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Qualitative Results for Nonlinear Integro-Dynamic Equations via Integral Inequalities

Martin Bohner[1] · Pallavi S. Scindia[2] · Sanket Tikare[3]

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Abstract
In this paper, a nonlinear integro-dynamic equation on time scales with local initial condition is considered. The purpose of this paper is to prove existence and uniqueness of solutions and to investigate qualitative properties of solutions of this equation such as boundedness, dependence of solutions on initial conditions, functions, and parameters, and Ulam stability. The analysis is based on the Krasnoselskiĭ fixed point theorem and Gronwall-type dynamic inequalities. For the illustrative purpose of our main results, examples on a nonstandard time scale domain are provided.

Keywords Integro-dynamic equations · Gronwall inequality · Time scales · Existence and uniqueness · Dependence of solutions · Hyers–Ulam stability · Hyers–Ulam–Rassias stability

Mathematics Subject Classification 34A12 · 34D20 · 34N05 · 45J05 · 47H10

1 Introduction

The theory of time scales is a recently developed area. The objective of this theory is to unify the existing theory of continuous and discrete calculus and extend these theories to hybrid continuous-discrete domains. Dynamic equations on time scales have the

✉ Sanket Tikare
sankettikare@rjcollege.edu.in; sanket.tikare@gmail.com

Martin Bohner
bohner@mst.edu

Pallavi S. Scindia
pus.maths@coep.ac.in; scindiaps@gmail.com

1 Department of Mathematics and Statistics, Missouri S&T, Rolla, MO 65409-0020, USA

2 Department of Mathematics, College of Engineering Pune, Pune, Maharashtra 411 005, India

3 Department of Mathematics, Ramniranjan Jhunjhunwala College, Mumbai, Maharashtra 400 086, India

potential to describe both continuous and discrete processes. Hence, it became possible to study continuous-discrete time hybrid processes as a single whole. The study of dynamic equations has inspired numerous researchers and has become a prominent field of research. On the other hand, equations that contain both derivatives and integrals of the unknown function, called integro-differential equations, have been taken up by a large number of researchers, and the study of these equations has grown so extensively that it is now one of the most important subjects in the field of mathematical analysis. Indeed, there are many practical situations in which integro-differential equations give a significantly better model than differential or integral equations. Integro-differential equations appear quite naturally in mathematical modelling of several real-world phenomena including cancer treatment [13], ecology [25], engineering [34], epidemiology [16], finance [6, 22], and image processing [1]. Motivated by a large number of important applications in many real-world processes, the theory of integro-differential equations is now well developed.

Recently, there is a trend of studying integro-differential equations in the time scale domain, so-called integro-dynamic equations. This is mainly because of the great use of integro-dynamic equations in describing several nonlocal continuous-discrete hybrid phenomena. Adivar and Raffoul [3], using the topological degree method and Schaefer's fixed point theorem, deduced the existence of periodic solutions of nonlinear system of integro-dynamic equations on periodic time scales and provided several applications to scalar integro-dynamic equations. Also, Adivar et al. [2] employed Schauder's fixed point theorem to generalise and improve results of [17, 23] for systems of nonlinear Volterra integro-differential and integro-difference equations for the time scale domain. Their work requires less restrictive assumptions than the one for continuous and discrete cases. Lupulescu et al. [19] studied several qualitative properties, including asymptotic stabilities and boundedness of solutions of Volterra integro-dynamic equations. Also, Younus and Rahman [40] obtained results of controllability, observability, and stability for a linear system of regressive Volterra integro-dynamic equations. Sevinik-Adigüzel et al. [27] proved the existence of a unique solution of Volterra integro-dynamic equations employing a suitable fixed point theorem in the setting of a complete b-metric space.

Xing et al. [38], by combining the monotone iterative method with contraction mapping principle, studied classical solutions of the nonlinear integro-dynamic equation on a time scale $\mathbb{T}$ of the type

$$\begin{aligned} x^{\Delta}(t) &= \mathcal{F}\left(t, x(t), \int_0^t \mathcal{K}(t,s)x(s)\Delta s\right), \quad t \in J, \\ x(0) &= x_0, \end{aligned}$$

where $J = [0, a] \cap \mathbb{T}$ for $0, a \in \mathbb{T}$, $\mathcal{F}\colon J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{K}\colon J \times J \to \mathbb{R}$ are continuous functions.

In 2011, an improvement and generalization of the results of [38] was offered by Liu et al. [18]. By replacing $p \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ on time scales with $p \in L^1_{\mathbb{T}}(\mathbb{T}, \mathbb{R})$, they extended the exponential function e_p and studied weak solutions of the nonlinear integro-dynamic equation of the type

$$x^{\Delta}(t) + p(t)x^{\sigma}(t) = \mathcal{F}\left(t, x(t), \int_0^t \mathcal{K}(t,s)\mathcal{W}(s, x(s))\Delta s\right), \quad t \in \mathbb{T},\ a \le t < b,$$
$$x(a) = x_0,$$

where $\mathbb{T}$ is a bounded time scale with $a = \inf \mathbb{T}$, $b = \sup \mathbb{T}$, and the functions $\mathcal{F}\colon \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\mathcal{W}\colon \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ are Δ-measurable in $\mathbb{T}$, locally Lipschitz continuous, and $\mathcal{K} \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T} \times \mathbb{R}, \mathbb{R})$.

The notion of Ulam stability has a wide range of applications in various fields, including numerical analysis, optimization, control theory, neural network etc., when it is quite difficult to find an analytical solution. Ulam stability theory provides an essential tool for investigating the analytical solution as well as a reliable method to approximately solve any dynamic problem. Thus, to study approximate behavior of dynamic equations, one needs to study the aspect of Ulam-type stability. Ulam stability for integro-differential equations have been studied in several papers, [10, 14, 15, 26, 35] to mention a few. Further, many authors have investigated various stability of integro-dynamic equations [4, 24, 39, 40]. Shah and Zada [30] established Ulam stability of a nonlinear Volterra integro-dynamic equation and its adjoint equation by the integrating factor method. It is noteworthy that, recently, in a series of papers, Shah et al. have performed an interesting Ulam stability analysis of several equations including nonlinear impulsive delay dynamic equations on time scales [31, 32], mixed integral dynamic equations with instantaneous and noninstantaneous impulses [28], nonlinear Volterra impulsive integro-delay dynamic equations [33], nonlinear Hammerstein impulsive integro-dynamic equations with delay [29], and nonlinear delay differential equations with fractional integrable impulses [41]. Meanwhile, Hoa et al. [5, 12, 20, 36] investigated several interesting properties, including Ulam stability and data dependence of solutions of various types of fuzzy fractional differential equations.

It is well known that solutions of dynamic equations on time scales depend on the data like initial conditions, functions, and parameters (constants) which appear on the right-hand side of the equation. Evidently, we get different solutions to the same dynamic equation for different data. Thus, from an application and theoretical point of view, it is interesting and reasonably important to know how a solution of a dynamic equation changes if these data change slightly. To the best of our information, up till now, very few papers are available in the literature discussing data dependence of integro-dynamic equations. All these works made us motivated to study qualitative aspects of nonlinear integro-dynamic equations. In this paper, we made an attempt to investigate results concerning existence, uniqueness, data dependence, and Ulam-type stability of nonlinear integro-dynamic equations (NIDEs) of the form

$$x^{\Delta}(t) + p(t)x^{\sigma}(t) = \mathcal{F}\left(t, x(t), \int_{t_0}^t \mathcal{H}(t, s, x(s))\Delta s\right), \quad t \in \mathbb{I}^{\kappa}, \tag{1}$$

subject to the initial condition

$$x(t_0) = x_0 \in \mathbb{R}^n, \tag{2}$$

where $\mathbb{I} := [t_0, T] \cap \mathbb{T}$ with a time scale $\mathbb{T} \subset \mathbb{R}$, $t_0, T \in \mathbb{T}$ with $t_0 < T$, $x : \mathbb{I} \to \mathbb{R}^n$ is an unknown function to be determined, $x^\sigma = x \circ \sigma$, x^Δ is the delta derivative of x, $p : \mathbb{I} \to \mathbb{R}$ is regressive and rd-continuous, $\mathcal{F} : \mathbb{I} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is rd-continuous in its first variable and continuous in its second and third variables, $\mathcal{H} : \mathbb{I} \times \mathbb{I} \times \mathbb{R}^n \to \mathbb{R}^n$ is rd-continuous in its first and second variables and continuous in its third variable. In addition, both $\mathcal{F}$ and $\mathcal{H}$ are assumed to be nonlinear functions.

The innovative idea in the present paper is to consider a new type of nonlinear integro-dynamic equations on time scales (1)–(2) and then, by employing a Gronwall-type dynamic inequality, investigate all qualitative properties mentioned above, with just Lipschitz-type conditions on functions $\mathcal{F}$ and $\mathcal{H}$ involved in the equation. The main advantage of employing the Gronwall-type dynamic inequality is that it demands fewer restrictions on the functions involved in the equation than any other approach.

The present paper is divided into six sections. After this introduction, in Sect. 2, preliminary notions pertinent to this paper are given. In Sect. 3, we derive an equivalent integral equation to the dynamic problem (1)–(2) and prove existence and uniqueness of solutions to the dynamic problem (1)–(2). Results of boundedness and data dependence of solutions are derived in Sect. 4. Section 5 deals with the investigation of Ulam stability for the dynamic problem (1)–(2). Here, with the help of MATLAB® code, we provide a graph depicting the solution. Finally, conclusions and remarks on further study are added in Sect. 6.

2 Preliminaries

In what follows, $\mathbb{T}$ denotes a time scale, which is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. Below we summarize some notions connected to the theory of time scales, which are pertinent to the present paper. These materials are standard and can be found in [7, 8]. For a given time scale $\mathbb{T}$, we derive a new set $\mathbb{T}^\kappa$ as follows: $\mathbb{T}^\kappa = \mathbb{T} \setminus \{\max \mathbb{T}\}$ if $\max \mathbb{T} < \infty$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2.1 We say that $f : \mathbb{T} \to \mathbb{R}^n$ is delta differentiable at $t \in \mathbb{T}^\kappa$ if there exists $f^\Delta(t) \in \mathbb{R}^n$, a so-called delta derivative of f, with the following property: For any $\varepsilon > 0$ there is a neighbourhood N of t such that

$$\| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\|_n \le \varepsilon \left|\sigma(t) - s\right| \text{ for all } s \in N.$$

Definition 2.2 We say that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous if it is continuous at every right-dense point or maximal point in $\mathbb{T}$ and its left sided limits exist at left-dense points in $\mathbb{T}$. The symbol $\mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^n)$ will be used for the set of all such functions.

