



Bifurcation of Limit Cycles of a Piecewise Smooth Hamiltonian System

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Abstract

In order to get the lower bound of the number of limit cycles for near-Hamiltonian systems, one often faces the difficulty in verifying the independence of the coefficients of some polynomials. The difficulty is mainly coming from the tedious iterative computation. In the present paper, we provide an approach to verify the independence which largely reduces calculation and illustrate this method by perturbing a piecewise smooth Hamiltonian system with a homoclinic loop. Using this method, we prove that the maximal number of limit cycles of this system is $n + \lfloor \frac{n+1}{2} \rfloor$, and this bound can be reached.

Keywords Lower bound · Melnikov function · Limit cycle · Homoclinic loop

1 Introduction and Main Results

Mutational behavior after slow change is ubiquitous in natural and artificial systems and is usually described by piecewise smooth mathematical models. For a long time, this research has attracted much attention in the field of nonlinear science. As a part of nonlinear dynamical systems, piecewise smooth dynamical systems are applied in many fields of applied science and engineering, such as collision vibration system in mechanical engineering, stick slip vibration system with dry friction, circuit system with controllable switch, see [1, 8, 17].

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Similar to smooth differential systems, the number and distribution of limit cycles is one of the important problems of non-smooth differential systems. So far as we know, there are two basic methods to study the number of limit cycles. One is the Melnikov function method developed in [6, 14] and the other is the averaging method established in [12, 13]. By using the above two methods, many researchers have been extensively studied the upper or lower bound of the number of limit cycles for the following piecewise smooth near-Hamiltonian system

$$\begin{cases} \dot{x} = H_y^+(x, y) + \varepsilon f^+(x, y), \\ \dot{y} = -H_x^+(x, y) + \varepsilon g^+(x, y), & x \geq 0, \\ \dot{x} = H_y^-(x, y) + \varepsilon f^-(x, y), \\ \dot{y} = -H_x^-(x, y) + \varepsilon g^-(x, y), & x < 0, \end{cases} \tag{1.1}$$

where $0 < |\varepsilon| \ll 1$, $H^\pm(x, y)$ are polynomials of x and y of degree $m + 1$ and

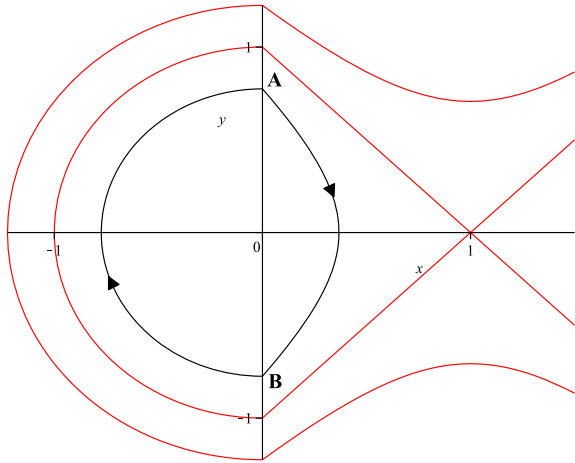
$$f^\pm(x, y) = \sum_{i+j=0}^n a_{i,j}^\pm x^i y^j, \quad g^\pm(x, y) = \sum_{i+j=0}^n b_{i,j}^\pm x^i y^j, \quad a_{i,j}^\pm, b_{i,j}^\pm \in \mathbb{R}, \quad i, j \in \mathbb{N}.$$

For the purpose of getting the upper bound, one usually analyzed the algebraic structure of the corresponding first order Melnikov function $M(h)$ of system (1.1) with the help of Picard-Fuchs equations. Then, the upper bound is obtained by comprehensively applying the methods in the literatures [7, 9, 22, 26] or argument principle or Chebyshev criterion etc, see [2, 5, 10, 16, 20, 21, 23, 25] that have used these methods. Fortunately, the independence of the coefficients of the coefficient polynomials of the generators of $M(h)$ does not need to be verified. Because this is very intricate. However, this is inevitable if you want to get the lower bound. In [11, 14, 24], the authors got the lower bound of the number of limit cycles bifurcating from the period annulus of system (1.1) with a homoclinic loop or heteroclinic loop or eye-figure loop. In order to get the independence of the coefficients, they took some special perturbed polynomials $f^\pm(x, y)$ and $g^\pm(x, y)$ for the sake of simplifying the calculation. In [4, 15, 18, 19], the authors obtained the lower bound for the general perturbed polynomials. But, the proof of the independence of the coefficients involves a lot of computations. With that in mind, in the present paper, we provide a way to verify the independence of coefficients which largely reduces calculation and illustrate this method with a concrete example.

