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Notes on Multiple Periodic Solutions for Second Order Hamiltonian Systems

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Abstract
In this paper, we study the multiplicity of periodic solutions for the second order Hamiltonian systems $\ddot{u} + \nabla F(t, u) = 0$ with the boundary condition $u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0$, where the potential F is either subquadratic $k(t)$-concave or subquadratic $\mu(t)$-convex. Based on the reduction method and a three-critical-point theorem due to Brezis and Nirenberg, we obtain the multiplicity results, which complement and sharply improve some related results in the literature.

Keywords Periodic solution · Second order Hamiltonian systems · Subquadratic · Local linking · Sobolev's inequality

Mathematics Subject Classification 34B15 · 34C25 · 58E05

1 Introduction and Main Results

Consider the second order Hamiltonian systems

$$\begin{cases} \ddot{u}(t) + \nabla F(t, u(t)) = 0 \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0$ and $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, x)| \leq a(|x|)b(t), \qquad |\nabla F(t, x)| \leq a(|x|)b(t)$$

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for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

With the aid of variational methods, the existence and multiplicity of periodic solutions of problem (1.1) have been extensively investigated in the literature during the past two decades, see [3–5],[7–16, 18–26] and references therein. Many solvability conditions are obtained, such as: the convexity conditions (see [14, 17, 19] and the references therein); the subquadratic conditions (including the sublinear nonlinearity case, see [18, 19] the references therein); the superquadratic conditions (see [6, 26] and the references therein) and the asymptotically linear conditions (see [20, 26, 29] and the references therein).

Lazer, Landesman and Meyers in [8] consider a minimax theorem for a C^2 functional φ defined on a Hilbert space H. They prove the existence of a critical point characterized by $\varphi(u_0) = \max_{v \in V} \min_{w \in W} \varphi(v + w)$, where V, W are subspaces of H, V is finite-dimensional and $H = V \bigoplus W$. This result has been generalized by Castro and Lazer [4], supposing a weaker condition that $\varphi(v) \to -\infty$ as $\|v\| \to \infty$, $v \in V$. Manasevich [12] extend the result to infinite-dimensional cases by using a global inversion theorem. See also Bates and Ekeland [2] and Manasevich [13] for related results. Recently, by means of the reduction method, a perturbation argument and the least action principle, Tang and Wu [19] greatly improve the result of Lazer et al. [8] in three aspects: requiring the spaces being reflexive Banach space instead of Hilbert space; requiring the functionals being C^1 instead of C^2; and using much weaker convexity of the functionals. As a main application, they successively studied the existence of periodic solutions of problem (1.1) with subquadratic convex potential, with subquadratic $\mu(t)$-convex potential and with subquadratic $k(t)$-concave potential, which unifies and significantly improves the recent results in [14, 24, 28]. Here we say that the potential F is $\mu(t)$-convex, if there exists $\mu \in L^1(0, T; \mathbb{R})$ such that $F(t, x) - \frac{1}{2}\mu(t)|x|^2$ in convex in x for a.e. $t \in [0, T]$. It is worth pointing out that the reduction technique, which has been motivated in earlier papers by Amann [1] and Thews [21] to discuss the existence and multiplicity of solutions for nonlinear equations, is rather powerful. For its applications on semilinear elliptic equations, we refer the readers to [24].

In the present paper, based on the reduction method and some abstract critical point theorem, i.e., the three-critical-point theorem proposed by Brezis and Nirenberg, we shall study the multiplicity of periodic solutions of (1.1) with subquadratic $k(t)$-concave or subquadratic $\mu(t)$-convex potential, which complements the results mentioned above and improves the corresponding results in Wu [24] and Zhao and Wu [27].

1.1 The Subquadratic $k(t)$-Concave Case

For the subquadratic $k(t)$-concave case, we make the following hypotheses:

- (A_1) There exists $k \in L^1(0, T; \mathbb{R}^+)$ with $\int_0^T k(t)dt < 12/T$ such that $-F(t, x) + \frac{1}{2}k(t)|x|^2$ is convex in x for a.e. $t \in [0, T]$.
- (A_2) $\int_0^T F(t, x)dt \to +\infty$ as $|x| \to +\infty$, $x \in \mathbb{R}^N$.
- (A_3) There exists $\delta > 0$ such that $F(t, x) - F(t, 0) \leq 0$ for all $|x| \leq \delta$ and a.e. $t \in [0, T]$.

(A_4) There exists $k \in L^1(0, T; \mathbb{R}^+)$ with $k(t) \leq \omega^2$ for a.e. $t \in [0, T]$ and

$$\text{meas}\left\{t \in [0, T] : k(t) < \omega^2\right\} > 0,$$

such that $-F(t, x) + \frac{1}{2}k(t)|x|^2$ is convex in x for a.e. $t \in [0, T]$, where $\omega = 2\pi\backslash T$.

Theorem 1.1 *Assume that the potential F satisfies assumptions (A) and (A_1)-(A_3). Then problem (1.1) possesses at least three distinct solutions.*

Remark 1.1 Theorem 1.1 generalizes [27, Theorem 1], where the existence of one nonzero solution is obtained under the assumptions (A), (A_2) and

(A_1') There exists $k \in L^1(0, T; \mathbb{R}^+)$ with $0 < \int_0^T k(t)dt < 12/T$ such that $|\nabla F(t, x_1) - \nabla F(t, x_2)| \leq k(t)|x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

(A_3') For a.e. $t \in [0, T]$, $\nabla F(t, 0) = 0$, and there exists $\delta > 0$ such that $F(t, x) - F(t, 0) < 0$ for all $x \in \mathbb{R}^N$ with $0 < |x| \leq \delta$ and for a.e. $t \in [0, T]$.

