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Algebraic Integrability of Planar Polynomial Vector Fields by Extension to Hirzebruch Surfaces

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Abstract

We study algebraic integrability of complex planar polynomial vector fields $X = A(x, y)(\partial/\partial x) + B(x, y)(\partial/\partial y)$ through extensions to Hirzebruch surfaces. Using these extensions, each vector field X determines two infinite families of planar vector fields that depend on a natural parameter which, when X has a rational first integral, satisfy strong properties about the dicriticity of the points at the line $x = 0$ and of the origin. As a consequence, we obtain new necessary conditions for algebraic integrability of planar vector fields and, if X has a rational first integral, we provide a region in $\mathbb{R}^2_{\geq 0}$ that contains all the pairs (i, j) corresponding to monomials $x^i y^j$ involved in the generic invariant curve of X.

Keywords Planar polynomial vector fields · Rational first integrals · Hirzebuch surfaces

Mathematics Subject Classification 34C05 · 32S65 · 14J26

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1 Introduction

The study of algebraic solutions of ordinary differential equations goes back to the 19th century [50] and the problem of deciding whether all their solutions are algebraic was completely solved for linear homogeneous differential equations of the second order with rational coefficients in the mid 20th century (see [33] for example). The next step was to consider polynomial differential equations of first order and first degree. Remarkable mathematicians as Darboux [17], Poincaré [44–47], Painlevé [41] and Autonne [3] were authors of seminal papers in this topic. One of the main proposed problems was to characterize those complex planar differential systems which are algebraically integrable. As said, this problem is more than a century old and, despite the fact that much progress has been made, it remains unsolved. The question of deciding about algebraic integrability is related to other very interesting problems as to bound the number of limit cycles of a (real) differential system [37–39] or the center problem [19, 49].

When a complex planar polynomial differential system is algebraically integrable, with primitive rational first integral f/g, $f, g \in \mathbb{C}[x, y]$ (see Definition 2.1), all their invariant curves are algebraic and their irreducible components are those of the elements of the pencil of curves with equations $\alpha f + \beta g = 0$, where $(\alpha : \beta)$ runs over the complex projective line [55]. Only finitely many pairs $(\alpha : \beta)$ (named remarkable values) give rise to reducible polynomials $\alpha f + \beta g$ in $\mathbb{C}[x, y]$. Remarkable values are of importance for the phase portrait of the system [12, 24]. Jouanolou in [32] (see also [16] and [13–15]) proved that the existence of enough algebraic invariant curves for the planar differential system implies its algebraic integrability.

Poincaré [44–47], considering an algebraically integrable planar differential system and looking for a rational first integral, posed the problem of giving an upper bound on the degree of the first integral which depends only on the degree of the differential system. Lins-Neto in [34] proved that there is no such a bound even for families of systems where the analytic type and the number of singularities of the associated vector field remain constant. This problem has attracted a lot of attention and one can find many articles related to it (even in higher dimension) [6, 8, 10, 11, 20, 26, 28, 42, 43, 47, 52–56]. The literature also contains several algorithms which, in specific cases, allow us to compute first integrals [4, 21–23, 25, 28].

We are concerned with rational first integrals of planar systems; however other interesting classes of first integrals (like elementary, Liouvillian or Darboux first integrals) are being studied (see for instance the survey [40] and references therein).

Many of the recent advances on algebraic integrability and the Poincaré problem have been obtained by considering foliations of the projective plane as one can see in several previous references. Since the involved foliations have singularities, a useful tool for addressing integrability and related problems consists of successively blowing up the projective plane and the blown up surfaces at the ordinary singularities of the foliation to reach a transformed foliation with at most simple singularities. It is well-known that the relatively minimal models of smooth complex rational surfaces Z are the complex projective plane $\mathbb{P}^2$ and the complex Hirzebruch surfaces $\mathbb{F}_\delta$, $\delta \neq 1$ being a nonnegative integer. This means that Z can be obtained from $\mathbb{P}^2$ or $\mathbb{F}_\delta$ after finitely many blowups.

In this paper, instead of considering foliations on $\mathbb{P}^2$, we propose to extend complex planar polynomial vector fields X to foliations on Hirzebruch surfaces $\mathbb{F}_\delta$, $\delta \geq 0$. In this way, we have an extension $\mathcal{F}_X^\delta$ for each nonnegative integer δ corresponding to a complex Hirzebruch surface. We hope that this procedure will contribute to substantial progress towards the above described problems.

Section 2 briefly recalls the essentials about algebraically integrable complex planar vector fields and foliations on surfaces, while Subsection 3.1 does the same with the basics about Hirzebruch surfaces. Keeping in mind the case of the projective plane (see [7] for instance), a foliation on a Hirzebruch surface can be given in a simple way as an affine vector field, or an affine differential 1-form, in four variables, where the coefficients are bigraded homogeneous polynomials which satisfy certain conditions with respect to radial vector fields, or Euler-type conditions (see the first part of Subsection 3.2).

Algorithm 3.2 considers a planar vector field X (in the variables x and y) and extends X to a bigraded homogeneous affine 1-form defining a foliation $\mathcal{F}_X^\delta$ on each Hirzebruch surface $\mathbb{F}_\delta$, where we take homogeneous coordinates $(X_0, X_1; Y_0, Y_1)$. These foliations are the key objects in our *first main result*, *Theorem* 4.2, which proves that when $X \neq c\frac{\partial}{\partial y}$, $c \in \mathbb{C} \setminus \{0\}$, is algebraically integrable, there exists a nonnegative integer δ_1 forcing a very special local behaviour of the extended foliations $\mathcal{F}_X^\delta$ according to the position of δ with respect to δ_1. More specifically, the point $(0, 1; 1, 0)$ cannot be a dicritical singularity of $\mathcal{F}_X^{\delta_1}$ but it is always a dicritical singularity of $\mathcal{F}_X^\delta$ whenever $\delta < \delta_1$, and $(0, 1; 0, 1)$ is the unique dicritical singularity of $\mathcal{F}_X^\delta$ belonging to the curve $X_0 = 0$ when $\delta > \delta_1$.

This result can be translated to the language of planar vector fields avoiding the use of Hirzebruch surfaces as we state in *Corollary* 4.4. It assigns to a vector field X two families of infinitely many vector fields $\{X_{1,0}^\delta\}_{\delta \geq 0}$ and $\{X_{1,1}^\delta\}_{\delta \geq 0}$ and states that, when X is algebraically integrable, the vector fields in these families must satisfy specific conditions of dicriticity at the origin and the points in the line $x = 0$. In this way, *algebraic integrability of X forces strong conditions on other derived planar vector fields giving rise to unknown necessary conditions for algebraic integrability.* It is worthwhile to add that our procedure discards the existence of rational first integrals for vector fields that do not satisfy the conditions in [27, Corollary 5] and that, recently, differential Galois theory has also be used for giving necessary conditions of integrability [2]. As before mentioned, a somewhat related problem (in the real setting) is the searching of algebraic limit cycles of planar polynomial vector fields. These limit cycles only exist when the vector field has no rational first integral [36] but the first integrals can be used for computing limit cycles when piecewise differential systems are considered [37].

The last section, Section 5, contains our *second main result*, *Theorem* 5.2. It considers an algebraically integrable vector field X and that obtained by swapping their variables X', and uses their extensions to Hirzebruch surfaces to determine a region in $\mathbb{R}^2_{\geq 0}$ where all the pairs (i, j) corresponding to monomials $x^i y^j$ with nonzero coefficient of the generic algebraic invariant curve g of X are included. *Corollary* 5.3 is deduced from Theorem 5.2 and when $\delta_1 = 0$ (respectively, $\delta_1' = 0$) gives a bound on the degree of the first integral of X depending on the value δ_1' (respectively, δ_1)

and the maximum degrees of the monomials x^i and y^j appearing in g with nonzero coefficient.

2 Preliminaries

2.1 Algebraically Integrable Complex Planar Polynomial Vector Fields

Let $\mathbb{C}[x, y]$ be the ring of polynomials in two variables x and y with complex coefficients and let $\mathbb{C}(x, y)$ be its quotient field. Consider a planar polynomial differential system

$$\dot{x} = A(x, y), \quad \dot{y} = B(x, y), \tag{2.1}$$

where $A(x, y), B(x, y) \in \mathbb{C}[x, y]$ are coprime or, equivalently, the planar vector field

$$X = A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y}. \tag{2.2}$$

This planar vector field can also be determined by the differential 1-form

$$\omega_X := B(x, y)dx - A(x, y)dy.$$

A *rational first integral of the system* (2.1) (*or of* X) is a rational function $f \in \mathbb{C}(x, y) \setminus \mathbb{C}$ satisfying:

$$X(f) = A(x, y)\frac{\partial f}{\partial x} + B(x, y)\frac{\partial f}{\partial y} = 0$$

(or, alternatively, $\omega_X \wedge df = 0$). We say that (2.1) (or X) is *algebraically integrable* if it admits a rational first integral f.

A rational function $h = h_1/h_2$ is said to be *reduced* when h_1 and h_2 are coprime. The degree of a rational function h, $\deg(h)$, is the maximum of the degrees of h_1 and h_2. Moreover, $h \in \mathbb{C}(x, y)$ is said to be *composite* if it can be written as $h = u \circ h'$, where $h' \in \mathbb{C}(x, y) \setminus \mathbb{C}$ and $u \in \mathbb{C}(t)$ with $\deg(u) \geq 2$. Otherwise h is said to be *noncomposite*.

