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# Bilinear Auto-Bäcklund Transformations and Similarity Reductions for a (3+1)-dimensional Generalized Yu-Toda-Sasa-Fukuyama System in Fluid Mechanics and Lattice Dynamics

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**Abstract**
Recent investigations on the liquids and lattices are both active. In this paper, with symbolic computation, we consider a (3+1)-dimensional generalized Yu-Toda-Sasa-Fukuyama system for the interfacial waves in a two-layer liquid or elastic waves in a lattice, with two sets of the bilinear auto-Bäcklund transformations hereby built up. Moreover, we construct one set of the similarity reductions, from that system to a known ordinary differential equation. As for the amplitude or elevation of the relevant wave, our results rely on the coefficients in that system.

**Keywords** Two-layer liquid · Lattice · (3+1)-dimensional generalized Yu-Toda-Sasa-Fukuyama system · Bilinear auto-Bäcklund transformations · Similarity reductions · Symbolic computation

**Mathematics Subject Classification** 37K35 · 37N10 · 35Q35 · 76B15

## 1 Introduction

Fluid mechanics and lattice dynamics deal with fluid behaviors/interactions under various forces and with the vibrations of atoms inside crystals, basic to such fields as the oceanic, atmospheric, mineral and solid state sciences [1–8]. In connection with

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fluid mechanics and lattice dynamics, in this paper, we consider a (3+1)-dimensional generalized Yu-Toda-Sasa-Fukuyama system for the interfacial waves in a two-layer liquid or elastic waves in a lattice [7] (and references therein), i.e.,

$$(h_1 v_t + h_2 v_{xxz} + h_4 v_x u_z + h_5 v v_z)_x + h_3 v_{yy} + h_6 v_{xy} + h_7 v_{xz} + h_8 v_{yz} = 0, \quad \text{(1a)}$$
$$u_x = v, \quad \text{(1b)}$$

where $t$ is the scaled temporal coordinate, $x$, $y$ and $z$ denote the scaled spatial coordinates, $v(x, y, z, t)$ indicates the amplitude or elevation of the relevant wave, while $v(x, y, z, t)$ and $u(x, y, z, t)$ mean two real differentiable functions as for $x$, $y$, $z$ and $t$ [7]. In addition, $h_\Upsilon$'s are the real constants, with $\Upsilon = 1, ..., 8$.

Some special cases of System (1) were studied previously [8–17]: for instance,

1. when $h_6 = h_7 = h_8 = 0$, describing the interfacial waves in a two-layer liquid, a (3+1)-dimensional Yu-Toda-Sasa-Fukuyama system [8], i.e.,

$$(h_1 v_t + h_2 v_{xxz} + h_4 v_x u_z + h_5 v v_z)_x + h_3 v_{yy} = 0, \quad \text{(2a)}$$
$$u_x = v; \quad \text{(2b)}$$

2. when $h_1 = h_2 = 1$, $h_5 = 6$, $h_4 = h_6 = h_7 = h_8 = 0$ and $z = x$, describing the long water waves and small-amplitude surface waves with the weak nonlinearity, weak dispersion and weak perturbation in a fluid[1], a (2+1)-dimensional generalized Kadomtsev-Petviashvili equation [9, 10],

$$(v_t + v_{xxx} + 6 v v_x)_x + h_3 v_{yy} = 0; \quad \text{(3)}$$

3. when $h_1 = h_2 = h_3 = h_5 = 1$, $h_4 = h_6 = h_7 = h_8 = 0$ and $z = x$, describing the weakly transverse water waves in the long wave regime with small surface tension, a (2+1)-dimensional Kadomtsev-Petviashvili equation [11, 12],

$$(v_t + v_{xxx} + v v_x)_x + v_{yy} = 0; \quad \text{(4)}$$

4. when $h_1 = h_2 = 1$, $h_5 = -6$, $h_3 = h_4 = h_6 = h_7 = h_8 = 0$ and $z = x$, describing the long waves in shallow water under the gravity, waves in a nonlinear lattice, ion-acoustic and magneto-acoustic waves in a plasma, and also applying to nonlinear optics[2] and quantum mechanics, a (1+1)-dimensional Korteweg-de Vries equation [13, 14],

$$v_t + v_{xxx} - 6 v v_x = 0; \quad \text{(5)}$$

5. when $h_1 = h_5 = 1$, $h_2 = \frac{1}{4}$, $h_4 = \frac{1}{2}$ and $h_3 = h_6 = h_7 = h_8 = 0$, describing the (2+1)-dimensional interaction between a Riemann wave propagating along the $z$

[1] Other fluid-mechanics studies have been shown, e.g., in Refs. [18–34].