Since, by condition (A_1'), we have

$$\begin{aligned}(\nabla(-F(t, x)) - \nabla(-F(t, y)), x - y) &\geq -|\nabla F(t, x) - \nabla F(t, y)||x - y| \\ &\geq -k(t)|x - y|^2\end{aligned}$$

for all $x, y \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, it follows that $-F(t, x) + \frac{1}{2}k(t)|x|^2$ is convex in x for a.e. $t \in [0, T]$. Hence (A_1') implies (A_1). In addition, the condition $\nabla F(t, 0) = 0$ for a.e. $t \in [0, T]$ (see assumption (A_3')) is deleted and our conclusion is better. There are functions F satisfying Theorem 1.1 but not satisfying the results in [14, 24, 27, 28]. For example, let

$$F(t, x) = \frac{1}{2}\alpha(t)|x|^2 - |x|^{\frac{3}{2}}, \qquad \forall x \in \mathbb{R}^N \text{ and a.e. } t \in [0, T],$$

where $\alpha \in L^\infty(0, T; \mathbb{R})$ with

$$\frac{6}{T} \leq \int_0^T \alpha^-(t)dt < \int_0^T \alpha^+(t)dt < \frac{12}{T},$$

and $\alpha^\pm(t)=\max\{\pm\alpha(t), 0\}$. Then F satisfies Theorem 1.1 with $k = \alpha^+$. But it does not satisfy the corresponding results in [14, 24, 27, 28], for $\int_0^T |\alpha(t)|dt > 12/T$ and F is not convex in x for $t \in [0, T]$ with $\alpha^-(t) > 0$.

Theorem 1.2 *The conclusion of Theorem 1.1 holds if we replace (A_1) by (A_4).*

Remark 1.2 Theorem 1.2 complements [19, Theorem 5.1], where the existence of one solution was obtained under the assumptions (A), (A_2) and (A_4). There are functions F satisfying Theorem 1.2 but not satisfying Theorem 1.1. For example, let

$$F(t, x) = \frac{2}{3T}\omega^2\left(\frac{3T}{4} - t\right)|x|^2 - |x|^{\frac{3}{2}}, \qquad \forall x \in \mathbb{R}^N \text{ and a.e. } t \in [0, T].$$

Then $\alpha(t) = \frac{4}{3T}\omega^2\left(\frac{3T}{4} - t\right)$ for $t \in [0, T]$,

$$\alpha^+(t) = \begin{cases} \alpha(t), & t \in \left[0, \frac{3T}{4}\right], \\ 0, & t \in \left[\frac{3T}{4}, T\right], \end{cases} \qquad \alpha^-(t) = \begin{cases} 0, & t \in \left[0, \frac{3T}{4}\right], \\ -\alpha(t), & t \in \left[\frac{3T}{4}, T\right], \end{cases}$$

and

$$-F(t,x) + \frac{1}{2}\alpha^+(t)|x|^2 = \frac{1}{2}\alpha^-(t)|x|^2 + |x|^{\frac{3}{2}}$$

for $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. It is easy to check that F satisfies the conditions of Theorem 1.2 with $k = \alpha^+$. However, since

$$\int_0^T \alpha^+(t)dt = \int_0^{\frac{3T}{4}} \frac{4}{3T}\omega^2\left(\frac{3T}{4} - t\right)dt = \frac{3\pi^2}{2T} > \frac{12}{T},$$

it does not satisfy condition (A_1) and hence not satisfy Theorem 1.1.

1.2 The Subquadratic $\mu(t)$-Convex Case

For the subquadratic $\mu(t)$-convex case, we make the following hypotheses:

(A_5) There exists $\mu \in L^1(0, T; \mathbb{R})$ with $\int_0^T \mu(t)dt > 0$ such that $F(t,x) - \frac{1}{2}\mu(t)|x|^2$ is convex in x for a.e. $t \in [0, T]$.

(A_6) There exist $\alpha \in L^1(0, T; \mathbb{R}^+)$ with $\int_0^T \alpha(t)dt < 12/T$ and $\gamma \in L^1(0, T; \mathbb{R}^+)$ such that

$$F(t,x) \le \frac{1}{2}\alpha(t)|x|^2 + \gamma(t), \qquad \forall x \in \mathbb{R}^N \text{ and a.e. } t \in [0, T]. \tag{1.2}$$

(A_7) There exist $\delta > 0$ and an integer $k \ge 1$ such that

$$\frac{1}{2}k^2\omega^2|x|^2 \le F(t,x) - F(t,0) \le \frac{1}{2}(k+1)^2\omega^2|x|^2$$

for all $|x| \le \delta$ and a.e. $t \in [0, T]$.

(A_8) There exists $\gamma \in L^1(0, T; \mathbb{R}^+)$ such that $F(t,x) \le \frac{1}{2}\omega^2|x|^2 + \gamma(t)$ for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, and meas $\left\{t \in [0, T] : F(t,x) - \frac{1}{2}\omega^2|x|^2 \to -\infty \text{ as} |x| \to \infty\right\} > 0$.

(A_9) There exists $\alpha \in L^\infty(0, T; \mathbb{R}^+)$ with $\alpha(t) \le \omega^2$ for a.e. $t \in [0, T]$ and

$$\text{meas}\left\{t \in [0, T] : \alpha(t) < \omega^2\right\} > 0,$$

such that $\limsup_{|x| \to \infty} |x|^{-2}F(t,x) \le \frac{1}{2}\alpha(t)$ uniformly for a.e. $t \in [0, T]$.

Theorem 1.3 *Assume that assumptions (A) and* (A_5)*-*(A_7) *are satisfied. Then problem (1.1) possesses at least three distinct solutions.*

Remark 1.3 Theorem 1.3 extends [24, Theorem 2.2]; there Wu consider problem (1.1) under the hypothesis (A), (A_5) and

(A_6') There exist $f,\ g\ \in\ L^1(0,T;\mathbb{R}^+)$ with $\int_0^T f(t)dt\ <\ 12/T$ such that $|\nabla F(t,x)| \le f(t)|x| + g(t)$ for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

(A_7') There exist $\delta > 0$ and an integer $k \ge 1$ such that $\frac{1}{2}k^2\omega^2|x|^2 \le F(t,x) \le \frac{1}{2}(k+1)^2\omega^2|x|^2$ for all $|x| \le \delta$ and a.e. $t \in [0,T]$.