An algebraically integrable differential system (2.1) admits a noncomposite reduced rational first integral f. Any rational function of the form $h = u \circ f, u \in \mathbb{C}(t) \setminus \mathbb{C}$, is also a rational first integral of (2.1); in addition, all the reduced rational first integrals of (2.1) are of this form (see [4, Theorem 10] for a proof). As a consequence, noncomposite reduced rational first integrals coincide with rational first integrals of minimal degree.

Definition 2.1 A rational first integral of the system (2.1) (or of X) is named *primitive* if it is reduced and noncomposite.

Let $h(x, y)$ be a nonzero polynomial in $\mathbb{C}[x, y]$. The algebraic curve with equation $h(x, y) = 0$ is an *invariant algebraic curve* of the system (2.1) (or of X) if $X(h) = k(x, y)h(x, y)$ for some polynomial $k(x, y)$.

Let $f = \frac{f_1(x,y)}{f_2(x,y)}$ be a primitive rational first integral of (2.1). Then the curves in $\mathbb{C}^2$ of the pencil $\alpha_1 f_1(x, y) + \alpha_2 f_2(x, y) = 0$, $(\alpha_1 : \alpha_2) \in \mathbb{P}^1$, are reduced and irreducible with the exception of those corresponding to finitely many values in $\mathbb{P}^1$ (see, for instance, [32, Chapter 2, Theorem 3.4.6]). The solutions of (2.1) are algebraic and are given by the irreducible components of the curves in this pencil (irreducible algebraic invariant curves). Abusing the notation, in this paper, the expression $\alpha_1 f_1(x, y) + \alpha_2 f_2(x, y)$, regarded as a polynomial in $\mathbb{C}(\alpha_1, \alpha_2)[x, y]$, where α_1, α_2 are also considered variables, will be named the *generic algebraic invariant curve* of X (associated to f).

2.2 Singular Foliations on Surfaces

Many interesting results about algebraic integrability of (singular) planar vector fields come from the study of singular foliations on the projective plane. This happens because these foliations are determined by homogeneous vector fields which extend the field of directions defined by the original planar vector field to the projective plane. In this article, we will use Hirzebruch surfaces instead of the projective plane.

We start by briefly recalling what a foliation is and, also, by providing some additional definitions and results we will use.

A (singular) *foliation* $\mathcal{F}$ on a smooth complex projective surface S can be defined by a family of pairs $\{(U_i, v_i)\}_{i\in I}$, given by an open covering $\{U_i\}_{i\in I}$ of S and nonvanishing holomorphic vector fields v_i on U_i such that, for any three indices i, j, k in I:

(1) $v_i = g_{ij} v_j$ on $U_i \cap U_j$, where g_{ij} is a nowhere vanishing holomorphic function on $U_i \cap U_j$.
(2) $g_{ij} g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$.

The *singular set* $\mathrm{Sing}(\mathcal{F})$ of $\mathcal{F}$ is the subset of S such that $\mathrm{Sing}(\mathcal{F}) \cap U_i$ is the set of zeros of v_i for all $i \in I$. The points in $\mathrm{Sing}(\mathcal{F})$ are called *singularities* of $\mathcal{F}$ and, when all the vector fields v_i have isolated zeros, we say that $\mathcal{F}$ has *isolated singularities* (and, in this case, $\mathrm{Sing}(\mathcal{F})$ is discrete). All the foliations considered in this paper have isolated singularities.

A foliation $\mathcal{F}$ as above can also be defined by using 1-forms. Then it is given by a family $\{(U_i, \omega_i)\}_{i\in I}$, where $\{U_i\}_{i\in I}$ is an open covering of S, ω_i is a nonzero holomorphic differential 1-form defined on U_i and, for indices i, j and k in I, it holds that

$$\omega_i = f_{ij} \omega_j \text{ on } U_i \cap U_j,$$

for some nowhere vanishing holomorphic function f_{ij} on $U_i \cap U_j$ and $f_{ij} f_{jk} = f_{ik}$ on $U_i \cap U_j \cap U_k$.

As said before, any foliation on the complex projective plane $\mathbb{P}^2$ is algebraic in the sense that it is the extension of the field of directions induced by a planar polynomial vector field; in fact, such a foliation can be described by a homogeneous polynomial vector field on $\mathbb{C}^3$, or by a homogeneous polynomial differential 1-form on $\mathbb{C}^3$ satisfying certain condition (see [7, 30]). In [29], it is shown that foliations on a Hirzebruch surface behave similarly. We explain it in the next section.

3 Foliations on Hirzebruch Surfaces

We desire to study singular complex planar polynomial vector fields through foliations on Hirzebruch surfaces. For this reason we start with a brief introduction to this class of surfaces.

3.1 Hirzebruch Surfaces

Let δ be a nonnegative integer. For each value δ, there is a rational surface $\mathbb{F}_\delta$ named the δth complex Hirzebruch surface. It can be defined as the quotient of the Cartesian product $\left(\mathbb{C}^2 \setminus \{\mathbf{0}\}\right) \times \left(\mathbb{C}^2 \setminus \{\mathbf{0}\}\right)$ by the following action on the algebraic torus $(\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$:

$$(\lambda, \mu) : (X_0, X_1; Y_0, Y_1) \mapsto (\lambda X_0, \lambda X_1; \mu Y_0, \lambda^{-\delta}\mu Y_1),$$

where $(X_0, X_1; Y_0, Y_1)$ are coordinates in $\left(\mathbb{C}^2 \setminus \{\mathbf{0}\}\right) \times \left(\mathbb{C}^2 \setminus \{\mathbf{0}\}\right)$.

$\mathbb{F}_\delta$ is a ruled algebraic surface which has the structure of a toric variety. In fact, it is the *rational scroll* $\mathbb{F}(0, \delta)$ as defined in [48, Chapter 2], where one can find more details on these surfaces.

The homogeneous coordinate ring of $\mathbb{F}_\delta$ is the polynomial ring in four variables $\mathbb{C}[X_0, X_1, Y_0, Y_1]$, where the variables are bigraded as follows: $\deg X_0 = \deg X_1 = (1, 0)$, $\deg(Y_0) = (0, 1)$ and $\deg Y_1 = (-\delta, 1)$.

The above given action $\mapsto$ provides a surjective map

$$\varphi : \left(\mathbb{C}^2 \setminus \{\mathbf{0}\}\right) \times \left(\mathbb{C}^2 \setminus \{\mathbf{0}\}\right) \to \mathbb{F}_\delta,$$

which allows us to consider the following four affine open sets of $\mathbb{F}_\delta$:

$$U_{ij} := \left\{\varphi(X_0, X_1; Y_0, Y_1) \mid X_i \neq 0 \text{ and } Y_j \neq 0\right\},$$

where $0 \leq i, j \leq 1$, giving rise to an open cover of $\mathbb{F}_\delta$.

Now, on U_{00},

$$\varphi(X_0, X_1; Y_0, Y_1) = \varphi\left(1, \frac{X_1}{X_0}, 1, \frac{X_0^\delta Y_1}{Y_0}\right),$$

and one can identify U_{00} with $\mathbb{C}^2$ after setting $\varphi(X_0, X_1; Y_0, Y_1) = (x_{00}, y_{00})$, where $x_{00} = \frac{X_1}{X_0}$ and $y_{00} = \frac{X_0^\delta Y_1}{Y_0}$. Analogous procedures can be carried out for identifying the remaining sets U_{ij} with $\mathbb{C}^2$ (with affine coordinates (x_{ij}, y_{ij})) and one gets the change of coordinates maps, in the overlaps of the sets U_{ij}, which provide the global structure of $\mathbb{F}_\delta$. For instance, the change of coordinates in the intersection $U_{00} \cap U_{10}$ is

$$(x_{00}, y_{00}) \to \left(\frac{1}{x_{00}}, x_{00}^\delta y_{00}\right) = (x_{10}, y_{10}),$$

where $\left(x_{10} = \frac{X_0}{X_1}, y_{10} = \frac{X_1^\delta Y_1}{Y_0}\right)$ are local coordinates in the set U_{10}.

To conclude this subsection, we notice that the divisor class group of $\mathbb{F}_\delta$ [31] is generated by the linear equivalence classes of two divisors F and M such that $F^2 = 0$, $M^2 = \delta$ and $F \cdot M = 1$. Moreover, the effective classes are those of the divisors of the form $d_1 F + d_2 M$, where d_1 and d_2 are integers such that $d_1 + \delta d_2 \geq 0$ and $d_2 \geq 0$. Finally, the nonzero elements of the vector space $H^0(\mathbb{F}_\delta, \mathcal{O}_{\mathbb{F}_\delta}(d_1 F + d_2 M))$ of global sections of the sheaf associated with an effective divisor $d_1 F + d_2 M$ correspond to the set of bigraded homogeneous polynomials $F(X_0, X_1, Y_0, Y_1) \in \mathbb{C}[X_0, X_1, Y_0, Y_1]$ of bidegree (d_1, d_2). These polynomials are nonzero linear combinations of monomials $X_0^a X_1^b Y_0^c Y_1^d$ such that $a + b - \delta d = d_1$ and $c + d = d_2$.