Definition 2.3 We say that $F : \mathbb{T} \to \mathbb{R}^n$ is an antiderivative of $f : \mathbb{T} \to \mathbb{R}^n$ if $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. In this case, we define the Cauchy delta integral of f by

$$\int_{t_0}^{t} f(s)\Delta s := F(t) - F(t_0), \quad \text{where } t_0 \in \mathbb{T}.$$

Definition 2.4 We say that $p\colon \mathbb{T} \to \mathbb{R}$ is regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$, where $\mu : \mathbb{T} \to \mathbb{R}^+$ is the graininess function. The symbol $\mathcal{R}(\mathbb{T}, \mathbb{R})$ will be used for the set of all rd-continuous regressive functions.

Definition 2.5 We say that $p\colon \mathbb{T} \to \mathbb{R}$ is positively regressive if $1+\mu(t)p(t) > 0$ for all $t \in \mathbb{T}$. The symbol $\mathcal{R}^+(\mathbb{T}, \mathbb{R})$ will be used for the set of all rd-continuous positively regressive functions.

Definition 2.6 For $p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$, the generalized exponential function $e_p(t, s)$ on the time scale $\mathbb{T}$ is defined as

$$e_p(t,s) := \begin{cases} \exp\left(\int_s^t \left(\frac{\mathrm{Log}|1+\mu(\tau)p(\tau)|}{\mu(\tau)}\right)\Delta\tau\right) & \text{if } \mu(\tau) \neq 0, \\ \exp\left(\int_s^t p(\tau)\Delta\tau\right) & \text{if } \mu(\tau) = 0. \end{cases}$$

For $p, q \in \mathcal{R}(\mathbb{T}, \mathbb{R})$, we define the following.

$$p \oplus q := p + q + \mu pq, \quad \ominus p := \frac{-p}{1+\mu p}, \quad p \ominus q := p \oplus (\ominus q).$$

Remark 2.1 Below we state some of the properties of the exponential function that are used in our investigation. For $p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$ and $r, s, t \in \mathbb{T}$, we have

(i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
(ii) $e_p(s, t) = e_{\ominus p}(t, s)$;
(iii) $e_p(t, s) = 1/e_p(s, t)$;
(iv) $e_p(t, r)e_p(r, s) = e_p(t, s)$.

We recall the extended Gronwall inequality, which is an essential tool in our investigation.

Theorem 2.1 (See [37, Theorem 2]) *Let* $y, f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^+)$ *with* f *a nondecreasing function and* $g, h \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$ *with* $g \geq 0$, $h \geq 0$. *If*

$$y(t) \leq f(t) + \int_a^t h(s)\left[y(s) + \int_a^s g(\tau)y(\tau)\Delta\tau\right]\Delta s \quad \text{for } t \in \mathbb{T}^\kappa,$$

then the following two inequalities hold:

(a) $y(t) \leq f(t)\left[1 + \int_a^t h(s)e_{h+g}(s, a)\Delta s\right]$ *for* $t \in \mathbb{T}^\kappa$.
(b) $y(t) \leq f(t)e_{h+g}(t, a)$ *for* $t \in \mathbb{T}^\kappa$.

In particular, if $f(t) \equiv 0$, *then* $y(t) \equiv 0$ *for* $t \in \mathbb{T}^\kappa$.

Theorem 2.2 (Arzelà–Ascoli theorem (See [42, Lemma 4])) *A subset of* $C_{\mathrm{rd}}(\mathbb{I}, \mathbb{R})$ *which is both equicontinuous and bounded is relatively compact.*

Theorem 2.3 (Krasnoselskiĭ fixed point theorem (See [21, Theorem 11.2])) *Let* B *be a Banach space,* $C \subset B$ *nonempty, closed and convex. Let* $F_1, F_2\colon C \to B$ *be such that:*

(i) *F_1 is continuous and $F_1(C)$ is relatively compact.*
(ii) *F_2 is a contraction.*
(iii) *$F_1(x) + F_2(y) \in C$ for all $x, y \in C$.*

Then there exists $\overline{x} \in C$ such that $F_1(\overline{x}) + F_2(\overline{x}) = \overline{x}$.

In this paper, we employ the following notations. Let $C_{rd}(\mathbb{I}, \mathbb{R}^n)$ be the family of all rd-continuous functions defined on $\mathbb{I}$ and taking values in $\mathbb{R}^n$, which is a Banach space when coupled with the norm $\|\cdot\|$ defined as $\|x\| := \sup_{t\in\mathbb{I}} \|x(t)\|_n$. We let

$$E := \sup_{s,t\in\mathbb{I}} |e_{\ominus p}(t,s)| > 0$$

and

$$\eta := \sup_{t\in\mathbb{I}} \int_{t_0}^{t} |e_{\ominus p}(t,s)|\Delta s > 0.$$

3 Existence and Uniqueness of Solutions

Lemma 3.1 *Let $p \in \mathcal{R}(\mathbb{I}, \mathbb{R})$, $t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}^n$, $\mathcal{F} \in C_{rd}(\mathbb{I} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, and $\mathcal{H} \in C_{rd}(\mathbb{I} \times \mathbb{I} \times \mathbb{R}^n, \mathbb{R}^n)$. Then, x is a solution of (1)–(2) if and only if x is a solution of the delta integral equation*

$$x(t) = e_{\ominus p}(t,t_0)x_0 + \int_{t_0}^{t} e_{\ominus p}(t,s)\mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s,\tau,x(\tau))\Delta\tau\right)\Delta s, \quad t \in \mathbb{I}^{\kappa}. \tag{3}$$

Proof Suppose x is a solution of (1)–(2). Then, multiplying (1) by $e_p(t,t_0)$, we get

$$(e_p(\cdot,t_0)x)^{\Delta}(t) = e_p(t,t_0)\mathcal{F}\left(t, x(t), \int_{t_0}^{t} \mathcal{H}(t,\tau,x(\tau))\Delta\tau\right).$$

Now, integrating the above equation from t_0 to t and then multiplying both sides by $e_{\ominus p}(t,t_0)$, we obtain (3). Conversely, suppose that (3) holds. Letting $t = t_0$ in (3), we find that (2) holds. Then, multiplying (3) by $e_p(t,t_0)$, we obtain

$$e_p(t,t_0)x(t) = x(t_0) + \int_{t_0}^{t} e_p(s,t_0)\mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s,\tau,x(\tau))\Delta\tau\right)\Delta s.$$

Now, differentiating the above equation, we get

$$(e_p(\cdot,t_0)x)^{\Delta}(t) = e_p(t,t_0)\mathcal{F}\left(t, x(t), \int_{t_0}^{t} H(t,\tau,x(\tau))\Delta\tau\right).$$

That is, (1) holds. This completes the proof. □

Throughout this paper, we use following hypotheses to obtain our results:

(H_1) Let $p \in \mathcal{R}(\mathbb{I}, \mathbb{R})$.

(H_2) Let $\mathcal{F}\colon \mathbb{I} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be rd-continuous in its first variable and continuous in its second and third variables such that

$$\|\mathcal{F}(t, x_1, y_1) - \mathcal{F}(t, x_2, y_2)\|_n \le L_{\mathcal{F}}(t)(\|x_1 - x_2\|_n + \|y_1 - y_2\|_n) \tag{4}$$

for all $t \in \mathbb{I}$ and $x_i, y_i, \in \mathbb{R}^n$ $(i = 1, 2)$, where $L_{\mathcal{F}} \in \mathcal{R}^+(\mathbb{I}, \mathbb{R}^+)$.

(H_3) Let $\mathcal{H}\colon \mathbb{I} \times \mathbb{I} \times \mathbb{R}^n \to \mathbb{R}^n$ be rd-continuous in its first and second variables and continuous in its third variable such that

$$\|\mathcal{H}(t, s, x_1) - \mathcal{H}(t, s, x_2)\|_n \le L_{\mathcal{H}}(s)\|x_1 - x_2\|_n \tag{5}$$

for all $t, s \in \mathbb{I}$ and $x_i \in \mathbb{R}^n$ $(i = 1, 2)$, where $L_{\mathcal{H}} \in \mathcal{R}^+(\mathbb{I}, \mathbb{R}^+)$.

(H_4) $\eta < \dfrac{1}{L^*_{\mathcal{F}}(1 + L^*_{\mathcal{H}}(T - t_0))}$, where $L^*_{\mathcal{F}} := \sup_{t\in\mathbb{I}} L_{\mathcal{F}}(t)$ and $L^*_{\mathcal{H}} := \sup_{s\in\mathbb{I}} L_{\mathcal{H}}(s)$.

(H_5) $E < \dfrac{1}{2L^*_{\mathcal{F}}(T - t_0)(1 + L^*_{\mathcal{H}}(T - t_0))}$, with $L^*_{\mathcal{F}}$ and $L^*_{\mathcal{H}}$as in (H_4).

Theorem 3.2 *Suppose* $(H_1) - (H_4)$ *hold. If*

$$M_{\mathcal{F}} := \sup\{\|\mathcal{F}(s, 0, \psi)\|_n \colon\ s \in \mathbb{I},\ \psi \in \mathbb{R}^n\} < \infty, \tag{6}$$

then the dynamic problem (1)–(2) *has a unique solution in* $\mathrm{C_{rd}}(\mathbb{I}, \mathbb{R}^n)$.

