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Existence of Ground State Solutions for Generalized Quasilinear Schrödinger Equations with Asymptotically Periodic Potential

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Abstract

This article is concerned with the existence of positive ground state solutions for an asymptotically periodic quasilinear Schrödinger equation. By using a Nehari-type constraint, we get the existence results which improve the ones in Shi and Chen (Comput Math Appl 71:849–858, 2016). Moreover, we give an application of our results, which extends the results in Li (Commun Pure Appl Anal 14:1803–1816, 2015).

Keywords Quasilinear Schrödinger equation · Ground state solutions · Asymptotically periodic · Nehari manifold

1 Introduction and Main Result

We are concerned with the existence of solutions for the following generalized quasilinear Schrödinger equation

$$-\operatorname{div}(f^2(u)\nabla u) + f(u)f'(u)|\nabla u|^2 + V(x)u = g(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $f \in C^1(\mathbb{R}, \mathbb{R}^+)$ is even, $f'(s) \geq 0$ for all $s \geq 0$, the potential $V(x)$ is positive. Solutions of (1.1) are related to the solitary wave solutions for quasilinear Schrödinger

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equation of the form

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi + W(x)\psi - g(x,\psi) - \Delta\rho(|\psi|^2)\rho'(|\psi|^2)\psi, \tag{1.2}$$

where $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, and ρ is a real function. The form of (1.2) have been derived as models of several physical phenomena corresponding to various types of $\rho(s)$, see [14, 15, 18] for an explanation. Seeking solutions of the type stationary waves, namely, the solutions of the form $\psi(t,x) = \exp(-iEt)u(x)$, $E \in \mathbb{R}$ and u is a real function, equation (1.2) can be reduce to the corresponding equation of elliptic type

$$-\Delta u + V(x)u - \Delta\rho(u^2)\rho'(u^2)u = g(x,u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $V(x) = W(x) - E$ is the new potential function. If $f^2(u) = 1 + \frac{[\rho(u^2)']^2}{2}$, equation (1.3) turns into equation (1.1) (see [19]).

If we take $\rho(s) = s$, i.e., $f^2(u) = 1 + 2u^2$, we get the superfluid film equation in plasma physics

$$-\Delta u + V(x)u - \Delta(u^2)u = g(x,u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

If we set $\rho(s) = \sqrt{1+s}$, i.e., $f^2(u) = 1 + \frac{u^2}{2(1+u^2)}$, we get the equation

$$-\Delta u + V(x)u - \frac{u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2}) = g(x,u), \quad x \in \mathbb{R}^N, \tag{1.5}$$

which models the self-channeling of a high-power ultrashort laser in matter (see [5]).

Problem (1.4) has been studied by many authors. To the best of our knowledge, the first existence results for problem (1.4) due to [18], where the authors obtained the existence results by using a constrained minimization argument. Since then, there are many results for problem (1.4) depending on the different assumptions on the potential V, such as radially symmetric potential, coercive potential, periodic potential, and so on (see [4, 11, 12, 14, 15, 17, 22, 23] and references therein).

The results of problem (1.5) are not too many, one can see references [2, 3, 6, 7, 20] for details. Under some appropriate assumptions on the nonlinear term, some results are obtained by different methods, such as a change of variables (see [2, 3, 7, 20]) and a perturbation method (see [6]). Especially, Chu and Liu in [3] studied problem (1.5) for the case $g(x,u) = \mu g(u)$, $\mu > 0$ is a parameter. They proved that (1.5) has at least a positive solution by using the monotonicity trick and a priori estimate. It is a little surprising that no condition is assumed on the nonlinear term $g(u)$ near infinity.

We point out that problem (1.4) and (1.5) are special cases in equation (1.1). A natural question is whether there is a unified method to research equation (1.1) with general functions $f(u)$? Fortunately, Shen and Wang in [19] have given an affirmative answer and obtained the existence of positive solution for (1.1) with a general function $f(u)$. Since then, some results on general equations have appeared, such as [5, 9, 21]. In [5], they found the related critical exponents for equation (1.1) and obtained the solitary wave solutions by using a change of variables and the variational argument. In [9], by employing the minimax theorems, they got the existence results of the positive solution. Moreover, they gave two applications of their results, which improved the results in [1]. Reference [21] established the existence of positive solutions for equation (1.1) with asymptotically periodic potential. The methods they used are the mountain mass theorem and the concentration compactness principle. In this paper, we will use the variable replacement in [19] to study equation (1.1) with asymptotically periodic potential, which is different from that in [21].