We point out that (A_6') is stronger than (A_6). This is because by (A) and (A_6'), we deduce that

$$\begin{aligned} F(t,x) &= \int_0^1 (\nabla F(t,sx),x)ds + F(t,0)\\ &\le \int_0^1 (f(t)|sx| + g(t))|x|ds + a(0)b(t)\\ &\le \frac{1}{2}f(t)|x|^2 + g(t)|x| + a(0)b(t)\\ &\le \frac{1}{2}f(t)|x|^2 + \frac{1}{2}g(t)\left(\frac{12-T|f|_1}{T(|g|_1+1)}|x|^2 + \frac{T(|g|_1+1)}{12-T|f|_1}\right) + a(0)b(t), \end{aligned}$$

which is just (1.2) with

$$\alpha(t) = f(t) + \frac{12-T|f|_1}{T(|g|_1+1)}g(t) \text{ and } \gamma(t) = \frac{T(|g|_1+1)}{2(12-T|f|_1)}g(t) + a(0)b(t),$$

where $|\cdot|_1$ denotes the usual norm of $L^1(0,T)$. Moreover, (A_7') implies that $F(t,0) = 0$ for a.e. $t \in [0,T]$, which gives (A_7) directly. Hence Theorem 1.3 implies [24, Theorem2.2]. There are functions F satisfying Theorem 1.3 but not satisfying [24, Theorem 2.2]. For example, let

$$F(t,x) = \begin{cases} \frac{1}{2}b(t)|x|^2, & |x| \le 1/2,\\ \frac{1}{2}(2b(t)-\mu(t))|x|^2 + \frac{1}{3}(\mu(t)-b(t))|x|^3 + \frac{1}{4}(\mu(t)-b(t))|x| - \frac{1}{24}(\mu(t)-b(t)), & \\ & 1/2 \le |x| \le 1,\\ \frac{1}{2}\mu(t)|x|^2 + \frac{3}{4}(b(t)-\mu(t))|x| + \frac{7}{24}(\mu(t)-b(t)), & |x| \ge 1, \end{cases}$$

where $b \in L^\infty(0,T;\mathbb{R})$ with $k^2\omega^2 \le b(t) \le (k+1)^2\omega^2$ $(k \ge 1)$ for a.e. $t \in [0,T]$, and $\mu \in L^\infty(0,T;\mathbb{R})$ with $\mu(t) \le \omega^2$ for a.e. $t \in [0,T]$, $\frac{6}{T} \le \int_0^T \mu^-(t)dt < \int_0^T \mu^+(t)dt \le \frac{12}{T}$, and $\mu^\pm = max\,\{\pm\mu(t), 0\}$. Then one has

$$F(t,x) \le \frac{1}{2}\mu(t)|x|^2 + p(t)|x| + c_1, \qquad \forall x \in \mathbb{R}^N \text{ and a.e. } t \in [0,T],$$

where $p(t) = 3(b(t) - \mu(t))/4 \in L^\infty(0, T; \mathbb{R}^+)$ and $c_1 = 2(k+1)^2\omega^2$, and hence

$$\begin{aligned} F(t,x) &\le \frac{1}{2}\mu^+(t)|x|^2 + p(t)|x| + c_1 \\ &\le \frac{1}{2}\mu^+(t)|x|^2 + \frac{1}{2}p(t)\left(\frac{12 - T|\mu^+|_1}{T(|p|_1+1)}|x|^2 + \frac{T(|p|_1+1)}{12 - T|\mu^+|_1}\right) + c_1, \end{aligned}$$

which is just (1.2) with

$$\alpha(t) = \mu^+(t) + \frac{12 - T|\mu^+|_1}{T(|p|_1+1)}p(t) \text{ and } \gamma(t) = \frac{T(|p|_1+1)}{2(12 - T|\mu^+|_1)}p(t) + c_1.$$

Thus F satisfies all the conditions of Theorem 1.3. But it does not satisfy the corresponding conditions of [24, Theorem 2.2], since $\int_0^T |\mu(t)|dt > 12/T$ and (A_6') does not hold.

Theorem 1.4 *The conclusion of Theorem 1.3 holds if we replace* (A_6) *by* (A_8).

Remark 1.4 There are functions F satisfying Theorem 1.4 but not satisfying Theorem 1.3 and [24, Theorem 2.2]. For example, let

$$F(t,x) = \begin{cases} \frac{1}{2}b(t)|x|^2, & |x| \le 1/2, \\ \frac{1}{2}(2b(t)-\mu(t))|x|^2 + \frac{1}{3}(\mu(t)-b(t))|x|^3 + \frac{1}{4}(\mu(t)-b(t))|x| - \frac{1}{24}(\mu(t)-b(t)), & \\ & 1/2 \le |x| \le 1, \\ \frac{1}{2}\mu(t)|x|^2 + \frac{3}{4}(b(t)-\mu(t))|x| + \frac{7}{24}(\mu(t)-b(t)), & |x| \ge 1, \end{cases}$$

where $b \in L^\infty(0, T; \mathbb{R})$ with $\omega^2 \le b \le 4\omega^2$ for a.e. $t \in [0, T]$, and $\mu \in L^\infty(0, T; \mathbb{R})$ with $\mu(t) \le \omega^2$ for a.e. $t \in [0, T]$, $\int_0^T \mu(t)dt > 0$, meas$\{t \in [0, T] : \mu^+(t) < \omega^2\} > 0$ and $b(t) - \mu(t) \le 4(\omega^2 - \mu^+(t))$ for a.e. $t \in [0, T]$. Hence we obtain

