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Considering the Shallow Water of a Wide Channel or an Open Sea Through a Generalized (2+1)-dimensional Dispersive Long-wave System

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Abstract
Under investigation in this paper is a generalized (2+1)-dimensional dispersive long-wave system, describing the nonlinear and dispersive long gravity waves in two horizontal directions in the shallow water of a wide channel of finite depth or an open sea. Via symbolic computation, we derive the same bilinear forms as those reported, but through a different method. Four sets of the similarity reductions are obtained, each of which leads to a known ordinary differential equation. The results rely on the coefficients in the original system, with respect to the horizontal velocity and wave elevation above the undisturbed water surface.

Keywords Oceanic water waves · Generalized (2+1)-dimensional dispersive long-wave system · Bilinear forms · Hirota method · Similarity reductions · Symbolic computation

1 Introduction

Studies on fluids have been reported [1–14]. For investigating the nonlinear and dispersive long gravity waves in two horizontal directions, especially those in the shallow water of a wide channel or an open sea with finite depth, Ref. [15] has proposed the following generalized (2+1)-dimensional dispersive long-wave system:

$$u_{yt} + \alpha \left[v_{xx} + \frac{1}{2} \left(u^2 \right)_{xy} \right] = 0 \ , \tag{1a}$$

$$v_t + \alpha \left(uv + \beta u + \delta^2 u_{xy} \right)_x = 0 \ , \tag{1b}$$

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with $u(x, y, t)$ as the horizontal velocity, $v(x, y, t)$ as the wave elevation above the undisturbed water surface, $u(x, y, t)$ and $v(x, y, t)$ as the real differentiable functions in respect of the variables x, y and t, the subscripts as the partial derivatives, $\alpha \neq 0$, β and $\delta \neq 0$ implying the real constants, while t and (x, y) denoting the time and propagation plane, separately. Also in Ref. [15], some special cases which can report the applications of System (1) have been listed.

Ref. [15] has derived two sets of the bilinear forms of System (1), i.e.,

$$(D_t \pm \alpha\delta D_x^2) f \cdot g = 0, \tag{2a}$$

$$\left[D_y D_t \pm \alpha\delta D_x^2 D_y \pm \frac{\alpha}{\delta}(\theta_4 + \beta) D_x \right] f \cdot g = 0, \tag{2b}$$

in which θ_4 indicates a real constant, $f(x, y, t)$ and $g(x, y, t)$ imply the C^∞ functions of x, y and t, while D_x, D_y and D_t represent the Hirota operators defined as [16]

$$\begin{aligned} &D_x^m D_y^r D_t^n f(x, y, t) \cdot g(x, y, t) \\ &\equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^r \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x, y, t)\, g(x', y', t') \Big|_{x'=x,\, y'=y,\, t'=t} \end{aligned},$$

with x', y' and t' denoting the formal variables, while m, r and n meaning three non-negative integers. Besides, Ref. [15] has also obtained certain scaling transformations, hetero-Bäcklund transformations and N-soliton solutions for System (1), where N is a positive integer. For System (1), Ref. [17] has constructed certain hetero- and auto-Bäcklund transformations with some soliton solutions, while Ref. [18] has given out some similarity reductions.[1]

To System (1), contributions of this paper could be introduced in the following aspects:

- **Background:** Nowadays, many nonlinear evolution equations/systems have been put into use in some physical studies, e.g., optical fibers, fluids and plasmas [17–28].
- **Motivations:** On the one hand, we plan to construct the same bilinear forms as Bilinear Forms (2) with a different method, to confirm the correctness of Bilinear Forms (2). On the other hand, we would like to find out more similarity reductions, which link System (1) to some ordinary differential equations (ODEs), to complement the existing results.
- **Novelty and outlines:** Bäcklund transformations and solutions of System (1) could be derived via the bilinear forms [15]. In comparison with the Bell polynomials in Ref. [15], the Hirota method may give rise to more potential bilinear forms [29]. Besides, similarity reductions in this paper, which are different from those in Ref. [18], might fit some other situations.
- **Originality:** To date, for System (1), similarity reductions different from those in Ref. [18] have not been investigated. In Sect. 2, we will derive two sets of the