[2] Other nonlinear-optics investigations have been found, e.g., in Refs. [35–48].

axis and a long wave propagating along the $x$ axis, a (2+1)-dimensional integrable Calogero-Bogoyavlenskii-Schiff system [15, 16],

$$v_t + \frac{1}{4} v_{xxz} + \frac{1}{2} v_x u_z + v v_z = 0, \tag{6a}$$

$$u_x = v; \tag{6b}$$

6. when $h_1 = -4$, $h_2 = 1$, $h_3 = 3$, $h_4 = 2$, $h_5 = 4$ and $h_6 = h_7 = h_8 = 0$, describing the interfacial waves in a two-layer liquid or elastic quasiplane waves in a lattice, a (3+1)-dimensional Yu-Toda-Sasa-Fukuyama system [17], i.e.,

$$(-4v_t + v_{xxz} + 2v_x u_z + 4v v_z)_x + 3v_{yy} = 0, \tag{7a}$$

$$u_x = v \quad . \tag{7b}$$

For System (1), currently interesting, under the coefficient constraints

$$h_1 = 1, \qquad h_4 = h_5, \tag{8}$$

on account of the transformations

$$v = \frac{6h_2}{h_4} (\ln f)_{xx}, \qquad u = \frac{6h_2}{h_4} (\ln f)_x, \tag{9}$$

Shen et al. [7] have presented a bilinear form, i.e.,

$$\left( D_x D_t + h_2 D_x^3 D_z + h_3 D_y^2 + h_6 D_x D_y + h_7 D_x D_z + h_8 D_y D_z \right) f \cdot f = 0, \tag{10}$$

where $f$ stands for a real differentiable function in respect of $x$, $y$, $z$ and $t$, while the bilinear notations $D_x$, $D_y$, $D_z$ and $D_t$ are explained in the Appendix. Besides, bilinear auto-Bäcklund transformation[3], breather and periodic solutions for System (1) have been worked out [7].

Hereby with symbolic computation [54–58], on the one hand, for System (1), we will build up two sets of the bilinear auto-Bäcklund transformations, which are different from the one presented in Ref. [7], through the Hirota method [10, 59–62]. On the other hand, we will construct a set of the similarity reductions for System (1).

## 2 Bilinear Auto-Bäcklund Transformations for System (1)

The bilinear notations $D_x$, $D_y$, $D_z$ and $D_t$ can be found in the Appendix.

Because of the existing bilinear form under coefficient constraints, i.e., (10) under (8), employing the Hirota method, assuming that $g$ stands for another solution of

[3] Other auto-Bäcklund transformations have been reported, e.g., in Refs. [49–51]. Besides, hetero-Bäcklund transformations, also named the non-auto-Bäcklund transformations, could been seen, e.g., in Refs. [52, 53].

Form (10) and taking into account the expression[4]

$$
\begin{aligned}
& f^2\Big[\Big(D_xD_t+h_2D_x^3D_z+h_3D_y^2+h_6D_xD_y+h_7D_xD_z+h_8D_yD_z\Big)\,g\cdot g\Big]\\
&\quad -g^2\Big[\Big(D_xD_t+h_2D_x^3D_z+h_3D_y^2+h_6D_xD_y+h_7D_xD_z\\
&\quad +h_8D_yD_z\Big)\,f\cdot f\Big]=0,
\end{aligned}
\tag{11}
$$

we get

$$
\begin{aligned}
& f^2\Big[\Big(D_xD_t+h_2D_x^3D_z+h_3D_y^2+h_6D_xD_y+h_7D_xD_z+h_8D_yD_z\Big)\,g\cdot g\Big]\\
&\quad -g^2\Big[\Big(D_xD_t+h_2D_x^3D_z+h_3D_y^2+h_6D_xD_y+h_7D_xD_z+h_8D_yD_z\Big)\,f\cdot f\Big]\\
&=\Big[f^2\big(D_xD_t\,g\cdot g\big)-g^2\big(D_xD_t\,f\cdot f\big)\Big]\\
&\quad +h_2\Big[f^2\big(D_x^3D_z\,g\cdot g\big)-g^2\big(D_x^3D_z\,f\cdot f\big)\Big]\\
&\quad +h_3\Big[f^2\big(D_y^2\,g\cdot g\big)-g^2\big(D_y^2\,f\cdot f\big)\Big]\\
&\quad +h_6\Big[f^2\big(D_xD_y\,g\cdot g\big)-g^2\big(D_xD_y\,f\cdot f\big)\Big]\\
&\quad +h_7\Big[f^2\big(D_xD_z\,g\cdot g\big)-g^2\big(D_xD_z\,f\cdot f\big)\Big]\\
&\quad +h_8\Big[f^2\big(D_yD_z\,g\cdot g\big)-g^2\big(D_yD_z\,f\cdot f\big)\Big].
\end{aligned}
$$