Next we study foliations on Hirzebruch surfaces showing that they have simple bigraded expressions.

3.2 Foliations on Hirzebruch surfaces. Bigraded Expressions and Reduction of Singularities

In this subsection, we show that a foliation on a Hirzebruch surface can be given by an affine vector field (or an affine differential 1-form) determined by suitable bigraded homogeneous polynomials in the variables X_0, X_1, Y_0, Y_1. This will ease our study of planar vector fields through foliations on Hirzebruch surfaces.

Let $\mathcal{F}$ be a foliation on a Hirzebruch surface. By [29, Section 3], $\mathcal{F}$ can be given through an affine vector field or through an affine 1-form. Indeed, on the one hand, $\mathcal{F}$ is determined by an affine vector field V as follows:

$$V = V_0 \frac{\partial}{\partial X_0} + V_1 \frac{\partial}{\partial X_1} + W_0 \frac{\partial}{\partial Y_0} + W_1 \frac{\partial}{\partial Y_1},$$

where V_0, V_1, W_0 and W_1 are bigraded homogeneous polynomials without nonconstant common factors (not all of them equal to 0),

$$V_0, V_1 \in H^0\left(\mathbb{F}_\delta, \mathcal{O}_{\mathbb{F}_\delta}((d_1+1)F + d_2 M)\right),\ W_0 \in H^0\left(\mathbb{F}_\delta, \mathcal{O}_{\mathbb{F}_\delta}(d_1 F + (d_2+1)M)\right),$$

$W_1 \in H^0\left(\mathbb{F}_\delta, \mathcal{O}_{\mathbb{F}_\delta}((d_1-\delta)F + (d_2+1)M)\right)$, d_1 and d_2 being integers such that $d_2 \geq 0$ and $d_1 \geq 0$ (respectively, $d_1 \geq -1$) when $\delta = 0$ (respectively, otherwise). Moreover, $\mathcal{F}$ is uniquely determined up to the addition of multiples of the radial vector fields

$$R_1 := X_0 \frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1} - \delta Y_1 \frac{\partial}{\partial Y_1} \quad \text{and}$$
$$R_2 := Y_0 \frac{\partial}{\partial Y_0} + Y_1 \frac{\partial}{\partial Y_1}.$$

On the other hand, $\mathcal{F}$ is also uniquely determined by an affine differential 1-form Ω:

$$\Omega = A_0 dX_0 + A_1 dX_1 + B_0 dY_0 + B_1 dY_1,$$

where A_0, A_1, B_0 and B_1 are bigraded homogeneous polynomials without nonconstant common factors (not all of them equal to 0),

$$A_0, A_1 \in H^0\left(\mathbb{F}_\delta, \mathcal{O}_{\mathbb{F}_\delta}((d_1-\delta+1)F+(d_2+2)M)\right),$$

$$B_0 \in H^0\left(\mathbb{F}_\delta, \mathcal{O}_{\mathbb{F}_\delta}((d_1-\delta+2)F+(d_2+1)M)\right)$$

and $B_1 \in H^0\left(\mathbb{F}_\delta, \mathcal{O}_{\mathbb{F}_\delta}((d_1+2)F+(d_2+1)M)\right)$, d_1 and d_2 being integers as above, which satisfy the following two conditions, called Euler-type conditions:

$$\Omega(R_1) = A_0X_0 + A_1X_1 - \delta B_1Y_1 = 0 \text{ and}$$
$$\Omega(R_2) = B_0Y_0 + B_1Y_1 = 0.$$

Notice that, considering the before defined open cover $\{U_{ij}\}_{0\leq i,j\leq 1}$, affine vector fields or affine 1-forms allow us to get families $\{(U_k, v_k)\}_{k\in I}$ or $\{(U_k, \omega_k)\}_{k\in I}$ (where $I = \{(i,j) \mid 0 \leq i, j \leq 1\}$) defining $\mathcal{F}$. More specifically, in the case of 1-forms (which we will preferably use), the restriction of $\mathcal{F}$ to an affine open subset U_{ij} gives rise to the foliation on $U_{ij} \cong \mathbb{C}^2$ induced by the 1-form

$$A_{i'}(x_0, x_1, y_0, y_1)dx_{ij} + B_{j'}(x_0, x_1, y_0, y_1)dy_{ij},$$

where $\{i'\} := \{0,1\} \setminus \{i\}$, $\{j'\} := \{0,1\} \setminus \{j\}$ and, for each $\ell \in \{0,1\}$, $x_\ell := x_{ij}$ (respectively, $x_\ell := 1$) if $\ell \neq i$ (respectively, otherwise) and $y_\ell := y_{ij}$ (respectively, $y_\ell := 1$) if $\ell \neq j$ (respectively, otherwise).

Our foliations have isolated singularities. One of the main techniques for treating these singularities is the blowup of the surface (corresponding to the foliation) at the singular points of the foliation and the consideration of the strict transform of the foliation on the blown up surface. Some references about the blowup procedure for foliations on smooth projective surfaces are [1, 18, 51] (see also [9]). Roughly speaking, blowing up a smooth complex surface S, at a point p, consists of replacing that point p by a complex projective line regarded as the set of limit directions at p. The obtained surface is defined by local charts and usually denoted by $\mathrm{Bl}_p(S)$.

As explained, a foliation $\mathcal{G}$ on S can be locally given by differential 1-forms. These forms, after blowing up at a singularity p, define differential 1-forms on affine charts of $\mathrm{Bl}_p(S)$ which, after gluing, give rise to a foliation on the surface $\mathrm{Bl}_p(S)$ named the strict transform $\tilde{\mathcal{G}}$ of $\mathcal{G}$. A detailed description of this procedure is given in [21, Section 4].

Assume that $\mathcal{G}$ is given at p, in local coordinates x and y, by a local differential 1-form $\omega = A(x,y)dx + B(x,y)dy$. Then, we define the multiplicity of $\mathcal{G}$ at p as the nonnegative integer m corresponding to the first nonvanishing jet $\omega_m := a_m(x,y)dx + b_m(x,y)dy$ of ω. Notice that $a_m(x,y)$ and $b_m(x,y)$ are homogeneous polynomials in two variables of degree m and that p is a singularity of $\mathcal{G}$ if and only if $m \geq 1$. If this is the case, defining $d(x,y) := xa_m(x,y) + yb_m(x,y)$, we say that p is a *terminal dicritical singularity* of $\mathcal{G}$ whenever $d(x,y)$ is the zero polynomial. In addition, p is a *simple singularity* if $m = 1$ and the matrix

$$\begin{pmatrix} \frac{\partial b_1}{\partial x} & \frac{\partial b_1}{\partial y} \\ -\frac{\partial a_1}{\partial x} & -\frac{\partial a_1}{\partial y} \end{pmatrix}$$

has two eigenvalues λ_1, λ_2 such that the product $\lambda_1\lambda_2$ does not vanish and their quotient is not a positive rational number, or $\lambda_1\lambda_2 = 0$ and $\lambda_1^2 + \lambda_2^2$ does not vanish. Nonsimple singularities are named to be *ordinary*. Notice that a terminal dicritical singularity is an ordinary singularity.

The singularities of a foliation $\mathcal{G}$ with isolated singularities can be reduced by blowing up at the ordinary singular points of $\mathcal{G}$ and at those belonging to its successive strict transforms. The following result summarizes well-known facts on reduction of foliations and terminal dicritical singularities (see [5, 51] and [21, Theorem 1 and Proposition 1]).

Theorem 3.1 *Let $\mathcal{G}$ be a foliation on a smooth projective surface S. Then:*

(1) There is a sequence of finitely many point blowups, $\pi : Z \to S$, such that the strict transform of $\mathcal{G}$ on Z has no ordinary singularity.
(2) A singularity p of $\mathcal{G}$ is not terminal dicritical if and only if the exceptional divisor of the surface $\mathrm{Bl}_p(S)$ is invariant by the strict transform of $\mathcal{G}$ on $\mathrm{Bl}_p(S)$.

Let p be a point in S. The points in the exceptional divisor E_p obtained after blowing up at p are named points in the *first infinitesimal neighbourhood* of p. Assuming that we blow up at some points in E_p, the points in the new created exceptional divisors are in the *second infinitesimal neighbourhood* of p. Inductively one defines the *kth infinitesimal neighbourhood* of p, $k \geq 1$. A point q is *infinitely near* p if either $q = p$ or it belongs to some kth infinitesimal neighbourhood of p. Finally, a point q is *infinitely near* S if it is infinitely near a point in S.

In addition, given two infinitely near S points p and q, q *is proximate to* p if q belongs to E_p or to any of its strict transforms. When q is proximate to p and $q \notin E_p$, q is named *satellite* and, otherwise, it is called *free*.

Let $\mathcal{G}$ be a foliation and π a map as in Theorem 3.1. The set $\mathcal{C}_{\mathcal{G}} = \{p_1, \ldots, p_n\}$ of blowup centers of π is called the *singular configuration* of $\mathcal{G}$, and its elements *infinitely near ordinary singularities* of $\mathcal{G}$. Notice that $\mathcal{C}_{\mathcal{G}}$ is formed by the ordinary singularities of $\mathcal{G}$ and those of the successive strict transforms of $\mathcal{G}$ by the sequence of blowups π. An *infinitely near ordinary singularity* $p_i \in \mathcal{C}_{\mathcal{G}}$ is named an (infinitely near) *dicritical singularity* if there is a point $p_j \in \mathcal{C}_{\mathcal{G}}$ which is infinitely near p_i and such that p_j is a terminal dicritical singularity of the strict transform of $\mathcal{G}$ on the surface containing p_j.