$$F(t,x) \le \frac{1}{2}\mu(t)|x|^2 + p(t)|x| + c_1, \qquad \forall x \in \mathbb{R}^N \text{ and a.e. } t \in [0, T],$$

where $p(t) = 3(b(t) - \mu(t))/4 \in L^\infty(0, T; \mathbb{R}^+)$ and $c_1 = 8\omega^2$. Then one has

$$\begin{aligned} F(t,x) &\le \frac{1}{2}\mu^+(t)|x|^2 + p(t)|x| + c_1 \\ &\le \frac{1}{2}(\mu^+(t) + \frac{1}{6}p(t))|x|^2 + 3p(t) + c_1 \end{aligned}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. Consequently, F satisfies all the assumptions of Theorem 1.4. But in the case $\int_0^T \mu^+(t)dt > 12/T$, it does not satisfy the conditions of Theorem 1.3 and [24, Theorem 2.2].

Theorem 1.5 *The conclusion of Theorem 1.3 holds if we replace* (A_6) *by* (A_9).

Remark 1.5 Theorem 1.5 is a complement of Theorem 4.2 in [19], where the existence of one solution of (1.1) was proved under conditions (A), (A_5) and (A_9). There are functions F satisfying Theorem 1.5 but not satisfying the results mentioned above. For example, let

$$F(t,x)=\begin{cases}\frac{1}{2}b(t)|x|^2, & |x|\le 1/2,\\ \frac{1}{2}(2b(t)-\mu(t))|x|^2+\frac{1}{3}(\mu(t)-b(t))|x|^3+\frac{1}{4}(\mu(t)-b(t))|x|-\frac{1}{24}(\mu(t)-b(t)), & \\ & 1/2\le|x|\le 1,\\ \frac{1}{2}\mu(t)|x|^2+\frac{3}{4}(b(t)-\mu(t))|x|+\frac{7}{24}(\mu(t)-b(t)), & |x|\ge 1,\end{cases}$$

where $b\in L^\infty(0,T;\mathbb{R}^+)$ with $k^2\omega^2\le b(t)\le(k+1)^2\omega^2$ $(k\ge 2)$ for a.e. $t\in[0,T]$, and $\mu\in L^\infty(0,T;\mathbb{R})$ with $\mu(t)\le\omega^2$ for a.e. $t\in[0,T]$, $\int_0^T\mu(t)dt>0$ and

$$\text{meas}\left\{t\in[0,T]:\mu(t)<\omega^2\right\}>0.$$

Then F satisfies Theorem 1.5 with $\alpha=\mu^+(t)$. However, in the case

$$\text{meas}\left\{t\in[0,T]:\mu(t)=\omega^2\right\}>0$$

and $\int_0^T\mu^+(t)dt>12/T$, it does not satisfy the conditions of Theorems 1.3, 1.4 and [24, Theorem 2.2].

The paper is organized as follows. Section 2 is devoted to some related preliminaries. In Sect. 3, we show that the reduction functional ψ is coercive and satisfies the (PS) condition. Then we apply the three-critical-point theorem to prove the theorems.

2 Preliminaries

Under assumption (A), the energy functional associated to problem (1.1) given by

$$\varphi(u)=-\frac{1}{2}\int_0^T|\dot{u}(t)|^2dt+\int_0^T[F(t,u(t))-F(t,0)]dt \tag{2.1}$$

is continuously differentiable and weakly lower semi-continuous on H_T^1, where

$$H_T^1=\left\{u:[0,T]\to\mathbb{R}^N\ \middle|\ \begin{array}{l}u\text{ is absolutely continuous,}\\ u(0)=u(T)\text{ and }\dot{u}\in L^2(0,T;\mathbb{R}^N)\end{array}\right\}$$

is a Hilbert space with the norm defined by

$$\|u\|=\left(\int_0^T|u(t)|^2dt+\int_0^T|\dot{u}(t)|^2dt\right)^{1/2}.$$

Furthermore, we have

$$\langle \varphi'(u), v\rangle = -\int_0^T (\dot{u}(t), \dot{v}(t))dt + \int_0^T (\nabla F(t, u(t)), v(t))dt$$

for all $u, v \in H_T^1$. It is well known that the weak solutions of problem (1.1) correspond to the critical points of φ (see [14]).

In view of [14, Proposition 1.3], for $u \in \widetilde{H}_T^1 = \left\{u \in H_T^1 : \int_0^T u(t)dt = 0\right\}$, we have

$$\int_0^T |u(t)|^2 dt \le \frac{T^2}{4\pi^2}\int_0^T |\dot{u}(t)|^2 dt \qquad \text{(Wirtinger's inequality)}$$

and

$$\|u\|_\infty^2 \le \frac{T}{12}\int_0^T |\dot{u}(t)|^2 dt \qquad \text{(Sobolev's inequality)},$$

which implies that

$$\|u\|_\infty \le C\|u\| \tag{2.2}$$

for all $u \in H_T^1$ and some $C > 0$, where $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$.