[1] Note that ODE (14) and ODE (15d) in Ref. [18] are wrong, and we need to correct them to $p'' - \frac{3}{2}p^2 - \frac{1}{2}p^3 + (\phi_1 z + \phi_2 - 1)p + (\phi_1 z + \phi_2 - \phi_3) = 0$.

bilinear forms, which are the same as those in Ref. [15], but through a different method, i.e., the Hirota method [16, 30–33]. In Sect. 3, with symbolic computation[2] [34–38], we will obtain four sets of the similarity reductions for System (1), which are different from those in Ref. [18]. Conclusions will be given in Sect. 4.

- **Significance and potential applications:** This paper could be of some use for the future studies on the nonlinear and dispersive long gravity waves in two horizontal directions, especially those in the shallow water of a wide channel or an open sea with finite depth.

2 Two Sets of the Bilinear Forms for System (1) through the Hirota Method

Since our goal is to construct some bilinear forms for System (1) in respect of $f(x, y, t)$ and $g(x, y, t)$, the Hirota method brings about the assumptions

$$u(x, y, t) = \zeta_1 \left[\ln\left(\frac{f}{g}\right)\right]_x \quad , \tag{3a}$$

$$v(x, y, t) = \zeta_2 \left[\ln\left(\frac{f}{g}\right)\right]_{xy} + \zeta_3 [\ln(fg)]_{xy} + \zeta_4 \quad , \tag{3b}$$

where ζ_2 and ζ_4 are two real constants, while ζ_1 and ζ_3 imply two real non-zero constants.

Integrating Eq. (1a) once in respect of x and y, respectively, with the integration function vanishing, we get

$$\zeta_1 \left[\ln\left(\frac{f}{g}\right)\right]_t + \alpha\zeta_2 \left[\ln\left(\frac{f}{g}\right)\right]_{xx} + \alpha\zeta_3 [\ln(fg)]_{xx} + \frac{1}{2}\alpha\zeta_1^2 \left\{\left[\ln\left(\frac{f}{g}\right)\right]_x\right\}^2 = 0 \quad . \tag{4}$$

To bring in the Hirota operators, based on the following formulae [16]:

$$\left[\ln\left(\frac{f}{g}\right)\right]_x = \frac{D_x f \cdot g}{fg} \quad , \tag{5a}$$

$$\left[\ln\left(\frac{f}{g}\right)\right]_t = \frac{D_t f \cdot g}{fg} \quad , \tag{5b}$$

$$[\ln(fg)]_{xx} = \frac{D_x^2 f \cdot g}{fg} - \left(\frac{D_x f \cdot g}{fg}\right)^2 \quad , \tag{5c}$$

[2] More relevant studies on symbolic computation could been found in Refs. [39–53].

with the assumption that

$$\zeta_2 = 0 \ , \qquad \zeta_3 = \frac{1}{2}\zeta_1^2 \ , \tag{6}$$

we convert Eq. (1a) into

$$\left(D_t + \frac{1}{2}\alpha\zeta_1 D_x^2\right) f \cdot g = 0 \ . \tag{7}$$

Similarly, we integrate Eq. (1b) once in respect of x with the integration function vanishing, to find

$$\begin{aligned}
&\zeta_2\left[\ln\left(\frac{f}{g}\right)\right]_{yt} + \zeta_3\left[\ln(fg)\right]_{yt} + \alpha\zeta_1\zeta_2\left[\ln\left(\frac{f}{g}\right)\right]_x\left[\ln\left(\frac{f}{g}\right)\right]_{xy} \\
&+ \alpha\zeta_1\zeta_3\left[\ln\left(\frac{f}{g}\right)\right]_x\left[\ln(fg)\right]_{xy} + \alpha\zeta_1\zeta_4\left[\ln\left(\frac{f}{g}\right)\right]_x \\
&+ \alpha\beta\zeta_1\left[\ln\left(\frac{f}{g}\right)\right]_x + \alpha\zeta_1\delta^2\left[\ln\left(\frac{f}{g}\right)\right]_{xxy} = 0 \ .
\end{aligned} \tag{8}$$