Then, making use of the exchange formulae[5] [10]

$$
F^2\left(D_xD_t\,G\cdot G\right)-G^2\left(D_xD_t\,F\cdot F\right)=2\,D_x\left(D_t\,G\cdot F\right)\cdot\left(FG\right), \tag{12a}
$$

$$
\begin{aligned}
& F^2\left(D_x^3D_z\,G\cdot G\right)-G^2\left(D_x^3D_z\,F\cdot F\right)\\
&\quad =2\,D_z\left(D_x^3\,G\cdot F\right)\cdot\left(FG\right)-6\,D_x\left(D_xD_z\,G\cdot F\right)\cdot\left(D_x\,G\cdot F\right),
\end{aligned}
\tag{12b}
$$

$$
\begin{aligned}
& F^2\left(D_x^3D_z\,G\cdot G\right)-G^2\left(D_x^3D_z\,F\cdot F\right)\\
&\quad =\frac{1}{2}\,D_z\left(D_x^3\,G\cdot F\right)\cdot\left(FG\right)+\frac{3}{2}\,D_x\left(D_x^2D_z\,G\cdot F\right)\cdot\left(FG\right)\\
&\qquad -3\,D_x\left(D_xD_z\,G\cdot F\right)\cdot\left(D_x\,G\cdot F\right)-\frac{3}{2}\,D_x\left(D_x^2\,G\cdot F\right)\cdot\left(D_z\,G\cdot F\right)\\
&\qquad -\frac{3}{2}\,D_z\left(D_x^2\,G\cdot F\right)\cdot\left(D_x\,G\cdot F\right),
\end{aligned}
\tag{12c}
$$

$$
F^2\left(D_y^2\,G\cdot G\right)-G^2\left(D_y^2\,F\cdot F\right)=2\,D_y\left(D_y\,G\cdot F\right)\cdot\left(FG\right), \tag{12d}
$$

$$
F^2\left(D_xD_y\,G\cdot G\right)-G^2\left(D_xD_y\,F\cdot F\right)=2\,D_x\left(D_y\,G\cdot F\right)\cdot\left(FG\right), \tag{12e}
$$

$$
F^2\left(D_xD_y\,G\cdot G\right)-G^2\left(D_xD_y\,F\cdot F\right)=2\,D_y\left(D_x\,G\cdot F\right)\cdot\left(FG\right), \tag{12f}
$$

$$
F^2\left(D_xD_z\,G\cdot G\right)-G^2\left(D_xD_z\,F\cdot F\right)=2\,D_x\left(D_z\,G\cdot F\right)\cdot\left(FG\right), \tag{12g}
$$

[4] similar to those in Refs. [63, 64]

[5] with $F$ and $G$ as the real differentiable functions of $x$, $y$, $z$ and $t$

$$F^2\,(D_xD_z\,G\cdot G)-G^2\,(D_xD_z\,F\cdot F)=2\,D_z\,(D_x\,G\cdot F)\cdot(FG)\,, \tag{12h}$$
$$F^2\,\big(D_yD_z\,G\cdot G\big)-G^2\,\big(D_yD_z\,F\cdot F\big)=2\,D_y\,(D_z\,G\cdot F)\cdot(FG)\,, \tag{12i}$$
$$F^2\,\big(D_yD_z\,G\cdot G\big)-G^2\,\big(D_yD_z\,F\cdot F\big)=2\,D_z\,\big(D_y\,G\cdot F\big)\cdot(FG)\,, \tag{12j}$$

we could build up two sets of the bilinear auto-Bäcklund transformations for System (1):

**Set 1:**

The exchange formulae, i.e., (12a), (12b), (12d), (12f), (12g) and (12j), bring about

$$\begin{aligned}
&f^2\Big[\Big(D_xD_t+h_2D_x^3D_z+h_3D_y^2+h_6D_xD_y+h_7D_xD_z+h_8D_yD_z\Big)\,g\cdot g\Big]\\
&\quad-g^2\Big[\Big(D_xD_t+h_2D_x^3D_z+h_3D_y^2+h_6D_xD_y+h_7D_xD_z+h_8D_yD_z\Big)\,f\cdot f\Big]\\
&=\Big[\,f^2\big(D_xD_t\,g\cdot g\big)-g^2\big(D_xD_t\,f\cdot f\big)\Big]\\
&\quad+h_2\Big[\,f^2\big(D_x^3D_z\,g\cdot g\big)-g^2\big(D_x^3D_z\,f\cdot f\big)\Big]\\
&\quad+h_3\Big[\,f^2\big(D_y^2\,g\cdot g\big)-g^2\big(D_y^2\,f\cdot f\big)\Big]\\
&\quad+h_6\Big[\,f^2\big(D_xD_y\,g\cdot g\big)-g^2\big(D_xD_y\,f\cdot f\big)\Big]\\
&\quad+h_7\Big[\,f^2\big(D_xD_z\,g\cdot g\big)-g^2\big(D_xD_z\,f\cdot f\big)\Big]\\
&\quad+h_8\Big[\,f^2\big(D_yD_z\,g\cdot g\big)-g^2\big(D_yD_z\,f\cdot f\big)\Big]\\
&=2\,D_x\big(D_t\,g\cdot f\big)\cdot\big(fg\big)+h_2\Big[2\,D_z\big(D_x^3\,g\cdot f\big)\cdot\big(fg\big)\\
&\quad-6\,D_x\big(D_xD_z\,g\cdot f\big)\cdot\big(D_x\,g\cdot f\big)\Big]+2\,h_3\,D_y\big(D_y\,g\cdot f\big)\cdot\big(fg\big)\\
&\quad+2\,h_6\,D_y\big(D_x\,g\cdot f\big)\cdot\big(fg\big)+2\,h_7\,D_x\big(D_z\,g\cdot f\big)\cdot\big(fg\big)\\
&\quad+2\,h_8\,D_z\big(D_y\,g\cdot f\big)\cdot\big(fg\big)\\
&=2\,D_x\Big[\big(D_t+h_7D_z\big)\,g\cdot f\Big]\cdot\big(fg\big)+2\,D_z\Big[\big(h_2D_x^3+h_8D_y\big)\,g\cdot f\Big]\cdot\big(fg\big)\\
&\quad-6\,h_2\,D_x\big(D_xD_z\,g\cdot f\big)\cdot\big(D_x\,g\cdot f\big)+2\,D_y\Big[\big(h_3D_y+h_6D_x\big)\,g\cdot f\Big]\cdot\big(fg\big).
\end{aligned}$$