Consider the following perturbed piecewise smooth Hamiltonian system

$$\begin{cases} \dot{x} = y + \varepsilon f^+(x, y), \\ \dot{y} = x - 1 + \varepsilon g^+(x, y), & x \geq 0, \\ \dot{x} = y + \varepsilon f^-(x, y), \\ \dot{y} = -x + \varepsilon g^-(x, y), & x < 0. \end{cases} \tag{1.2}$$

Fig. 1 The phase portrait of system (1.2) with $\varepsilon = 0$



The corresponding Hamiltonian functions for system (1.2) $_{\varepsilon=0}$ are

$$H^+(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + x, \quad x \geq 0, \tag{1.3}$$

and

$$H^-(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2, \quad x < 0. \tag{1.4}$$

When $\varepsilon = 0$, system (1.2) has a family of periodic orbits as follows

$$\begin{aligned} \Gamma_h &= \{(x, y) | H^+(x, y) = h, x \geq 0\} \cup \{(x, y) | H^-(x, y) = h, x < 0\} \\ &:= \Gamma_h^+ \cup \Gamma_h^- \end{aligned}$$

with $h \in (0, \frac{1}{2})$. As h tends to 0, Γ_h approaches to the origin, and as h tends to $\frac{1}{2}$, Γ_h approaches to a homoclinic loop passing through the saddle point $(1,0)$, see Fig. 1.

By [6, 14], corresponding to periodic orbits $\{\Gamma_h | h \in (0, \frac{1}{2})\}$, system (1.2) has the first order Melnikov function described by

$$M(h) = \int_{\Gamma_h^+} g^+(x, y)dx - f^+(x, y)dy + \int_{\Gamma_h^-} g^-(x, y)dx - f^-(x, y)dy, \tag{1.5}$$

and the number of zeros of $M(h)$ controls the number of limit cycles of system (1.2) if $M(h)$ is not identically zero. In [10] the authors posed the following conjecture.

Conjecture *By using the first order Melnikov function, the maximal number of limit cycles of system (1.2) bifurcating from the period annulus around the origin is $n + \lfloor \frac{n+1}{2} \rfloor$.*

We confirm the conjecture with the following theorem.

Theorem 1.1 *By using the first order Melnikov function, the number of limit cycles of system (1.2) bifurcating from the period annulus around the origin is not more than $n + \lfloor \frac{n+1}{2} \rfloor$, and this bound can be reached.*

This paper is organized as follows. In Sect. 2, we obtain the detailed expression of the first order Melnikov function $M(h)$ and verify the independence of the coefficients of the coefficient polynomials of the generators of $M(h)$ by using mathematical induction. Section 3 is devoted to the proof of Theorem 1.1. Finally, conclusion is drawn in Sect. 4.

2 The Algebraic Structure of the First Order Melnikov Function

In order to estimate the number of zeros of the first order Melnikov function $M(h)$, one should study the algebraic structure of $M(h)$. To this end, we denote

$$I_{i,j}(h) = \int_{\Gamma_h^+} x^i y^j dy, \quad J_{i,j}(h) = \int_{\Gamma_h^-} x^i y^j dy, \quad h \in \left(0, \frac{1}{2}\right).$$

Since the orbits Γ_h^\pm are symmetric with respect to the x -axis, $I_{i,2j+1}(h) = J_{i,2j+1}(h) \equiv 0$. So we only need to consider $I_{i,2j}(h)$ and $J_{i,2j}(h)$.

The next Lemma shows that $M(h)$ can be expressed as a combination of some generator integrals with polynomial coefficients and the coefficients of these polynomials can be taken as free parameters.

Lemma 2.1 *For $h \in (0, \frac{1}{2})$ and any $n \in \mathbb{N}$ it holds that*

$$M(h) = \sqrt{h} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_i h^i + \left(\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \beta_i h^i \right) I_{1,0}(h) + h \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_i h^i, \tag{2.1}$$

where α_i, β_i and γ_i are constants and can be chosen arbitrarily.