According to Formulae (5) and the following formulae [16]:

$$[\ln(fg)]_{xy} = \frac{D_x D_y f \cdot g}{fg} - \frac{D_x f \cdot g}{fg}\frac{D_y f \cdot g}{fg} \ , \tag{9a}$$

$$[\ln(fg)]_{yt} = \frac{D_y D_t f \cdot g}{fg} - \frac{D_y f \cdot g}{fg}\frac{D_t f \cdot g}{fg} \ , \tag{9b}$$

$$\begin{aligned}
\left[\ln\left(\frac{f}{g}\right)\right]_{xxy} &= \frac{D_x^2 D_y f \cdot g}{fg} - 2\frac{D_x D_y f \cdot g}{fg}\frac{D_x f \cdot g}{fg} - \frac{D_x^2 f \cdot g}{fg}\frac{D_y f \cdot g}{fg} \\
&\quad + 2\left(\frac{D_x f \cdot g}{fg}\right)^2 \frac{D_y f \cdot g}{fg} \ ,
\end{aligned} \tag{9c}$$

Eqs. (7) and (8) give rise to

$$\zeta_1 = \pm 2\delta \ , \qquad \left[D_y D_t \pm \alpha\delta D_x^2 D_y \pm \frac{\alpha}{\delta}(\zeta_4 + \beta) D_x\right] f \cdot g = 0 \ . \tag{10}$$

Based on the above derivation, we are able to come up with the theorem:

Theorem 2.1 *In brief, via Assumptions (3), we construct the following bilinear forms for System (1) via the Hirota method:*

$$(D_t \pm \alpha\delta D_x^2) f \cdot g = 0 \ , \tag{11a}$$

$$\left[D_y D_t \pm \alpha\delta D_x^2 D_y \pm \frac{\alpha}{\delta}(\zeta_4 + \beta) D_x\right] f \cdot g = 0 \ , \tag{11b}$$

which are the same as Bilinear Forms (2) when $\zeta_4 = \theta_4$.

3 Four Sets of the Similarity Reductions for System (1)

For obtaining some similarity reductions, we give rise to the assumptions[3]

$$u(x,y,t)=\theta(x,y,t)+\omega(x,y,t)\,p\left[z(x,y,t)\right], \tag{12a}$$
$$v(x,y,t)=\gamma(x,y,t)+\kappa(x,y,t)\,q\left[z(x,y,t)\right], \tag{12b}$$

where $\theta(x,y,t)$, $\omega(x,y,t)\neq 0$, $\gamma(x,y,t)$, $\kappa(x,y,t)\neq 0$ and $z(x,y,t)\neq 0$ imply some real differentiable functions to be determined, while $p[z(x,y,t)]$ and $q[z(x,y,t)]$ are two real differentiable functions of z.

Making use of symbolic computation and inserting Assumptions (12) into System (1), we obtain that

$$\begin{aligned}&\chi_0 pp''+\chi_0 p'^2+\chi_1 p''+\chi_2 pp'+\chi_3 p'+\chi_4 p+\chi_5 p^2\\&\quad+\chi_6 q''+\chi_7 q'+\chi_8 q+\chi_9=0\ ,\end{aligned} \tag{13a}$$
$$\begin{aligned}&\tau_0 p'''+\tau_1 p''+\tau_2 p'+\tau_3 p+\tau_4 q'+\tau_5 q\\&\quad+\tau_6 p'q+\tau_6 pq'+\tau_7 pq+\tau_8=0\ ,\end{aligned} \tag{13b}$$