Proof **Existence:** Consider a subset $B_r \subset \mathrm{C_{rd}}$ such that

$$B_r := \{x \in \mathrm{C_{rd}}(\mathbb{I}, \mathbb{R}^n)\colon\ \|x\| \le r\},$$

where $r = 2(E\|x(t_0)\|_n + \eta M_{\mathcal{F}})$. Next, define the operator $\mathcal{W}\colon B_r \to \mathrm{C_{rd}}$ by

$$\mathcal{W}[x](t) = e_{\ominus p}(t, t_0)x(t_0) + \int_{t_0}^{t} e_{\ominus p}(t, s)\mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, x(\tau))\Delta\tau\right)\Delta s. \tag{7}$$

Clearly, this opeator $\mathcal{W}$ is well-defined on B_r. According to Lemma 3.1, a fixed point of $\mathcal{W}$ is a solution of (1)–(2). In order to employ the Krasnoselskiĭ fixed point theorem, Theorem 2.3, we express (7) as

$$\mathcal{W}[x](t) = \mathcal{W}_1[x](t) + \mathcal{W}_2[x](t),$$

where

$$\mathcal{W}_1[x](t) := e_{\ominus p}(t, t_0)x(t_0) + \int_{t_0}^{t} e_{\ominus p}(t, s)\mathcal{F}\left(s, 0, \int_{t_0}^{s} \mathcal{H}(s, \tau, 0)\Delta\tau\right)\Delta s \tag{8}$$

and

$$\mathcal{W}_2[x](t) := \int_{t_0}^{t} e_{\ominus p}(t,s)\left[\mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s,\tau,x(\tau))\Delta\tau\right) - \mathcal{F}\left(s, 0, \int_{t_0}^{s} \mathcal{H}(s,\tau,0)\Delta\tau\right)\right]\Delta s. \tag{9}$$

We show that $\mathcal{W}_1\colon B_r \to \mathrm{C_{rd}}$ is completely continuous and $\mathcal{W}_2\colon B_r \to \mathrm{C_{rd}}$ is a contraction. Further, we show that if $x, y \in B_r$, then $\mathcal{W}_1[x] + \mathcal{W}_2[y] \in B_r$. The proof will be given in the following steps.

Assertion 1. $\mathcal{W}_1\colon B_r \to \mathrm{C_{rd}}$ is completely continuous.

It is clear that $\mathcal{W}_1$ is continuous. We show that $\mathcal{W}_1$ is bounded. For $x \in B_r$

$$\begin{aligned}
&\|\mathcal{W}_1[x]\| \\
&\quad = \sup_{t\in\mathbb{I}} \left\| e_{\ominus p}(t,t_0)x(t_0) + \int_{t_0}^{t} e_{\ominus p}(t,s)\mathcal{F}\left(s,0,\int_{t_0}^{s}\mathcal{H}(s,\tau,0)\Delta\tau\right)\Delta s \right\|_n \\
&\quad \le \sup_{t\in\mathbb{I}} \left\{ |e_{\ominus p}(t,t_0)|\|x(t_0)\|_n + \int_{t_0}^{t} |e_{\ominus p}(t,s)| \left\| \mathcal{F}\left(s,0,\int_{t_0}^{s}\mathcal{H}(s,\tau,0)\Delta\tau\right)\right\|_n \Delta s \right\} \\
&\quad \le E\|x(t_0)\|_n + \eta M_{\mathcal{F}}.
\end{aligned}$$

Thus, $\mathcal{W}_1[x]$ is bounded for $x \in B_r$. Next, for equicontinuity of $\mathcal{W}_1$, let $t_1, t_2 \in \mathbb{I}$ and $x \in B_r$. Then

$$\begin{aligned}
&\|\mathcal{W}_1[x](t_2) - \mathcal{W}_1[x](t_1)\|_n \\
&\quad \le |e_{\ominus p}(t_2,t_0) - e_{\ominus p}(t_1,t_0)|\|x(t_0)\|_n \\
&\qquad + \left\| \int_{t_0}^{t_2} e_{\ominus p}(t_2,s)\mathcal{F}\left(s,0,\int_{t_0}^{s}\mathcal{H}(s,\tau,0)\Delta\tau\right)\Delta s \right. \\
&\qquad \left. - \int_{t_0}^{t_1} e_{\ominus p}(t_1,s)\mathcal{F}\left(s,0,\int_{t_0}^{s}\mathcal{H}(s,\tau,0)\Delta\tau\right)\Delta s \right\|_n \\
&\quad \le |e_{\ominus p}(t_2,t_0) - e_{\ominus p}(t_1,t_0)|\|x(t_0)\|_n \\
&\qquad + |e_{\ominus p}(t_2,t_0) - e_{\ominus p}(t_1,t_0)| \int_{t_0}^{t_1} |e_p(s,t_0)| \left\|\mathcal{F}\left(s,0,\int_{t_0}^{s}\mathcal{H}(s,\tau,0)\Delta\tau\right)\right\|_n \Delta s \\
&\qquad + |e_{\ominus p}(t_2,t_0)| \int_{\min\{t_1,t_2\}}^{\max\{t_1,t_2\}} |e_p(s,t_0)| \left\|\mathcal{F}\left(s,0,\int_{t_0}^{s}\mathcal{H}(s,\tau,0)\Delta\tau\right)\right\|_n \Delta s \\
&\quad \le |e_{\ominus p}(t_2,t_0) - e_{\ominus p}(t_1,t_0)| \left(\|x(t_0)\|_n + M_{\mathcal{F}} \int_{t_0}^{t_1} |e_p(s,t_0)|\Delta s \right) \\
&\qquad + M_{\mathcal{F}} \int_{\min\{t_1,t_2\}}^{\max\{t_1,t_2\}} |e_{\ominus p}(t_2,s)|\Delta s \\
&\quad \le |e_{\ominus p}(t_2,t_0) - e_{\ominus p}(t_1,t_0)| \left(\|x(t_0)\|_n + \eta M_{\mathcal{F}}\right) + EM_{\mathcal{F}}|t_2 - t_1|.
\end{aligned}$$

It is seen that the right-hand side of the above inequality tends to zero as $|t_2 - t_1| \to 0$. Thus, $\mathcal{W}_1[x]$ is equicontinuous for $x \in B_r$, and a standard application of the Arzelà–Ascoli theorem, Theorem 2.2, guarantees that $\mathcal{W}_1$ is compact and subsequently, is completely continuous.

Assertion 2. $\mathcal{W}_2 \colon B_r \to \mathrm{C}_{\mathrm{rd}}$ is a contraction.

For $x, y \in B_r$, we have

$$
\begin{aligned}
&\|\mathcal{W}_2[x](t) - \mathcal{W}_2[y](t)\|_n \\
&\quad \le \int_{t_0}^{t} |e_{\ominus p}(t,s)| \left\| \mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s,\tau,x(\tau))\Delta\tau\right) \right. \\
&\qquad \left. - \mathcal{F}\left(s, y(s), \int_{t_0}^{s} \mathcal{H}(s,\tau,y(\tau))\Delta\tau\right) \right\|_n \Delta s \\
&\quad \le \int_{t_0}^{t} |e_{\ominus p}(t,s)| L_{\mathcal{F}}(s) \left(\|x(s) - y(s)\|_n + \int_{t_0}^{s} L_{\mathcal{H}}(\tau)\|x(\tau) - y(\tau)\|_n \Delta\tau \right) \Delta s \\
&\quad \le \int_{t_0}^{t} |e_{\ominus p}(t,s)| L_{\mathcal{F}}(s)(1 + L^*_{\mathcal{H}}(s - t_0)) \|x(s) - y(s)\|_n \Delta s \\
&\quad \le L^*_{\mathcal{F}}(1 + L^*_{\mathcal{H}}(T - t_0)) \int_{t_0}^{t} |e_{\ominus p}(t,s)| \|x(s) - y(s)\|_n \Delta s \\
&\quad \le L^*_{\mathcal{F}}(1 + L^*_{\mathcal{H}}(T - t_0)) \int_{t_0}^{t} |e_{\ominus p}(t,s)| \Delta s \|x - y\| \\
&\quad \le \eta L^*_{\mathcal{F}}(1 + L^*_{\mathcal{H}}(T - t_0)) \|x - y\|.
\end{aligned}
$$

In view of (H_4), we obtain that $\mathcal{W}_2$ is contraction.

Assertion 3. If $x, y \in B_r$, then $\mathcal{W}_1[x] + \mathcal{W}_2[y] \in B_r$.

Let $x, y \in B_r$. Then for $t \in \mathbb{I}$

$$
\begin{aligned}
&\|\mathcal{W}_1[x](t) + \mathcal{W}_2[y](t)\|_n \\
&\quad = \left\| e_{\ominus p}(t,t_0)x(t_0) + \int_{t_0}^{t} e_{\ominus p}(t,s) \mathcal{F}\left(s, 0, \int_{t_0}^{s} \mathcal{H}(s,\tau,0)\Delta\tau\right) \Delta s \right. \\
&\qquad + \int_{t_0}^{t} e_{\ominus p}(t,s) \left[\mathcal{F}\left(s, y(s), \int_{t_0}^{s} \mathcal{H}(s,\tau,y(\tau))\Delta\tau\right) \right. \\
&\qquad \left. \left. - \mathcal{F}\left(s, 0, \int_{t_0}^{s} \mathcal{H}(s,\tau,0)\Delta\tau\right) \right] \Delta s \right\|_n \\
&\quad \le |e_{\ominus p}(t,t_0)| \|x(t_0)\|_n + \int_{t_0}^{t} |e_{\ominus p}(t,s)| \left\| \mathcal{F}\left(s, 0, \int_{t_0}^{s} \mathcal{H}(s,\tau,0)\Delta\tau\right) \right\|_n \Delta s \\
&\qquad + \int_{t_0}^{t} |e_{\ominus p}(t,s)| \left\| \mathcal{F}\left(s, y(s), \int_{t_0}^{s} \mathcal{H}(s,\tau,y(\tau))\Delta\tau\right) \right. \\
&\qquad \left. - \mathcal{F}\left(s, 0, \int_{t_0}^{s} \mathcal{H}(s,\tau,0)\Delta\tau\right) \right\|_n \Delta s \\
&\quad \le E\|x(t_0)\|_n + \eta M_{\mathcal{F}} + (\eta L^*_{\mathcal{F}}(1 + L^*_{\mathcal{H}}(T - t_0)))r.
\end{aligned}
$$

Using the definition of r and (H$_4$), we write

$$\|\mathcal{W}_1[x](t)+\mathcal{W}_2[y](t)\|_n \leq r \quad \text{for all } t \in \mathbb{I}.$$

This gives

$$\|\mathcal{W}_1[x]+\mathcal{W}_2[y]\| \leq r \quad \text{for } x, y \in B_r.$$

This shows that $\mathcal{W}_1[x]+\mathcal{W}_2[y] \in B_r$ for $x, y \in B_r$.
Thus, all the conditions of Theorem 2.3 are hold and we can deduce that the operator $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$ has a fixed point in B_r which is a solution of (1)–(2). It remains to prove the uniqueness of the solution of (1)–(2).
Uniqueness: Assume that $x, y \in B_r$ are two solutions of (1)–(2). Then for any $t \in \mathbb{I}$, (H$_2$) and (H$_3$) yield the estimates

$$\begin{aligned}
&\|x(t)-y(t)\|_n \\
&\quad \leq \int_{t_0}^{t} \left|e_{\ominus p}(t,s)\right| \left\| \mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s,\tau,x(\tau))\Delta\tau\right)\right. \\
&\qquad \left. -\mathcal{F}\left(s, y(s), \int_{t_0}^{s} \mathcal{H}(s,\tau,y(\tau))\Delta\tau\right)\right\|_n \Delta s \\
&\quad \leq \int_{t_0}^{s} EL_{\mathcal{F}}(s)\left(\|x(s)-y(s)\|_n + \int_{t_0}^{s} L_{\mathcal{H}}(\tau)\,\|x(\tau)-y(\tau)\|_n\,\Delta\tau\right)\Delta s.
\end{aligned}$$

Now, in view Theorem 2.1, we obtain

$$\|x(t)-y(t)\|_n \leq 0 \quad \text{for all } t \in \mathbb{I},$$

that is, $x = y$. This completes the proof. □

Remark 3.1 If $\mathcal{H} = 0$, then Theorem 3.2 coincides with [9, Theorem 4.1].