Proof Let D be the interior of $\Gamma_h^+ \cup \overrightarrow{BA}$, see Fig. 1. Using the Green’s Formula, one has

$$\begin{aligned} \int_{\Gamma_h^+} x^i y^j dx &= \oint_{\Gamma_h^+ \cup \overrightarrow{BA}} x^i y^j dx - \int_{\overrightarrow{BA}} x^i y^j dx \\ &= \oint_{\Gamma_h^+ \cup \overrightarrow{BA}} x^i y^j dx = j \iint_D x^i y^{j-1} dx dy, \\ \int_{\Gamma_h^+} x^{i+1} y^{j-1} dy &= \oint_{\Gamma_h^+ \cup \overrightarrow{BA}} x^{i+1} y^{j-1} dy = -(i+1) \iint_D x^i y^{j-1} dx dy. \end{aligned}$$

Thus,

$$\int_{\Gamma_h^+} x^i y^j dx = -\frac{j}{i+1} \int_{\Gamma_h^+} x^{i+1} y^{j-1} dy. \tag{2.2}$$

Similarly, one has

$$\int_{\Gamma_h^-} x^i y^j dx = -\frac{j}{i+1} \int_{\Gamma_h^-} x^{i+1} y^{j-1} dy. \tag{2.3}$$

By (1.5), (2.2) and (2.3), one obtains

$$\begin{aligned} M(h) &= \int_{\Gamma_h^+} \sum_{i+j=0}^n b_{i,j}^+ x^i y^j dx - \int_{\Gamma_h^+} \sum_{i+j=0}^n a_{i,j}^+ x^i y^j dy \\ &\quad + \int_{\Gamma_h^-} \sum_{i+j=0}^n b_{i,j}^- x^i y^j dx - \int_{\Gamma_h^-} \sum_{i+j=0}^n a_{i,j}^- x^i y^j dy \\ &= - \sum_{i+j=1, j \geq 1}^n \frac{j}{i+1} b_{i,j}^+ \int_{\Gamma_h^+} x^{i+1} y^{j-1} dy - \sum_{i+j=0}^n a_{i,j}^+ \int_{\Gamma_h^+} x^i y^j dy \\ &\quad - \sum_{i+j=1, j \geq 1}^n \frac{j}{i+1} b_{i,j}^- \int_{\Gamma_h^-} x^{i+1} y^{j-1} dy - \sum_{i+j=0}^n a_{i,j}^- \int_{\Gamma_h^-} x^i y^j dy \\ &= \sum_{i+j=0}^n \xi_{i,j} I_{i,j}(h) + \sum_{i+j=0}^n \eta_{i,j} J_{i,j}(h), \end{aligned}$$

where

$$\xi_{i,j} = \begin{cases} -a_{i,j}^+ - \frac{j+1}{i} b_{i-1,j+1}^+, & 1 \leq i \leq n, 1 \leq i+j \leq n, \\ -a_{i,j}^+, & i=0, 0 \leq j \leq n, \end{cases}$$

$$\eta_{i,j} = \begin{cases} -a_{i,j}^- - \frac{j+1}{i} b_{i-1,j+1}^-, & 1 \leq i \leq n, 1 \leq i+j \leq n, \\ -a_{i,j}^-, & i=0, 0 \leq j \leq n. \end{cases}$$

It is easy to check that $\xi_{i,j}$ and $\eta_{i,j}$ can be taken as free parameters.

□

Now we claim that

$$\sum_{i+j=0}^n \xi_{i,j} I_{i,j}(h) = \left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \bar{\alpha}_i h^i \right) I_{0,0}(h) + \left(\sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \bar{\beta}_j h^j \right) I_{1,0}(h), \tag{2.4}$$

where $\bar{\alpha}_i, i = 0, 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$ and $\bar{\beta}_j, j = 0, 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$ can be taken as free parameters.