Lemma 2.1 *If $\alpha \in L^\infty(0, T; \mathbb{R}^+)$ such that $\alpha(t) \le \omega^2$ for a.e. $t \in [0, T]$ and*

$$meas\left\{t \in [0, T] : \alpha(t) < \omega^2\right\} > 0,$$

then there exists $a < 1$ such that

$$\int_0^T \alpha(t)|u|^2 dt \le a\int_0^T |\dot{u}|^2 dt, \qquad \forall u \in \widetilde{H}_T^1.$$

Proof We prove this assertion by contradiction. In fact, if not, there exists a sequence $(u_n) \subset \widetilde{H}_T^1$ such that

$$\int_0^T \alpha(t)|u_n|^2 dt > \left(1 - \frac{1}{n}\right)\int_0^T |\dot{u}_n|^2 dt, \qquad \forall n \in \mathbb{N},$$

which implies that $u_n \ne 0$ for all n. By the homogeneity of the above inequality, we may assume that $\int_0^T |\dot{u}_n|^2 dt = 1$ and

$$\int_0^T \alpha(t)|u_n|^2 dt \ge 1 - \frac{1}{n}, \qquad \forall n \in \mathbb{N}. \tag{2.3}$$

It follows from the weak compactness of the unit ball of $\widetilde{H}_T^1$ that there exists a subsequence, still denoted by (u_n), such that

$$u_n \rightharpoonup u \qquad \text{in } \widetilde{H}_T^1,$$
$$u_n \to u \qquad \text{in } C(0, T; \mathbb{R}^N).$$

Combining this with (2.3), we obtain

$$\int_0^T \alpha(t)|u|^2 dt \geq 1,$$

and then,

$$1 \geq \int_0^T |\dot{u}|^2 dt \geq \omega^2 \int_0^T |u|^2 dt \geq \int_0^T \alpha(t)|u|^2 dt \geq 1.$$

Hence we obtain

$$1 = \int_0^T |\dot{u}|^2 dt = \omega^2 \int_0^T |u|^2 dt$$

and

$$\int_0^T (\omega^2 - \alpha(t))|u|^2 dt = 0,$$

which yields that $u = a \cos \omega t + b \sin \omega t$, $a, b \in \mathbb{R}^N$, $u \neq 0$ and $u = 0$ on a positive measure subset. This contradicts the fact that $u = a \cos \omega t + b \sin \omega t$ only has finite zeros if $u \neq 0$. □

To end this section, we state the reduction method developed by Tang and Wu [19, Lemma 2.1] and the three-critical-point theorem proposed by Brezis and Nirenberg (see [3, Theorem4]).

Proposition 2.1 (see [19]) *Suppose that V is a reflexive Banach space and W is a Banach space, $\varphi \in C^1(V \times W, \mathbb{R})$. Assume that there exists $\mu > 0$ such that*

$$\langle D_1\varphi(v_1, w) - D_1\varphi(v_2, w), v_1 - v_2\rangle \leq -\mu\|v_1 - v_2\|^2 \tag{2.4}$$

for all $v_1, v_2 \in V$ and $w \in W$. Then there exists a map $\theta \in C(W, V)$ such that $\theta(w)$ is the unique maxmum of $\varphi(\cdot, w)$ for all $w \in W$ and the functional ψ, given by

$$\psi(w) = \varphi(\theta(w), w) = \sup_{v \in V} \varphi(v, w),$$

is continuously differentiable and

$$\psi'(w) = D_2\varphi(\theta(w), w), \quad \forall w \in W.$$

Moreover, $(\theta(w), w)$ *is a critical point of* φ *if and only if* w *is a critical point of* ψ.

Remark 2.1 When V and W are Hilbert spaces, Proposition 2.1 is a corollary of Amann [1, Theorem 2.3].

Proposition 2.2 (Brezis and Nirenberg [3]) *Let* X *be a Banach space with a direct sum decomposition* $X = X_1 \bigoplus X_2$ *with* $\dim X_2 < +\infty$ *and let* ψ *be a* C^1*-functional on* X *with* $\psi(0) = 0$*, satisfying the (PS) condition. Assume that, for some* $\delta_0 > 0$,

$$\psi(u) \geq 0, \qquad \forall u \in X_1 \text{ with } \|u\| \leq \delta_0$$

and

$$\psi(u) \leq 0, \qquad \forall u \in X_2 \text{ with } \|u\| \leq \delta_0.$$

Assume also that ψ *is bounded from below and* $\inf_{u\in X} \psi(u) < 0$*. Then* ψ *has at least two nonzero critical points.*

3 Proofs of the Theorems

First, we consider the subquadratic $k(t)$-concave case. Let φ be the functional introduced in (2.1) and decompose the Hilbert space H_T^1 as $H_T^1 = V \bigoplus W$ with

$$V = \widetilde{H}_T^1 \qquad \text{and} \qquad W = \mathbb{R}^N.$$

Proof of Theorem 1.1 The proof will be divided into several steps.

(1) By condition (A_1), since

$$(\nabla(-F(t, x)) - \nabla(-F(t, y)), x - y) \geq -k(t)|x - y|^2, \qquad \forall x, y \in \mathbb{R}^N,$$

we obtain, for $v_1, v_2 \in \widetilde{H}_T^1$ and $x \in \mathbb{R}^N$,

$$\begin{aligned}
&\langle D_1\varphi(v_1, x) - D_1\varphi(v_2, x), v_1 - v_2\rangle \\
&= -\int_0^T |\dot{v}_1 - \dot{v}_2|^2 dt + \int_0^T (\nabla F(t, v_1 + x) - \nabla F(t, v_2 + x), v_1 - v_2) dt \\
&\leq -\int_0^T |\dot{v}_1 - \dot{v}_2|^2 dt + \int_0^T k(t)|v_1 - v_2|^2 dt \\
&\leq -\int_0^T |\dot{v}_1 - \dot{v}_2|^2 dt + \int_0^T k(t) dt \cdot \|v_1 - v_2\|_\infty^2 \\
&\leq -\left(1 - \frac{T}{12}\int_0^T k(t) dt\right) |\dot{v}_1 - \dot{v}_2|_2^2,
\end{aligned}$$

that is, φ satisfies (2.4) with $\mu = 1-(T/12)\int_0^T k(t)dt$. Applying Proposition 2.1, we obtain a continuous map $\theta : \mathbb{R}^N \to \widetilde{H}_T^1$ and a C^1-functional $\psi : \mathbb{R}^N \to \mathbb{R}$ such that

$$\psi(x) = \varphi(\theta(x) + x) = \sup_{v \in \widetilde{H}_T^1} \varphi(v + x).$$

It suffices to find two nonzero critical points of ψ.