Corollary 3.1 *Suppose* $(H_1)-(H_3)$ *and* (H_5) *hold. If* (6) *holds, then the dynamic problem* (1)–(2) *has a unique solution.*

Remark 3.2 Both Theorem 3.2 and Corollary 3.1 also hold if (H$_2$) and (H$_3$) are replaced by the following.

(H$_{L2}$) Let $\mathcal{F}\colon \mathbb{I}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ be rd-continuous in its first variable and continuous in its second and third variables such that

$$\|\mathcal{F}(t,x_1,y_1)-\mathcal{F}(t,x_2,y_2)\|_n \leq L_{\mathcal{F}}(t)(\|x_1-x_2\|_n + \|y_1-y_2\|_n) \tag{10}$$

for all $t \in \mathbb{I}$ and $\|x_i\|_n < r$, $\|y_i\|_n < r$ $(i = 1, 2)$, with $L_{\mathcal{F}} \in \mathcal{R}^+(\mathbb{I}, \mathbb{R}^+)$.

(H$_{L3}$) Let $\mathcal{H}\colon \mathbb{I}\times\mathbb{I}\times\mathbb{R}^n \to \mathbb{R}^n$ be rd-continuous in first and second variables and continuous in third variable such that

$$\|\mathcal{H}(t,s,x_1)-\mathcal{H}(t,s,x_2)\|_n \leq L_{\mathcal{H}}(s)\|x_1-x_2\|_n \tag{11}$$

for all $t, s \in \mathbb{I}$ and $\|x_i\|_n < r$, $\|y_i\|_n < r$ $(i = 1, 2)$, with $L_{\mathcal{H}} \in \mathcal{R}^+(\mathbb{I}, \mathbb{R}^+)$.

Remark 3.3 If $\|\mathcal{F}(t, u, v)\|_n \leq K$ for all $t \in \mathbb{I}$ and $u, v \in \mathbb{R}^n$, then based on Theorem 3.2, it is seen that any solution x of (1)–(2) satisfies

$$\begin{aligned}
&\|x(t_2) - x(t_1)\|_n \\
&= \left\| e_{\ominus p}(t_2, t_0)x(t_0) + \int_{t_0}^{t_2} e_{\ominus p}(t_2, s)\mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, x(\tau))\Delta\tau\right) \Delta s \right.\\
&\quad \left. - e_{\ominus p}(t_1, t_0)x(t_0) - \int_{t_0}^{t_1} e_{\ominus p}(t_1, s)\mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, x(\tau))\Delta\tau\right) \Delta s \right\|_n \\
&\leq |e_{\ominus p}(t_2, t_0) - e_{\ominus p}(t_1, t_0)| \|x(t_0)\|_n \\
&\quad + \left\| \int_{t_0}^{t_2} e_{\ominus p}(t_2, s)\mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, x(\tau))\Delta\tau\right) \Delta s \right.\\
&\quad \left. - \int_{t_0}^{t_1} e_{\ominus p}(t_1, s)\mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, x(\tau))\Delta\tau\right) \Delta s \right\|_n \\
&\leq |e_{\ominus p}(t_2, t_0) - e_{\ominus p}(t_1, t_0)| \|x(t_0)\|_n \\
&\quad + |e_{\ominus p}(t_2, t_0) - e_{\ominus p}(t_1, t_0)| \int_{t_0}^{t_1} |e_p(s, t_0)| \left\| \mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, x(\tau))\Delta\tau\right) \right\|_n \Delta s \\
&\quad + |e_{\ominus p}(t_2, t_0)| \int_{\min\{t_1,t_2\}}^{\max\{t_1,t_2\}} |e_p(s, t_0)| \left\| \mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, x(\tau))\Delta\tau\right) \right\|_n \Delta s \\
&\leq |e_{\ominus p}(t_2, t_0) - e_{\ominus p}(t_1, t_0)| \left(\|x(t_0)\|_n + EK(T - t_0)\right) + EK|t_2 - t_1| \quad \text{for all } t_1, t_2 \in \mathbb{I}.
\end{aligned}$$

We now give an example of an NIDE of the type (1) such that all conditions of Theorem 3.2 are satisfied.

Example 3.3 Let $\mathbb{T}$ be the time scale defined as

$$\mathbb{T} := \bigcup_{k=0}^{\infty} [2k, 2k + 1].$$

This time scale appears in mathematical models of population dynamics of certain species that reproduce at discrete time intervals and whose life span is one unit of time. Consider the NIDE

$$x^{\Delta}(t) + x^{\sigma}(t) = L_{\mathcal{F}} \sin\left(x(t) + \int_{t_0}^{t} (\sin x(s) + \cos x(s) + 2t)\Delta s \right), \quad t \in [t_0, T]_{\mathbb{T}}^{\kappa}, \tag{12}$$

with $x_0 = 0$, where

$$L_{\mathcal{F}} \in \left(0, \frac{1}{(2e)^{m+1}(2m+1)(4m+3)}\right). \tag{13}$$

We take $t_0 = 0$, $T = 2m + 1$, $m \in \mathbb{N}$. Here $p(t) \equiv 1$. So $p \in \mathcal{R}$, and (H_1) holds. Moreover,

$$\mathcal{H}(t, s, x) = \sin x + \cos x + 2t \;\text{ and }\; \mathcal{F}(t, x, y) = L_{\mathcal{F}} \sin(x + y).$$

We note that (H_2) satisfied, since

$$\begin{aligned} |\mathcal{F}(t, x_1(t), y_1(t)) - \mathcal{F}(t, x_2(t), y_2(t))| &= L_{\mathcal{F}} |\sin(x_1(t) + y_1(t)) - \sin(x_2(t) + y_2(t))| \\ &\le L_{\mathcal{F}} |x_1(t) - x_2(t) + y_1(t) - y_2(t)| \\ &\le L_{\mathcal{F}} (|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|). \end{aligned}$$

Further,

$$\begin{aligned} |\mathcal{H}(t, s, x_1(s)) - \mathcal{H}(t, s, x_2(s))| &\le |\sin x_1(s) - \sin x_2(s)| + |\cos x_1(s) - \cos x_2(s)| \\ &\le 2|x_1(s) - x_2(s)|. \end{aligned}$$

Hence, (H_3) is satisfied with $L_{\mathcal{H}} = 2$. Now, since

$$\eta = \sup_{t \in [0, 2m+1]_{\mathbb{T}}} \int_0^t e_{\ominus 1}(t, s)\Delta s = (2e)^{m+1}(2m + 1),$$

we find that (H_4) satisfied with $L_{\mathcal{H}} = 2$ and $L_{\mathcal{F}}$ given in (13). Thus, all conditions of Theorem 3.2 are satisfied and therefore, by Theorem 3.2, the NIDE (12) has a unique solution on $[0, 2m + 1]_{\mathbb{T}}$.

4 Estimate and Data Dependence for Solutions

Throughout the rest of the paper, we denote $\alpha = EL_{\mathcal{F}} + L_{\mathcal{H}}$.
Our first result in this section concerns the *a priori* estimate for the possible solutions to (1)–(2).

Theorem 4.1 *Suppose* $(H_1) - (H_3)$ *hold. If* x *is the solution of* (1)–(2) *defined by Theorem* 3.2*, then*

$$\|x\| \le (E\|x_0\|_n + \eta M_{\mathcal{F}})\, e_{\alpha}(T, t_0). \tag{14}$$

Proof According to Lemma 3.1, $x(t)$ is given by

$$x(t) = e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^{t} e_{\ominus p}(t, s)\mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, x(\tau))\Delta\tau\right)\Delta s.$$

We rewrite the above equation as

$$\begin{aligned} x(t) = e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^{t} e_{\ominus p}(t, s)\left[\mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, x(\tau))\Delta\tau\right)\right. \\ \left. -\mathcal{F}\left(s, 0, \int_{t_0}^{s} \mathcal{H}(s, \tau, 0)\Delta\tau\right)\right]\Delta s \\ + \int_{t_0}^{t} e_{\ominus p}(t, s)\mathcal{F}\left(s, 0, \int_{t_0}^{s} \mathcal{H}(s, \tau, 0)\Delta\tau\right)\Delta s. \end{aligned}$$

Now, (H_2) and (H_3) yields

$$\|x(t)\|_n \leq E\|x_0\|_n + \eta M_{\mathcal{F}} + \int_{t_0}^{t} EL_{\mathcal{F}}(s)\left(\|x(s)\|_n + \int_{t_0}^{s} L_{\mathcal{H}}(\tau)\|x(\tau)\|_n\Delta\tau\right)\Delta s.$$

Finally, employing the Gronwall inequality given in Theorem 2.1 (b) and keeping in mind the increasing nature of the exponential function in the first argument, we obtain

$$\|x(t)\|_n \leq (E\|x_0\|_n + \eta M_{\mathcal{F}})\, e_\alpha(T, t_0),$$

and (14) follows easily. □

Remark 4.1 Under the conditions of Theorem 4.1 with the same calculation, and keeping in mind the inequality from Theorem 2.1 (a), we infer that the solution x of (1)–(2) satisfies the estimate

$$\|x\| \leq (E\|x_0\| + \eta M_{\mathcal{F}})\left[1 + \int_{t_0}^{T} EL_{\mathcal{F}}(s)e_\alpha(s, t_0)\Delta s\right].$$

Example 4.2 From Example 3.3, according to Lemma 3.1, the unique solution of NIDE (12) is given by

$$\begin{aligned} x(t) &= L_{\mathcal{F}} \int_0^t e_{\ominus 1}(t, s)\sin\left(x(s) + \int_0^s [\sin(x(\tau)) + \cos(x(\tau)) + 2s]\Delta\tau\right)\Delta s \\ &= L_{\mathcal{F}} \int_0^t e_{\ominus 1}(t, s)\sin\left(x(s) + \int_0^s \sin(x(\tau))\Delta\tau + \int_0^s \cos(x(\tau))\Delta\tau + 2s^2\right)\Delta s. \end{aligned}$$

Keeping in mind (13) and the boundedness of 'sine' function, we obtain

$$\|x\| \leq L_{\mathcal{F}}(2e)^{m+1}(2m + 1)$$

$$< \frac{1}{4m+3}.$$

Hence the solution of NIDE (12) is bounded and the bound is $\dfrac{1}{4m+3}$.