Under the coefficient constraints, i.e., (8), assumptions that

$$D_x\Big[\big(D_t+h_7D_z\big)\,g\cdot f\Big]\cdot\big(fg\big)=0, \tag{13a}$$
$$D_z\Big[\big(h_2D_x^3+h_8D_y\big)\,g\cdot f\Big]\cdot\big(fg\big)=0, \tag{13b}$$
$$D_x\big(D_xD_z\,g\cdot f\big)\cdot\big(D_x\,g\cdot f\big)=0, \tag{13c}$$
$$D_y\Big[\big(h_3D_y+h_6D_x\big)\,g\cdot f\Big]\cdot\big(fg\big)=0, \tag{13d}$$

develop into

$$v=\frac{6h_2}{h_4}(\ln f)_{xx}, \tag{14a}$$
$$u=\frac{6h_2}{h_4}(\ln f)_{x}, \tag{14b}$$

$$v_0 = \frac{6h_2}{h_4}(\ln g)_{xx}, \tag{14c}$$

$$u_0 = \frac{6h_2}{h_4}(\ln g)_{x}, \tag{14d}$$

$$\left(D_t + h_7 D_z\right) g \cdot f = 0, \tag{14e}$$

$$\left(h_2 D_x^3 + h_8 D_y\right) g \cdot f = 0, \tag{14f}$$

$$D_x D_z\, g \cdot f = \mu_1 D_x\, g \cdot f, \tag{14g}$$

$$\left(h_3 D_y + h_6 D_x\right) g \cdot f = 0, \tag{14h}$$

with $u_0$ and $v_0$ as another set of the solutions of System (1), while $\mu_1$ as a real constant.

**Theorem 2.1** *Equations (14) comprise a set of the bilinear auto-Bäcklund transformations*[6] *for System (1), because of their mutual consistency.*

**Corollary 2.1** *Describing certain interfacial waves in a two-layer liquid or elastic quasiplane waves in a lattice, modelling the amplitude or elevation of the relevant wave, Bilinear Auto-Bäcklund Transformations (14) depend on $h_2$, $h_3$, $h_4$, $h_6$, $h_7$ and $h_8$, the coefficients in System (1), under (8), the coefficient constraints.*

In order to confirm the mutual consistence of Bäcklund Transformations (14), using symbolic computation, under the variable-coefficient constraints

$$h_7 \neq 0, \qquad h_2 h_3 h_6 h_8 > 0, \tag{15}$$

and with the choice of

$$\mu_1 = \sigma_1, \tag{16}$$

we could find certain analytic solutions of Bäcklund Transformations (14), i.e.,

$$v = 0, \tag{17a}$$

$$u = 0, \tag{17b}$$

$$v_0 = \frac{6h_2}{h_4}\left\{\ln\left[1 + \delta_1 \exp\left(\pm\sqrt{\frac{h_6 h_8}{h_2 h_3}}x \mp \frac{h_6}{h_3}\sqrt{\frac{h_6 h_8}{h_2 h_3}}y + \sigma_1 z - h_7 \sigma_1 t\right)\right]\right\}_{xx}, \tag{17c}$$

$$u_0 = \frac{6h_2}{h_4}\left\{\ln\left[1 + \delta_1 \exp\left(\pm\sqrt{\frac{h_6 h_8}{h_2 h_3}}x \mp \frac{h_6}{h_3}\sqrt{\frac{h_6 h_8}{h_2 h_3}}y + \sigma_1 z - h_7 \sigma_1 t\right)\right]\right\}_{x}, \tag{17d}$$

where $\sigma_1$ denotes a real non-zero constant, while $\delta_1$ represents a positive constant.

**Theorem 2.2** *There stand Analytic Solutions (17c) and (17d)*[7] *for System (1).*

[6] The plural form is used here, because of the existence of $\mu_1$ (which is as-yet-undetermined).

[7] The plural form is used here, because of the existence of $\sigma_1$ and $\delta_1$ (which are as-yet-undetermined) and of the fact that we get a family of the solutions.