in which

$$\chi_0=\alpha\omega^2 z_x z_y\ , \tag{14a}$$
$$\chi_1=\omega z_t z_y+\alpha\theta\omega z_x z_y\ , \tag{14b}$$
$$\chi_2=2\alpha\omega\omega_y z_x+2\alpha\omega\omega_x z_y+\alpha\omega^2 z_{xy}\ , \tag{14c}$$
$$\begin{aligned}\chi_3=&\ \omega_t z_y+z_t\omega_y+\omega z_{yt}+\alpha\omega\theta_y z_x+\alpha\omega z_y\theta_x\\&+\alpha\theta\omega_y z_x+\alpha\theta\omega_x z_y+\alpha\theta\omega z_{xy},\end{aligned} \tag{14d}$$
$$\chi_4=\omega_{yt}+\alpha\omega_y\theta_x+\alpha\theta_y\omega_x+\alpha\omega\theta_{xy}+\alpha\theta\omega_{xy}\ , \tag{14e}$$
$$\chi_5=\alpha\omega_y\omega_x+\alpha\omega\omega_{xy}\ , \tag{14f}$$
$$\chi_6=2\alpha\kappa z_x^2\ , \tag{14g}$$
$$\chi_7=2\alpha z_x\kappa_x+\alpha\kappa z_{xx}\ , \tag{14h}$$
$$\chi_8=\alpha\kappa_{xx}\ , \tag{14i}$$
$$\chi_9=\theta_{yt}+\alpha\theta_y\theta_x+\alpha\theta\theta_{xy}+\alpha\gamma_{xx}\ , \tag{14j}$$
$$\tau_0=\alpha\delta^2\omega z_y z_x^2\ , \tag{14k}$$
$$\tau_1=\alpha\delta^2\omega_y z_x^2+2\alpha\delta^2 z_y z_x\omega_x+2\alpha\delta^2\omega z_x z_{xy}+\alpha\delta^2\omega z_y z_{xx}\ , \tag{14l}$$
$$\begin{aligned}\tau_2=&\ \alpha\beta\omega z_x+\alpha\gamma\omega z_x+2\alpha\delta^2\omega_x z_{xy}+2\alpha\delta^2 z_x\omega_{xy}+\alpha\delta^2\omega_y z_{xx}\\&+\alpha\delta^2 z_y\omega_{xx}+\alpha\delta^2\omega z_{xxy}\ ,\end{aligned} \tag{14m}$$
$$\tau_3=\alpha\omega\gamma_x+\alpha\beta\omega_x+\alpha\gamma\omega_x+\alpha\delta^2\omega_{xxy}\ , \tag{14n}$$
$$\tau_4=\kappa z_t+\alpha\theta\kappa z_x\ , \tag{14o}$$
$$\tau_5=\kappa_t+\alpha\kappa\theta_x+\alpha\theta\kappa_x\ , \tag{14p}$$

[3] similar to those in Refs. [54–61]

$$\tau_6 = \alpha\kappa\omega z_x \quad , \tag{14q}$$

$$\tau_7 = \alpha\omega\kappa_x + \alpha\kappa\omega_x \quad , \tag{14r}$$

$$\tau_8 = \gamma_t + \alpha\theta\gamma_x + \alpha\beta\theta_x + \alpha\gamma\theta_x + \alpha\delta^2\theta_{xxy}, \tag{14s}$$

χ_i's ($i = 0, ..., 9$) and τ_j's ($j = 0, ..., 8$) are some real differentiable functions with respect to x, y and t, while the prime sign means d/dz. Because $p(z)$ and $q(z)$ are the functions of z only, we are able to convert Eq. (13) into a set of the ODEs in respect of $p(z)$ and $q(z)$. Each set of $\theta(x, y, t)$, $\omega(x, y, t)$, $\gamma(x, y, t)$, $\kappa(x, y, t)$ and $z(x, y, t)$ could lead to, at least, a similarity reduction of System (1). In this paper, we consider the case of $z_x z_y \neq 0$, so that $\chi_0 \neq 0$ and $\tau_0 \neq 0$, to obtain that

$$\chi_i = \Omega_i(z)\chi_0 \quad , \qquad \tau_j = \Gamma_j(z)\tau_0 \quad , \tag{15}$$

with $\Omega_i(z)$'s and $\Gamma_j(z)$'s as some real to-be-determined functions of z only.