In fact, differentiating $H^+(x, y) = h$ both sides with respect to y , one obtains

$$y - x \frac{\partial x}{\partial y} + \frac{\partial x}{\partial y} = 0. \tag{2.5}$$

Multiplying (2.5) by the one-form $x^i y^{j-1} dy$ and integrating over Γ_h^+ , one obtains the relation

$$I_{i,j} = \frac{j-1}{i+1} I_{i+1,j-2} - \frac{j-1}{i+2} I_{i+2,j-2}, \quad i \geq 0, j \geq 1. \tag{2.6}$$

Similarly, multiplying $H^+(x, y) = h$ both sides by $x^{i-2} y^j dy$ and integrating over Γ_h^+ , one gets another relation

$$I_{i,j} = -2hI_{i-2,j} + 2I_{i-1,j} + I_{i-2,j+2}, \quad i \geq 2, j \geq 0. \tag{2.7}$$

Elementary manipulations reduce equations (2.6) and (2.7) to

$$I_{i,j} = -\frac{i}{i+j+1} \left[2hI_{i-2,j} - \frac{2i+j-1}{i-1} I_{i-1,j} \right], \quad i \geq 2, j \geq 0 \tag{2.8}$$

and

$$I_{i,j} = \frac{j-1}{i+j+1} \left[2hI_{i,j-2} - \frac{i}{i+1} I_{i+1,j-2} \right], \quad i \geq 0, j \geq 1. \tag{2.9}$$

Now we will prove the claim by induction on n . Without loss of generality, we only prove the claim if n is an even number (the claim can be proved similarly if n is an odd number). Indeed, a direct computation using the above two equalities (2.8) and (2.9) gives

$$\begin{cases} I_{0,2}(h) = \frac{2}{3}hI_{0,0}(h), \\ I_{2,0}(h) = -\frac{4}{3}hI_{0,0}(h) + 2I_{1,0}(h), \\ I_{1,2}(h) = \frac{1}{6}hI_{0,0}(h) + \left(\frac{1}{2}h - \frac{1}{4}\right)I_{1,0}(h), \\ I_{3,0}(h) = -\frac{5}{2}hI_{0,0}(h) - \left(\frac{3}{2}h - \frac{15}{4}\right)I_{1,0}(h). \end{cases} \tag{2.10}$$

Hence, one has for $n = 2, 3$

$$\begin{aligned} \sum_{i+j=0}^2 \xi_{i,j} I_{i,j}(h) &= \left[\left(\frac{2}{3}\xi_{0,2} - \frac{4}{3}\xi_{2,0} \right) h + \xi_{0,0} \right] I_{0,0}(h) + (\xi_{1,0} + 2\xi_{2,0}) I_{1,0}(h), \\ \sum_{i+j=0}^3 \xi_{i,j} I_{i,j}(h) &= \left[\left(\frac{2}{3}\xi_{0,2} - \frac{4}{3}\xi_{2,0} + \frac{1}{6}\xi_{1,2} - \frac{5}{2}\xi_{3,0} \right) h + \xi_{0,0} \right] I_{0,0}(h) \\ &\quad + \left[\left(\frac{1}{2}\xi_{1,2} - \frac{3}{2}\xi_{3,0} \right) h - \frac{1}{4}\xi_{1,2} + \frac{15}{4}\xi_{3,0} + \xi_{1,0} + 2\xi_{2,0} \right] I_{1,0}(h). \end{aligned}$$

That is, the claim holds for $n = 2, 3$.

Now assume that the claim holds for all $i + j \leq n - 1$. Then, taking $(i, j) = (0, n), (2, n - 2), \dots, (n - 2, 2)$ in (2.9) and $(i, j) = (n, 0)$ in (2.8), respectively, one has

$$\begin{pmatrix} I_{0,n}(h) \\ I_{2,n-2}(h) \\ I_{4,n-4}(h) \\ \vdots \\ I_{n-2,2}(h) \\ I_{n,0}(h) \end{pmatrix} = \frac{1}{n+1} \begin{pmatrix} 2(n-1)hI_{0,n-2}(h) \\ 2(n-3)(hI_{2,n-4}(h) - \frac{1}{3}I_{3,n-4}(h)) \\ 2(n-5)(hI_{4,n-6}(h) - \frac{2}{3}I_{5,n-6}(h)) \\ \vdots \\ 2(hI_{n-2,0}(h) - \frac{n-2}{2(n-1)}I_{n-1,0}(h)) \\ -n(2hI_{n-2,0}(h) - \frac{2n-1}{n-1}I_{n-1,0}(h)) \end{pmatrix}. \tag{2.11}$$