3.3 Extending a Planar Vector Field to a Foliation on a Hirzebruch Surface

In this section we describe how any complex planar vector field can be extended to a foliation on any Hirzebruch surface $\mathbb{F}_\delta$. For this purpose, we consider a complex planar vector field X as in (2.2) and the following algorithm, whose inputs are a nonnegative integer δ and a differential 1-form $A(x, y)dx + B(x, y)dy$ (with $A(x, y)$ and $B(x, y)$ in $\mathbb{C}[x, y]$ and coprime) defining X, and whose ouputs are four bigraded homogeneous polynomials $A_{\delta,0}, A_{\delta,1}, B_{\delta,0}, B_{\delta,1} \in \mathbb{C}[X_0, X_1, Y_0, Y_1]$ of suitable bidegrees.

Algorithm 3.2 **Input:** A pair (δ, ω), where $\delta \in \mathbb{Z}_{\geq 0}$ and $\omega = A(x, y)dx + B(x, y)dy$ $(A(x, y), B(x, y) \in \mathbb{C}[x, y]$ coprime).

Output: $A_{\delta,0}, A_{\delta,1}, B_{\delta,0}, B_{\delta,1} \in \mathbb{C}[X_0, X1, Y_0, Y_1]$.

(1) Write the rational functions $A\left(\frac{X_1}{X_0}, \frac{X_0^\delta Y_1}{Y_0}\right)$ and $B\left(\frac{X_1}{X_0}, \frac{X_0^\delta Y_1}{Y_0}\right)$ as reduced rational fractions $\frac{X_0^{\alpha_0} A_{\delta,1}}{X_0^{\alpha_1} Y_0^{\alpha_2}}$ and $\frac{X_0^{\beta_0} B_{\delta,1}}{X_0^{\beta_1} Y_0^{\beta_2}}$, respectively, where $(\alpha_0, \alpha_1, \alpha_2), (\beta_0, \beta_1, \beta_2) \in \mathbb{Z}^3_{\geq 0}$ and $A_{\delta,1}$ and $B_{\delta,1}$ are bigraded homogeneous polynomials in $\mathbb{C}[X_0, X_1, Y_0, Y_1]$ of respective bidegrees $(a_1, a_2) := (\alpha_1 - \alpha_0, \alpha_2)$ and $(b_1, b_2) := (\beta_1 - \beta_0, \beta_2)$ such that $A_{\delta,1}, \; B_{\delta,1}$ and $X_0 Y_0$ are pairwise coprime.
(2) Let $m_1 := a_1 - b_1 + 1 + \delta$. If $m_1 > 0$, then $B_{\delta,1} := X_0^{m_1} B_{\delta,1}$; otherwise, $A_{\delta,1} := X_0^{-m_1} A_{\delta,1}$.
(3) Let $m_2 := a_2 - b_2 - 1$. If $m_2 > 0$, then $B_{\delta,1} := Y_0^{m_2} B_{\delta,1}$; otherwise, $A_{\delta,1} := Y_0^{-m_2} A_{\delta,1}$.
(4) Let $b := 0$ if Y_0 divides $B_{\delta,1}$, and $b := 1$ otherwise. Set $B_{\delta,1} := Y_0^b B_{\delta,1}$ and $A_{\delta,1} := Y_0^b A_{\delta,1}$.
(5) Let $a := 0$ if X_0 divides $\delta Y_1 B_{\delta,1} - X_1 A_{\delta,1}$ and $a := 1$ otherwise. Set $A_{\delta,1} := X_0^a A_{\delta,1}$ and $B_{\delta,1} := X_0^a B_{\delta,1}$.
(6) Set $A_{\delta,0} := \frac{\delta Y_1 B_{\delta,1} - X_1 A_{\delta,1}}{X_0}$ and $B_{\delta,0} := \frac{-Y_1 B_{\delta,1}}{Y_0}$.

Lemma 3.3 *Fix $\delta \in \mathbb{Z}_{\geq 0}$. Let $\omega_X = A(x, y)dx + B(x, y)dy$ be a differential 1-form defining a planar vector field X, and let $A_{\delta,0}, A_{\delta,1}, B_{\delta,0}$ and $B_{\delta,1}$ be the polynomials of $\mathbb{C}[X_0, X_1, Y_0, Y_1]$ obtained as the output of Algorithm 3.2 from the input given by the pair (δ, ω_X). Then $A_{\delta,0}, A_{\delta,1}, B_{\delta,0}$ and $B_{\delta,1}$ are bigraded homogeneous polynomials with respective bidegrees $(d_1 - \delta + 1, d_2 + 2)$, $(d_1 - \delta + 1, d_2 + 2)$, $(d_1 - \delta + 2, d_2 + 1)$ and $(d_1 + 2, d_2 + 1)$ for some integers d_1, d_2. Moreover they satisfy the equalities*

$$X_0 A_{\delta,0} + X_1 A_{\delta,1} - \delta Y_1 B_{\delta,1} = 0 \quad \text{and} \quad Y_0 B_{\delta,0} + Y_1 B_{\delta,1} = 0 \tag{3.1}$$

and have no nonconstant common factor.

Proof Notice that the polynomials $A_{\delta,1}$ and $B_{\delta,1}$ obtained in Step (1) of Algorithm 3.2 are coprime and have respective bidegrees (a_1, a_2) and (b_1, b_2). A straightforward but tedious study of the different possibilities that may appear in Algorithm 3.2 shows the existence of integers d_1, d_2 such that the bidegree of $A_{\delta,1}$ is $(d_1 - \delta + 1, d_2 + 2)$ and the bidegree of $B_{\delta,1}$ is $(d_1 + 2, d_2 + 1)$. As an example: if $m_1, m_2 > 0$ (in Steps (2) and (3)), $b = 0$ in Step (4) and $a = 1$ in Step (5); then $A_{\delta,1}$ and $B_{\delta,1}$ have respective bidegrees $(a_1 + 1, a_2)$ and $(a_1 + 2 + \delta, a_2 - 1)$. Hence $d_1 = a_1 + \delta$ and $d_2 = a_2 - 2$ in this case.

The polynomials $A_{\delta,1}$ and $B_{\delta,1}$ obtained after applying the steps from (1) to (5) satisfy that X_0 (respectively, Y_0) divides $\delta Y_1 B_{\delta,1} - X_1 A_{\delta,1}$ (respectively, $B_{\delta,1}$). Therefore the rational functions $A_{\delta,0}$ and $B_{\delta,0}$ defined in Step (6) are polynomials and their bidegrees coincide with those given in the statement. In addition, Equalities (3.1) hold trivially.

It is easily derived from the algorithm that the only two possible common factors of the output polynomials are X_0 and Y_0. Let us see that none of them can be such a common factor. The polynomials $A_{\delta,1}$ and $B_{\delta,1}$ obtained in Step (1) do not share factors with X_0Y_0. After steps (2) and (3), at most one of them ($A_{\delta,1}$ and $B_{\delta,1}$) has X_0 (respectively, Y_0) as a factor. On the one hand, in Step (4) we ensure that either Y_0 does not divide $A_{\delta,1}$, or Y_0 divides $B_{\delta,1}$ but Y_0^2 does not (what implies that Y_0 does not divide $B_{\delta,0}$ after Step (6)). On the other hand, in Step (5) we force X_0 to divide $\delta Y_1 B_{\delta,1} - X_1 A_{\delta,1}$ (but X_0^2 does not); then, after Step (6), X_0 does not divide $A_{\delta,0}$. □

Consider a nonnegative integer δ and identify the affine plane $\mathbb{C}^2$ with the open subset U_{00} of $\mathbb{F}_\delta$. Then, as a consequence of Section 3.2 and Lemma 3.3, we deduce the following result.

Proposition 3.4 *Let δ be a nonnegative integer and $\omega_X = A(x, y)dx + B(x, y)dy$ a differential 1-form defining a complex planar polynomial vector field X. Let*

$$A_{\delta,0}, A_{\delta,1}, B_{\delta,0}, B_{\delta,1} \in \mathbb{C}[X_0, X_1, Y_0, Y_1]$$

be the output of Algorithm 3.2 when its input is the pair (δ, ω_X). Then the bigraded homogeneous affine differential 1-form

$$\Omega_\delta := A_{\delta,0}dX_0 + A_{\delta,1}dX_1 + B_{\delta,0}dY_0 + B_{\delta,1}dY_1$$

defines a foliation on the Hirzebruch surface $\mathbb{F}_\delta$, with isolated singularities, whose restriction to the open set U_{00} gives the 1-form in two variables that determines the vector field X.

The foliation obtained from the pair (δ, ω_X) by Proposition 3.4 is called the *extension of the vector field X* to the Hirzebruch surface $\mathbb{F}_\delta$ and it is denoted by $\mathcal{F}_X^\delta$.

