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Symbolically Computing the Shallow Water via a (2+1)-Dimensional Generalized Modified Dispersive Water-Wave System: Similarity Reductions, Scaling and Hetero-Bäcklund Transformations

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Abstract

For the water waves, people consider some dispersive systems. Describing the nonlinear and dispersive long gravity waves travelling along two horizontal directions in the shallow water of uniform depth, we now symbolically compute a (2+1)-dimensional generalized modified dispersive water-wave system. With respect to the height of the water surface and horizontal velocity of the water wave, with symbolic computation, we work out (1) a set of the scaling transformations, (2) a set of the hetero-Bäcklund transformations, from that system to a known linear partial differential equation, and (3) four sets of the similarity reductions, each of which is from that system to a known ordinary differential equation. We pay attention that our hetero-Bäcklund transformations and similarity reductions rely on the coefficients in that system.

Keywords Shallow water · Nonlinear and dispersive long gravity waves · (2+1)-dimensional generalized modified dispersive water-wave system · Symbolic computation · Scaling transformation · Hetero-Bäcklund transformations · Similarity reductions

Mathematics Subject Classification 37K35 · 37N10 · 35Q35 · 76B15

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1 Introduction

As yet, for the sake of studying the water waves (including the soliton considerations) [1–14], people have used such systems/equations as a variable-coefficient nonlinear dispersive-wave system describing the long gravity water waves in a shallow oceanic environment [9], a variable-coefficient generalized dispersive water-wave system describing the long weakly-nonlinear and weakly-dispersive surface waves of variable depth in the shallow water [10, 11], a generalized (2+1)-dimensional dispersive long-wave system describing the nonlinear and dispersive long gravity waves in two horizontal directions on the shallow water of an open sea or a wide channel of finite depth [12], types of the nonlocal Boussinesq equations in the water waves [15, 16], an extended Kadomtsev-Petviashvili equation in a fluid [17], a (3+1)-dimensional Kadomtsev-Petviashvili equation [18], (2+1)- and (3+1)-dimensional extended shallow water wave equations [19], (2+1)- and (3+1)-dimensional shallow water wave equations [20], shallow water equations from the generalized Camassa-Holm framework [21], coupled Ramani and Nizhnik-Novikov-Veselov systems [22] and nonlinear time fractional partial differential equations [23]. Other relevant systems and/or models in fluid mechanics have been reported, e.g., in Refs. [24–28].

Another example, which we purpose to investigate, is a (2+1)-dimensional generalized modified dispersive water-wave system describing the nonlinear and dispersive long gravity waves travelling along two horizontal directions in the shallow water of uniform depth, i.e.,

$$u_{yt} + \alpha u_{xxy} - 2\alpha v_{xx} - \beta u u_{xy} - \beta u_x u_y = 0, \tag{1a}$$

$$v_t - \alpha v_{xx} - \beta\,(uv)_x = 0, \tag{1b}$$

with $u(x, y, t)$ meaning the height of the water surface, $v(x, y, t)$ indicating the horizontal velocity of the water wave, both $u(x, y, t)$ and $v(x, y, t)$ being the real differentiable functions, α and β implying the real non-zero constants, while the subscripts being the partial derivatives as for the scaled space variables x, y and time variable t.

There have existed some special cases of System (1), as follows:

- when $\alpha = -1$ and $\beta = -2$, describing the nonlinear and dispersive long gravity waves travelling along two horizontal directions in the shallow water of uniform depth, a (2+1)-dimensional Broer-Kaup-Kupershmidt system [29–31], i.e.,

$$u_{yt} - u_{xxy} + 2v_{xx} + 2u u_{xy} + 2u_x u_y = 0, \tag{2a}$$

$$v_t + v_{xx} + 2\,(uv)_x = 0, \tag{2b}$$

  with $u(x, y, t)$ meaning the height of the water surface, $v(x, y, t)$ indicating the horizontal velocity of the water wave, while x, y and t being the scaled space variables and time variable, separately [29];
- when $\alpha = 1$ and $\beta = 2$, describing the nonlinear and dispersive long gravity waves travelling along two horizontal directions in the shallow water of uniform