**Corollary 2.2** *Describing certain interfacial waves in a two-layer liquid or elastic quasiplane waves in a lattice, modelling the amplitude or elevation of the relevant wave, Analytic Solutions (17c) and (17d) depend on* $h_2$, $h_3$, $h_4$, $h_6$, $h_7$ *and* $h_8$, *the coefficients in System (1), under (8) and (15), the coefficient constraints.*

**Set 2:**

The exchange formulae, i.e., (12a), (12c), (12d), (12e), (12h) and (12i), result in

$$
\begin{aligned}
& f^2\Big[\Big(D_xD_t + h_2D_x^3D_z + h_3D_y^2 + h_6D_xD_y + h_7D_xD_z + h_8D_yD_z\Big)\, g\cdot g\Big] \\
&\quad -g^2\Big[\Big(D_xD_t + h_2D_x^3D_z + h_3D_y^2 + h_6D_xD_y + h_7D_xD_z + h_8D_yD_z\Big)\, f\cdot f\Big] \\
&= \big[f^2\big(D_xD_t\, g\cdot g\big) - g^2\big(D_xD_t\, f\cdot f\big)\big] \\
&\quad + h_2\big[f^2\big(D_x^3D_z\, g\cdot g\big) - g^2\big(D_x^3D_z\, f\cdot f\big)\big] \\
&\quad + h_3\big[f^2\big(D_y^2\, g\cdot g\big) - g^2\big(D_y^2\, f\cdot f\big)\big] \\
&\quad + h_6\big[f^2\big(D_xD_y\, g\cdot g\big) - g^2\big(D_xD_y\, f\cdot f\big)\big] \\
&\quad + h_7\big[f^2\big(D_xD_z\, g\cdot g\big) - g^2\big(D_xD_z\, f\cdot f\big)\big] \\
&\quad + h_8\big[f^2\big(D_yD_z\, g\cdot g\big) - g^2\big(D_yD_z\, f\cdot f\big)\big] \\
&= 2\, D_x\big(D_t\, g\cdot f\big)\cdot\big(fg\big) + \frac{1}{2}\, h_2\big[D_z\big(D_x^3\, g\cdot f\big)\cdot\big(fg\big) \\
&\quad + 3\, D_x\big(D_x^2D_z\, g\cdot f\big)\cdot\big(fg\big) - 3\, D_x\big(D_x^2\, g\cdot f\big)\cdot\big(D_z\, g\cdot f\big) \\
&\quad - 6\, D_x\big(D_xD_z\, g\cdot f\big)\cdot\big(D_x\, g\cdot f\big) - 3\, D_z\big(D_x^2\, g\cdot f\big)\cdot\big(D_x\, g\cdot f\big)\big] \\
&\quad + 2\, h_3\, D_y\big(D_y\, g\cdot f\big)\cdot\big(fg\big) + 2\, h_6\, D_x\big(D_y\, g\cdot f\big)\cdot\big(fg\big) \\
&\quad + 2\, h_7\, D_z\big(D_x\, g\cdot f\big)\cdot\big(fg\big) + 2\, h_8\, D_y\big(D_z\, g\cdot f\big)\cdot\big(fg\big) \\
&= \frac{1}{2}\, D_x\big[\big(4D_t + 3h_2D_x^2D_z + 4h_6D_y\big)\, g\cdot f\big]\cdot\big(fg\big) \\
&\quad + \frac{1}{2}\, D_z\big[\big(h_2D_x^3 + 4h_7D_x\big)\, g\cdot f\big]\cdot\big(fg\big) \\
&\quad + 2\, D_y\big[\big(h_3D_y + h_8D_z\big)\, g\cdot f\big]\cdot\big(fg\big) - \frac{3}{2}\, h_2\, D_z\big(D_x^2\, g\cdot f\big)\cdot\big(D_x\, g\cdot f\big) \\
&\quad - \frac{3}{2}\, h_2\, D_x\big(D_x^2\, g\cdot f\big)\cdot\big(D_z\, g\cdot f\big) - 3\, h_2\, D_x\big(D_xD_z\, g\cdot f\big)\cdot\big(D_x\, g\cdot f\big).
\end{aligned}
$$

Assumptions that

$$D_x\big[\big(4D_t + 3h_2D_x^2D_z + 4h_6D_y\big)\, g\cdot f\big]\cdot\big(fg\big) = 0, \tag{18a}$$

$$D_z\big[\big(h_2D_x^3 + 4h_7D_x\big)\, g\cdot f\big]\cdot\big(fg\big) = 0, \tag{18b}$$

$$D_y\big[\big(h_3D_y + h_8D_z\big)\, g\cdot f\big]\cdot\big(fg\big) = 0, \tag{18c}$$

