)}x(\sigma(t)) \qquad \text{for } t \in [t_{\lambda}, \infty)_{\mathbb{T}}. \tag{20}$$

Using the Pötzsche chain rule to get

$$\begin{aligned}(y^{\beta}(t))^{\Delta} &= \beta\left(\int_0^1 \left[y(t) + h\mu(t)y^{\Delta}(t)\right]^{\beta-1} dh\right) y^{\Delta}(t) \\ &= \beta\left(\int_0^1 \left[(1-h)\,y(t) + hy\left(\sigma(t)\right)\right]^{\beta-1} dh\right) y^{\Delta}(t) \\ &> \begin{cases} \beta y^{\beta-1}\left(\sigma(t)\right) y^{\Delta}(t), & 0 < \beta \leq 1, \\ \beta y^{\beta-1}(t) y^{\Delta}(t), & \beta \geq 1. \end{cases}\end{aligned}$$

If $0 < \beta \leq 1$, then

$$x^{\Delta}(t) < -\lambda^{\beta}\varphi(t)b(t) - \beta\frac{y^{\Delta}(t)}{y(\sigma(t))}\left(\frac{y(\sigma(t))}{y(t)}\right)^{\beta} x(\sigma(t));$$

and if $\beta \geq 1$, then

$$x^{\Delta}(t) \leq -\lambda^{\beta}\varphi(t)b(t) - \beta\frac{y^{\Delta}(t)}{y(\sigma(t))}\frac{y(\sigma(t))}{y(t)}x(\sigma(t)).$$

Note that $y(t)$ is strictly increasing and $a^{\frac{1}{\beta}}y^{\Delta}$ is strictly decreasing, we see that for $\beta > 0$ and $t \in [t_{\lambda}, \infty)_{\mathbb{T}}$,

$$\begin{aligned}x^{\Delta}(t) &\leq -\lambda^{\beta}\varphi(t)b(t) - \beta\frac{y^{\Delta}(t)}{y(\sigma(t))}x(\sigma(t)) \\ &= -\lambda^{\beta}\varphi(t)b(t) - \beta a^{-\frac{1}{\beta}}(t)x^{\frac{\beta+1}{\beta}}(\sigma(t)).\end{aligned} \tag{21}$$

Multiplying by $\dfrac{t^{\beta+1}}{a(t)}$ and integrating from T to $\sigma(t) \in [t_{\lambda}, \infty)_{\mathbb{T}}$, we obtain

$$\int_T^{\sigma(t)} \frac{\tau^{\beta+1}}{a(\tau)}x^{\Delta}(\tau)\Delta\tau \leq -\lambda^{\beta}\int_T^{\sigma(t)} \frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau - \beta\int_T^{\sigma(t)} \left[\frac{\tau^{\beta}x^{\sigma}(\tau)}{a(\tau)}\right]^{\frac{\beta+1}{\beta}}\Delta\tau. \tag{22}$$

Now for any $\varepsilon > 0$, there exists $t \geq t_{\lambda}$ such that

$$\frac{t}{\sigma(t)} \geq l - \varepsilon \text{ and } t^{\beta}\left(\frac{x(t)}{a(t)}\right)^{\sigma} \geq a_{*} - \varepsilon \qquad \text{for } t \in [t_{\lambda}, \infty)_{\mathbb{T}}, \tag{23}$$

where

$$l=\liminf_{t\to\infty}\frac{t}{\sigma(t)}\text{ and }a_{*}=\liminf_{t\to\infty}t^{\beta}\left(\frac{x\left(t\right)}{a(t)}\right)^{\sigma}.$$

It follows from (22) that

$$\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}x^{\Delta}(\tau)\Delta\tau\leq-\lambda^{\beta}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau$$
$$-\beta\left(l-\varepsilon\right)^{\beta+1}\int_{T}^{\sigma(t)}\left[\left(\frac{\tau^{\beta}}{a(\tau)}x(\tau)\right)^{\sigma}\right]^{\frac{\beta+1}{\beta}}\Delta\tau.$$

Using integration by parts, we obtain

$$\begin{aligned}\left(\frac{t^{\beta+1}}{a(t)}x(t)\right)^{\sigma}\leq&\frac{T^{\beta+1}}{a(T)}x(T)-\lambda^{\beta}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau\\&+\int_{T}^{\sigma(t)}\left(\frac{\tau^{\beta+1}}{a(\tau)}\right)^{\Delta}x^{\sigma}\left(\tau\right)\Delta\tau\\&-\beta\left(l-\varepsilon\right)^{\beta+1}\int_{T}^{\sigma(t)}\left[\left(\frac{\tau^{\beta}}{a(\tau)}x(\tau)\right)^{\sigma}\right]^{\frac{\beta+1}{\beta}}\Delta\tau.\end{aligned}\tag{24}$$