We now turn our attention to the results concerning dependence of solutions on various quantities. Below we prove the dependence of solutions on initial conditions. For this, first we consider the dynamic equation

$$y^{\Delta}(t) + p(t)y^{\sigma}(t) = \mathcal{F}\left(t, y(t), \int_{t_0}^{t} \mathcal{H}(t, s, y(s))\Delta s\right), \quad t \in \mathbb{I}^{\kappa}, \tag{15}$$

subject to the initial condition

$$y(t_0) = y_0 \in \mathbb{R}^n. \tag{16}$$

Theorem 4.3 (Dependence on initial conditions) *Assume that the functions $\mathcal{F}$ and $\mathcal{H}$ in NIDEs (1) and (15) satisfy (H_2) and (H_3). If x and y are the solutions of (1)–(2) and (15)–(16), respectively, then the inequality*

$$\|x - y\| \le E\|x_0 - y_0\| e_{\alpha}(T, t_0) \tag{17}$$

holds. Additionally, if $\|x_0 - y_0\|_n \le \delta$ for some $\delta > 0$, then we have

$$\|x - y\| \le E\delta e_{\alpha}(T, t_0). \tag{18}$$

Proof In view of Lemma 3.1, the solutions x and y of (1)–(2) and (15)–(16), respectively, are given by

$$x(t) = e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^{t} e_{\ominus p}(t, s)\mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, x(\tau))\Delta\tau\right)\Delta s$$

and

$$y(t) = e_{\ominus p}(t, t_0)y_0 + \int_{t_0}^{t} e_{\ominus p}(t, s)\mathcal{F}\left(s, y(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, y(\tau))\Delta\tau\right)\Delta s.$$

Then for each $t \in \mathbb{I}$, we have

$$\begin{aligned}
&\|x(t) - y(t)\|_n \\
&\quad \le |e_{\ominus p}(t, t_0)|\,\|x_0 - y_0\|_n + \int_{t_0}^{t} \left|e_{\ominus p}(t, s)\right| \left\| \mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, x(\tau))\Delta\tau\right)\right. \\
&\qquad \left. - \mathcal{F}\left(s, y(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, y(\tau))\Delta\tau\right)\right\|_n \Delta s.
\end{aligned}$$

Now, (H_2) and (H_3) lead to

$$\begin{aligned}
&\|x(t) - y(t)\|_n \\
&\quad \leq E\,\|x_0 - y_0\|_n \\
&\qquad + \int_{t_0}^{t} E\, L_{\mathcal{F}}(s) \left(\|x(s) - y(s)\|_n + \int_{t_0}^{s} L_{\mathcal{H}}(\tau)\, \|x(\tau) - y(\tau))\|_n\, \Delta\tau \right) \Delta s.
\end{aligned}$$

Employing the Gronwall inequality given in Theorem 2.1 (b) and keeping in mind the increasing nature of the exponential function in the first argument, we obtain, for all $t \in \mathbb{I}$

$$\|x(t) - y(t)\|_n \leq E\,\|x_0 - y_0\|_n\, e_\alpha(T, t_0).$$

Indeed, (17) and (18) follow from the above inequality. □

Remark 4.2 Under the conditions of Theorem 4.3 with the same calculation, and keeping in mind the inequality from Theorem 2.1 (a), we obtain the inequality

$$\|x - y\| \leq E\delta \left[1 + \int_{t_0}^{T} E\, L_{\mathcal{F}}(s) e_\alpha(s, t_0) \Delta s \right],$$

where $\|x_0 - y_0\|_n \leq \delta$ for some $\delta > 0$.

Next, to obtain a result concerning the dependency of solution on functions involved in the dynamic equation, we consider a variant form of the original NIDEs

$$z^\Delta(t) + p(t) z^\sigma(t) = \widehat{\mathcal{F}}\left(t, z(t), \textstyle\int_{t_0}^{t} \widehat{\mathcal{H}}(t, s, z(s)) \Delta s\right), \quad t \in \mathbb{I}^\kappa, \tag{19}$$

subject to the initial condition

$$z(t_0) = z_0 \in \mathbb{R}^n. \tag{20}$$

Theorem 4.4 (Dependence on functions) *Assume that the functions $\mathcal{F}$ and $\mathcal{H}$ in NIDE* (1) *satisfy* (H_2) *and* (H_3) *. Further, assume that there exists a constant $S > 0$ such that*

$$\|\mathcal{F}(t, u_1, v_1) - \widehat{\mathcal{F}}(t, u_2, v_2)\| \leq S$$

for all $t \in \mathbb{I}$ and $u_i, v_i \in \mathbb{R}^n$ ($i = 1, 2$). If x and z are the solutions of (1)–(2) *and* (19)–(20)*, respectively, then*

$$\|x - z\| \leq (E\delta + \eta S)\, e_\alpha(T, t_0), \tag{21}$$

where $\|x_0 - z_0\|_n \leq \delta$ for some $\delta > 0$.

Proof Since x and z are the solutions of (1)–(2) and (19)–(20), respectively, using Lemma 3.1, we write for $t \in \mathbb{I}$,

$$
\begin{aligned}
&\|x(t)-z(t)\|_n \\
&\quad \le |e_{\ominus p}(t,t_0)|\,\|x_0-z_0\|_n \\
&\qquad + \int_{t_0}^{t} \left|e_{\ominus p}(t,s)\right| \left\| \mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s,\tau,x(\tau))\Delta\tau\right)\right. \\
&\qquad \left. -\widehat{\mathcal{F}}\left(s, z(s), \int_{t_0}^{s} \widehat{\mathcal{H}}(s,\tau,z(\tau))\Delta\tau\right)\right\|_n \Delta s \\
&\quad \le |e_{\ominus p}(t,t_0)|\,\|x_0-z_0\|_n \\
&\qquad + \int_{t_0}^{t} \left|e_{\ominus p}(t,s)\right| \left\| \mathcal{F}\left(s, x(s), \int_{t_0}^{s} \mathcal{H}(s,\tau,x(\tau))\Delta\tau\right)\right. \\
&\qquad \left. -\mathcal{F}\left(s, z(s), \int_{t_0}^{s} \mathcal{H}(s,\tau,z(\tau))\Delta\tau\right)\right\|_n \Delta s \\
&\qquad + \int_{t_0}^{t} \left|e_{\ominus p}(t,s)\right| \left\| \mathcal{F}\left(s, z(s), \int_{t_0}^{s} \mathcal{H}(s,\tau,z(\tau))\Delta\tau\right)\right. \\
&\qquad \left. -\widehat{\mathcal{F}}\left(s, z(s), \int_{t_0}^{s} \widehat{\mathcal{H}}(s,\tau,z(\tau))\Delta\tau\right)\right\|_n \Delta s \\
&\quad \le E\,\|x_0-z_0\|_n \\
&\qquad + \int_{t_0}^{t} E L_{\mathcal{F}}(s)\left(\|x(s)-z(s)\|_n + \int_{t_0}^{s} L_{\mathcal{H}}(\tau)\,\|x(\tau)-z(\tau))\|_n\,\Delta\tau\right)\Delta s \\
&\qquad + S\int_{t_0}^{t} |e_{\ominus p}(t,s)|\Delta s \\
&\quad \le E\,\|x_0-z_0\|_n + \int_{t_0}^{t} E L_{\mathcal{F}}(s)\,(\|x(s)-z(s)\|_n \\
&\qquad + \int_{t_0}^{s} L_{\mathcal{H}}(\tau)\,\|x(\tau)-z(\tau))\|_n\,\Delta\tau\Big)\,\Delta s + \eta S.
\end{aligned}
$$

Employing the Gronwall inequality given in Theorem 2.1 (b) and keeping in mind the increasing nature of the exponential function in the first argument, we obtain, for all $t \in \mathbb{I}$

$$\|x(t)-z(t)\|_n \le (E\,\|x_0-z_0\|_n + \eta S)\, e_\alpha(T,t_0), \tag{22}$$

and (21) follows easily. □

Remark 4.3 Under the conditions of Theorem 4.3 with the same calculation, and keeping in mind the inequality from Theorem 2.1 (a), we obtain the inequality

$$\|x-z\| \le (E\delta + \eta S)\left[1 + \int_{t_0}^{T} E\, L_{\mathcal{F}}(s) e_\alpha(s,t_0)\Delta s\right],$$

where $\|x_0-z_0\|_n \le \delta$ for some $\delta > 0$.

Finally, to prove the result related with dependence on parameters, we consider NIDEs involving parameters of the form

$$x^{\Delta}(t) + p(t)x^{\sigma}(t) = \mathcal{F}\left(t, \gamma_1, x(t), \int_{t_0}^{t} \mathcal{H}(t, s, x(s))\Delta s\right), \quad t \in \mathbb{I}^{\kappa}; \tag{23}$$

$$x^{\Delta}(t) + p(t)x^{\sigma}(t) = \mathcal{F}\left(t, \gamma_2, x(t), \int_{t_0}^{t} \mathcal{H}(t, s, x(s))\Delta s\right), \quad t \in \mathbb{I}^{\kappa}, \tag{24}$$

subject to the initial condition

$$x(t_0) = x_0 \in \mathbb{R}^n, \tag{25}$$

where $\gamma_1, \gamma_2 \in \mathbb{R}$.