$$D_z\big(D_x^2\, g\cdot f\big)\cdot\big(D_x\, g\cdot f\big) = 0, \tag{18d}$$

$$D_x\big(D_x^2\, g\cdot f\big)\cdot\big(D_z\, g\cdot f\big) = 0, \tag{18e}$$

$$D_x\big(D_xD_z\, g\cdot f\big)\cdot\big(D_x\, g\cdot f\big) = 0, \tag{18f}$$

give rise to

$$v = \frac{6h_2}{h_4}(\ln f)_{xx}, \tag{19a}$$

$$u = \frac{6h_2}{h_4}(\ln f)_{x}, \tag{19b}$$

$$v_0 = \frac{6h_2}{h_4}(\ln g)_{xx}, \tag{19c}$$

$$u_0 = \frac{6h_2}{h_4}(\ln g)_{x}, \tag{19d}$$

$$\left(4D_t + 3h_2D_x^2D_z + 4h_6D_y\right) g \cdot f = 0, \tag{19e}$$

$$\left(h_2D_x^3 + 4h_7D_x\right) g \cdot f = 0, \tag{19f}$$

$$\left(h_3D_y + h_8D_z\right) g \cdot f = 0, \tag{19g}$$

$$D_x^2\, g \cdot f = \mu_2 D_x\, g \cdot f, \tag{19h}$$

$$D_x^2\, g \cdot f = \mu_3 D_z\, g \cdot f, \tag{19i}$$

$$D_x D_z\, g \cdot f = \mu_4 D_x\, g \cdot f, \tag{19j}$$

with $\mu_2$, $\mu_3$ and $\mu_4$ as three real constants.

**Theorem 2.3** *Equations (19) comprise a set of the bilinear auto-Bäcklund transformations for System (1), because of their mutual consistency.*

**Corollary 2.3** *Describing certain interfacial waves in a two-layer liquid or elastic quasiplane waves in a lattice, modelling the amplitude or elevation of the relevant wave, Bilinear Auto-Bäcklund Transformations (19) depend on $h_2$, $h_3$, $h_4$, $h_6$, $h_7$ and $h_8$, the coefficients in System (1), under (8), the coefficient constraints.*

In order to confirm the mutual consistence of Bäcklund Transformations (19), making use of symbolic computation, under the variable-coefficient constraints

$$h_2h_7 < 0, \qquad h_3h_8 \neq 0, \qquad 3h_3h_7 + h_6h_8 \neq 0, \tag{20}$$

and with the choices of

$$\mu_2 = \pm 2\sqrt{-\frac{h_7}{h_2}}, \qquad \mu_3 = -\frac{4h_7}{h_2\sigma_2}, \qquad \mu_4 = \sigma_2, \tag{21}$$

we are able to obtain some analytic solutions of Bäcklund Transformations (19), i.e.,

$$v = 0, \tag{22a}$$

$$u = 0, \tag{22b}$$

$$v_0 = \frac{6h_2}{h_4}\left\{ \ln\left[1 + \delta_2 \exp\left( \pm 2\sqrt{-\frac{h_7}{h_2}}x - \frac{h_8\sigma_2}{h_3}y + \sigma_2 z + \frac{3h_3h_7 + h_6h_8}{h_3}\sigma_2 t\right)\right]\right\}_{xx}, \tag{22c}$$

$$u_0 = \frac{6h_2}{h_4}\left\{\ln\left[1+\delta_2\exp\left(\pm 2\sqrt{-\frac{h_7}{h_2}}x - \frac{h_8\sigma_2}{h_3}y + \sigma_2 z + \frac{3h_3h_7+h_6h_8}{h_3}\sigma_2 t\right)\right]\right\}_x, \tag{22d}$$

where $\sigma_2$ means a real non-zero constant, while $\delta_2$ denotes a positive constant.

**Theorem 2.4** *There stand Analytic Solutions (22c) and (22d) for System (1).*

**Corollary 2.4** *Describing certain interfacial waves in a two-layer liquid or elastic quasiplane waves in a lattice, modelling the amplitude or elevation of the relevant wave, Analytic Solutions* (22c) *and (22d) depend on* $h_2$, $h_3$, $h_4$, $h_6$, $h_7$ *and* $h_8$, *the coefficients in System (1), under (8) and (20), the coefficient constraints.*

## 3 Similarity Reductions for System (1)

Our purpose is to build up some similarity reductions for System (1) following the similar approaches to the ones in Refs. [65–73] in the form

$$u(x, y, z, t) = \theta(x, y, z, t) + \alpha(x, y, z, t)p[r(x, y, z, t)], \tag{23a}$$
$$v(x, y, z, t) = \delta(x, y, z, t) + \kappa(x, y, z, t)q[r(x, y, z, t)], \tag{23b}$$

with $p(r)$ and $q(r)$ implying the real differentiable functions, while $\theta(x, y, z, t)$, $\alpha(x, y, z, t)$, $\delta(x, y, z, t)$, $\kappa(x, y, z, t)$ and $r(x, y, z, t)$ standing for the real differentiable functions to be determined.