Utilizing the quotient rule and applying the Pötzsche chain rule, we see

$$\left(\frac{\tau^{\beta+1}}{a(\tau)}\right)^{\Delta}=\frac{\left(\tau^{\beta+1}\right)^{\Delta}}{a^{\sigma}(\tau)}-\frac{\tau^{\beta+1}a^{\Delta}(\tau)}{a(\tau)a^{\sigma}(\tau)}\tag{25}$$
$$\leq\frac{\left(\tau^{\beta+1}\right)^{\Delta}}{a^{\sigma}(\tau)}\tag{26}$$
$$\leq(\beta+1)\left(\frac{\tau^{\beta}}{a(\tau)}\right)^{\sigma}\tag{27}$$
$$\leq(\beta+1)\frac{\sigma^{\beta}\left(\tau\right)}{a(\tau)}.\tag{28}$$

Hence

$$\begin{aligned}\left(\frac{t^{\beta+1}}{a(t)}x(t)\right)^{\sigma}\leq&\frac{T^{\beta+1}}{a(T)}x(T)-\lambda^{\beta}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau\\&+\int_{T}^{\sigma(t)}\Bigg[(\beta+1)\left(\frac{\tau^{\beta}}{a(\tau)}x(\tau)\right)^{\sigma}\\&-\beta\left(l-\varepsilon\right)^{\beta+1}\left[\left(\frac{\tau^{\beta}}{a(\tau)}x(\tau)\right)^{\sigma}\right]^{\frac{\beta+1}{\beta}}\Bigg]\Delta\tau.\end{aligned}$$

Using the inequality

$$Bu - Au^{\frac{\beta+1}{\beta}} \leq \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}} \frac{B^{\beta+1}}{a^{\beta}}, \tag{29}$$

with $A = \beta\,(l-\varepsilon)^{\beta+1}$, $B = \beta + 1$ and $u = \left(\frac{\tau^{\beta}}{a(\tau)} x(\tau)\right)^{\sigma}$, we obtain

$$\begin{aligned} \left(\frac{t^{\beta+1}}{a(t)} x(t)\right)^{\sigma} \leq & \frac{T^{\beta+1}}{a(T)} x(T) - \lambda^{\beta} \int_{T}^{\sigma(t)} \frac{\tau^{\beta+1}}{a(\tau)} \varphi(\tau) b(\tau) \Delta\tau \\ & + \frac{1}{(l-\varepsilon)^{\beta(\beta+1)}} \left(\sigma\,(t) - T\right). \end{aligned} \tag{30}$$

Dividing by t, we get

$$\begin{aligned} t^{\beta} \left(\frac{x(t)}{a(t)}\right)^{\sigma} & \leq \frac{1}{t} \left(\frac{t^{\beta+1}}{a(t)} x(t)\right)^{\sigma} \\ & \leq \frac{\frac{T^{\beta+1}}{a(T)} x(T)}{t} - \frac{\lambda^{\beta}}{t} \int_{T}^{\sigma(t)} \frac{\tau^{\beta+1}}{a(\tau)} \varphi(\tau) b(\tau) \Delta\tau \\ & \quad + \frac{1}{(l-\varepsilon)^{\beta(\beta+1)}} \left(\frac{\sigma\,(t)}{t} - \frac{T}{t}\right) \\ & \leq \frac{\frac{T^{\beta+1}}{a(T)} x(T)}{t} - \frac{\lambda^{\beta}}{t} \int_{T}^{\sigma(t)} \frac{\tau^{\beta+1}}{a(\tau)} \varphi(\tau) b(\tau) \Delta\tau \\ & \quad + \frac{1}{(l-\varepsilon)^{\beta(\beta+1)}} \left(\frac{1}{l-\varepsilon} - \frac{T}{t}\right). \end{aligned}$$

Taking the lim sup as $t \to \infty$ to get

$$R \leq -\liminf_{t\to\infty} \frac{\lambda^{\beta}}{t} \int_{T}^{\sigma(t)} \frac{\tau^{\beta+1}}{a(\tau)} \varphi(\tau) b(\tau) \Delta\tau + \frac{1}{(l-\varepsilon)^{\beta(\beta+1)+1}},$$

where

$$R := \limsup_{t\to\infty} t^{\beta} \left(\frac{x(t)}{a(t)}\right)^{\sigma}. \tag{31}$$