depth, a (2+1)-dimensional modified dispersive water-wave system [32], i.e.,

$$u_{yt} + u_{xxy} - 2v_{xx} - 2uu_{xy} - 2u_x u_y = 0, \tag{3a}$$

$$v_t - v_{xx} - 2\,(uv)_x = 0 \;\; ; \tag{3b}$$

- when $\alpha = \frac{1}{2}$, $\beta = 1$ and $y = x$, describing the long waves in the shallow water, a (1+1)-dimensional Broer-Kaup system [33, 34], i.e.,

$$u_t + \frac{1}{2}u_{xx} - v_x - uu_x = 0, \tag{4a}$$

$$v_t - \frac{1}{2}v_{xx} - (uv)_x = 0, \tag{4b}$$

  with $u(x, t)$ standing for the scaled wave horizontal velocity, while $v(x, t)$ related to the wave horizontal velocity and wave height [34];
- when $\alpha = -1$, $\beta = -2$ and $y = x$, describing the long waves in the shallow water, a (1+1)-dimensional Broer-Kaup-Kupershmidt system [29, 35], i.e.,

$$u_t - u_{xx} + 2v_x + 2uu_x = 0, \tag{5a}$$

$$v_t + v_{xx} + 2\,(uv)_x = 0 \;\; . \tag{5b}$$

Using symbolic computation [36–43], we aim to construct out a set of the scaling transformations, a set of the hetero-Bäcklund transformations as well as four sets of the similarity reductions for System (1). By the way, more symbolic-computation results can be seen, e.g., in Refs. [44–54].

2 Scaling and Hetero-Bäcklund Transformations for System (1)

Scaling transformations can help us find certain assumptions, so as to make us construct, e.g., some hetero-Bäcklund transformations [55, 56] or bilinear forms [9, 56, 57].

We work out a set of the scaling transformations

$$x \to \rho^1 x, \qquad y \to \rho^{\xi} x, \qquad t \to \rho^2 t, \qquad u \to \rho^{-1} u, \qquad v \to \rho^{-1-\xi} v, \tag{6}$$

and then make the assumptions that

$$u(x, y, t) = \eta_1 w_x(x, y, t) + \eta_2, \qquad v(x, y, t) = \eta_3 w_{xy}(x, y, t), \tag{7}$$

with $\rho > 0$ standing for a positive constant, ξ implying an integer, while $\eta_1 \neq 0$, η_2 and $\eta_3 \neq 0$ indicating three real constants.

We employ symbolic computation and Assumptions (7), integrate Eq. (1b) once in relation to x and y, separately, with the integration functions vanishing and decide on

$$\alpha = \frac{1}{2}\beta\eta_1, \tag{8}$$

to get the following Bell-polynomial expression:

$$Y_t(w) - \alpha Y_{2x}(w) - \beta \eta_2 Y_x(w) = 0, \tag{9}$$

with the Bell polynomials defined as [58, 59]

$$Y_{mx,ry,nt}(w) \equiv Y_{m,r,n}(w_{0,0,0}, \cdots, w_{0,0,n}, \cdots, w_{0,r,0}, \cdots, w_{0,r,n}, \cdots, \\ w_{m,r,0}, \cdots, w_{m,r,n}) = e^{-w} \partial_x^m \partial_y^r \partial_t^n e^w,$$

$w(x, y, t)$ as a C^∞ function with respect of x, y and t, $w_{k,g,l} = \partial_x^k \partial_y^g \partial_t^l w$ ($k = 0, \cdots, m$, $g = 0, \cdots, r$, $l = 0, \cdots, n$), while m, r and n as three non-negative integers.

Similarly, making use of symbolic computation and Assumptions (7), integrating Eq. (1a) once in relation to x and y, separately, with the integration functions vanishing and choosing that

$$\alpha\,(\eta_1 - 2\eta_3) = -\frac{1}{2}\beta\eta_1^2, \tag{11}$$

help us find a Bell-polynomial expression, i.e.,

$$Y_t(w) - \alpha Y_{2x}(w) - \beta \eta_2 Y_x(w) = 0, \tag{12}$$

which is the same as Bell-Polynomial Expression (9).