Theorem 4.5 (Dependence on parameters) *Let* (H_3) *hold. Assume that there exist* $\Omega,\ \Omega_\gamma > 0$ *and* $\widetilde{L}_{\mathcal{F}} \in \mathcal{R}^+(\mathbb{I}, \mathbb{R}^+)$ *such that*

$$\|\mathcal{F}(t, \gamma_i, u_1, v_1) - \mathcal{F}(t, \gamma_i, u_2, v_2)\|_n \leq \Omega_\gamma \widetilde{L}_{\mathcal{F}}(t)\left(\|u_1 - u_2\|_n + \|v_1 - v_2\|_n\right)$$

and

$$\|\mathcal{F}(t, \gamma_1, u_1, v_1) - \mathcal{F}(t, \gamma_2, u_1, v_1)\|_n \leq \Omega\,|\gamma_1 - \gamma_2|$$

for all $t \in \mathbb{I}$, $\gamma_i \in \mathbb{R}$ *and* $u_i, v_i \in \mathbb{R}^n$ $(i = 1, 2)$*. If* x_1 *and* x_2 *are the solutions of* (23)–(25) *and* (24)–(25)*, respectively, then*

$$\|x_1 - x_2\| \leq \eta\Omega|\gamma_1 - \gamma_2| e_{\widetilde{\alpha}}(T, t_0), \quad \textit{where } \widetilde{\alpha} = E\Omega_\gamma \widetilde{L}_{\mathcal{F}} + L_{\mathcal{H}}. \tag{26}$$

Proof Since x_1 and x_2 are the solutions of (23)–(25) and (24)–(25), respectively, using Lemma 3.1, we can write for all $t \in \mathbb{I}$,

$$\begin{aligned}
&\|x_1(t) - x_2(t)\|_n \\
&\quad = \left\| \int_{t_0}^{t} e_{\ominus p}(t, s)\left[\mathcal{F}\left(s, \gamma_1, x_1(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, x_1(\tau))\Delta\tau\right)\right.\right. \\
&\qquad \left.\left. -\mathcal{F}\left(s, \gamma_2, x_2(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, x_2(\tau))\Delta\tau\right)\right]\Delta s \right\|_n.
\end{aligned}$$

Then

$$\begin{aligned}
&\|x_1(t) - x_2(t)\|_n \\
&\quad \leq \int_{t_0}^{t} |e_{\ominus p}(t, s)| \left\| \mathcal{F}\left(s, \gamma_1, x_1(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, x_1(\tau))\Delta\tau\right)\right. \\
&\qquad \left. -\mathcal{F}\left(s, \gamma_2, x_2(s), \int_{t_0}^{s} \mathcal{H}(s, \tau, x_2(\tau))\Delta\tau\right)\right\|_n \Delta s
\end{aligned}$$

$$\leq \int_{t_0}^{t} |e_{\ominus p}(t,s)| \left\| \mathcal{F}\left(s, \gamma_1, x_1(s), \int_{t_0}^{s} \mathcal{H}(s,\tau,x_1(\tau))\Delta\tau\right) \right.$$
$$\left. -\mathcal{F}\left(s, \gamma_2, x_1(s), \int_{t_0}^{s} \mathcal{H}(s,\tau,x_1(\tau))\Delta\tau\right) \right\|_n \Delta s$$
$$+ \int_{t_0}^{t} |e_{\ominus p}(t,s)| \left\| \mathcal{F}\left(s, \gamma_2, x_1(s), \int_{t_0}^{s} \mathcal{H}(s,\tau,x_1(\tau))\Delta\tau\right) \right.$$
$$\left. -\mathcal{F}\left(s, \gamma_2, x_2(s), \int_{t_0}^{s} \mathcal{H}(s,\tau,x_2(\tau))\Delta\tau\right) \right\|_n \Delta s$$
$$\leq \Omega|\gamma_1 - \gamma_2| \int_{t_0}^{t} |e_{\ominus p}(t,s)|\Delta s$$
$$+ \int_{t_0}^{t} |e_{\ominus p}(t,s)|\Omega_\gamma \widetilde{L}_{\mathcal{F}}(s) \left(\|x_1(s) - x_2(s)\|_n + \int_{t_0}^{s} L_{\mathcal{H}}(\tau)\|x_1(\tau) - x_2(\tau)\|_n \right) \Delta s$$
$$\leq \eta\Omega|\gamma_1 - \gamma_2|$$
$$+ \int_{t_0}^{t} E\Omega_\gamma \widetilde{L}_{\mathcal{F}}(s) \left(\|x_1(s) - x_2(s)\|_n + \int_{t_0}^{s} L_{\mathcal{H}}(\tau)\|x_1(\tau) - x_2(\tau)\|_n \right) \Delta s.$$

Employing the Gronwall inequality given in Theorem 2.1 (b) and keeping in mind the increasing nature of the exponential function in the first argument, we obtain (26). □

Remark 4.4 Under the conditions of Theorem 4.5 with the same calculation, and keeping in mind the inequality from Theorem 2.1 (a), we obtain the inequality

$$\|x_1 - x_2\| \leq \eta\Omega|\gamma_1 - \gamma_2| \left[1 + \int_{t_0}^{T} E\Omega_\gamma \widetilde{L}_{\mathcal{F}}(s) e_{\widetilde{\alpha}}(s, t_0)\Delta s \right].$$

5 Ulam Stability

In this section, we shall investigate Ulam stability for NIDE (1). For this, first we shall introduce the following definitions.

Definition 5.1 We say that NIDE (1) has Hyers–Ulam stability if there exists a real number $C_{\mathcal{F}} > 0$ such that for each $\varepsilon > 0$ and for each $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I}, \mathbb{R}^n)$ satisfying

$$\left\| y^\Delta(t) + p(t)y^\sigma(t) - \mathcal{F}\left(t, y(t), \int_{t_0}^{t} \mathcal{H}(t,s,y(s))\Delta s\right) \right\|_n \leq \varepsilon \quad \text{for all } t \in \mathbb{I}^\kappa, \tag{27}$$

there exists a solution $x \in \mathrm{C}_{\mathrm{rd}}(\mathbb{I}, \mathbb{R}^n)$ of (1) with

$$\|y(t) - x(t)\|_n \leq \varepsilon C_{\mathcal{F}} \quad \text{for all } t \in \mathbb{I}.$$

Here $C_{\mathcal{F}}$ is a so-called HUS constant.

Definition 5.2 We say that NIDE (1) has generalised Hyers–Ulam stability if there exists a function $\theta_{\mathcal{F}} \in \mathrm{C}(\mathbb{R}^+, \mathbb{R}^+)$ with $\theta_{\mathcal{F}}(0) = 0$, such that for each $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I}, \mathbb{R}^n)$ satisfying (27), there exists a solution $x \in \mathrm{C}_{\mathrm{rd}}(\mathbb{I}, \mathbb{R}^n)$ of (1) with

$$\|y(t) - x(t)\|_n \leq \theta_{\mathcal{F}}(\varepsilon) \quad \text{for all } t \in \mathbb{I}.$$

Definition 5.3 Let $\mathcal{N}$ be a family of positive, nondecreasing rd-continuous real-valued functions defined on $\mathbb{I}$. We say that NIDE (1) has Hyers–Ulam–Rassias stability of type $\mathcal{N}$ if for each $\psi \in \mathcal{N}$, there exists $C_{\mathcal{F},\psi} > 0$ such that for each $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I}, \mathbb{R}^n)$ satisfying

$$\left\| y^{\Delta}(t) + p(t)y^{\sigma}(t) - \mathcal{F}\left(t, y(t), \int_{t_0}^{t} \mathcal{H}(t, s, y(s))\Delta s\right) \right\|_n \leq \varepsilon\psi(t) \quad \text{for all } t \in \mathbb{I}^{\kappa}, \tag{28}$$

there exists a solution $x \in \mathrm{C}_{\mathrm{rd}}(\mathbb{I}, \mathbb{R}^n)$ of (1) with

$$\|y(t) - x(t)\|_n \leq C_{\mathcal{F},\psi}\, \varepsilon\, \psi(t) \quad \text{for all } t \in \mathbb{I}.$$

Here $C_{\mathcal{F},\psi}$ is a so-called $\mathrm{HURS}_{\mathcal{N}}$ constant.

Definition 5.4 Let $\mathcal{N}$ be a family of positive, nondecreasing rd-continuous real-valued functions defined on $\mathbb{I}$. We say that NIDE (1) has generalised Hyers–Ulam–Rassias stability of type $\mathcal{N}$ if for each $\psi \in \mathcal{N}$, there exists $C_{\mathcal{F},\psi} > 0$ such that for each $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I}, \mathbb{R}^n)$ satisfying

$$\left\| y^{\Delta}(t) + p(t)y^{\sigma}(t) - \mathcal{F}\left(t, y(t), \int_{t_0}^{t} \mathcal{H}(t, s, y(s))\Delta s\right) \right\|_n \leq \psi(t) \quad \text{for all } t \in \mathbb{I}^{\kappa}, \tag{29}$$

there exists a solution $x \in \mathrm{C}_{\mathrm{rd}}(\mathbb{I}, \mathbb{R}^n)$ of (1) with

$$\|y(t) - x(t)\|_n \leq C_{\mathcal{F},\psi}\, \psi(t) \quad \text{for all } t \in \mathbb{I}.$$

Here $C_{\mathcal{F},\psi}$ is a so-called $\mathrm{GHURS}_{\mathcal{N}}$ constant.

Remark 5.1 A function $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I}, \mathbb{R}^n)$ is a solution of (28) if there exists a function $\mathcal{G} \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{I}, \mathbb{R}^n)$ (which depends on y) such that

(i) $\|\mathcal{G}(t)\|_n \leq \varepsilon\psi(t) \quad$ for all $t \in \mathbb{I}$,
(ii) $y^{\Delta}(t) + p(t)y^{\sigma}(t) = \mathcal{F}\left(t, y(t), \int_{t_0}^{t} \mathcal{H}(t, s, y(s))\Delta s\right) + \mathcal{G}(t) \quad$ for all $t \in \mathbb{I}^{\kappa}$.

Theorem 5.1 *Let (H_1) hold and the functions $\mathcal{F}$ and $\mathcal{H}$ in NIDE (1) satisfy (H_2) and (H_3). Assume that there exists $\lambda > 0$ such that for every $\psi \in \mathcal{N}$ and for all $t \in \mathbb{I}$,*

$$\int_{t_0}^{t} |e_{\ominus p}(t, s)|\psi(s)\Delta s \leq \lambda\psi(t). \tag{30}$$

Then the NIDE (1) has Hyers–Ulam–Rassias stability of type $\mathcal{N}$ with HURS$_{\mathcal{N}}$ constant $\lambda e_\alpha(T, t_0)$.