Since $\varepsilon > 0$ and $0 < \lambda < 1$ are arbitrary, we get

$$R \leq -\liminf_{t\to\infty} \frac{1}{t} \int_{T}^{\sigma(t)} \frac{\tau^{\beta+1}}{a(\tau)} \varphi(\tau) b(\tau) \Delta\tau + \frac{1}{l^{\beta(\beta+1)+1}}. \tag{32}$$

Multiplying both sides of (21) by $\frac{t^{\beta+1}}{a(t)}$, we obtain

$$\begin{aligned}\frac{t^{\beta+1}}{a(t)}x^{\Delta}(t) &\leq -\lambda^{\beta}\frac{t^{\beta+1}}{a(t)}\varphi(t)b(t)-\beta\left(\frac{t^{\beta}}{a(t)}x^{\sigma}(t)\right)^{\frac{\beta+1}{\beta}}\\ &\leq -\lambda^{\beta}\frac{t^{\beta+1}}{a(t)}\varphi(t)b(t)-\beta\left(t^{\beta}\left(\frac{x(t)}{a(t)}\right)^{\sigma}\right)^{\frac{\beta+1}{\beta}}.\end{aligned}$$

Using (23) gives

$$\frac{t^{\beta+1}}{a(t)}x^{\Delta}(t) \leq -\lambda^{\beta}\frac{t^{\beta+1}}{a(t)}\varphi(t)b(t)-\beta\left(a_{*}-\varepsilon\right)^{\frac{\beta+1}{\beta}} \text{ for } t\in[t_{\lambda},\infty)_{\mathbb{T}}. \tag{33}$$

Integrating (33) from T to $\sigma(t)\in[t_{\lambda},\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}x^{\Delta}(\tau)\Delta\tau &\leq -\lambda^{\beta}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau\\ &\quad -\beta\left(a_{*}-\varepsilon\right)^{\frac{\beta+1}{\beta}}\left(\sigma(t)-T\right).\end{aligned}$$

By integrating by parts, we conclude that

$$\begin{aligned}\left(\frac{t^{\beta+1}}{a(t)}x(t)\right)^{\sigma} &\leq \frac{T^{\beta+1}}{a(T)}x(T)+\int_{T}^{\sigma(t)}\left(\frac{\tau^{\beta+1}}{a(\tau)}\right)^{\Delta}x^{\sigma}(\tau)\Delta\tau\\ &\quad -\lambda^{\beta}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau-\beta\left(a_{*}-\varepsilon\right)^{\frac{\beta+1}{\beta}}\left(\sigma(t)-T\right).\end{aligned}$$

By using (27) , we get

$$\begin{aligned}\left(\frac{t^{\beta+1}}{a(t)}x(t)\right)^{\sigma} &\leq \frac{T^{\beta+1}}{a(T)}x(T)+(\beta+1)\int_{T}^{\sigma(t)}\left(\frac{\tau^{\beta}}{a(\tau)}x(\tau)\right)^{\sigma}\Delta\tau\\ &\quad -\lambda^{\beta}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau\\ &\quad -\beta\left(a_{*}-\varepsilon\right)^{\frac{\beta+1}{\beta}}\left(\sigma(t)-T\right)\\ &\leq \frac{T^{\beta+1}}{a(T)}x(T)+(\beta+1)\frac{R+\varepsilon}{(l-\varepsilon)^{\beta}}\left[\sigma(t)-T\right]\\ &\quad -\lambda^{\beta}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau\\ &\quad -\beta\left(a_{*}-\varepsilon\right)^{\frac{\beta+1}{\beta}}\left(\sigma(t)-T\right).\end{aligned} \tag{34}$$