Further, System (1) with the assumption

$$w(x, y, t) = \ln\left[h(x, y, t)\right], \tag{13}$$

develops into

$$h_t(x, y, t) - \alpha\, h_{xx}(x, y, t) - \beta\eta_2\, h_x(x, y, t) = 0, \tag{14}$$

in which $h(x, y, t)$ means a positive differentiable function.

Taking into consideration all the above, with symbolic computation, we end up with the following set of the hetero-Bäcklund transformations for System (1):

$$u(x, y, t) = \frac{2\alpha}{\beta}\,\frac{h_x(x, y, t)}{h(x, y, t)} + \eta_2, \tag{15a}$$

$$v(x, y, t) = \frac{2\alpha}{\beta}\left[\frac{h_{xy}(x, y, t)}{h(x, y, t)} - \frac{h_x(x, y, t)}{h(x, y, t)}\,\frac{h_y(x, y, t)}{h(x, y, t)}\right], \tag{15b}$$

$$h_t(x, y, t) - \alpha\, h_{xx}(x, y, t) - \beta\eta_2\, h_x(x, y, t) = 0\ . \tag{15c}$$

Eq. (15c) denotes a known linear partial differential equation, whose information has been reported [60, 61]. Moreover, with symbolic computation, we hereby present the following sample solutions for Eq. (15c):

$$h(x, y, t) = 1 + e^{\zeta_1 x + \zeta_2(y) + (\alpha \zeta_1^2 + \beta \eta_2 \zeta_1) t},$$

where ζ_1 is a real non-zero constant and $\zeta_2(y)$ is a real differentiable function of y.

Explanation with the relevant physics: Eqs. (15) stand for a set of the hetero-Bäcklund transformations, which can link the solutions $h(x, y, t)$ of Eq. (15c) and the solutions $u(x, y, t)$ and $v(x, y, t)$ of System (1). As for the nonlinear and dispersive long gravity waves travelling along two horizontal directions in the shallow water of uniform depth, with respect to $u(x, y, t)$, the height of the water surface, and $v(x, y, t)$, the horizontal velocity of the water wave, Hetero-Bäcklund Transformations (15) rely on α and β, the coefficients for System (1).

3 Four Sets of the Similarity Reductions for System (1)

Choosing the assumptions

$$u(x, y, t) = \theta(x, y, t) + \omega(x, y, t) p[z(x, y, t)], \tag{16a}$$

$$v(x, y, t) = \gamma(x, y, t) + \kappa(x, y, t) q[z(x, y, t)], \tag{16b}$$

which are similar to those in Refs. [62–67], thinking about the case of $z_x \neq 0$ and $z_y \neq 0$, and then, with symbolic computation, substituting Assumptions (16) into System (1), we find

$$\delta_0 p''' + \delta_1 \left(p p'' + p'^2 \right) + \delta_2 p p' + \delta_3 p^2 + \delta_4 q'' + \delta_5 q' + \delta_6 q + \delta_7 p'' + \delta_8 p' + \delta_9 p + \delta_{10} = 0, \tag{17a}$$

$$\psi_0 q'' + \psi_1 \left(p q' + p' q \right) + \psi_2 p q + \psi_3 q' + \psi_4 p' + \psi_5 q + \psi_6 p + \psi_7 = 0, \tag{17b}$$

in which

$$\delta_0 = \alpha \omega z_x^2 z_y, \tag{18a}$$

$$\delta_1 = -\beta \omega^2 z_x z_y, \tag{18b}$$

$$\delta_2 = -\beta \omega \left(2\omega_y z_x + 2\omega_x z_y + \omega z_{xy} \right), \tag{18c}$$

$$\delta_3 = -\beta \left(\omega_x \omega_y + \omega \omega_{xy} \right), \tag{18d}$$

$$\delta_4 = -2\alpha \kappa z_x^2, \tag{18e}$$

$$\delta_5 = -2\alpha \left(2\kappa_x z_x + \kappa z_{xx} \right), \tag{18f}$$

$$\delta_6 = -2\alpha \kappa_{xx}, \tag{18g}$$

$$\delta_7 = \alpha z_x \left(\omega_y z_x + 2\omega_x z_y \right) + \omega \left(z_y z_t - \beta \theta z_x z_y + 2\alpha z_x z_{xy} + \alpha z_y z_{xx} \right), \tag{18h}$$