Therefore, by the induction hypothesis and (2.11), one obtains

$$\begin{aligned} \sum_{i+j=0}^n \xi_{i,j} I_{i,j}(h) &= \sum_{i+j=0}^{n-1} \xi_{i,j} I_{i,j}(h) + \sum_{i+j=n} \xi_{i,j} I_{i,j}(h) \\ &= \left(\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \tilde{\alpha}_i h^i \right) I_{0,0}(h) + \left(\sum_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} \tilde{\beta}_j h^j \right) I_{1,0}(h) \\ &\quad + \xi_{0,n} \frac{2(n-1)}{n+1} h I_{0,n-2}(h) + \xi_{2,n-2} \frac{2(n-3)}{n+1} \\ &\quad \times \left(h I_{2,n-4}(h) - \frac{1}{3} I_{3,n-4}(h) \right) \\ &\quad + \dots - \xi_{n,0} \frac{n}{n+1} \left(2h I_{n-2,0}(h) - \frac{2n-1}{n-1} I_{n-1,0}(h) \right) \\ &\triangleq \left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \tilde{\alpha}_i h^i \right) I_{0,0}(h) + \left(\sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \tilde{\beta}_j h^j \right) I_{1,0}(h), \end{aligned} \tag{2.12}$$

where $\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\alpha}_i$ and $\tilde{\beta}_i$ are constants.

Next, we prove that $\tilde{\alpha}_i, i = 0, 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$ and $\tilde{\beta}_j, j = 0, 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$ can be taken as free parameters. In fact, by the induction hypothesis, one has $\tilde{\alpha}_i, i = 0, 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$ and $\tilde{\beta}_j, j = 0, 1, 2, \dots, \lfloor \frac{n-2}{2} \rfloor$ are independent of each other. That is, the determinant of the following Jacobian matrix

$$\mathbf{A} = \frac{\partial \left(\tilde{\alpha}_{\lfloor \frac{n-1}{2} \rfloor}, \dots, \tilde{\alpha}_0, \tilde{\beta}_{\lfloor \frac{n-2}{2} \rfloor}, \dots, \tilde{\beta}_0 \right)}{\partial \left(\xi_{i_0, j_{\lfloor \frac{n-1}{2} \rfloor}}, \dots, \xi_{i_{\lfloor \frac{n-1}{2} \rfloor}, j_0}, \xi_{k_0, l_{\lfloor \frac{n-2}{2} \rfloor}}, \dots, \xi_{k_{\lfloor \frac{n-2}{2} \rfloor}, l_0} \right)}$$

is different from zero, here the sum of subscripts of $\xi_{i,j}$ in the above Jacobian matrix is less than or equal to $n - 1$. A directly calculation implies the following Jacobian

matrix

$$\begin{aligned} \mathbf{C} &= \frac{\partial \left(\bar{\alpha}_{\lfloor \frac{n-1}{2} \rfloor}, \dots, \bar{\alpha}_0, \bar{\beta}_{\lfloor \frac{n-2}{2} \rfloor}, \dots, \bar{\beta}_0, \bar{\alpha}_{\lfloor \frac{n}{2} \rfloor} \right)}{\partial \left(\xi_{i_0, j_{\lfloor \frac{n-1}{2} \rfloor}}, \dots, \xi_{i_{\lfloor \frac{n-1}{2} \rfloor}, j_0}, \xi_{k_0, l_{\lfloor \frac{n-2}{2} \rfloor}}, \dots, \xi_{k_{\lfloor \frac{n-2}{2} \rfloor}, l_0}, \xi_{0, n} \right)} \\ &= \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \frac{2^{\lfloor \frac{n}{2} \rfloor}}{n+1} \end{pmatrix}, \end{aligned}$$

where $\mathbf{0}$ is a row vector and \mathbf{B} is a column vector. It is easy to get that

$$|\mathbf{C}| = \frac{2^{\lfloor \frac{n}{2} \rfloor}}{n+1} |\mathbf{A}| \neq 0,$$

which yields $\bar{\alpha}_i, i = 0, 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$ and $\bar{\beta}_j, j = 0, 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$ can be taken as free parameters.

In a similar way, one can prove that

$$\sum_{i+j=0}^n \eta_{i,j} J_{i,j}(h) = \left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \hat{\alpha}_i h^i \right) J_{0,0}(h) + \left(\sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \hat{\beta}_j h^j \right) J_{1,0}(h), \quad (2.13)$$

where $\hat{\alpha}_i, i = 0, 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$ and $\hat{\beta}_j, j = 0, 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$ can be taken as free parameters.