Dividing both sides by t, we have

$$t^{\beta}\left(\frac{x(t)}{a(t)}\right)^{\sigma} \leq \frac{1}{t}\left(\frac{t^{\beta+1}}{a(t)}x(t)\right)^{\sigma} \leq \frac{\frac{T^{\beta+1}}{a(T)}x(T)}{t} + (\beta+1)\frac{R+\varepsilon}{(l-\varepsilon)^{\beta}}\left[\frac{\sigma(t)}{t} - \frac{T}{t}\right]$$
$$-\frac{\lambda^{\beta}}{t}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau - \beta\left(a_{*}-\varepsilon\right)^{\frac{\beta+1}{\beta}}\left[\frac{\sigma(t)}{t} - \frac{T}{t}\right]$$
$$\leq \frac{\frac{T^{\beta+1}}{a(T)}x(T)}{t} + (\beta+1)\frac{R+\varepsilon}{(l-\varepsilon)^{\beta}}\left[\frac{1}{l-\varepsilon} - \frac{T}{t}\right]$$
$$-\frac{\lambda^{\beta}}{t}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau - \beta\left(a_{*}-\varepsilon\right)^{\frac{\beta+1}{\beta}}\left[1 - \frac{T}{t}\right].$$

Taking the lim sup as $t \to \infty$ and using (31) , we get

$$R \leq (\beta+1)\frac{R+\varepsilon}{(l-\varepsilon)^{\beta+1}} - \liminf_{t\to\infty}\frac{\lambda^{\beta}}{t}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau - \beta\left(a_{*}-\varepsilon\right)^{\frac{\beta+1}{\beta}}.$$

Since $\varepsilon > 0$ and $0 < \lambda < 1$ are arbitrary, we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau \leq \frac{R}{l^{\beta+1}}\left(\beta+1-l^{\beta+1}\right) - \beta a_{*}^{\frac{\beta+1}{\beta}}. \qquad (35)$$

Substituting (32) into (35) , we achieve

$$\liminf_{t\to\infty}\frac{1}{t}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau \leq \frac{1}{l^{\beta(\beta+1)+1}}\left(1-\frac{l^{\beta+1}}{\beta+1}\right),$$

which contradicts the condition (16) . The proof is completed. □

Theorem 2 *Suppose that* (14) *holds. If* $l > 0$ *and*

$$\liminf_{t\to\infty}\frac{1}{\sigma(t)}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau > \frac{1}{l^{\beta(\beta+1)}}\left(1-\frac{l^{\beta}}{\beta+1}\right), \qquad (36)$$

for sufficiently large $T \in [t_0,\infty)_{\mathbb{T}}$, *then all solutions of Eq.* (1) *are oscillatory.*

Proof Assume y is a nonoscillatory solution of Eq. (1) on $[t_0,\infty)_{\mathbb{T}}$. Then, let $y(t) > 0$ and $y(\lambda(t)) > 0$ on $[t_0,\infty)_{\mathbb{T}}$, without loss of generality. From Lemma 1 , we see that

$$\left[a(t)\phi_{\beta}\left(y^{\Delta}(t)\right)\right]^{\Delta} < 0 \text{ and } y^{\Delta}(t) > 0 \quad \text{for } t \geq t_0.$$

As shown in the proof of Theorem 1, (30) and (34) hold for sufficiently large $t \in [t_0, \infty)_{\mathbb{T}}$. Dividing both sides of (30) by $\sigma(t)$, we obtain

$$\begin{aligned} t^{\beta}\left(\frac{x(t)}{a(t)}\right)^{\sigma} &\leq \left(\frac{t^{\beta}}{a(t)}x(t)\right)^{\sigma} \leq \frac{\frac{T^{\beta+1}}{a(T)}x(T)}{\sigma(t)} - \frac{\lambda^{\beta}}{\sigma(t)}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau \\ &\quad + \frac{1}{(l-\varepsilon)^{\beta(\beta+1)}}\left(1-\frac{T}{\sigma(t)}\right) \\ &\leq \frac{\frac{T^{\beta+1}}{a(T)}x(T)}{\sigma(t)} - \frac{\lambda^{\beta}}{\sigma(t)}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau \\ &\quad + \frac{1}{(l-\varepsilon)^{\beta(\beta+1)}}\left(1-\frac{T}{\sigma(t)}\right). \end{aligned}$$

Taking the lim sup as $t \to \infty$ to obtain

$$R \leq -\liminf_{t\to\infty}\frac{\lambda^{\beta}}{\sigma(t)}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau + \frac{1}{(l-\varepsilon)^{\beta(\beta+1)}}.$$

Since $\varepsilon > 0$ and $0 < \lambda < 1$ are arbitrary, we get inequality

$$R \leq -\liminf_{t\to\infty}\frac{1}{\sigma(t)}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau + \frac{1}{l^{\beta(\beta+1)}}. \tag{37}$$