$$\delta_8 = \left(\omega_t z_y + \omega_y z_t + \omega z_{yt} \right) - \beta \left(\theta_y \omega z_x + \theta \omega_y z_x + \theta_x \omega z_y + \theta \omega_x z_y + \theta \omega z_{xy} \right) + \alpha \left(2\omega_x z_{xy} + 2\omega_{xy} z_x + \omega_y z_{xx} + \omega_{xx} z_y + \omega z_{xxy} \right), \tag{18i}$$

$$\delta_9 = \omega_{yt} - \beta\left(\theta_x\omega_y + \theta_y\omega_x + \theta_{xy}\omega + \theta\omega_{xy}\right) + \alpha\omega_{xxy}, \tag{18j}$$

$$\delta_{10} = \theta_{yt} - \beta\left(\theta_x\theta_y + \theta\theta_{xy}\right) - 2\alpha\gamma_{xx} + \alpha\theta_{xxy}, \tag{18k}$$

$$\psi_0 = -\alpha\kappa z_x^2, \tag{18l}$$

$$\psi_1 = -\beta\omega\kappa z_x, \tag{18m}$$

$$\psi_2 = -\beta\left(\omega\kappa_x + \omega_x\kappa\right), \tag{18n}$$

$$\psi_3 = -2\alpha\kappa_x z_x + \kappa\left(z_t - \beta\theta z_x - \alpha z_{xx}\right), \tag{18o}$$

$$\psi_4 = -\beta\gamma\omega z_x, \tag{18p}$$

$$\psi_5 = \kappa_t - \beta\left(\theta\kappa_x + \theta_x\kappa\right) - \alpha\kappa_{xx}, \tag{18q}$$

$$\psi_6 = -\beta\left(\gamma_x\omega + \gamma\omega_x\right), \tag{18r}$$

$$\psi_7 = \gamma_t - \beta\left(\theta\gamma_x + \theta_x\gamma\right) - \alpha\gamma_{xx}, \tag{18s}$$

with $\theta(x, y, t)$, $\omega(x, y, t) \neq 0$, $\gamma(x, y, t)$, $\kappa(x, y, t) \neq 0$ and $z(x, y, t) \neq 0$ as the real to-be-determined differentiable functions, $p(z)$ and $q(z)$ as the real differentiable functions, while the prime sign as d/dz.

Seeing that Eqs. (17) with Expressions (18) stand for a set of the ordinary differential equations (ODEs) as for $p(z)$ and $q(z)$, we require the ratios of different derivatives and powers of $p(z)$ and $q(z)$ to represent the functions of z only, i.e.,

$$\delta_i = \Omega_i(z)\delta_0, \qquad \psi_j = \Gamma_j(z)\psi_0, \tag{19}$$

with $\Omega_i(z)$'s ($i = 0, ..., 10$) and $\Gamma_j(z)$'s ($j = 0, ..., 7$) as some real to-be-determined functions of z only. Hence, a set of the conditions for $\theta(x, y, t)$, $\omega(x, y, t) \neq 0$, $\gamma(x, y, t)$, $\kappa(x, y, t) \neq 0$ and $z(x, y, t) \neq 0$ are built up, for which any set of the solutions could develop into, at least, a similarity reduction.

On account of the second freedom in Remark 3 in Ref. [62], Eqs. (19) with $i = 1, 4$ and $j = 1$ come to

$$\omega(x, y, t) = \pm\frac{\alpha}{\beta}z_x, \qquad \kappa(x, y, t) = \mp\frac{\alpha}{2\beta}z_x z_y, \tag{20a}$$

$$\Omega_1(z) = \mp 1, \qquad \Omega_4(z) = 1, \qquad \Gamma_1(z) = \pm 1 \ . \tag{20b}$$

Since the first freedom in Remark 3 in Ref. [62] helps us transform Eqs. (19) with $i = 2$ into

$$z(x, y, t) = \lambda_1 x + \lambda_2 y + \lambda_3 t + \lambda_4, \qquad \Omega_2(z) = 0, \tag{21}$$

Eqs. (19) with $i = 3, 5, 6$ and $j = 2$ bring about

$$\Omega_3(z) = \Omega_5(z) = \Omega_6(z) = \Gamma_2(z) = 0, \tag{22}$$

with $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$ and λ_4 denoting the real constants.