Observe that $\bar{\alpha}_i$ and $\bar{\beta}_j$ are expressed by $\xi_{i,j}$ and $\hat{\alpha}_i$ and $\hat{\beta}_j$ are expressed by $\eta_{i,j}$, one has that $\bar{\alpha}_i, \bar{\beta}_j, \hat{\alpha}_i$ and $\hat{\beta}_j$ are independent of each other. A straightforward calculation yields that

$$I_{0,0}(h) = -2\sqrt{2h}, \quad J_{0,0}(h) = 2\sqrt{2h}, \quad J_{1,0}(h) = -\pi h. \quad (2.14)$$

In view of (2.4), (2.13) and (2.14), one gets the equality (2.1), where

$$\alpha_i = 2\sqrt{2}(\hat{\alpha}_i - \bar{\alpha}_i), \quad \beta_i = \bar{\beta}_i, \quad \gamma_i = -\pi \hat{\beta}_i,$$

which means that α_i, β_i and γ_i can be chosen arbitrarily. This ends the proof. \diamond

Remark 2.1 In the proof of Lemma 2.1, we have verified that the coefficients of the coefficient polynomials of $I_{0,0}(h), I_{1,0}(h), J_{0,0}(h)$ and $J_{1,0}(h)$ are independent of each other under general polynomial perturbations by using mathematical induction. Compared with the verification processes in the previous literatures, the calculation in this paper is simpler.

3 Proof of Theorem 1.1

In order to obtain the lower bound of the number of zeros of $M(h)$, we resort to a result of Coll, Gasull and Prohens published in [3]. We review this result here for the convenience of the reader.

Lemma 3.1 [3]. *Consider $p + 1$ linearly independent analytical functions $f_i : U \rightarrow \mathbb{R}$, $i = 0, 1, 2, \dots, p$, where $U \in \mathbb{R}$ is an interval. Suppose that there exists $j \in \{0, 1, \dots, p\}$ such that f_j has constant sign. Then there exists $p + 1$ constants δ_i , $i = 0, 1, \dots, p$, such that $f(x) = \sum_{i=0}^p \delta_i f_i(x)$ has at least p simple zeros in U .*

To apply the above Lemma 3.1 one should show that the first order Melnikov function $M(h)$ can be expressed as a combination of some linearly independent functions. To this end, let us start with some preliminary considerations.

Let $u = \sqrt{h}$, $h \in (0, \frac{1}{2})$. Then (2.1) can be written as

$$\begin{aligned} M(u) &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_i u^{2i+1} + \left(\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \beta_i u^{2i} \right) I_{1,0}(u^2) + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_i u^{2i+2} \\ &= \sum_{i=1}^{n+1} \delta_i u^i + \left(\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \beta_i u^{2i} \right) I_{1,0}(u^2), \end{aligned}$$

where δ_i are constants and can be chosen as free parameters. Therefore, one finds that

$$M(u) = \sum_{i=1}^{n+1} \delta_i u^i + \left(\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \beta_i u^{2i} \right) \varphi(u), \tag{3.1}$$

in view of

$$I_{1,0}(h) = -\sqrt{2h} + \frac{1}{2}(2h - 1) \ln \frac{1 - \sqrt{2h}}{1 + \sqrt{2h}},$$

where

$$\varphi(u) = -\sqrt{2}u + \frac{1}{2}(2u^2 - 1) \ln \frac{1 - \sqrt{2}u}{1 + \sqrt{2}u}.$$

It is easy to check that $\varphi(u)$ satisfies the following differential equation

$$(2u^2 - 1)\varphi'(u) = 4u\varphi(u) + 4\sqrt{2}u^2. \tag{3.2}$$

In order to show the linear independence of generating functions of $M(u)$ in (3.1), we extend $M(u)$ to a complex domain. From now on, we suppose that u is complex.