Dividing both sides by of (34) $\sigma(t)$, we obtain

$$\begin{aligned} t^{\beta}\left(\frac{x(t)}{a(t)}\right)^{\sigma} &\leq \left(\frac{t^{\beta}x(t)}{a(t)}\right)^{\sigma} \\ &\leq \frac{\frac{T^{\beta+1}}{a(T)}x(T)}{\sigma(t)} + (\beta+1)\frac{R+\varepsilon}{(l-\varepsilon)^{\beta}}\left[1-\frac{T}{\sigma(t)}\right] \\ &\quad - \frac{\lambda^{\beta}}{\sigma(t)}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau - \beta\left(a_{*}-\varepsilon\right)^{\frac{\beta+1}{\beta}}\left[1-\frac{T}{\sigma(t)}\right]. \end{aligned}$$

Taking the lim sup as $t \to \infty$ and utilizing (31), we see

$$R \leq (\beta+1)\frac{R+\varepsilon}{(l-\varepsilon)^{\beta}} - \liminf_{t\to\infty}\frac{\lambda^{\beta}}{\sigma(t)}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau - \beta\left(a_{*}-\varepsilon\right)^{\frac{\beta+1}{\beta}}.$$

Since $\varepsilon > 0$ and $0 < \lambda < 1$ are arbitrary, we have

$$\liminf_{t\to\infty}\frac{1}{\sigma(t)}\int_{T}^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau \leq \frac{R}{l^{\beta}}\left(\beta+1-l^{\beta}\right) - \beta a_{*}^{\frac{\beta+1}{\beta}}. \tag{38}$$

Substituting (37) into (38) , we achieve

$$\liminf_{t\to\infty}\frac{1}{\sigma(t)}\int_T^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau\le\frac{1}{l^{\beta(\beta+1)}}\left(1-\frac{l^\beta}{\beta+1}\right),$$

which contradicts the condition (36) . The proof is completed. □

Example 1 Consider the dynamic equations of second-order for $t\in[t_0,\infty)_{\mathbb{T}}$,

$$y^{\Delta\Delta}(t)+\frac{\alpha}{t^2}y(t)=0 \tag{39}$$

and

$$y^{\Delta\Delta}(t)+\frac{\alpha}{t^2}y(\sigma(t))=0, \tag{40}$$

where $l=\liminf_{t\to\infty}t/\sigma(t)>0$ and $\alpha>0$ is a constant. It is not difficult to derive that all solutions of (39) and (40) are oscillatory if $\alpha>\frac{1}{l^3}\left(1-\frac{l^2}{2}\right)$ or $\alpha>\frac{1}{l^2}\left(1-\frac{l}{2}\right)$ by using Theorems 1 and 2 respectively.

Theorem 3 *Suppose that* (14) *holds. If* $l>0$ *and*

$$\liminf_{t\to\infty}\frac{1}{t}\int_T^{t}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau>\frac{1}{l^{\beta(\beta+1)}}\left(1-\frac{l^\beta}{\beta+1}\right), \tag{41}$$

for sufficiently large $T\in[t_0,\infty)_{\mathbb{T}}$, *then all solutions of Eq.* (1) *are oscillatory.*

Proof Assume y is a nonoscillatory solution of Eq. (1) on $[t_0,\infty)_{\mathbb{T}}$. Then, let $y(t)>0$ and $y(k(t))>0$ on $[t_0,\infty)_{\mathbb{T}}$, without loss of generality. From Lemma 1, we see that

$$\left[a(t)\phi_\beta\left(y^\Delta(t)\right)\right]^\Delta<0 \text{ and } y^\Delta(t)>0 \quad \text{for } t\ge t_0.$$

As shown in the proof of Theorem 1, (21) holds for sufficiently large $t\in[t_0,\infty)_{\mathbb{T}}$. Multiply (21) by $\dfrac{t^{\beta+1}}{a(t)}$ and integrating from T to $t\in[t_\lambda,\infty)_{\mathbb{T}}$, we get

$$\begin{aligned}\int_T^t\frac{\tau^{\beta+1}}{a(\tau)}x^\Delta(\tau)\Delta\tau&\le-\lambda^\beta\int_T^t\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau-\beta\int_T^t\left[\frac{\tau^\beta x^\sigma(\tau)}{a(\tau)}\right]^{\frac{\beta+1}{\beta}}\Delta\tau\\&\le-\lambda^\beta\int_T^t\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau-\beta\int_T^t\left[\tau^\beta\left(\frac{x(\tau)}{a(\tau)}\right)^\sigma\right]^{\frac{\beta+1}{\beta}}\Delta\tau.\end{aligned}$$