Proof Let $y \in C^1_{rd}(\mathbb{I}, \mathbb{R}^n)$ satisfy (28). Then, by Remark 5.1, we have

$$y^\Delta(t) + p(t)y^\sigma(t) = \mathcal{F}\left(t, y(t), \int_{t_0}^t \mathcal{H}(t, s, y(s))\Delta s\right) + \mathcal{G}(t) \quad \text{for all } t \in \mathbb{I}^\kappa.$$

Now, Lemma 3.1 implies that

$$\begin{aligned} y(t) &= e_{\ominus p}(t, t_0)\ y(t_0) + \int_{t_0}^t e_{\ominus p}(t, s)\left[\mathcal{F}\left(s, y(s), \int_{t_0}^s \mathcal{H}(s, \tau, y(\tau))\Delta\tau\right) + \mathcal{G}(s)\right]\Delta s, \\ &= e_{\ominus p}(t, t_0)\ y(t_0) + \int_{t_0}^t e_{\ominus p}(t, s)\mathcal{F}\left(s, y(s), \int_{t_0}^s \mathcal{H}(s, \tau, y(\tau))\Delta\tau\right)\Delta s \\ &\quad + \int_{t_0}^t e_{\ominus p}(t, s)\ \mathcal{G}(s)\Delta s \end{aligned}$$

for all $t \in \mathbb{I}$. Then

$$\begin{aligned} &\left\| y(t) - e_{\ominus p}(t, t_0)\ y(t_0) - \int_{t_0}^t e_{\ominus p}(t, s)\mathcal{F}\left(s, y(s), \int_{t_0}^s \mathcal{H}(s, \tau, y(\tau))\Delta\tau\right)\Delta s \right\|_n \\ &\quad \le \varepsilon \int_{t_0}^t |e_{\ominus p}(t, s)|\psi(s)\Delta s. \end{aligned}$$

Thus, in view of (30), we obtain

$$\begin{aligned} &\left\| y(t) - e_{\ominus p}(t, t_0)\ y(t_0) - \int_{t_0}^t e_{\ominus p}(t, s)\mathcal{F}\left(s, y(s), \int_{t_0}^s \mathcal{H}(s, \tau, y(\tau))\Delta\tau\right)\Delta s \right\|_n \\ &\quad \le \varepsilon\lambda\psi(t) \end{aligned} \tag{31}$$

for all $t \in \mathbb{I}$. Let $x \in C_{rd}(\mathbb{I}, \mathbb{R}^n)$ be the solution of the dynamic problem

$$\begin{aligned} x^\Delta(t) + p(t)x^\sigma(t) &= \mathcal{F}\left(t, x(t), \int_{t_0}^t \mathcal{H}(t, s, x(s))\Delta s\right), \quad t \in \mathbb{I}^\kappa, \\ x(t_0) &= y(t_0). \end{aligned}$$

Then, we have

$$\begin{aligned} x(t) &= e_{\ominus p}(t, t_0)\, y(t_0) \\ &\quad + \int_{t_0}^t e_{\ominus p}(t, s)\mathcal{F}\left(s, x(s), \int_{t_0}^s \mathcal{H}(s, \tau, x(\tau))\Delta\tau\right)\Delta s \quad \text{for all } t \in \mathbb{I}. \end{aligned} \tag{32}$$

From (31), (32), and using (H_2) and (H_3), we have

$$
\begin{aligned}
&\|y(t)-x(t)\|_n \qquad (33)\\
&\leq \left\| y(t)-e_{\ominus p}(t,t_0)y(t_0)-\int_{t_0}^{t} e_{\ominus p}(t,s)\mathcal{F}\left(s,y(s),\int_0^s \mathcal{H}(s,\tau,y(\tau))\Delta\tau\right)\Delta s\right\|_n\\
&\quad+\int_{t_0}^{t}|e_{\ominus p}(t,s)|\left\|\mathcal{F}\left(s,y(s),\int_{t_0}^{s}\mathcal{H}(s,\tau,y(\tau))\Delta\tau\right)\right.\\
&\quad\left.-\mathcal{F}\left(s,x(s),\int_{t_0}^{s}\mathcal{H}(s,\tau,x(\tau))\Delta\tau\right)\right\|_n\Delta s\\
&\leq \varepsilon\lambda\psi(t)+\int_{t_0}^{t}EL_{\mathcal{F}}(s)\left(\|y(s)-x(s)\|_n+\int_{t_0}^{s}L_{\mathcal{H}}(\tau)\left[\|y(\tau)-x(\tau)\|_n\right]\Delta\tau\right)\Delta s. \qquad (34)
\end{aligned}
$$

Employing the Gronwall inequality given in Theorem 2.1 (b) and keeping in mind the increasing nature of the exponential function in the first argument, we obtain

$$\|y(t)-x(t)\|_n \leq \varepsilon\lambda\psi(t)e_\alpha(T,t_0).$$

This yields

$$\|y(t)-x(t)\|_n \leq \varepsilon C_{\mathcal{F},\psi}\psi(t) \quad \text{for all } t\in\mathbb{I},$$

where $C_{\mathcal{F},\psi} := \lambda e_\alpha(T,t_0)$. Thus NIDE (1) has Hyers–Ulam–Rassias stability of type $\mathcal{N}$ with HURS$_{\mathcal{N}}$ constant $\lambda e_\alpha(T,t_0)$. □

Corollary 5.1 *Let* (H_1) *hold and the functions* $\mathcal{F}$ *and* $\mathcal{H}$ *in NIDE* (1) *satisfy the conditions of Theorem 5.1. Then NIDE* (1) *has generalized Hyers–Ulam–Rassias stability of type* $\mathcal{N}$ *with GHURS*$_{\mathcal{N}}$ *constant* $\lambda e_\alpha(T,t_0)$.

Proof Taking $\varepsilon = 1$ in the proof of Theorem 5.1 we obtain that NIDE (1) has generalized Hyers–Ulam–Rassias stability of type $\mathcal{N}$ with GHURS$_{\mathcal{N}}$ constant $\lambda e_\alpha(T,t_0)$. □

Remark 5.2 Employing the inequality from Theorem 2.1 (a) to (33) we infer that NIDE (1) has Hyers–Ulam–Rassias stability of type $\mathcal{N}$ as well as generalized Hyers–Ulam–Rassias stability of type $\mathcal{N}$ with

$$\lambda\left[1+\int_{t_0}^{T}EL_{\mathcal{F}}(s)e_\alpha(T,t_0)\Delta s\right]$$

being both the HURS$_{\mathcal{N}}$ and GHURS$_{\mathcal{N}}$ constant.

Corollary 5.2 *Let* (H_1) *hold and the functions* $\mathcal{F}$ *and* $\mathcal{H}$ *in NIDE* (1) *satisfies* (H_2) *and* (H_3). *Then NIDE* (1) *has Hyers–Ulam stable stability with HUS constant* $\eta e_\alpha(T,t_0)$.

Proof Take $\psi(t)\equiv 1$. Then

$$\int_{t_0}^{t}|e_{\ominus p}(t,s)|\psi(s)\Delta s \leq \eta \quad \text{for all } t\in\mathbb{I},$$

and from the proof of Theorem 5.1, we obtain

$$\|y(t) - x(t)\|_n \le \varepsilon C_{\mathcal{F}} \quad \text{for all } t \in \mathbb{I}. \tag{35}$$

This proves that (1) has Hyers–Ulam stability with HUS constant $C_{\mathcal{F}} := \eta e_\alpha(T, t_0)$. ◻

Corollary 5.3 *Let* (H_1) *hold and the functions* $\mathcal{F}$ *and* $\mathcal{H}$ *in* (1) *satisfy* (H_2) *and* (H_3). *Then* (1) *has generalized Hyers–Ulam stability of type* $\mathcal{N}$.

Proof Define $\theta_{\mathcal{F}}(\varepsilon) := \varepsilon\, C_{\mathcal{F}}$. Clearly $\theta_{\mathcal{F}} \in \mathrm{C}(\mathbb{R}^+, \mathbb{R}^+)$ and $\theta_{\mathcal{F}}(0) = 0$. Then (35) takes the form,

$$\|y(t) - x(t)\| \le \theta_{\mathcal{F}}(\varepsilon) \quad \text{for all } t \in \mathbb{I}.$$

This proves that NIDE (1) has generalized Hyers–Ulam stability of type $\mathcal{N}$. ◻

Remark 5.3 Employing the inequality from Theorem 2.1 (a) to (33) we infer that NIDE (1) has Hyers–Ulam stability as well as generalized Hyers–Ulam stability with

$$\eta \left[1 + \int_{t_0}^{T} E L_{\mathcal{F}}(s) e_\alpha(T, t_0) \Delta s\right]$$

being the HUS constant.

The following example illustrates the results obtained in this section.