Because the first freedom in Remark 3 in Ref. [62] makes us simplify Eqs. (19) with $j = 4$ to

$$\gamma(x, y, t) = 0, \qquad \Gamma_4(z) = 0, \tag{23}$$

Eqs. (19) with $j = 6, 7$ indicate

$$\Gamma_6(z) = \Gamma_7(z) = 0\ . \tag{24}$$

Until now, with symbolic computation, there exist two choices of $\theta(x, y, t)$ making $\Omega_7(z)$ represent a constant:

Choice 1:
According to the first freedom in Remark 3 in Ref. [62], Eqs. (19) with $j = 7$ make for

$$\theta(x, y, t) = \frac{\lambda_3}{\beta\lambda_1}, \qquad \Omega_7(z) = 0, \tag{25}$$

as a result that Eqs. (19) with $i = 8, 9, 10$ and $j = 3, 5$ can be simplified into

$$\Omega_8(z) = \Omega_9(z) = \Omega_{10}(z) = \Gamma_3(z) = \Gamma_5(z) = 0\ . \tag{26}$$

So far, System (1) could be transformed into the following ODEs:

$$p''' \mp \left(pp'' + p'^2\right) + q'' = 0, \tag{27a}$$

$$q'' \pm \left(p'q + pq'\right) = 0\ . \tag{27b}$$

Thus, simplifying each of two sets of ODEs (27) into an ODE makes us find

$$q = -p' \pm \frac{1}{2}p^2 + \phi_1 z + \phi_2, \tag{28}$$

and considering ODEs (27)-(28) helps us obtain

$$p'' - \frac{1}{2}p^3 \mp (\phi_1 z + \phi_2)\, p + (\phi_3 - \phi_1) = 0, \tag{29}$$

with ϕ_1, ϕ_2 and ϕ_3 as the real constants of integration.
With symbolic computation, we end up with two sets of the similarity reductions for System (1), i.e.,

$$u(x, y, t) = \frac{\lambda_3}{\beta\lambda_1} \pm \frac{\alpha}{\beta}\lambda_1 p[z(x, y, t)], \tag{30a}$$

$$v(x, y, t) = \mp\frac{\alpha}{2\beta}\lambda_1\lambda_2 \left\{-p'[z(x, y, t)] \pm \frac{1}{2}p[z(x, y, t)]^2 + \phi_1 z + \phi_2\right\}, \tag{30b}$$

$$z(x, y, t) = \lambda_1 x + \lambda_2 y + \lambda_3 t + \lambda_4, \tag{30c}$$

$$p'' - \frac{1}{2}p^3 \mp (\phi_1 z + \phi_2)\, p + (\phi_3 - \phi_1) = 0 \ \ . \tag{30d}$$

ODEs (30d) stand for two known ODEs, each of which has been investigated in Refs. [68, 69]. Right now, with symbolic computation, choosing $\phi_1 = \phi_2 = \phi_3 = 0$, we can get the following sample solutions for ODEs (30d):

$$p(z) = \pm\frac{2}{z + \zeta_3},$$

where ζ_3 is a real constant.

Explanation with the relevant physics: As for the nonlinear and dispersive long gravity waves travelling along two horizontal directions in the shallow water of uniform depth, with respect to $u(x, y, t)$, the height of the water surface, and $v(x, y, t)$, the horizontal velocity of the water wave, Similarity Reductions (30) rely on α and β, the coefficients for System (1). Two sets of Similarity Reductions (30) appear, as a result of the existence of the "±" signs.