Progressing as in the proof of Theorem 1, we arrive that

$$\frac{t^{\beta+1}}{a(t)}x(t) \leq \frac{T^{\beta+1}}{a(T)}x(T) - \lambda^{\beta}\int_T^t \frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau + \frac{1}{(l-\varepsilon)^{\beta(\beta+1)}}(t-T). \tag{42}$$

Dividing both sides by t, we obtain

$$\begin{aligned} t^{\beta}\left(\frac{x(t)}{a(t)}\right)^{\sigma} \leq \frac{t^{\beta}}{a(t)}x(t) &\leq \frac{\frac{T^{\beta+1}}{a(T)}x(T)}{t} - \frac{\lambda^{\beta}}{t}\int_T^t \frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau \\ &\quad + \frac{1}{(l-\varepsilon)^{\beta(\beta+1)}}\left(1-\frac{T}{t}\right) \\ &\leq \frac{\frac{T^{\beta+1}}{a(T)}x(T)}{t} - \frac{\lambda^{\beta}}{t}\int_T^t \frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau \\ &\quad + \frac{1}{(l-\varepsilon)^{\beta(\beta+1)}}\left(1-\frac{T}{t}\right). \end{aligned}$$

Taking the lim sup as $t \to \infty$, we see

$$R \leq -\liminf_{t\to\infty}\frac{\lambda^{\beta}}{t}\int_T^t \frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau + \frac{1}{(l-\varepsilon)^{\beta(\beta+1)}}.$$

Since $\varepsilon > 0$ and $0 < \lambda < 1$ are arbitrary, we get inequality

$$R \leq -\liminf_{t\to\infty}\frac{1}{t}\int_T^{\sigma(t)} \frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau + \frac{1}{l^{\beta(\beta+1)}}. \tag{43}$$

Again, multiplying (21) by $\frac{t^{\beta+1}}{a(t)}$, we get

$$\begin{aligned} \frac{t^{\beta+1}}{a(t)}x^{\Delta}(t) &\leq -\lambda^{\beta}\frac{t^{\beta+1}}{a(t)}\varphi(t)b(t) - \beta\left(\frac{t^{\beta}x^{\sigma}(t)}{a(t)}\right)^{\frac{\beta+1}{\beta}} \\ &\leq -\lambda^{\beta}\frac{t^{\beta+1}}{a(t)}\varphi(t)b(t) - \beta\left(\left(t^{\beta}\frac{x(t)}{a(t)}\right)^{\sigma}\right)^{\frac{\beta+1}{\beta}}\left(\frac{t}{\sigma(t)}\right)^{\beta+1}. \end{aligned} \tag{44}$$

Progressing as in the proof of Theorem 1, we arrive that

$$R \leq (\beta+1)\frac{R+\varepsilon}{(l-\varepsilon)^{\beta}} - \liminf_{t\to\infty}\frac{\lambda^{\beta}}{t}\int_T^t \frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau.$$

Since $\varepsilon > 0$ and $0 < \lambda < 1$ are arbitrary, we get

$$\liminf_{t\to\infty}\frac{1}{t}\int_T^t \frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau \leq \frac{R}{l^{\beta}}\left(\beta+1-l^{\beta}\right). \tag{45}$$

Substituting (43) into (45) , we achieve

$$\liminf_{t\to\infty} \frac{1}{t} \int_T^t \frac{\tau^{\beta+1}}{a(\tau)} \varphi(\tau) b(\tau) \Delta\tau \leq \frac{1}{l^{\beta(\beta+1)}} \left(1 - \frac{l^\beta}{\beta+1}\right),$$

which contradicts the condition (41) . The proof is completed. □

Example 2 Consider a nonlinear dynamic equation of second-order for $t \in [t_0, \infty)_{\mathbb{T}}$,

$$\left[a(t)\phi_\beta\left(y^\Delta(t)\right)\right]^\Delta + \frac{\gamma a(t)}{tk^\beta(t)} \phi_\beta\left(y(k(t))\right) = 0, \quad k(t) \leq t, \tag{46}$$

where $\gamma > 0$ is a constant and $l = \liminf_{t\to\infty} t/\sigma(t) > 0$. Let $b(t) = \gamma a(t)/(tk^\beta(t))$. Therefore,

$$\liminf_{t\to\infty} \frac{1}{t} \int_T^t \frac{\tau^{\beta+1}}{a(\tau)} \varphi(\tau) b(\tau) \Delta\tau = \liminf_{t\to\infty} \gamma \left(1 - \frac{T}{t}\right) = \gamma.$$