(2) It follows from the definition of ψ and condition (A_2) that

$$\psi(x) \geq \varphi(x) = \int_0^T [F(t, x) - F(t, 0)]dt \to +\infty \qquad \text{as } |x| \to \infty.$$

Hence ψ is coercive on $\mathbb{R}^N$, and then ψ is bounded from below and satisfies the (PS) condition, i.e., $(u_n) \subset W$ possesses a convergent subsequence whenever $\{\psi(u_n)\}$ is bounded and $\psi'(u_n) \to 0$ as $n \to \infty$.

(3) Set $W_1 = \{0\}$ and $W_2 = \mathbb{R}^N$. The continuity of θ implies that there exists $\delta_0 > 0$ small enough such that

$$\|\theta(x) + x\|_\infty \leq C\|\theta(x) + x\| \leq \delta$$

for all $x \in W_2$ with $|x| \leq \delta_0$. Thus, using (A_3), we obtain

$$\psi(x) = \varphi(\theta(x) + x) \leq \int_0^T [F(t, \theta(x) + x) - F(t, 0)]dt \leq 0 \tag{3.1}$$

for all $x \in W_2$ with $|x| \leq \delta_0$. On the other hand, noting

$$\psi(0) \geq \varphi(0) = 0,$$

we have

$$\psi(w) \geq 0, \qquad \forall w \in W_1 \text{ with } \|w\| \leq \delta_0.$$

This, jointly with (3.1), shows that φ has a local linking at 0 with respect to (W_1, W_2).

(4) If $\inf_{x \in \mathbb{R}^N} \psi(x) = 0$, then all $x \in W_2$ with $|x| \leq \delta_0$ are minimums of ψ, which implies that ψ has infinitely many critical points. If $\inf_{x \in \mathbb{R}^N} \psi(x) < 0$, Proposition 2.2 implies that ψ has at least two nontrivial critical points. In any case, we can find two nonzero critical points x_1, x_2 of ψ. Consequently, $u_1 = \theta(x_1) + x_1$ and $u_2 = \theta(x_2) + x_2$ are two nontrivial solutions of problem (1.1). This completes the proof.

□

Proof of Theorem 1.2 The proof is identical to that of Theorem 1.1 except that now (2.4) follows from condition (A_4). Indeed, using Lemma 2.1, we obtain

$$\begin{aligned}\langle D_1\varphi(v_1,x)-D_1\varphi(v_2,x),v_1-v_2\rangle &\le -\int_0^T|\dot{v}_1-\dot{v}_2|^2dt+\int_0^T k(t)|v_1-v_2|^2dt\\ &\le -(1-a)|\dot{v}_1-\dot{v}_2|_2^2\end{aligned}$$

for all $v_1, v_2 \in \widetilde{H}_T^1$ and $x \in \mathbb{R}^N$. Therefore, (2.4) holds with $\mu = 1 - a$. □

Next we consider the subquadratic $\mu(t)$-convex case. Let $\Phi = -\varphi$ and decompose the Hilbert space H_T^1 as $H_T^1 = V \bigoplus W$ with

$$V=\mathbb{R}^N \qquad \text{and} \qquad W=\widetilde{H}_T^1.$$

Proof of Theorem 1.3 We divide the proof into several steps.

(1) It follows from (A_5) that

$$(\nabla F(t,x)-\nabla F(t,y),x-y)\ge \mu(t)|x-y|^2, \qquad \forall x,y\in\mathbb{R}^N,$$

and hence,

$$\begin{aligned}\langle D_1\Phi(x_1,w)-D_1\Phi(x_2,w),x_1-x_2\rangle &= -\int_0^T(\nabla F(t,x_1+w)-\nabla F(t,x_2+w),x_1-x_2)dt\\ &\le -\int_0^T\mu(t)|x_1-x_2|^2dt\\ &\le -\frac{1}{T}\int_0^T\mu(t)dt\|x_1-x_2\|^2, \end{aligned}\tag{3.2}$$

for all $x_1, x_2 \in \mathbb{R}^N$ and $w \in \widetilde{H}_T^1$. Therefore, (2.4) holds with $\mu = T^{-1}\int_0^T \mu(t)dt$. In view of Proposition 2.1, we obtain a continuous map $\theta : \widetilde{H}_T^1 \to \mathbb{R}^N$ and a C^1-functional $\psi : \widetilde{H}_T^1 \to \mathbb{R}$ such that

$$\psi(w)=\Phi(\theta(w)+w)=\sup_{x\in\mathbb{R}^N}\Phi(x+w).$$

It suffices to find two nonzero critical points of ψ.

(2) Since the reduced functional ψ is defined on an infinite dimensional subspace, it is difficult to verify the coerciveness and (PS) condition for ψ. To overcome this difficulty, we need to study the properties of θ.

We claim that the functional θ obtained above is bounded, i.e., it maps bounded sets into bounded sets. Indeed, note that $D_1\Phi(\theta(w), w) = 0$. Setting $x_1 = \theta(w)$

and $x_2 = 0$ in (3.2), we obtain

$$\begin{aligned}\frac{1}{T}\int_0^T \mu(t)dt\|\theta(w)\|^2 &\le -\langle D_1\Phi(\theta(w), w) - D_1\Phi(0, w), \theta(w)\rangle \\ &= \langle D_1\Phi(0, w), \theta(w)\rangle \\ &= -\int_0^T (\nabla F(t, w), \theta(w))dt \\ &\le \max_{s\in[0, C\|w\|]} a(s)\int_0^T b(t)dt \cdot C\|\theta(w)\|\end{aligned}$$

by assumption (A) and (2.2). Hence the desired result follows from the last inequality.