If $f = \frac{f_1(x,y)}{f_2(x,y)}$ is a reduced rational function and $\delta \in \mathbb{Z}_{\geq 0}$, then there exist two coprime bigraded homogeneous polynomials $F_1, F_2 \in \mathbb{C}[X_0, X_1, Y_0, Y_1]$ of the same bidegree such that the following equality of rational functions holds:

$$f(X_1/X_0, X_0^\delta Y_1/Y_0) = \frac{F_1}{F_2}.$$

By [35, Proposition 1.6], a planar vector field X has f as a rational first integral if and only if the function F_1/F_2 is a rational first integral of the foliation $\mathcal{F}_X^\delta$.

4 Necessary Conditions for Algebraic Integrability

This section is devoted to study the behaviour of algebraically integrable complex planar polynomial vector fields X through their extensions to foliations $\mathcal{F}_X^\delta$ on Hirzebruch surfaces. We show in Theorem 4.2 that the points with coordinates $(0, 1; 0, 1)$

(respectively, $(0, 1; 1, 0)$) in each surface $\mathbb{F}_\delta$ are dicritical singularities of $\mathcal{F}_X^\delta$ whenever $\delta > \delta_1$ (respectively, $\delta < \delta_1$) for a fixed nonnegative integer δ_1 which is the minimum nonnegative integer such that $(0, 1; 1, 0)$ is not a dicritical singularity of $\mathcal{F}_X^\delta$. This result can be reformulated in terms of planar vector fields depending on a nonnegative integer parameter which gives rise to a new technique for discarding the existence of a rational first integral of a vector field (see Corollary 4.4). We start with a lemma which we will use in the proof of the forthcoming Theorem 4.2.

Lemma 4.1 *Let X be an algebraically integrable complex planar vector field. Let $f = \frac{f_1(x,y)}{f_2(x,y)}$ be a primitive rational first integral of X and $g(x, y) = \alpha f_1(x, y) + \beta f_2(x, y) \in \mathbb{C}(\alpha, \beta)[x, y]$ the associated generic algebraic invariant curve of X. Then $X \neq c\frac{\partial}{\partial y}$ for all $c \in \mathbb{C} \setminus \{0\}$ if and only if $g(x, y) \notin \mathbb{C}(\alpha, \beta)[x]$.*

Proof $X = c\frac{\partial}{\partial y}$ for some $c \in \mathbb{C} \setminus \{0\}$ if and only if the foliation on $\mathbb{C}^2$ defined by X is determined by the 1-form $\omega_X = dx$. This means that the function x is a first integral of this foliation, that is, $f_1(x, y)$ and $f_2(x, y)$ are polynomials in $\mathbb{C}[x]$ of degree 1. This is equivalent to say that $g(x, y) \in \mathbb{C}(\alpha, \beta)[x]$ because the polynomial of $\mathbb{C}[x, y]$ obtained after replacing, in $g(x, y)$, α and β by general complex numbers, must be irreducible. □

Theorem 4.2 *Let X be an algebraically integrable complex planar polynomial vector field such that $X \neq c\frac{\partial}{\partial y}$ for all $c \in \mathbb{C} \setminus \{0\}$. For each $\delta \in \mathbb{Z}_{\geq 0}$, consider the foliation $\mathcal{F}_X^\delta$ given by the extension of X to the Hirzebruch surface $\mathbb{F}_\delta$. Then, there exists a nonnegative integer δ_1 satisfying the following conditions:*

(i) For all integers δ such that $\delta > \delta_1$, the point $(0, 1; 0, 1) \in \mathbb{F}_\delta$ is the unique dicritical singularity of $\mathcal{F}_X^\delta$ belonging to the curve $X_0 = 0$.
(ii) For all nonnegative integer δ such that $\delta < \delta_1$, the point $(0, 1; 1, 0) \in \mathbb{F}_\delta$ is a dicritical singularity of $\mathcal{F}_X^\delta$.
(iii) The point $(0, 1; 1, 0) \in \mathbb{F}_{\delta_1}$ is not a dicritical singularity of $\mathcal{F}_X^{\delta_1}$.

Proof Let $f = \frac{f_1(x,y)}{f_2(x,y)}$ be a primitive rational first integral of X. Then the associated generic algebraic invariant curve of X is $g(x, y) = \alpha f_1(x, y) + \beta f_2(x, y) \in \mathbb{C}(\alpha, \beta)[x, y]$. Let us write $g(x, y) = \sum g_{ij} x^i y^j$, where the coefficients g_{ij} are homogeneous linear polynomials in α, β. Let d_x (respectively, d_y) be the degree in the variable x (respectively, y) of $g(x, y)$, that is, the degree of g when it is regarded as a polynomial in x (respectively, y) with coefficients in $\mathbb{C}(\alpha, \beta, y)$ (respectively, $\mathbb{C}(\alpha, \beta, x)$). Denote by d_x^0 (respectively, d_y^0) the degree of $g(x, 0)$ (respectively, $g(y, 0)$). Notice that $d_y > 0$ by Lemma 4.1.

We can write $g(x, y)$ as the sum of four polynomials A, B, C and D (with variables x, y and coefficients in $\mathbb{C}(\alpha, \beta)$) as showed in the following displayed formula:

$$g(x, y) = \underbrace{\sum_{i=0}^{d_x^0} g_{i0} x^i}_{=A} + \underbrace{\sum_{j=1}^{d_y^0} g_{0j} y^j}_{=B} + \underbrace{\sum_{1 \leq i \leq d_x^0} \sum_{j=1}^{d'_y} g_{ij} x^i y^j}_{=C} + \underbrace{\sum_{i > d_x^0} \sum_{j=1}^{d''_y} g_{ij} x^i y^j}_{=D}. \tag{4.1}$$

Denote by Coeff(h) the set of nonzero coefficients h_{ij} of a polynomial

$$h(x, y) = \sum h_{ij} x^i y^j \in \mathbb{C}(\alpha, \beta)[x, y].$$

Also, consider the following set of nonnegative rational numbers:

$$\Gamma = \left\{ \frac{i - d_x^0}{j} \mid j > 0 \text{ and } g_{ij} \in \operatorname{Coeff}(g) \right\} \cap \mathbb{Q}_{\geq 0}. \tag{4.2}$$

Let δ be an arbitrary nonnegative integer. Consider the Hirzebruch surface $\mathbb{F}_\delta$ and identify $\mathbb{C}^2$ with the open set U_{00} of $\mathbb{F}_\delta$ as showed in Subsection 3.1. Then, replacing in Equation (4.1), x by X_1/X_0 and y by $X_0^\delta Y_1/Y_0$, and multiplying by suitable powers X_0^a and Y_0^b, we obtain an irreducible bigraded homogeneous polynomial $G_\delta(X_0, X_1; Y_0, Y_1) \in \mathbb{C}(\alpha, \beta)[X_0, X_1, Y_0, Y_1]$ of bidegree (a, b).

$$G_\delta(X_0, X_1; Y_0, Y_1) := X_0^a Y_0^b \cdot \left(g_{00} + \sum_{i=1}^{d_x^0} g_{i0} \frac{X_1^i}{X_0^i} + \sum_{j=1}^{d_y^0} g_{0j} \frac{X_0^{\delta j} Y_1^j}{Y_0^j} + \sum_{1 \leq i \leq d_x^0} \sum_{j=1}^{d_y'} g_{ij} \frac{X_0^{\delta j - i} X_1^i Y_1^j}{Y_0^j} \right.$$
$$\left. + \sum_{i > d_x^0} \sum_{j=1}^{d_y''} g_{ij} \frac{X_0^{\delta j - i} X_1^i Y_1^j}{Y_0^j} \right).$$

The polynomial G_δ is not divisible by neither X_0 nor Y_0, $b = d_y = \max\{d_y^0, d_y', d_y''\} > 0$ and $a = a' + d_x^0$, with $a' \in \mathbb{Z}_{\geq 0}$. Therefore,

$$G_\delta(X_0, X_1; Y_0, Y_1) = X_0^{a'} \cdot \left(g_{00} X_0^{d_x^0} Y_0^{d_y} + \sum_{i=1}^{d_x^0} g_{i0} X_0^{d_x^0 - i} X_1^i Y_0^{d_y} + \sum_{j=1}^{d_y^0} g_{0j} X_0^{\delta j + d_x^0} Y_0^{d_y - j} Y_1^j \right.$$
$$\left. + \sum_{1 \leq i \leq d_x^0} \sum_{j=1}^{d_y'} g_{ij} X_0^{\delta j + d_x^0 - i} X_1^i Y_0^{d_y - j} Y_1^j + \sum_{i > d_x^0} \sum_{j=1}^{d_y''} g_{ij} X_0^{\delta j + d_x^0 - i} X_1^i Y_0^{d_y - j} Y_1^j \right).$$

Notice that, in the above expression between parentheses, negative exponents may only appear in the last block of summations.