Thinking of the case of $r_x = r_y = r_z = 0$, $\alpha(x, y, z, t) \neq 0$, $\kappa(x, y, z, t) \neq 0$, $p[r(x, y, z, t)] \neq 0$ and $q[r(x, y, z, t)] \neq 0$, employing symbolic computation and substituting Assumptions (23) into System (1), we obtain

$$\begin{aligned} & h_1\kappa_x s(t)q' + h_4\left(\alpha_{xz}\kappa_x + \alpha_z\kappa_{xx}\right)pq + h_5\left(\kappa_x\kappa_z + \kappa\kappa_{xz}\right)q^2 \\ & \quad + h_4\left(\alpha_{xz}\delta_x + \alpha_z\delta_{xx}\right)p + \left[h_1\kappa_{xt} + h_2\kappa_{xxxz} + h_3\kappa_{yy} + h_4\left(\theta_z\kappa_{xx} + \theta_{xz}\kappa_x\right)\right. \\ & \quad \left. + h_5\left(\kappa_x\delta_z + \kappa_z\delta_x + \kappa\delta_{xz} + \kappa_{xz}\delta\right) + h_6\kappa_{xy} + h_7\kappa_{xz} + h_8\kappa_{yz}\right]q \\ & \quad + h_1\delta_{xt} + h_2\delta_{xxxz} + h_3\delta_{yy} + h_4\left(\theta_z\delta_{xx} + \theta_{xz}\delta_x\right) + h_5\left(\delta_x\delta_z + \delta\delta_{xz}\right) \\ & \quad + h_6\delta_{xy} + h_7\delta_{xz} + h_8\delta_{yz} = 0, \end{aligned} \tag{24a}$$

$$-\kappa q + \alpha_x p + (\theta_x - \delta) = 0, \tag{24b}$$

with the apostrophe indicating the differentiation with respect to $r$, while $s(t) = dr(t)/dt$.

Once it is required that Eqs. (24) stand for a couple of the real ordinary differential equations (ODEs) with respect to $p(r)$ and $q(r)$, the ratios of the coefficients of different derivatives and powers of $p(r)$ and $q(r)$ must represent certain real functions as for $r$ only.

We then make use of the coefficients of $q'$ in Eq. (24a) and $q$ in Eq. (24b), respectively, as the normalizing coefficients in Eqs. (24), to get

$$\Omega_1(r)h_1\kappa_x s(t) = h_4\left(\alpha_{xz}\kappa_x + \alpha_z\kappa_{xx}\right), \tag{25a}$$

$$\Omega_2(r)h_1\kappa_x s(t) = h_5\left(\kappa_x\kappa_z + \kappa\kappa_{xz}\right), \tag{25b}$$

$$\Omega_3(r)h_1\kappa_x s(t) = h_4\left(\alpha_{xz}\delta_x + \alpha_z\delta_{xx}\right), \tag{25c}$$

$$\begin{aligned}\Omega_4(r)h_1\kappa_x s(t) = {} & h_1\kappa_{xt} + h_2\kappa_{xxxz} + h_3\kappa_{yy} + h_4\left(\theta_z\kappa_{xx} + \theta_{xz}\kappa_x\right) + h_6\kappa_{xy} \\ & + h_5\left(\kappa_x\delta_z + \kappa_z\delta_x + \kappa\delta_{xz} + \kappa_{xz}\delta\right) + h_7\kappa_{xz} + h_8\kappa_{yz},\end{aligned} \tag{25d}$$

$$\begin{aligned}\Omega_5(r)h_1\kappa_x s(t) = {} & h_1\delta_{xt} + h_2\delta_{xxxz} + h_3\delta_{yy} + h_4\left(\theta_z\delta_{xx} + \theta_{xz}\delta_x\right) \\ & + h_5\left(\delta_x\delta_z + \delta\delta_{xz}\right) + h_6\delta_{xy} + h_7\delta_{xz} + h_8\delta_{yz},\end{aligned} \tag{25e}$$

and

$$-\Gamma_1(r)\kappa = \alpha_x, \tag{26a}$$

$$-\Gamma_2(r)\kappa = \theta_x - \delta, \tag{26b}$$

with $\Omega_i(r)$'s and $\Gamma_j(r)$'s meaning certain real to-be-determined functions as for $r$, while $i = 1, ..., 5$ and $j = 1, 2$.

On the basis of the remarks in Ref. [65], a set of the conditions for $\theta(x, y, z, t)$, $\alpha(x, y, z, t)$, $\delta(x, y, z, t)$, $\kappa(x, y, z, t)$ and $r(t)$ are figured, any solution of which could come to, at least, a similarity reduction.

On account of the second freedom in Remark 3 in Ref. [65], Eqs. (25b) and (26a) bring about

$$\alpha(x, y, z, t) = -\int \kappa(x, y, z, t)dx, \qquad h_5 \neq 0, \tag{27a}$$

$$\kappa(x, y, z, t) = \frac{h_1}{h_5}s(t)z + \eta_0(x, y, t), \qquad \Omega_2(r) = \Gamma_1(r) = 1, \tag{27b}$$

and then Eq. (25a) results in

$$\eta_0(x, y, t) = \eta_1(y, t)x + \eta_2(y, t), \qquad \Omega_1(r) = -\frac{h_4}{h_5}, \tag{28}$$

with $\eta_0(x, y, t)$ as a real differentiable function of $x$, $y$ and $t$, while $\eta_1(y, t)$ and $\eta_2(y, t)$ as two real differentiable functions of $y$ and $t$.