Now letting $(w_n) \subset \widetilde{H}_T^1$ be a bounded sequence such that $\psi'(w_n) \to 0$ as $n \to \infty$, we show that (w_n) has a convergent subsequence. Without loss of generality, we can assume that

$$\begin{aligned} &w_n \rightharpoonup w \qquad \text{in } \widetilde{H}_T^1, \\ &w_n \to w \qquad \text{in } C(0, T; \mathbb{R}^N).\end{aligned}$$

Since (w_n) is bounded, then $\{\theta(w_n)\}$ and hence $\{\theta(w_n) + w_n\}$ are also bounded in H_T^1. Hence, taking

$$a_1 = \max_{s\in[0, C(\|\theta(w_n)+w_n\|)]} a(s), \qquad a_2 = \max_{s\in[0, C(\|\theta(w)+w\|)]} a(s),$$

we have

$$\begin{aligned}&|\dot{w}_n - \dot{w}|_2^2 \\ &\quad= \langle \psi'(w_n) - \psi'(w), w_n - w\rangle + \int_0^T (\nabla F(t, \theta(w_n) + w_n) - \nabla F(t, \theta(w) + w), w_n - w)dt \\ &\quad\le \|\psi'(w_n)\|\|w_n - w\| - \langle \psi'(w), w_n - w\rangle + \left(a_1\int_0^T b(t)dt + a_2\int_0^T b(t)dt\right)\|w_n - w\|_\infty \\ &\quad\to 0\end{aligned}$$

because of assumption (A) and (2.2). The equivalence of the L^2-norm for $\dot{w}$ and the H_T^1-norm on $\widetilde{H}_T^1$ implies that

$$w_n \to w \qquad \text{in } \widetilde{H}_T^1.$$

Hence the (PS) condition holds.

Moreover, it follows from (A_6) and Sobolev's inequality that

$$\begin{aligned}\Phi(w) &= \frac{1}{2}\int_0^T |\dot{w}|^2 dt - \int_0^T [F(t,w)-F(t,0)]dt \\ &\geq \frac{1}{2}\int_0^T |\dot{w}|^2 dt - \frac{1}{2}\int_0^T \alpha(t)|w|^2 dt - \int_0^T \gamma(t)dt + \int_0^T F(t,0)dt \\ &\geq \frac{1}{2}\int_0^T |\dot{w}|^2 dt - \frac{1}{2}\int_0^T \alpha(t)dt\cdot \|w\|_\infty^2 - \int_0^T \gamma(t)dt + \int_0^T F(t,0)dt \\ &\geq \frac{1}{2}\left(1-\frac{T}{12}\int_0^T \alpha(t)dt\right)|\dot{w}|_2^2 - \int_0^T \gamma(t)dt + \int_0^T F(t,0)dt\end{aligned}$$

for all $w \in \widetilde{H}_T^1$, which implies that

$$\Phi(w) \to +\infty \qquad \text{as } \|w\| \to \infty,\ \ w \in \widetilde{H}_T^1. \tag{3.3}$$

This, jointly with the definition of ψ, shows that

$$\psi(w) \to +\infty \qquad \text{as } \|w\| \to \infty,\ \ w \in \widetilde{H}_T^1.$$

So ψ is bounded from below.

(3) Letting

$$W_2 = \left\{ \sum_{j=1}^{k} (a_j \cos j\omega t + b_j \sin j\omega t) \,\middle|\, a_j, b_j \in \mathbb{R}^N,\ j = 1, 2, \cdots, k \right\}$$

and $W_1 = (\mathbb{R}^N \bigoplus W_2)^\perp$. Then $\widetilde{H}_T^1 = W_1 \bigoplus W_2$, and the reduction functional ψ has a local linking at 0 with respect to (W_1, W_2).
In fact, condition (A_7) and (2.2) imply that

$$\begin{aligned}\psi(w) &= \sup_{x\in\mathbb{R}^N} \Phi(x+w) \\ &\geq \Phi(w) \\ &= \frac{1}{2}\int_0^T |\dot{w}|^2 dt - \int_0^T [F(t,w)-F(t,0)]dt \\ &\geq \frac{1}{2}\int_0^T |\dot{w}|^2 dt - \frac{(k+1)^2\omega^2}{2}\int_0^T |w|^2 dt \\ &\geq 0\end{aligned}$$

for all $w \in W_1$ with $\|w\| \leq \delta/C$. Furthermore, by the continuity of θ, the map $w \longmapsto \theta(w) + w$ from $\widetilde{H}_T^1$ to H_T^1 is continuous. Hence, using (2.2), there exists a constant $\delta_0 \in (0, \delta/C)$ such that

$$\|\theta(w)+w\|_\infty \leq C\|\theta(w)+w\| \leq \delta$$

for $w \in \widetilde{H}_T^1$ with $\|w\| \le \delta_0$, so that, using (A_7), we obtain

$$\begin{aligned}\psi(w) &= \Phi(\theta(w)+w)\\ &= \frac{1}{2}\int_0^T |\dot{w}|^2 dt - \int_0^T [F(t,\theta(w)+w) - F(t,0)]dt\\ &\le \frac{1}{2}\int_0^T |\dot{w}|^2 dt - \frac{1}{2}k^2\omega^2 \int_0^T |\theta(w)+w|^2 dt\\ &\le 0 \end{aligned} \tag{3.4}$$

for $w \in W_2$ with $\|w\| \le \delta_0$. Thus, ψ has a local linking at 0 with respect to (W_1, W_2).

(4) It follows from (3.4) that $\inf_{w\in\widetilde{H}_T^1}\psi(w) \le 0$.
If $\inf_{w\in\widetilde{H}_T^1}\psi(w) = 0$, all $w \in W_2$ with $\|w\| \le \delta_0$ are minimums of ψ, which implies that ψ has infinitely many critical points. If $\inf_{w\in\widetilde{H}_T^1}\psi(w) < 0$, Proposition 2.2 implies that ψ has at least two nontrivial critical points. In any case, the functional ψ possesses two nonzero critical points, denoted by w_1, w_2. Consequently, $u_1 = \theta(w_1) + w_1$ and $u_2 = \theta(w_2) + w_2$ are two nonzero solutions of problem (1.1). This completes the proof.