Firstly let us assume that $\Gamma = \emptyset$. This implies that $i < d_x^0$ for all $g_{ij} \in \operatorname{Coeff}(g)$; so $D = 0$. Then $a' = 0$ and

$$G_\delta(X_0, X_1; Y_0, Y_1) = g_{00} X_0^{d_x^0} Y_0^{d_y} + \sum_{i=1}^{d_x^0} g_{i0} X_0^{d_x^0 - i} X_1^i Y_0^{d_y} + \sum_{j=1}^{d_y^0} g_{0j} X_0^{\delta j + d_x^0} Y_0^{d_y - j} Y_1^j$$
$$+ \sum_{i < d_x^0} \sum_{j=1}^{d_y'} g_{ij} X_0^{\delta j + d_x^0 - i} X_1^i Y_0^{d_y - j} Y_1^j.$$

Notice that $d_x^0 > 0$ because otherwise $B = 0$ and $C = 0$, what implies that $g(x, y) = g_{00}$ (a contradiction because, by Lemma 4.1, $d_y > 0$). Therefore $g_{d_x^0 0} \neq 0$. This shows that the point $(0, 1; 0, 1)$ is the unique point belonging to the intersection of the curves on $\mathbb{F}_\delta$ defined by the equations $X_0 = 0$ and $G_\delta(X_0, X_1; Y_0, Y_1) = 0$ or, equivalently, it is the unique dicritical singularity of $\mathcal{F}_X^\delta$ belonging to the curve defined by $X_0 = 0$ (independently of the value of δ). In this case $\delta_1 = 0$ is the integer satisfying the conditions given in the statement.

Let us assume now that $\Gamma \neq \emptyset$. Under this assumption, let us define

$$k := \max(\Gamma)$$

and distinguish the following three cases, depending on the value of δ:

Case 1: The set

$$\Delta := \{g_{ij} \in \mathrm{Coeff}(G_\delta) \mid j > 0 \ \text{ and } \delta j + d_x^0 - i < 0\} \tag{4.3}$$

is not empty.

The above condition shows that $\Delta \subseteq \mathrm{Coeff}(D)$ and $\delta < k$. Moreover,

$$a' = -\min\{\delta j + d_x^0 - i \mid g_{ij} \in \Delta\}.$$

Hence, the points of intersection between the curves on $\mathbb{F}_\delta$ defined by the equations $X_0 = 0$ and $G_\delta(X_0, X_1; Y_0, Y_1) = 0$ are the points $(0, 1; y_0, y_1)$ satisfying the following condition

$$\sum_{j=1}^{d_y''} g_{\delta j + a' + d_x^0, j}\, y_0^{d_y - j} y_1^j = 0.$$

In particular $(0, 1; 1, 0)$ belongs to that intersection and, as a consequence, $(0, 1; 1, 0)$ *is a dicritical singularity* of $\mathcal{F}_X^\delta$.

Case 2: The set Δ in (4.3) is empty and there exists $g_{lm} \in \mathrm{Coeff}(G_\delta)$ such that $m > 0$ and $\delta m + d_x^0 - l = 0$.

In this case, since Δ is empty, $a' = 0$ and $\delta \geq k$; moreover, since $\delta = \frac{l - d_x^0}{m} \in \Gamma$, we conclude that $\delta = k$. If $d_x^0 = 0$, then $C = 0$ and $g_{00} \neq 0$; hence $G_\delta(0, 1; 1, 0) \neq 0$, that is, $(0, 1; 1, 0) \in \mathbb{F}_k$ *is not a dicritical singularity of* $\mathcal{F}_X^k$. When $d_x^0 \neq 0$, the same thing happens because $g_{d_x^0 0} \neq 0$.

Case 3: $\delta j + d_x^0 - i > 0$ for all $g_{ij} \in \mathrm{Coeff}(G_\delta)$ such that $j > 0$.

Then $a' = 0$ and $\delta > k$, and we distinguish the following subcases:

(3.1) If $d_x^0 > 0$, then $g_{d_x^0 0} \neq 0$ and $(0, 1; 0, 1)$ is the unique point where the curves with equations $G_\delta(X_0, X_1; Y_0, Y_1) = 0$ and $X_0 = 0$ meet. This means that $(0, 1; 0, 1)$ *is a dicritical singularity of* $\mathcal{F}_X^\delta$ *and the unique one belonging to the curve* $X_0 = 0$.

(3.2) If $d_x^0 = 0$, then G_δ has the following shape:

$$G_\delta(X_0, X_1; Y_0, Y_1) = g_{00} Y_0^{d_y} + \sum_{j=1}^{d_y^0} g_{0j} X_0^{\delta j} Y_0^{d_y - j} Y_1^j + X_0 H,$$

where $H \in \mathbb{C}(\alpha, \beta)[X_0, X_1, Y_0, Y_1]$. Since $\delta > k \geq 0$, it is clear that $g_{00} \neq 0$ (because, otherwise, X_0 would divide G_δ) and then $(0, 1; 0, 1)$ *is the unique dicritical singularity of* $\mathcal{F}_X^\delta$ *belonging to the curve* $X_0 = 0$.

Notice that cases 1, 2 and 3 correspond to the following situations: $\delta < k$, $\delta = k$ and $\delta > k$.

Finally, define $\delta_1 := \lceil k \rceil$ and let us see that this integer satisfies conditions (i), (ii) and (iii) of the statement.

If k is an integer, then cases 1 and 3 show that conditions (i) and (ii) are satisfied for $\delta_1 = k$. Hence, it only remains to show that $(0, 1; 1, 0) \in \mathbb{F}_k$ is not a dicritical singularity of $\mathcal{F}_X^k$; but the value $\delta = k$ corresponds to Case 2 and $(0, 1; 1, 0)$ is not a dicritical singularity of $\mathcal{F}_X^\delta$ in that case.

If k is not an integer then any $\delta \in \mathbb{Z}_{\geq 0}$ satisfies either Case 1 or Case 3; this fact shows that conditions (i), (ii) and (iii) hold. □

Remark 4.3 Let X be a complex planar vector field satisfying the conditions of Theorem 4.2. Then, the value δ_1 provided by that theorem is the minimum nonnegative integer δ such that the point $(0, 1; 1, 0) \in \mathbb{F}_\delta$ is not a dicritical singularity of $\mathcal{F}_X^\delta$.

For each $\delta \in \mathbb{Z}_{\geq 0}$, the point $(0, 1; 0, 1) \in \mathbb{F}_\delta$ (respectively, $(0, 1; 1, 0)$) belongs to the affine chart U_{11} (respectively, U_{10}), and the curve of $\mathbb{F}_\delta$ with equation $X_0 = 0$ does not meet neither U_{00} nor U_{01}. These facts allow us to write Theorem 4.2 in terms of the planar vector fields induced by the restriction of $\mathcal{F}_X^\delta$ to the charts U_{10} and U_{11}. Therefore, Theorem 4.2 can be reformulated without any reference to Hirzebruch surfaces as follows:

Corollary 4.4 *Let X be an algebraically integrable complex planar polynomial vector field such that $X \neq c\frac{\partial}{\partial y}$ for all $c \in \mathbb{C} \setminus \{0\}$. For each $\delta \in \mathbb{Z}_{\geq 0}$, let $A_{\delta,0}$, $A_{\delta,1}$, $B_{\delta,0}$ and $B_{\delta,1}$ be the polynomials of $\mathbb{C}[X_0, X_1, Y_0, Y_1]$ obtained as the output of Algorithm 3.2 from the input given by the pair (δ, ω_X). Consider the planar vector fields X_{10}^δ and X_{11}^δ defined, respectively, by the following differential 1-forms:*

$$\omega_{10}^\delta := A_{\delta,0}(x, 1, 1, y)dx + B_{\delta,1}(x, 1, 1, y)dy, \quad \textit{and}$$
$$\omega_{11}^\delta := A_{\delta,0}(x, 1, y, 1)dx + B_{\delta,0}(x, 1, y, 1)dy.$$

Let δ_1 be the minimum nonnegative integer such that the origin $(0, 0)$ is not a dicritical singularity of $X_{10}^{\delta_1}$. Then, for all $\delta > \delta_1$:

(a) the origin $(0, 0)$ is the unique dicritical singularity of X_{11}^δ in the line defined by $x = 0$, and
(b) the vector field X_{10}^δ has no dicritical singularity in the line defined by $x = 0$.

As a consequence of the above result we state the following corollary, which provides conditions forcing a planar vector field to be nonalgebraically integrable.

Corollary 4.5 *Let X be a complex planar polynomial vector field such that $X \neq c\frac{\partial}{\partial y}$ for all $c \in \mathbb{C} \setminus \{0\}$. For every $\delta \in \mathbb{Z}_{\geq 0}$, consider the planar vector fields X_{10}^{δ} and X_{11}^{δ} defined in Corollary 4.4. Let $\mathcal{N}$ be the set of nonnegative integers δ such that origin $(0,0)$ is not a dicritical singularity of X_{10}^{δ}. When $\mathcal{N} \neq \emptyset$, set $\delta_1 := \min \mathcal{N}$. Then, X is not algebraically integrable if at least one of the following conditions is satisfied:*

(a) *$\mathcal{N}$ is empty.*
(b) *$\mathcal{N}$ is not empty and there exists a positive integer $\delta > \delta_1$ such that either the origin $(0,0)$ is not a dicritical singularity of X_{11}^{δ}, or $(0,0)$ is a dicritical singularity of X_{11}^{δ} but not the unique one in the line defined by the equation $x = 0$.*
(c) *$\mathcal{N}$ is not empty and there exists a positive integer $\delta > \delta_1$ such that X_{10}^{δ} has a dicritical singularity in the line defined by $x = 0$.*

The following example gives a complex planar vector field which is not algebraically integrable, fact that we deduce from Corollary 4.5.