Denote $G(x,s) := \int_0^s g(x,t)dt$, we observe that the natural variational functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} f^2(u)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} G(x,u)dx,$$

corresponding to equation (1.1), may be not well defined in the space $H^1(\mathbb{R}^N)$. To find a suitable functional space to obtain the critical point corresponding to $J(u)$, we can use a change of variable constructed by Shen and Wang in [19], as

$$v := F(u) = \int_0^u f(t)dt.$$

After the change of variable, we get a new variational functional

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)|F^{-1}(v)|^2)dx - \int_{\mathbb{R}^N} G(x, F^{-1}(v))dx.$$

Then $I(v) = J(u) = J(F^{-1}(v))$ and I is well defined in $H^1(\mathbb{R}^N)$, $I \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ (see [5, 19]).

If u is a weak solution of problem (1.1), then it should satisfy

$$\int_{\mathbb{R}^N} \left[f^2(u)\nabla u \cdot \nabla\varphi + f(u)f'(u)|\nabla u|^2\varphi + V(x)u\varphi - g(x,u)\varphi \right] dx = 0, \quad (1.6)$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Let $\varphi = \frac{\psi}{f(u)}$, then it can be checked (see [19]) that (1.6) is equivalent to the following equality

$$\langle I'(v), \psi\rangle = \int_{\mathbb{R}^N} \left(\nabla v \cdot \nabla\psi + V(x)\frac{F^{-1}(v)}{f(F^{-1}(v))}\psi - \frac{g(x, F^{-1}(v))}{f(F^{-1}(v))}\psi \right) dx = 0. \quad (1.7)$$

Therefore, in order to find the solutions of problem (1.1), it suffices to study the existence of solutions of the following equation

$$-\triangle v + V(x)\frac{F^{-1}(v)}{f(F^{-1}(v))} = \frac{g(x, F^{-1}(v))}{f(F^{-1}(v))}, \quad x \in \mathbb{R}^N. \quad (1.8)$$

In the following, we consider the existence of ground state solutions for problem (1.8) with asymptotically periodic condition. Denote

$$\mathscr{F}_0 := \{k(x) : \forall \epsilon > 0,\ \lim_{|y|\to\infty} \text{meas}\{x \in B_1(y) : |k(x)| \geq \epsilon\} = 0\},$$
$$\mathscr{F} := \{h(x,s) : \forall \epsilon > 0,\ \lim_{|y|\to\infty} \text{meas}\{x \in B_1(y) : |h(x,s)| \geq \epsilon\} = 0$$
$$\text{uniformly for } |s| \text{ bounded}\}.$$

Then, we give some assumptions on the function $f(t)$, the potential $V(x)$ and the nonlinear term $g(x,t)$.

(f) $f \in C^1(\mathbb{R}, \mathbb{R}^+)$ is even, $f'(t) \geq 0$ for all $t \geq 0$, $f(0) = 1$, $\lim_{t\to+\infty} f(t) = a$ for some $a \geq 1$.

(V) $0 < V_{\min} \leq V(x) \leq V_0(x) \in L^\infty(\mathbb{R}^N)$, $V(x) - V_0(x) \in \mathscr{F}_0$, and V_0 satisfies $V_0(x+z) = V_0(x)$ for all $x \in \mathbb{R}^N$ and $z \in \mathbb{Z}^N$.

The function $g \in C(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R})$ satisfies

(g_1) $\lim_{t\to 0^+} \frac{g(x,t)}{t} = 0$ uniformly for $x \in \mathbb{R}^N$.
(g_2) $\lim_{t\to\infty} \frac{g(x,t)}{t^{2^*-1}} = 0$ uniformly for $x \in \mathbb{R}^N$.
(g_3) $t \mapsto \frac{g(x,t)}{f(t)F(t)}$ is nondecreasing on $(0, +\infty)$.
(g_4) there exists $g_0 \in C(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R}^+)$ such that

(1) $g(x,t) \geq g_0(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$ and $g(x,t) - g_0(x,t) \in \mathscr{F}$.
(2) $g_0(x+z,t) = g_0(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$ and $z \in \mathbb{Z}^N$.
(3) $t \mapsto \frac{g_0(x,t)}{f(t)F(t)}$ is nondecreasing on $(0, +\infty)$.
(4) $\lim_{t\to\infty} \frac{G_0(x,t)}{t^2} = +\infty$ uniformly for $x \in \mathbb{R}^N$.