Employment of Theorem 3 means that every solution of (46) is oscillatory if

$$\gamma > \frac{1}{l^{\beta(\beta+1)}} \left(1 - \frac{l^\beta}{\beta+1}\right).$$

Theorem 4 *Suppose that* (14) *holds. If* $l > 0$ *and*

$$\liminf_{t\to\infty} \frac{1}{\sigma(t)} \int_T^t \frac{\tau^{\beta+1}}{a(\tau)} \varphi(\tau) b(\tau) \Delta\tau > \frac{1}{l^{\beta(\beta+1)}} \left(1 - \frac{l^{\beta+1}}{\beta+1}\right), \tag{47}$$

for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$, *then every solution of Eq.* (1) *is oscillatory.*

Proof Assume y is a nonoscillatory solution of Eq. (1) on $[t_0, \infty)_{\mathbb{T}}$. Then, let $y(t) > 0$ and $y(k(t)) > 0$ on $[t_0, \infty)_{\mathbb{T}}$, without loss of generality. From Lemma 1, we see that

$$\left[a(t)\phi_\beta\left(y^\Delta(t)\right)\right]^\Delta < 0 \text{ and } y^\Delta(t) > 0 \quad \text{for } t \geq t_0.$$

As shown in the proof of Theorem 3, (42) and (44) hold for sufficiently large $t_\lambda \in [t_0, \infty)_{\mathbb{T}}$. Dividing both sides of (42) by $\sigma(t)$, we obtain

$$\begin{aligned} t^\beta \left(\frac{x(t)}{a(t)}\right)^\sigma (l - \varepsilon) &\leq \frac{\frac{T^{\beta+1}}{a(T)} x(T)}{\sigma(t)} \\ &\leq \frac{\frac{T^{\beta+1}}{a(T)} x(T)}{\sigma(t)} - \frac{\lambda^\beta}{\sigma(t)} \int_T^t \frac{\tau^{\beta+1}}{a(\tau)} \varphi(\tau) b(\tau) \Delta\tau \\ &\quad + \frac{1}{(l-\varepsilon)^{\beta(\beta+1)}} \left(\frac{t}{\sigma(t)} - \frac{T}{\sigma(t)}\right) \end{aligned}$$

$$\leq \frac{\frac{T^{\beta+1}}{a(T)}x(T)}{\sigma(t)} - \frac{\lambda^{\beta}}{\sigma(t)}\int_{T}^{t}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau + \frac{1}{(l-\varepsilon)^{\beta(\beta+1)}}\left(1-\frac{T}{\sigma(t)}\right).$$

Taking the lim sup as $t \to \infty$ we get

$$R(l-\varepsilon) \leq -\liminf_{t\to\infty}\frac{\lambda^{\beta}}{\sigma(t)}\int_{t}^{t}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau + \frac{1}{(l-\varepsilon)^{\beta(\beta+1)}}.$$

Since $0 < \lambda < 1$ and $\varepsilon > 0$ are arbitrary, we get inequality

$$Rl \leq -\liminf_{t\to\infty}\frac{1}{\sigma(t)}\int_{T}^{t}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau + \frac{1}{l^{\beta(\beta+1)}}. \tag{48}$$

Integrating the inequality (44) from t to $t \in [t, \infty)_{\mathbb{T}}$ to obtain

$$\begin{aligned}\int_{T}^{t}\frac{\tau^{\beta+1}}{a(\tau)}x^{\Delta}(\tau)\Delta\tau &\leq -\lambda^{\beta}\int_{T}^{t}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau \\ &\quad -\beta(l-\varepsilon)^{\beta+1}(a_{*}-\varepsilon)^{\frac{\beta+1}{\beta}}(t-t) \\ &\leq -\lambda^{\beta}\int_{T}^{t}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau.\end{aligned}$$

By integrating by parts, we obtain

$$\begin{aligned}\frac{t^{\beta+1}}{a(t)}x(t) \leq \frac{T^{\beta+1}}{a(t)}x(T) &+ \int_{T}^{t}\left(\frac{\tau^{\beta+1}}{a(\tau)}\right)^{\Delta}x^{\sigma}(\tau)\Delta\tau \\ &-\lambda^{\beta}\int_{T}^{t}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau.\end{aligned}$$