Example 4.6 Let X be the planar vector field defined by the differential 1-form

$$\omega = (xy + y^2 + 5x^3y)dx + (-x^2 - xy + y^3)dy.$$

We run Algorithm 3.2 using as input the pair (δ, ω). The output is

$$\begin{aligned}
A_{\delta,0} =& - X_0^2X_1^2Y_0^4Y_1 - 5X_1^4Y_0^4Y_1 - X_0^{\delta+3}X_1Y_0^3Y_1^2 - \delta X_0^2X_1^2Y_0^4Y_1 - \delta X_0^{\delta+3}X_1Y_0^3Y_1^2 \\
&+ \delta X_0^{3\delta+4}Y_0Y_1^4, \\
A_{\delta,1} =& X_0^3X_1Y_0^4Y_1 + 5X_0X_1^3Y_0^4Y_1 + X_0^{\delta+4}Y_0^3Y_1^2, \\
B_{\delta,0} =& X_0^3X_1^2Y_0^3Y_1 + X_0^{\delta+4}X_1Y_0^2Y_1^2 - X_0^{3\delta+5}Y_1^4, \text{ and} \\
B_{\delta,1} =& - X_0^3X_1^2Y_0^4 - X_0^{\delta+4}X_1Y_0^3Y_1 + X_0^{3\delta+5}Y_0Y_1^3.
\end{aligned}$$

The vector field X_{10}^{δ} introduced in Corollary 4.4 is given by the differential 1-form

$$\begin{aligned}
\omega_{10}^{\delta} =& (-5y - (1+\delta)x^2y - (1+\delta)x^{\delta+3}y^2 + \delta x^{3\delta+4}y^4)dx \\
&+ (-x^3 - x^{\delta+4}y + x^{3\delta+5}y^3)dy.
\end{aligned}$$

On the one hand, the origin is a simple singularity of X_{10}^{δ} for all $\delta \in \mathbb{Z}_{\geq 0}$ and then we deduce that $\delta_1 = 0$. On the other hand, the vector field X_{11}^{1} is defined by the differential 1-form

$$\omega_{11}^{1} = (-5y^4 - 2x^2y^4 - 2x^4y^3 + x^7y)dx + (x^3y^3 + x^5y^2 - x^8)dy.$$

Now, if we reduce the singularity $(0,0)$ of ω_{11}^{1} by successive blowups to get at most simple singularities (see Subsection 3.2) we see that the origin is not a dicritical singularity of X_{11}^{1}. Indeed, to reduce the singularity $(0,0)$ we have to blow up 17 infinitely near points $\{p_i\}_{i=1}^{17}$ which constitute a simple chain, where p_2 is proximate to p_1; p_3, p_4 and p_5 are proximate to p_2, and p_i is proximate to p_{i-1} for $6 \leq i \leq 17$. No point p_i is terminal dicritical, therefore $(0, 0)$ is not dicritical. As a consequence, X is not algebraically integrable by Part (b) of Corollary 4.5.

Remark 4.7 Corollary 4.5 allows us to discard the existence of a rational first integral for certain complex planar vector fields. Necessary conditions for algebraic integrability are given in [27, Corollary 5] but they can only be applied to differential forms $A(x, y)dx + B(x, y)dy$, where $A(x, y)$ and $B(x, y)$ have the same degree n and their homogeneous components of degree n are coprime. Example 4.6 does not satisfy those conditions, proving that the necessary conditions for algebraic integrability given in Corollary 4.4 are different from those in [27].

The conditions for algebraic integrability given in Theorem 4.2 (and Corollary 4.4) are necessary, but not sufficient, as the following example shows.

Example 4.8 Let X be the complex planar vector field defined by the differential 1-form

$$\omega = (y + xy)dx + (1 + xy^2 + x^2)dy.$$

The output of Algorithm 3.2 when the input is the pair $(1, \omega)$ is

$$\begin{aligned}
A_{1,0} &= X_0Y_0^3Y_1 - X_1Y_0^3Y_1 + X_0^2X_1Y_0Y_1^3,\\
A_{1,1} &= X_0Y_0^3Y_1 + X_1Y_0^3Y_1,\\
B_{1,0} &= -X_0^2Y_0^2Y_1 - X_1^2Y_0^2Y_1 - X_0^3X_1Y_1^3 \text{ and}\\
B_{1,1} &= X_0^2Y_0^3 + X_1^2Y_0^3 + X_0^3X_1Y_0Y_1^2.
\end{aligned}$$

and when the input is $(\delta \neq 1, \omega)$, it is

$$\begin{aligned}
A_{\delta,0} &= -X_0X_1Y_0^3Y_1 - X_1^2Y_0^3Y_1 + \delta X_0^2Y_0^3Y_1 + \delta X_1^2Y_0^3Y_1 + \delta X_0^{2\delta+1}X_1Y_0Y_1^3,\\
A_{\delta,1} &= X_0^2Y_0^3Y_1 + X_0X_1Y_0^3Y_1,\\
B_{\delta,0} &= -X_0^3Y_0^2Y_1 - X_0X_1^2Y_0^2Y_1 - X_0^{2\delta+2}X_1Y_1^3 \text{ and}\\
B_{\delta,1} &= X_0^3Y_0^3 + X_0X_1^2Y_0^3 + X_0^{2\delta+2}X_1Y_0Y_1^2.
\end{aligned}$$

These outputs define the foliations $\mathcal{F}_X^\delta$, $\delta \geq 0$.

For a start, $(0, 1; 1, 0) \in \mathbb{F}_0$ is a terminal dicritical singularity of $\mathcal{F}_X^0$ but $(0, 1; 1, 0) \in \mathbb{F}_1$ is not a singularity of $\mathcal{F}_X^1$.

Assume now that $\delta > 1$. The point $(0, 1; 0, 1) \in \mathbb{F}_\delta$ is the unique ordinary singularity of $\mathcal{F}_X^\delta$ belonging to the curve with equation $X_0 = 0$. Let us see that it is a dicritical singularity. Indeed, the restriction of $\mathcal{F}_X^\delta$ to the open set U_{11} of $\mathbb{F}_\delta$ determines a vector field which is given by the differential 1-form:

$$\omega^\delta := (-xy^3 + (\delta - 1)y^3 + \delta x^2y^3 + \delta x^{2\delta+1}y)dx - (x^3y^2 + xy^2 + x^{2\delta+2})dy.$$

The origin is a singularity of ω^δ. To reduce this singularity we have to blow up ω^δ and its strict transforms using changes of local coordinates of the type $(x = x', y = x'y')$; the strict transform of ω^δ after $n \leq \delta - 2$ blowups is

$$\tilde{\omega}^\delta(n) := (-xy^3 + (\delta - n - 1)y^3 + (\delta - n)x^2y^3 + (\delta - n)x^{2(\delta-n)+1}y)dx \\ -(x^3y^2 + xy^2 + x^{2(\delta-n)+2})dy.$$

In particular,

$$xa_3^\delta(n) + yb_3^\delta(n) = (\delta - n - 2)xy^3,$$

where $\tilde{\omega}_3^\delta(n) = a_3^\delta(n)dx + b_3^\delta(n)dy = (\delta - n - 1)y^3dx - xy^2dy$ is the first nonvanishing jet of $\tilde{\omega}^\delta(n)$. Therefore, after $n = \delta - 2$ blowups the origin becomes terminal dicritical and thus $(0, 1; 0, 1)$ is a dicritical singularity of $\mathcal{F}_X^\delta$.

Thus, we have just proved that the conditions given in the statement of Theorem 4.2 hold for $\delta_1 = 1$. However X is not algebraically integrable, as we are going to prove. Indeed, consider the complex projective plane $\mathbb{P}^2$ with projective coordinates $(\mathcal{X} : \mathcal{Y} : \mathcal{Z})$ and the foliation $\mathcal{F}$ on $\mathbb{P}^2$ defined by the homogeneous 1-form

$$\Omega = (-\mathcal{X}^3\mathcal{Z} - \mathcal{X}^2\mathcal{Y}\mathcal{Z} - 2\mathcal{X}\mathcal{Y}^2\mathcal{Z} - \mathcal{Y}\mathcal{Z}^3)d\mathcal{X} + (\mathcal{X}^3\mathcal{Z} \\ +\mathcal{X}^2\mathcal{Y}\mathcal{Z})d\mathcal{Y} + (\mathcal{X}^4 + \mathcal{X}^2\mathcal{Y}^2 + \mathcal{X}\mathcal{Y}\mathcal{Z}^2)d\mathcal{Z}$$

whose restriction to the open subset of $\mathbb{P}^2$ defined by $\mathcal{X} \neq 0$ (identified with $\mathbb{C}^2$) gives rise to the initial vector field X. Assume that X, and therefore $\mathcal{F}$, has a rational first integral f and let us see that we get a contradiction. With notation as in Subsection 3.2, consider the subset $\mathcal{D}_\mathcal{F} \subseteq \mathcal{C}_\mathcal{F}$ given by the (infinitely near) dicritical singularities of $\mathcal{F}$ and let $\pi_\mathcal{F} : Z_\mathcal{F} \to \mathbb{P}^2$ be the birational map obtained by blowing up the points in $\mathcal{D}_\mathcal{F}$. $\mathcal{D}_\mathcal{F}$ consists of 6 points, $p_1, \ldots, p_6$, such that $p_1 \in \mathbb{P}^2$, p_2 and p_6 belong to the first infinitesimal neighbourhood of p_1 and, for $i \in \{3, 4, 5\}$, p_i is a free point of the first infinitesimal neighbourhood of p_{i-1}.