We know that $\varphi(u)$ can be analytically extended to the complex domain $\Omega = \mathbb{C} \setminus \{u \in \mathbb{R} \mid |u| \geq \frac{\sqrt{2}}{2}\}$. For $|u| > \frac{\sqrt{2}}{2}$ and let $\varphi^\pm(u)$ denotes the analytic continuation of $\varphi(u)$ along an arc with $\pm \text{Im}(u) > 0$, respectively. Then by (3.2) the function $\varphi^\pm(u)$ satisfies

$$\varphi^+(u) - \varphi^-(u) = c(2u^2 - 1)\mathbf{i}, \quad u \in \left(-\infty, -\frac{\sqrt{2}}{2}\right) \cup \left(\frac{\sqrt{2}}{2}, +\infty\right), \quad (3.3)$$

where $\mathbf{i}^2 = -1$ and c is a nonzero real number.

The following Proposition plays a key role in estimating the lower bound of the number of zeros of $M(h)$.

Proposition 3.1 *The Melnikov function $M(u)$ in (3.1) can be represented as a linear combination of the following $n + \lfloor \frac{n+1}{2} \rfloor + 1$ linearly independent generating functions*

$$u, \dots, u^{n+1}, \varphi(u), u^2\varphi(u), \dots, u^{2\lfloor \frac{n-1}{2} \rfloor}\varphi(u).$$

Proof We assume that there exist constants $\sigma_1, \dots, \sigma_{n+1}, \mu_0, \mu_1, \dots, \mu_{\lfloor \frac{n-1}{2} \rfloor}$ such that

$$\begin{aligned} \overline{M}(u) &\triangleq \sigma_1 u + \dots + \sigma_{n+1} u^{n+1} + \mu_0 \varphi(u) \\ &\quad + \mu_1 u^2 \varphi(u) + \dots + \mu_{\lfloor \frac{n-1}{2} \rfloor} u^{2\lfloor \frac{n-1}{2} \rfloor} \varphi(u) \equiv 0. \end{aligned} \quad (3.4)$$

To show the linear independence of generating functions, we only need to prove that all the coefficients in (3.4) are zeros.

Since $\varphi(u)$ can be analytically extended to domains Ω , $M(h)$ can be analytically extended to the domain Ω . For $u \in (-\infty, -\frac{\sqrt{2}}{2}) \cup (\frac{\sqrt{2}}{2}, +\infty)$, by (3.3) and (3.4), one has that

$$\overline{M}^+(u) - \overline{M}^-(u) = c(2u^2 - 1)\mathbf{i} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \mu_i u^{2i} \equiv 0,$$

which implies $\mu_i = 0, i = 0, 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$. Then $\overline{M}(u)$ in (3.4) is simplified into the form

$$\overline{M}(u) \triangleq \sigma_1 u + \dots + \sigma_{n+1} u^{n+1} \equiv 0.$$

Taking for granted that the functions u, u^2, \dots, u^{n+1} are independent of each other, one obtains that $\sigma_i = 0, i = 1, 2, \dots, n + 1$. This ends the proof.

□

The following Lemma proves to be extremely useful in estimating the upper bound of the number of zeros of $M(h)$.

Lemma 3.2 Let $f^{(n)}(h)$ represents the n th-order derivative of $f(h)$. Then for $n \geq m + 1$ and any $m, n \in \mathbb{N}$, it holds that

$$\left(h^m \ln \frac{1 - \sqrt{2h}}{1 + \sqrt{2h}} \right)^{(n)} = \frac{P_{n-1}(h)}{h^{n-m-\frac{1}{2}}(2h - 1)^n},$$

where $P_{n-1}(h)$ is a polynomial of degree $n - 1$.

Proof It is easy to get that

$$\left(\ln \frac{1 - \sqrt{2h}}{1 + \sqrt{2h}} \right)^{(k)} = \frac{Q_{k-1}(h)}{h^{k-\frac{1}{2}}(2h - 1)^k},$$

in view of induction on k , where $Q_{k-1}(h)$ is a polynomial of degree $k - 1$. Hence, by Leibniz formula and the above equality, one finds that

$$\begin{aligned} \left(h^m \ln \frac{1 - \sqrt{2h}}{1 + \sqrt{2h}} \right)^{(n)} &= \sum_{k=0}^n C_n^k (h^m)^{(n-k)} \left(\ln \frac{1 - \sqrt{2h}}{1 + \sqrt{2h}} \right)^{(k)} \\ &= \sum_{k=0}^n \frac{C_n^k m(m-1) \cdots (m-n+k+1) Q_{k-1}(h)}{h^{n-m-\frac{1}{2}}(2h-1)^k} \\ &:= \frac{P_{n-1}(h)}{h^{n-m-\frac{1}{2}}(2h-1)^n}. \end{aligned}$$