Because we are searching for the positive solution, we can assume that $g(x,t) = g_0(x,t) = 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^-$. Now we state our main results.

Theorem 1.1 *Suppose that conditions (f), (V) and $(g_1) - (g_4)$ are satisfied, then problem (1.1) possesses a positive ground state solution.*

In the particular case: $V(x) = V_0(x)$, $g(x,t) = g_0(x,t)$, we can get a solution for the periodic problem from Theorem 1.1. That is, considering the problem

$$-\text{div}(f^2(u)\nabla u) + f(u)f'(u)|\nabla u|^2 + V_0(x)u = g_0(x,u), \quad x \in \mathbb{R}^N, \tag{1.9}$$

under the hypothesis:

(V_0) the function $V_0(x)$ satisfies $0 < \inf_{x\in\mathbb{R}^N} V_0(x) \leq V_0(x) \in L^\infty(\mathbb{R}^N)$ and $V_0(x+z) = V_0(x)$ for all $x \in \mathbb{R}^N$ and $z \in \mathbb{Z}^N$.

We can obtain the existence result for the periodic problem.

Corollary 1.2 *Suppose that (f) and (V_0) hold, $g_0(x,t) = g(x,t)$ satisfies $(g_1) - (g_4)$. Then equation (1.9) possesses a positive ground state solution.*

Remark 1.3 As far as we know, there are no other results concerning problem (1.1) where the potential $V(x)$ is asymptotically periodic except reference [21]. Here, we consider a new reformative condition which unify the asymptotic processes of V, g at infinity, which means $\mathscr{F}$ and $\mathscr{F}_0$ contain more elements than those in [21]. Moreover, in [21] the authors obtained the existence of nontrivial solutions for problem (1.1) by using the mountain pass theorem. Here, with the aid of a Nehari-type constraint, we consider the ground state solution, which has great physical interests.

Remark 1.4 To the best of our knowledge, even for the periodic case, our result for problem (1.1) is new. In [10], Li et al. studied the existence of infinitely many geometrically distinct solutions for problem (1.1). Our result is different from the result there.

Now, we give an application of Theorem 1.1.

For $f^2(u) = 1 + \frac{u^2}{2(1+u^2)}$, by a direct calculation, we know $f(u)$ satisfies condition (f) with $a = \sqrt{\frac{3}{2}}$, we can get the following results directly.

Theorem 1.5 *Suppose that conditions (V) and* $(g_1) - (g_4)$ *are satisfied, then problem* (1.5) *possesses a positive ground state solution.*

As a by-product of our calculations we can obtain a weak solution for the periodic problem.

Corollary 1.6 *Suppose that* (V_0) *holds,* $g_0(x,t) = g(x,t)$ *satisfies* $(g_1) - (g_4)$. *Then equation*

$$-\Delta u + V_0(x)u - \frac{u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2}) = g_0(x,u), \quad x \in \mathbb{R}^N,$$

possesses a positive ground state solution.

Remark 1.7 It is worth pointing out that there is no result for equation (1.5) when the potential is asymptotically periodic. For the periodic potential, there are references [7, 8], they discussed the following equation

$$-\Delta u + V_0(x)u - [\Delta(1+u^2)^{\alpha/2}]\frac{\alpha u}{2(1+u^2)^{(2-\alpha)/2}} = g_0(x,u), \tag{1.10}$$

where α is a parameter. Jalilian [7] considered equation (1.10) with $1.36 < \alpha \leq 2$ and proved that (1.10) had infinitely many geometrically distinct solutions. Li [8] proved the existence of a ground state solution for equation (1.10) with $1 \leq \alpha \leq 2$ if g_0 satisfies some conditions and

(g_5) $\mathfrak{g}(x,t) := \frac{1}{4\alpha}g_0(x,t)t - G_0(x,t) > 0$, $|g_0(x,t)|^\sigma \leq a_1\mathfrak{g}(x,t)|t|^\sigma$, for some $a_1 > 0$, $\sigma > \max\{1, \alpha - 1\}$ and for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ with t large enough.

In fact, (g_5) plays a crucial role in getting a bounded (PS) sequence. Here, we do not need such condition. Even for the periodic case for equation (1.5), our result is also new.