□

Proof of Theorem 1.4 The proof is similar to that of Theorem 1.3 with the exception that now (3.3) follows from condition (A_8). We verify this assertion by contradiction. If not, there exist $c_2 \in \mathbb{R}$ and a sequence $(u_n) \subset \widetilde{H}_T^1$ such that $\|u_n\| \to \infty$ as $n \to \infty$ and

$$\Phi(u_n) \le c_2, \qquad \forall n \in \mathbb{N}. \tag{3.5}$$

Set

$$H_1 = \mathbb{R}^N \bigoplus (\sin\omega t\mathbb{R}^N) \bigoplus (\cos\omega t\mathbb{R}^N).$$

Then $H_T^1 = H_1 \bigoplus H_1^{\perp}$, and for $u_n \in \widetilde{H}_T^1$, we can rewrite

$$u_n = (a_n\|u_n\|\cos\omega t + b_n\|u_n\|\sin\omega t) + w_n,$$

where $a_n, b_n \in \mathbb{R}^N$ and $w_n \in H_1^{\perp}$. It is obvious that

$$\int_0^T |\dot{w}_n|^2 dt \ge 4\omega^2 \int_0^T |w_n|^2 dt.$$

Hence, using (A_8), we have

$$\begin{aligned}c_2 &\geq \Phi(u_n)\\ &\geq \frac{1}{2}\int_0^T |\dot{u}_n|^2 dt - \frac{\omega^2}{2}\int_0^T |u_n|^2 dt - \int_0^T \gamma(t)dt + \int_0^T F(t,0)dt\\ &= \frac{1}{2}\int_0^T |\dot{w}_n|^2 dt - \frac{\omega^2}{2}\int_0^T |w_n|^2 dt - \int_0^T \gamma(t)dt + \int_0^T F(t,0)dt\\ &\geq \frac{3}{8}\int_0^T |\dot{w}_n|^2 dt - \int_0^T \gamma(t)dt + \int_0^T F(t,0)dt,\end{aligned}$$

which implies that (w_n) is bounded. Taking $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$, and hence (a_n) and (b_n) are bounded. Up to a subsequence, we can assume that

$$a_n \to a \quad \text{and} \quad b_n \to b \quad \text{as } n \to \infty,$$

for some $a, b \in \mathbb{R}$. By the boundedness of (w_n), one has $w_n/\|u_n\| \to 0$. Hence

$$v_n \to a\cos\omega t + b\sin\omega t \text{ in } H_T^1,$$

and $|a| + |b| \neq 0$, which yields that

$$v_n(t) \to a\cos\omega t + b\sin\omega t \text{ uniformly for a.e. } t \in [0, T]$$

by Sobolev's inequality. Therefore,

$$|u_n(t)| \to \infty \text{ as } n \to \infty$$

for a.e. $t \in [0, T]$, since $a\cos\omega t + b\sin\omega t$ only has finite zeros. Now set

$$E = \left\{ t \in [0, T] \,\middle|\, F(t, x) - \frac{1}{2}\omega^2|x|^2 \to -\infty \text{ as } |x| \to \infty \right\}.$$

It follows from Fatou's lemma (see [25]) that

$$\begin{aligned}\liminf_{n\to\infty} \Phi(u_n) &\geq \liminf_{n\to\infty} \int_0^T \left(\frac{\omega^2}{2}|u_n|^2 - F(t, u_n)\right) dt + \int_0^T F(t,0)dt\\ &\geq \liminf_{n\to\infty} \int_E \left(\frac{\omega^2}{2}|u_n|^2 - F(t, u_n)\right) dt - \int_0^T \gamma(t)dt + \int_0^T F(t,0)dt\\ &= +\infty,\end{aligned}$$

a contradiction with (3.5). Thus

$$\Phi(u) \to +\infty \text{ as } \|u\| \to \infty, \quad u \in \widetilde{H}_T^1.$$

□

Proof of Theorem 1.5 It follows from (A) and (A_9) that, for $\varepsilon \in (0, 1-a)$, there exists $M_\varepsilon > 0$ such that

$$F(t,x) \le \frac{1}{2}(\alpha(t)+\varepsilon\omega^2)|x|^2 + \max_{s\in[0,M_\varepsilon]} a(s)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. Hence, we obtain, by Lemma 2.1 and Wirtinger's inequality,

$$\begin{aligned}\Phi(w) &= \frac{1}{2}\int_0^T |\dot{w}|^2 dt - \int_0^T [F(t,w)-F(t,0)]dt \\ &\ge \frac{1}{2}\int_0^T |\dot{w}|^2 dt - \frac{1}{2}\int_0^T (\alpha(t)+\varepsilon\omega^2)|w|^2 dt - \max_{s\in[0,M_\varepsilon]} a(s)\int_0^T b(t)dt + \int_0^T F(t,0)dt \\ &\ge \frac{1}{2}(1-a-\varepsilon)|\dot{w}|_2^2 - \max_{s\in[0,M_\varepsilon]} a(s)\int_0^T b(t)dt + \int_0^T F(t,0)dt\end{aligned}$$

for all $w \in \widetilde{H}_T^1$. The equivalence of the norm $|\dot{w}|_2$ and the H_T^1-norm on $\widetilde{H}_T^1$ shows that

$$\Phi(w) \to +\infty \text{ as } \|w\| \to \infty, \quad w \in \widetilde{H}_T^1.$$

The remainder of the proof is the same as that of Theorem 1.3. Consequently, Theorem 1.5 holds. □

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