The proof is completed. □

Proof of Theorem 1.1 In accordance with Lemma 2.1, Lemma 3.1 and Proposition 3.1, one knows that $M(u)$ in (3.1) can have $n + \lfloor \frac{n+1}{2} \rfloor$ zeros on $(0, \frac{\sqrt{2}}{2})$, which means that $M(h)$ in (2.1) can have $n + \lfloor \frac{n+1}{2} \rfloor$ zeros on the interval $(0, \frac{1}{2})$. Therefore, system (1.2) can have $n + \lfloor \frac{n+1}{2} \rfloor$ limit cycles for $h \in (0, \frac{1}{2})$.

Next we want to obtain the upper bound of the number of limit cycles of system (1.2). (2.1) can be written as

$$\begin{aligned}
 M(h) &= \left(\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \beta_i h^i \right) \left(-\sqrt{2h} + \left(h - \frac{1}{2} \right) \ln \frac{1 - \sqrt{2h}}{1 + \sqrt{2h}} \right) \\
 &\quad + \sqrt{h} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_i h^i + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_i h^{i+1} \tag{3.5} \\
 &:= \left(h - \frac{1}{2} \right) \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \beta_i h^i \ln \frac{1 - \sqrt{2h}}{1 + \sqrt{2h}} + \sqrt{h} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \zeta_i h^i + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_i h^{i+1},
 \end{aligned}$$

where ζ_i are constants. Differentiating (3.5) $\lfloor \frac{n-1}{2} \rfloor + 2$ times using Lemma 3.2 gives that

$$\begin{aligned}
 M(\lfloor \frac{n-1}{2} \rfloor + 2)(h) &= \left(\sqrt{h} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \zeta_i h^i + \left(h - \frac{1}{2} \right) \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \beta_i h^i \ln \frac{1 - \sqrt{2h}}{1 + \sqrt{2h}} \right)^{\lfloor \frac{n-1}{2} \rfloor + 2} \\
 &= \frac{P_{\lfloor \frac{n}{2} \rfloor}(h)}{h^{\lfloor \frac{n-1}{2} \rfloor + \frac{3}{2}}} + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor + 2} C_{\lfloor \frac{n-1}{2} \rfloor + 2}^k \left(h - \frac{1}{2} \right)^{(k)} \\
 &\quad \times \left(\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \beta_i h^i \ln \frac{1 - \sqrt{2h}}{1 + \sqrt{2h}} \right)^{\lfloor \frac{n-1}{2} \rfloor + 2 - k} \\
 &= \frac{P_{\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor + 1}(h)}{h^{\lfloor \frac{n-1}{2} \rfloor + \frac{3}{2}} (2h - 1)^{\lfloor \frac{n-1}{2} \rfloor + 1}}.
 \end{aligned}$$

Thus $M(\lfloor \frac{n-1}{2} \rfloor + 2)(h)$ has at most $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor + 1$ zeros on $(0, \frac{1}{2})$. Therefore, by Rolle’s theorem, $M(h)$ has at most $n + \lfloor \frac{n+1}{2} \rfloor + 1$ zeros on $[0, \frac{1}{2})$. Notice that $M(0) = 0$ and hence $M(h)$ has at most $n + \lfloor \frac{n+1}{2} \rfloor$ zeros on $(0, \frac{1}{2})$. This completes the proof of Theorem 1.1. \square

4 Conclusion

The motivation of this work is to find a simple approach to verify the independence of the coefficients of the coefficient polynomials of the generators of the first order Melnikov function $M(h)$ of system (1.1). Because this is an essential step in estimating

the lower bound of the number of zeros of $M(h)$ and the existing methods for verifying independence are cumbersome.

To achieve our goal, we illustrate this approach by estimating the number of limit cycles of a near-Hamiltonian system with a homoclinic loop. Using this method, we have proved that this near-Hamiltonian system (1.2) has at most $n + \lfloor \frac{n+1}{2} \rfloor$ limit cycles and this number can be reached. This is a new result on the bound of the number of limit cycles for such system with a homoclinic loop and the method in this paper can be applied in the study of limit cycle bifurcations of integrable differential systems.

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Data Availability The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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