Example 5.2 Let

$$\mathbb{T} = \bigcup_{k=0}^{\infty} [2k, 2k+1]$$

and consider the NIDE

$$x^{\Delta}(t) + x^{\sigma}(t) = L_{\mathcal{F}}(x(t) + 3)^{1/2} + L_{\mathcal{F}} \int_{t_0}^{t} \frac{x(s)}{1 + x(s)} \Delta s, \quad t \in [t_0, T]_{\mathbb{T}}^{\kappa}, \tag{36}$$

with $x(t_0) = 0$, where

$$L_{\mathcal{F}} \in \left(0, \frac{1}{(2e)^{m+1}(2m+1)(2m+2)}\right). \tag{37}$$

We take $t_0 = 0$, $T = 2m + 1$, $m \in \mathbb{N}$. Here $p(t) \equiv 1$. So $p \in \mathcal{R}$, and (H_1) holds. Moreover,

$$\mathcal{H}(t, s, x) = \frac{x}{1 + x} \text{ and } \mathcal{F}(t, x, y) = L_{\mathcal{F}}((x + 3)^{1/2} + y),$$

We note that (H_2) is satisfied, since

$$
\begin{aligned}
&|\mathcal{F}(t, x_1(t), y_1(t)) - \mathcal{F}(t, x_2(t), y_2(t))| \\
&\quad = L_{\mathcal{F}}\left(\left|(x_1(t)+3)^{1/2} - (x_2(t)+3)^{1/2}\right| + |y_1(t) - y_2(t)|\right)
\end{aligned}
$$

and, by [7, Corollary 1.68], we can write

$$
\begin{aligned}
&|\mathcal{F}(t, x_1(t), y_1(t)) - \mathcal{F}(t, x_2(t), y_2(t))| \\
&\quad \le L_{\mathcal{F}}\left(\sup_{z\in\mathbb{R}}\left|\frac{z}{(z^2+3)^{1/2}}\right| |x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|\right) \\
&\quad \le L_{\mathcal{F}}(|x_1 - x_2| + |y_1 - y_2|).
\end{aligned}
$$

Also,

$$
\begin{aligned}
|\mathcal{H}(t, s, x_1(s)) - (t, s, x_2(s))| &= \left|\frac{x_1(s)}{1+x_1(s)} - \frac{x_2(s)}{1+x_2(s)}\right| \\
&= \left|\frac{x_1(s) - x_2(s)}{(1+x_1(s))(1+x_2(s))}\right| \\
&\le |x_1(s) - x_2(s)|.
\end{aligned}
$$

Hence, (H_3) is satisfied with $L_{\mathcal{H}} = 1$. Further, since

$$
\eta = \sup_{t\in[0,2m+1]_{\mathbb{T}}} \int_0^t e_{\ominus 1}(t, s)\Delta s = (2e)^{m+1}(2m+1),
$$

we find that (H_4) is satisfied with $L_{\mathcal{H}} = 1$ and $L_{\mathcal{F}}$ given in (37). Therefore, all the conditions of Theorem 3.2 are satisfied. Hence, NIDE (36) has a unique solution. In fact, by Lemma 3.1, this unique solution is given by

$$
x(t) = L_{\mathcal{F}} \int_0^t e_{\ominus 1}(t, s)\left((x(s)+3)^{1/2} + \int_0^s \frac{x(\tau)}{1+x(\tau)}\Delta\tau\right)\Delta s.
$$

Further, if $y \in C^1_{\mathrm{rd}}([0, 2m+1]_{\mathbb{T}}, \mathbb{R})$ satisfies

$$
\left|y^{\Delta}(t) + y^{\sigma}(t) - L_{\mathcal{F}}(y(t)+3)^{1/2} - \int_0^t \frac{y(s)}{1+y(s)}\Delta s\right| \le \varepsilon,
$$

then by Corollary 5.2, there exists a solution x of NIDE (36) satisfying

$$
|y(t) - x(t)| \le \varepsilon(2e)^{m+1}(2m+1)e_{\alpha}(2m+1, 0),
$$

where $\alpha = (2e)^{m+1}L_{\mathcal{F}} + 1$. Hence, NIDE (36) has Hyers–Ulam stability with HUS constant $(2e)^{m+1}(2m+1)e_{\alpha}(2m+1, 0)$. We refer to Fig. 1 for a picture of the solution

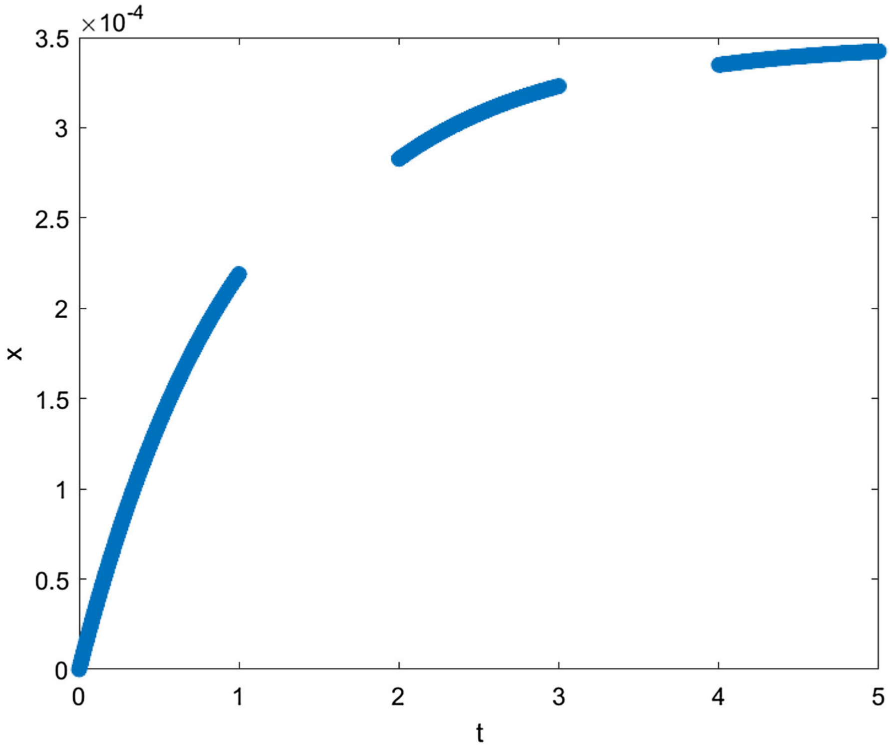

Fig. 1 The solution x of (36) with $m = 2$ and $L_{\mathcal{F}} = 0.0002$

of (36) when $m = 2$ and $L_{\mathcal{F}} = 0.0002$ (which satisfies (37)). As there is no software available to solve such problems on time scales, we explain now how we were able to depict this solution. We used the general-purpose MATLAB® code IDSOLVER by Gelmi and Jorquera [11], with slight modifications, in combination with manual calculations in order to account for the special structure of the time scale. To begin with, we replaced the last line in the IDSOLVER with

```
dy(n) =c(x, y(1)) + d(x) * quadl(@(s)
          k(x, s). * ys(s)./(1 + ys(s)), alpha(x), beta(x), TolQuad);
```

Step 1

First, we solved (36) on [0, 1], with the initial condition $x(0) = 0$, using the code

```
xinterval = [01];
n = 1;
InitCond = 0;
c = @(x, y)0.0002. * sqrt(y + 3) - y;
d = @(x)1;
```

```
k = @(x, s)0.0002;
alpha = @(x)0;
beta = @(x)x;
Tol = 1e − 30;
Flag = 0;
idsolver(xinterval, n, InitCond, c, d, k, alpha, beta, Tol, Flag)
```

This produced the left part of the graph in Fig. 1, as well as the values

$$x(1) \approx 0.000218985443791 =: \xi_1,$$
$$x(1^-) \approx 0.000218972695594 =: \xi_1^- \quad \text{with} \quad 1^- := 0.999899989999,$$

which were used to approximate

$$\lambda_1 := \int_0^1 \frac{x(s)}{1+x(s)} \Delta s \approx 5000 \left(\xi_1 + \frac{\xi_1 - \xi_1^-}{1-1^-} \right) - \sqrt{\xi_1 + 3}$$
$$\approx 0.00015930590286625453724869533495221.$$

Step 2
Next, we manually solved (36) for $t = 1$, with the initial condition $x(1) = \xi_1$, so

$$x(2) = \frac{\xi_1 + 0.0002 \left(\sqrt{\xi_1 + 3} + \lambda_1\right)}{2}$$
$$\approx 0.00028272005469256373461090156373461 =: \xi_2.$$

Step 3
Next, we solved (36) on [2, 3], i.e.,

$$x'(t) + x(t) = 0.0002\sqrt{x(t)+3} + 0.0002\lambda_2 + 0.0002 \int_2^t \frac{x(s)}{1+x(s)} \mathrm{d}s,$$

where

$$\lambda_2 := \int_0^2 \frac{x(s)}{1+x(s)} \Delta s = \lambda_1 + \frac{\xi_1}{1+\xi_1}$$
$$\approx 0.00037824340253172780167184558590068,$$

with the initial condition $x(2) = \xi_2$, using the same code as in Step 1, with obvious modifications in lines 1, 3, 4, and 7. This produced the middle part of the graph in Figure 1, as well as the values

$$x(3) \approx 0.000323061106953 =: \xi_3,$$
$$x(3^-) \approx 0.000323058756143 =: \xi_3^- \quad \text{with} \quad 3^- := 2.999899989999,$$

which were used to approximate

$$\lambda_3 := \int_0^3 \frac{x(s)}{1+x(s)}\Delta s = \lambda_2 + \int_2^3 \frac{x(s)}{1+x(s)}\Delta s$$
$$\approx 5000\left(\xi_3 + \frac{\xi_3 - \xi_3^-}{3-3^-}\right) - \sqrt{\xi_3+3}$$
$$\approx 0.00069021594828844142473145045426559.$$

Step 4

Next, we manually solved (36) for $t = 3$, with the initial condition $x(3) = \xi_3$, so

$$x(4) = \frac{\xi_3 + 0.0002\left(\sqrt{\xi_3+3} + \lambda_3\right)}{2}$$
$$\approx 0.00033481398154801175287459501175287 =: \xi_4.$$

Step 5

Next, we solved (36) on [4, 5], i.e.,

$$x'(t) + x(t) = 0.0002\sqrt{x(t)+3} + 0.0002\lambda_4 + 0.0002\int_4^t \frac{x(s)}{1+x(s)}\mathrm{d}s,$$

where

$$\lambda_4 := \int_0^4 \frac{x(s)}{1+x(s)}\Delta s = \lambda_3 + \frac{\xi_3}{1+\xi_3}$$
$$\approx 0.0010131727204691227659816244 2032,$$

with the initial condition $x(4) = \xi_4$, using the same code as in Step 1, with obvious modifications in lines 1, 3, 4, and 7. This produced the right part of the graph in Figure 1.

6 Concluding Remark

A new type of nonlinear integro-dynamic equations on time scales (1) was considered and several qualitative results are derived in an effective way. We have derived the existence of the solution of this equation using the fixed point theorem of Krasnoselskiĭ. The uniqueness of solution, dependence of solutions on various data, and Ulam stability are investigated mainly employing Gronwall-type dynamic inequalities. To illustrate the applicability of the main results of this paper, we have provided three examples on a nonstandard time scale domain. As there is no software available to solve such types of integro-dynamic equations on time scales, we rewrote a MATLAB®code from the literature, ran this code for one of the examples with some manual calculations, and then plotted the solution. The results obtained in this paper are new, original, and could be useful tools for researchers working in related areas. Moreover, we emphasize that other qualitative properties like oscillation and nonoscillation, asymptotic behaviour, and controllability of solutions would be an interesting topics for future research.

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Declarations

Conflict of interest The authors declare that there is no competing interests regarding the publication of this article.

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