Choice 2:

Based on the second freedom in Remark 3 in Ref. [62], Eqs. (19) with $j = 7$ give rise to

$$\theta(x, y, t) = \frac{\lambda_3 - \alpha\lambda_1^2}{\beta\lambda_1}, \qquad \Omega_7(z) = 1, \tag{31}$$

so that Eqs. (19) with $i = 8, 9, 10$ and $j = 3, 5$ can be transformed into

$$\Omega_8(z) = \Omega_9(z) = \Omega_{10}(z) = \Gamma_5(z) = 0, \qquad \Gamma_3(z) = -1\ . \tag{32}$$

Until now, System (1) could be simplified to the following ODEs:

$$p''' \mp \left(pp'' + p'^2\right) + q'' + p'' = 0, \tag{33a}$$

$$q'' \pm \left(p'q + pq'\right) - q' = 0 \ \ . \tag{33b}$$

Therefore, simplifying each of two sets of ODEs (33) into an ODE helps us work out

$$q = -p' \pm \frac{1}{2}p^2 - p + \phi_4 z + \phi_5, \tag{34}$$

and taking into consideration ODEs (33)-(34) makes us discover

$$p'' - \frac{1}{2}p^3 \pm \frac{3}{2}p^2 \mp (\phi_4 z + \phi_5 \pm 1)\, p + (\phi_4 z - \phi_4 + \phi_5 + \phi_6) = 0, \tag{35}$$

with ϕ_4, ϕ_5 and ϕ_6 as the real constants of integration.

With symbolic computation, we conclude with two more sets of the similarity reductions for System (1), i.e.,

$$u(x, y, t) = \frac{\lambda_3 - \alpha\lambda_1^2}{\beta\lambda_1} \pm \frac{\alpha}{\beta}\lambda_1 p[z(x, y, t)], \tag{36a}$$

$$v(x, y, t) = \mp\frac{\alpha}{2\beta}\lambda_1\lambda_2 \left\{-p'[z(x, y, t)] \pm \frac{1}{2}p[z(x, y, t)]^2 - p[z(x, y, t)] + \phi_4 z + \phi_5\right\}, \tag{36b}$$

$$z(x, y, t) = \lambda_1 x + \lambda_2 y + \lambda_3 t + \lambda_4, \tag{36c}$$

$$p'' - \frac{1}{2}p^3 \pm \frac{3}{2}p^2 \mp (\phi_4 z + \phi_5 \pm 1)\, p + (\phi_4 z - \phi_4 + \phi_5 + \phi_6) = 0\ . \tag{36d}$$

ODEs (36d) indicate two known ODEs, each of which has been investigated in Refs. [68, 69]. This time, with symbolic computation, selecting $\phi_4 = \phi_6 = 0$ and $\phi_5 = \pm\frac{1}{2}$, we are able to obtain the following sample solutions for ODEs (36d):

$$p(z) = \frac{2}{z + \zeta_4} \pm 1 \qquad \text{or} \qquad p(z) = -\frac{2}{z + \zeta_5} \pm 1,$$

where both ζ_4 and ζ_5 are the real constants.

Explanation with the relevant physics: As for the nonlinear and dispersive long gravity waves travelling along two horizontal directions in the shallow water of uniform depth, with respect to $u(x, y, t)$, the height of the water surface, and $v(x, y, t)$, the horizontal velocity of the water wave, Similarity Reductions (36) rely on α and β, the coefficients for System (1). Two sets of Similarity Reductions (36) appear, as a result of the existence of the "±" signs.

4 Conclusions

To sum up, we have briefly reviewed the recent developments on the shallow water and soliton consideration with analytic solutions, and with symbolic computation, studied System (1), i.e., a (2+1)-dimensional generalized modified dispersive water-wave system describing the nonlinear and dispersive long gravity waves travelling along two horizontal directions in the shallow water of uniform depth. With respect to $u(x, y, t)$, the height of the water surface, and $v(x, y, t)$, the horizontal velocity of the water wave, we have symbolically computed out Scaling Transformations (6), Hetero-Bäcklund Transformations (15), from System (1) to Eq. (15c), Similarity Reductions (30), from System (1) to ODEs (30d), and Similarity Reductions (36), from System (1) to ODEs (36d). We have paid attention that Eq. (15c) stands for a known linear partial differential equation, while each of ODEs (30d) and ODEs (36d), a known ODE. Hetero-Bäcklund Transformations (15), Similarity Reductions (30) and Similarity Reductions (36) have been obtained to rely on α and β, the coefficients for System (1).

We hope that Hetero-Bäcklund Transformations (15), Similarity Reductions (30) and Similarity Reductions (36), with their sample solutions and other solutions, could help the future shallow-water studies. Information about the future work for System (1) includes the Darboux transformation, inverse scattering transformation, etc.

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