Set $D_{\tilde{f}}$ a general fiber of $\tilde{f} := f \circ \pi_\mathcal{F} : Z_\mathcal{F} \to \mathbb{P}^1$ and $\tilde{\mathcal{F}}$ the strict transform of $\mathcal{F}$ by $\pi_\mathcal{F}$. Since the unique terminal dicritical singularities in $\mathcal{D}_\mathcal{F}$ are p_5 and p_6, the strict transforms on $Z_\mathcal{F}$ of the exceptional divisors E_{p_i}, $i \in \{2, 3, 4\}$, (respectively, E_{p_5} and E_{p_6}) are (respectively, are not) invariant by $\tilde{\mathcal{F}}$ (see Theorem 3.1(2)); then, by [28, Proposition 1(b)], the divisor $D_{\tilde{f}}$ is linearly equivalent to $dL^* - (\alpha + \beta)E_{p_1}^* - \alpha \sum_{i=2}^5 E_{p_i}^* - \beta E_{p_6}^*$ for some positive integers d, α, β, where L^* (respectively, $E_{p_i}^*$) denotes the pull-back on $Z_\mathcal{F}$ of a general line of $\mathbb{P}^2$ (respectively, the exceptional divisor E_{p_i}).

The lines C_1 and C_2 with respective equations $\mathcal{X} = 0$ and $\mathcal{Z} = 0$ are invariant curves of the foliation $\mathcal{F}$. Both lines pass through the point p_1, the strict transform of C_1 (respectively, C_2) on the surface obtained by blowing up at p_1 passes through p_2 (respectively, p_6) and p_3 does not belong to the strict transform of C_1. This means that the strict transform $\tilde{C}_1$ (respectively, $\tilde{C}_2$) of C_1 (respectively, C_2) on $Z_\mathcal{F}$ is linearly equivalent to the divisor $L^* - E_{p_1}^* - E_{p_2}^*$ (respectively, $L^* - E_{p_6}^*$). Again by [28, Proposition 1(b)] one has that $d - (\alpha + \beta) - \alpha = 0$ and $d - (\alpha + \beta) - \beta = 0$ and, therefore, $D_{\tilde{f}}$ is linearly equivalent to the divisor $dL^* - (2d/3)E^*_{p_1} - (d/3)\sum_{i=2}^{6} E^*_{p_i}$. A canonical divisor of the surface $Z_{\mathcal{F}}$ is $K_{Z_{\mathcal{F}}} = -3L^* + \sum_{i=1}^{6} E^*_{p_i}$. Applying Formula (3) in [28], we deduce that the canonical sheaf of the foliation $\tilde{\mathcal{F}}$ (see [28, Section 2]) is $\mathcal{O}_{Z_{\mathcal{F}}}(K_{\tilde{\mathcal{F}}})$, where $K_{\tilde{\mathcal{F}}} = 2L^* - E^*_{p_1} - E^*_{p_2} - E^*_{p_5} - E^*_{p_6}$. Thus, $(K_{Z_{\mathcal{F}}} - K_{\tilde{\mathcal{F}}}) \cdot D_{\tilde{f}} = -d \neq 0$, which is a contradiction with [28, Proposition 1(b) and Equality (2)].

5 The Newton Polytope of the Generic Algebraic Invariant Curve

Given a polynomial $f(x, y) = \sum a_{ij} x^i y^j \in \mathbb{C}[x, y]$, the *Newton polytope* of f, denoted by $\mathrm{Newt}(f)$, is the convex hull of the set $\{(i, j) \mid a_{ij} \neq 0\} \subseteq \mathbb{R}^2$.

Let X be an algebraically integrable complex planar polynomial vector field. Then, the Newton polytope $\mathrm{Newt}(g)$ of the generic algebraic invariant curve $g(x, y)$, associated to a primitive rational first integral f of X, does not depend on the choice of f.

Definition 5.1 The *Newton polytope* $\mathrm{Newt}(X)$ of an algebraically integrable complex planar polynomial vector field X is defined as $\mathrm{Newt}(g)$, where $g(x, y)$ is the generic algebraic invariant curve associated to any primitive rational first integral of X.

The following result studies the Newton polytope of a vector field as above.

Theorem 5.2 *Let $X = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}$ be an algebraically integrable complex planar polynomial vector field such that $X \neq c\frac{\partial}{\partial y}$ and $X \neq c\frac{\partial}{\partial x}$ for all $c \in \mathbb{C} \setminus \{0\}$. Consider the vector field X' obtained from X by swapping the variables x and y, that is,*

$$X' = b(y, x)\frac{\partial}{\partial x} + a(y, x)\frac{\partial}{\partial y}.$$

Let δ_1 (respectively, δ'_1) be the nonnegative integer introduced in Theorem 4.2 for the vector field X (respectively, X'). Then, with notation as in the proof of Theorem 4.2, $\mathrm{Newt}(X)$ *is contained in the following region:*

$$\left\{(u, v) \in \mathbb{R}^2_{\geq 0} \mid u \leq d_x^0 + \delta_1 v \text{ and } v \leq d_y^0 + \delta'_1 u\right\},$$

where $\mathbb{R}^2_{\geq 0}$ denotes the set of points of $\mathbb{R}^2$ with nonnegative coordinates.

Proof Let $f = \frac{f_1(x,y)}{f_2(x,y)}$ be a primitive rational first integral of X and set

$$g(x, y) := \alpha f_1(x, y) + \beta f_2(x, y) = \sum_{ij} g_{ij} x^i y^j \in \mathbb{C}(\alpha, \beta)[x, y]$$

the associated generic algebraic invariant curve of X as expressed in (4.1).

Keep notation as given in the proof of Theorem 4.2. If the set Γ defined in (4.2) is empty, then $\delta_1 = 0$ and $i \leq d_x^0$ for any nonzero coefficient g_{ij} of the generic invariant curve (see the proof of Theorem 4.2); therefore the inequality $i \leq d_x^0 + \delta_1 j$ holds trivially.

Assume now that Γ is not empty and let k be the maximum of Γ (notice that $\delta_1 = \lceil k \rceil$). Pick $g_{ij} \in \mathrm{Coeff}(g)$. If $j > 0$ and $\frac{i-d_x^0}{j} \geq 0$ then $\frac{i-d_x^0}{j} \in \Gamma$ and therefore

$$i \leq kj + d_x^0 \leq d_x^0 + \delta_1 j.$$

If $j > 0$ and $\frac{i-d_x^0}{j} < 0$ then $i < d_x^0 \leq d_x^0 + \delta_1 j$. Finally, if $j = 0$, $i \leq d_x^0$ by the definition of d_x^0.

Reasoning analogously with the vector field X' and since it is algebraically integrable with generic algebraic curve $g(y, x)$, it holds that $j \leq d_y^0 + \delta_1' i$ for all (i, j) such that $g_{ij} \in \mathrm{Coeff}(g)$. This concludes the proof. □

As a consequence of Theorem 5.2, we provide, under certain assumptions, a bound on the degree of a primitive rational first integral of an algebraically integrable planar vector field that depends only on the values δ_1, δ_1', d_x^0 and d_y^0.

Corollary 5.3 *With assumptions and notation as given in Theorem 5.2, suppose that $\delta_1 = 0$ (respectively, $\delta_1' = 0$). Then the degree of a primitive rational first integral of X is bounded from above by $(1 + \delta_1')d_x^0 + d_y^0$ (respectively, $(1 + \delta_1)d_y^0 + d_x^0$).*

Remark 5.4 Let X be an algebraically integrable planar vector field. Then the value d_x^0 (respectively, d_y^0) coincides with the total intersection number between a general integral algebraic invariant curve of X and the line $y = 0$ (respectively, $x = 0$).

We conclude this paper with a result about complex planar vector fields X having a rational first integral of a specific type. Firstly, notice that X has a primitive rational first integral of the form

$$\frac{a + xyH_1(x, y)}{b + xyH_2(x, y)}, \tag{5.1}$$

with $H_1, H_2 \in \mathbb{C}[x, y]$ and $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ if and only if $d_x^0 = d_y^0 = 0$.

Corollary 5.5 *Let X be a complex planar vector field and keep notation as given in Theorem 5.2.*

(a) If X has a primitive rational first integral of type (5.1), then the Newton polytope of X, $\mathrm{Newt}(X)$, is contained in the convex cone

$$\Psi_X := \left\{(u, v) \in \mathbb{R}^2_{\geq 0} \mid u \leq \delta_1 v \text{ and } v \leq \delta_1' u\right\},$$

which can be computed only from X.

(b) If $\delta_1 = 0$ or $\delta_1' = 0$, then X has no primitive rational first integral of type (5.1).

Proof Part (a) is straightforward from Theorem 5.2. Part (b) follows because, if X had a rational first integral of the form (5.1) and either $\delta_1 = 0$ or $\delta_1' = 0$, then, by Part (a), the set Ψ_X would be $\{(0, 0)\}$, which is a contradiction. □

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