By reason of the first freedom in Remark 3 in Ref. [65], Eqs. (25c) and (26b) make for

$$\theta(x, y, z, t) = \xi_0 x, \qquad \delta(x, y, z, t) = \xi_0, \qquad \Omega_3(r) = \Gamma_2(r) = 0, \tag{29}$$

so that Eqs. (25d) and (25e) develop into

$$\eta_1(y, t) = \xi_1, \qquad \eta_2(y, t) = \xi_2 y + \xi_3(t),$$

$$r(t)=\xi_4 t+\xi_5, \qquad \Omega_4(r)=\Omega_5(r)=0, \tag{30}$$

with $\xi_0$, $\xi_1 \neq 0$, $\xi_2 \neq 0$, $\xi_4 \neq 0$ and $\xi_5$ as the real constants, while $\xi_3(t)$ as a real function of $t$.

Until now, System (1) can be simplified to the following ODEs:

$$q'-\frac{h_4}{h_5}pq+q^2=0, \tag{31a}$$

$$p+q=0. \tag{31b}$$

For the purpose of transforming ODEs (31) into a single ODE, we get

$$p=-q, \tag{32}$$

and then find

$$q'+\left(1+\frac{h_4}{h_5}\right)q^2=0. \tag{33}$$

In short, under the variable-coefficient constraints

$$h_1 \neq 0, \qquad h_5 \neq 0, \qquad h_4+h_5 \neq 0, \tag{34}$$

we conclude with a set of the similarity reductions for System (1), written as

$$u(x,y,z,t)=\xi_0 x+\left[\frac{1}{2}\xi_1 x^2+\xi_2 xy+\xi_3(t)x+\frac{h_1}{h_5}\xi_4 xz\right]q[r(t)], \tag{35a}$$

$$v(x,y,z,t)=\xi_0+\left[\xi_1 x+\xi_2 y+\xi_3(t)+\frac{h_1}{h_5}\xi_4 z\right]q[r(t)], \tag{35b}$$

$$r(t)=\xi_4 t+\xi_5, \tag{35c}$$

with $q(r)$ satisfying

$$q'+\left(1+\frac{h_4}{h_5}\right)q^2=0. \tag{36}$$

ODE (36) indicates a known ODE, the information of which has been reported in Ref. [74].

**Theorem 3.1** *There lie Similarity Reductions (35) with ODE (36) for System (1).*

**Corollary 3.1** *Describing certain interfacial waves in a two-layer liquid or elastic quasiplane waves in a lattice, modelling the amplitude or elevation of the relevant wave, Similarity Reductions (35) with ODE (36) depend on $h_1$, $h_4$ and $h_5$, the coefficients in System (1), under (34), the coefficient constraints.*

## 4 Conclusions

To date, studies on the liquids and lattices have appeared interesting. In this paper, with symbolic computation, we have considered System (1), a (3+1)-dimensional generalized Yu-Toda-Sasa-Fukuyama system for the interfacial waves in a two-layer liquid or elastic waves in a lattice.

For System (1), we have built up Bilinear Auto-Bäcklund Transformations (14) and Bilinear Auto-Bäcklund Transformations (19), both of which are different from that in Ref. [7]. On the other hand, we have constructed Similarity Reductions (35) with ODE (36), i.e., from System (1) to a known ODE. As for the amplitude or elevation of the relevant wave, our results have been presented to rely on the coefficients in System (1).

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**Data Availibility** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

## Declarations

**Conflict of Interest** The authors declare that they have no conflict of interest.

## Appendix

The Hirota bilinear operators $D_x$, $D_y$, $D_z$ and $D_t$ have been defined as [10]

$$D_x^{m_1} D_y^{m_2} D_z^{m_3} D_t^{m_4} G(x, y, z, t) \cdot F(x, y, z, t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \tilde{x}}\right)^{m_1} \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial \tilde{y}}\right)^{m_2} \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \tilde{z}}\right)^{m_3} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \tilde{t}}\right)^{m_4} G(x, y, z, t)\, F(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})\Big|_{\tilde{x}=x,\, \tilde{y}=y,\, \tilde{z}=z,\, \tilde{t}=t}, \tag{A.1}$$

with $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ and $\tilde{t}$ indicating four formal variables, $G(x, y, z, t)$ denoting a $C^\infty$ function of $x$, $y$, $z$ and $t$, $F(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$ representing a $C^\infty$ function of $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ and $\tilde{t}$, while $m_1$, $m_2$, $m_3$ and $m_4$ implying four non-negative integers [10].

Recent applications of the Hirota bilinear operators include the ones to certain Bose-Einstein condensates with the dipole-dipole attractions and repulsions [75], liquids with the gas bubbles [76], time-dependent radiative transfer problems [77], nonlinear reduced fluid models for some plasmas [78] and ion-acoustic wave structures with the effects of magnetic fields in plasma physics [79]. Other recent vigorous references include, e.g., Refs. [22, 26, 27, 43, 80].

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