By using (27) , we get

$$\begin{aligned}\frac{t^{\beta+1}}{a(t)}x(t) &\leq \frac{T^{\beta+1}}{a(T)}x(T) + (\beta+1)\int_{T}^{t}\left(\frac{\tau^{\beta}}{a(\tau)}x(\tau)\right)^{\sigma}\Delta\tau \\ &\quad -\lambda^{\beta}\int_{T}^{t}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau \\ &\leq \frac{T^{\beta+1}}{a(T)}x(T) + (\beta+1)\frac{R+\varepsilon}{(l-\varepsilon)^{\beta}}[t-T] \\ &\quad -\lambda^{\beta}\int_{T}^{t}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau.\end{aligned}$$

Dividing both sides by $\sigma(t)$, we have

$$\begin{aligned} t^{\beta}\left(\frac{x(t)}{a(t)}\right)^{\sigma}(l-\varepsilon) &\leq \frac{t^{\beta+1}x(t)}{\sigma(t)\,a(t)}(l-\varepsilon) \\ &\leq \frac{\frac{T^{\beta+1}}{a(T)}x(T)}{\sigma(t)} + (\beta+1)\frac{R+\varepsilon}{(l-\varepsilon)^{\beta}}\left[\frac{t}{\sigma(t)} - \frac{T}{\sigma(t)}\right] \\ &\quad - \frac{\lambda^{\beta}}{\sigma(t)}\int_T^t \frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau \\ &\leq \frac{\frac{T^{\beta+1}}{a(T)}x(T)}{\sigma(t)} + (\beta+1)\frac{R+\varepsilon}{(l-\varepsilon)^{\beta}}\left[1 - \frac{T}{\sigma(t)}\right] \\ &\quad - \frac{\lambda^{\beta}}{\sigma(t)}\int_T^t \frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau. \end{aligned}$$

Taking the lim sup as $t \to \infty$ and utilizing (31), we get

$$Rl \leq (\beta+1)\frac{R+\varepsilon}{(l-\varepsilon)^{\beta}} - \liminf_{t\to\infty}\frac{\lambda^{\beta}}{\sigma(t)}\int_T^t \frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau.$$

Since $\varepsilon > 0$ and $0 < \lambda < 1$ are arbitrary, we see

$$\liminf_{t\to\infty}\frac{1}{\sigma(t)}\int_T^t \frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau \leq \frac{R}{l^{\beta}}\left(\beta+1-l^{\beta+1}\right). \tag{49}$$

Substituting (48) into (49), we achieve

$$\liminf_{t\to\infty}\frac{1}{\sigma(t)}\int_T^t \frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau \leq \frac{1}{l^{\beta(\beta+1)}}\left(1-\frac{l^{\beta+1}}{\beta+1}\right),$$

which contradicts the condition (47). The proof is completed. □

3 Discussions and Conclusions

(1) In this paper, several new Nehari-type criteria are presented that can be applied to Eq. (1) are valid for various types of time scales, e.g., $\mathbb{T} = \mathbb{R}, \mathbb{T} = \mathbb{Z}, \mathbb{T} = h\mathbb{Z}$ with $h > 0$, $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, etc. (see [9]).
(2) The results in this paper are including the both cases and also we do not need to assume $k(t) \geq t$ or $k(t) \leq t$, for all sufficiently large t.
(3) We note that Theorems 2 and 3 improve Theorem 4, namely, conditions (36) and (41) improve (47); see the following details:

$$\frac{1}{\sigma(t)}\int_T^{\sigma(t)}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau \geq \frac{1}{\sigma(t)}\int_T^{t}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau$$
$$\leq \frac{1}{t}\int_T^{t}\frac{\tau^{\beta+1}}{a(\tau)}\varphi(\tau)b(\tau)\Delta\tau$$

and

$$1-\frac{l^{\beta}}{\beta+1}\leq 1-\frac{l^{\beta+1}}{\beta+1}.$$

(4) It would be interesting to extend the sharp Nehari-type criterion that the solutions of the second-order Euler differential equation $y''(t)+\frac{\gamma}{t^2}y(t)=0$ are oscillatory when $\gamma>\frac{1}{4}$ to a second-order dynamic equation, see [32].

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Declarations

Conflict of interests The authors declare that they have no competing interests. There are not any non-financial competing interests (political, personal, religious, ideological, academic, intellectual, commercial or any other) to declare in relation to this manuscript.

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