For the sake of simplicity, we give out the assumption that[4]

$$z(x, y, t) = \lambda_1 x + \lambda_2 y + \lambda_3 t + \lambda_4 \quad , \tag{16}$$

with λ_1, λ_2 and λ_3 as the real non-zero constants, while λ_4 as a real constant. Substituting Eqs. (14q), (14k) and (16) into Eqs. (15) turns to

$$\kappa(x, y, t) = \delta^2\lambda_1\lambda_2 \quad , \qquad \Gamma_6(z) = 1 \quad . \tag{17}$$

According to the second freedom of Remark 3 in Ref. [62], Eq. (14g) results in

$$\omega(x, y, t) = \pm\delta\lambda_1 \quad , \qquad \Omega_6(z) = 1 \quad . \tag{18}$$

With the first freedom of Remark 3 in Ref. [62], Eq. (14b) leads to

$$\theta(x, y, t) = -\frac{\lambda_3}{\alpha\lambda_1} \quad , \qquad \Omega_1(z) = 0 \quad , \tag{19}$$

and Eq. (14m) helps us derive

$$\Gamma_2(z) = \frac{\beta + \gamma}{\delta^2\lambda_1\lambda_2} \quad . \tag{20}$$

Based on the first and the second freedom of Remark 3 in Ref. [62], respectively, we will obtain two branches of the results.

Branch 1: $\gamma(x, y, t) = -\beta \ , \quad \Gamma_2(z) = 0$

Inserting $\gamma(x, y, t) = -\beta$ into Eqs. (14) brings about

$$\Omega_2(z) = \Omega_3(z) = \Omega_4(z) = \Omega_5(z) = \Omega_7(z) = \Omega_8(z) = \Omega_9(z) = 0 \quad , \tag{21a}$$

[4] motivated by Refs. [54–58]

$$\Gamma_1(z) = \Gamma_3(z) = \Gamma_4(z) = \Gamma_5(z) = \Gamma_7(z) = \Gamma_8(z) = 0 \quad . \tag{21b}$$

Eqs. (13) can turn into

$$pp'' + p'^2 + q'' = 0 \quad , \tag{22a}$$
$$p''' + p'q + pq' = 0 \quad . \tag{22b}$$

Then we integrate ODE (22a) twice about z, to obtain

$$q = -\frac{1}{2}p^2 + \phi_1 z + \phi_2 \quad , \tag{23}$$

with ϕ_1 and ϕ_2 being two real constants of integration. Integrating ODE (22b) once in respect of z and considering ODE (23), we can transfer ODEs (22) to a simple ODE, written as

$$p'' - \frac{1}{2}p^3 + (\phi_1 z + \phi_2)p + \phi_3 = 0 \quad , \tag{24}$$

where ϕ_3 denotes a real constant of integration.

Thus, we derive two sets of the similarity reductions for System (1), i.e.,

$$u(x, y, t) = -\frac{\lambda_3}{\alpha\lambda_1} \pm \delta\lambda_1 p[z(x, y, t)] \quad , \tag{25a}$$
$$v(x, y, t) = -\beta - \delta^2\lambda_1\lambda_2 \left\{\frac{1}{2}p^2[z(x, y, t)] - \phi_1 z - \phi_2\right\} \quad , \tag{25b}$$
$$z(x, y, t) = \lambda_1 x + \lambda_2 y + \lambda_3 t + \lambda_4 \quad , \tag{25c}$$
$$p'' - \frac{1}{2}p^3 + (\phi_1 z + \phi_2)p + \phi_3 = 0 \quad . \tag{25d}$$

ODE (25d) is a known ODE, reported in Ref. [63].

Branch 2: $\gamma(x, y, t) = \delta^2\lambda_1\lambda_2 - \beta, \quad \Gamma_2(z) = 1$

When $\gamma(x, y, t) = \delta^2\lambda_1\lambda_2 - \beta$, we propose to derive

$$\Omega_2(z) = \Omega_3(z) = \Omega_4(z) = \Omega_5(z) = \Omega_7(z) = \Omega_8(z) = \Omega_9(z) = 0 \quad , \tag{26a}$$
$$\Gamma_1(z) = \Gamma_3(z) = \Gamma_4(z) = \Gamma_5(z) = \Gamma_7(z) = \Gamma_8(z) = 0 \quad . \tag{26b}$$

Eqs. (13) are converted into

$$pp'' + p'^2 + q'' = 0 \quad , \tag{27a}$$
$$p''' + p' + p'q + pq' = 0 \quad . \tag{27b}$$

Similarly, we integrate ODE (27a) twice about z to find

$$q = -\frac{1}{2}p^2 + \phi_4 z + \phi_5 \quad , \tag{28}$$

with ϕ_4 and ϕ_5 as two real constants of integration. Integrating ODE (27b) once about z and considering ODE (28) could develop into

$$p'' - \frac{1}{2}p^3 + (\phi_4 z + \phi_5 + 1)p + \phi_6 = 0 \ , \tag{29}$$

with ϕ_6 as a real constants of integration.

Thus, we require into another two sets of the similarity reductions for System (1), i.e.,

$$u(x, y, t) = -\frac{\lambda_3}{\alpha\lambda_1} \pm \delta\lambda_1 p[z(x, y, t)] \ , \tag{30a}$$

$$v(x, y, t) = \delta^2\lambda_1\lambda_2 - \beta - \delta^2\lambda_1\lambda_2 \left\{\frac{1}{2}p^2[z(x, y, t)] - \phi_4 z - \phi_5\right\} \ , \tag{30b}$$

$$z(x, y, t) = \lambda_1 x + \lambda_2 y + \lambda_3 t + \lambda_4, \tag{30c}$$

$$p'' - \frac{1}{2}p^3 + (\phi_4 z + \phi_5 + 1)p + \phi_6 = 0 \ . \tag{30d}$$

ODE (30d) is a known ODE, reported in Ref. [63].

With respect to the horizontal velocity and the wave elevation above the undisturbed water surface, we derive the following theorem about System (1), describing the nonlinear and dispersive long gravity waves in two horizontal directions in the shallow water of a wide channel of finite depth or an open sea.

Theorem 3.1 *Similarity Reductions (25) and Similarity Reductions (30), both of which are different from those in Ref. [18], depend on all the constant coefficients in System (1), i.e., α, β and δ. The reason why there are two sets of Similarity Reductions (25)/Similarity Reductions (30) is the existence of "$\pm$" sign.*

4 Discussions

We have noticed that both Similarity Reductions (25) and Similarity Reductions (30) are different from those in Ref. [18], while both ODE (25d) and ODE (30d) are the known ODEs. Our results have been shown to depend on α, β and δ, all the constant coefficients in System (1), and might be of some use in the studies on the nonlinear and dispersive long gravity waves in two horizontal directions in the shallow water of a wide channel of finite depth or an open sea.

5 Conclusions

As for a generalized (2+1)-dimensional dispersive long-wave system in respect of the horizontal velocity and the wave elevation above the undisturbed water surface, i.e., System (1), we have obtained the following:

- Two sets of the bilinear forms, i.e., Bilinear Forms (11), which are the same as Bilinear Forms (2), but through a different method, i.e., the Hirota method. Thus, the correctness of Bilinear Forms (2) can be confirmed.
- Four sets of the similarity reductions for System (1), i.e., Similarity Reductions (25), from System (1) to ODE (25d), and Similarity Reductions (30), from System (1) to ODE (30d).

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