Notation: In this paper, we use the following notations.

- $H^1(\mathbb{R}^N)$ is the usual Hilbert space endowed with the norm

$$\|u\|_H^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2 \right) dx.$$

- $L^s(\mathbb{R}^N)$ is the usual Banach space endowed with the norm

$$\|u\|_s^s = \int_{\mathbb{R}^N} |u|^s dx, \quad \forall s \in [1, +\infty).$$

- $\|u\|_\infty = \text{ess}\sup_{x\in\mathbb{R}^N} |u(x)|$ denotes the usual norm in $L^\infty(\mathbb{R}^N)$.
- $E = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty\}$ is endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2 \right) dx.$$

- $B_r(y) := \{x \in \mathbb{R}^N : |x - y| < r\}$.
- $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$.
- $|\Omega|$ denote the Lebesgue measure of the set Ω.
- $C, C_1, C_2, \cdots$ denote various positive (possibly different) constants.

2 Some Preliminary Results

Lemma 2.1 *The functions $f(t)$, $F(t)$, $g(x,t)$, $G(x,t)$ enjoy the following properties under the assumptions (f) and (g_3).*

(1) *$F(t)$ is uniquely defined and invertible, $F(t)$ and $F^{-1}(t)$ are odd;*
(2) $\frac{f'(t)t}{f(t)} \geq 0$ *for all* $t \in \mathbb{R}$*;*
(3) $1 \leq f(t) \leq a$ *and* $\frac{t}{a} \leq F^{-1}(t) \leq t$ *for all* $t \geq 0$*;*
(4) $\frac{F^{-1}(t)}{t} \to 1$ *as* $t \to 0$*;*
(5) $\frac{F^{-1}(t)}{t} \to \frac{1}{a}$ *as* $t \to \infty$*;*
(6) $\frac{g(x,t)F(t)}{f(t)} - 2G(x,t) \geq 0$ *for all* $t \geq 0$*;*
(7) *The function* $\frac{t}{f(t)F(t)}$ *is strictly decreasing for all* $t \geq 0$.

Proof The proof of the items (1) and (2) follow from the definition of F and the assumption (f) directly.

(3) By the mean value theorem, we know

$$F(t) = \int_0^t f(s)ds = f(\xi)t,$$

for some $\xi \in [0, t]$. Note that, f is nondecreasing and $F(t)$ is increasing, then

$$t = f(0)t \leq F(t) = f(\xi)t \leq at,$$

so that $\frac{t}{a} \leq F^{-1}(t) \leq t$.

The items (4) and (5) can be obtained by the L′Hospital rule immediately.

(6) Let $L(x,t) := \frac{g(x,t)F(t)}{f(t)} - 2G(x,t)$, by the condition (g_3), one has

$$\frac{\partial}{\partial t}L(x,t) = F^2(t)\frac{\partial}{\partial t}\left\{\frac{g(x,t)}{f(t)F(t)}\right\} \geq 0,$$

when $t \geq 0$. Then, $L(x,t)$ is non-decreasing in $(0,+\infty)$. Hence, $L(x,t) = \frac{g(x,t)F(t)}{f(t)} - 2G(x,t) \geq L(x,0) = 0$ for all $t \geq 0$.

(7) Let $l(t) = \frac{t}{f(t)F(t)}$. Since $f(t)$ is nondecreasing in $(0,+\infty)$, one has

$$0 \leq F(t) = \int_0^t f(s)ds \leq tf(t). \tag{2.1}$$

Then using item (2) and (2.1), we obtain

$$l'(t) = \frac{F(t) - tf(t) - \frac{f'(t)t}{f(t)}F(t)}{f(t)F^2(t)} \leq \frac{F(t) - tf(t)}{f(t)F^2(t)} \leq 0.$$

The above inequality proves item (7). □

Lemma 2.2 ([13]) *Suppose that condition (V) holds. Then there are two positive constants d_1 and d_2 such that $d_1\|u\|_H^2 \leq \|u\|^2 \leq d_2\|u\|_H^2$ for all $u \in E$.*

Remark 2.3 From the above Lemma 2.2 and the Sobolev embedding, we get that the embedding $E \hookrightarrow L^\alpha(\mathbb{R}^N)$ is continuous for any $\alpha \in [2, 2^*]$.

Lemma 2.4 *Assume that (f), (V), $(g_1)-(g_4)$ hold. If $\{u_n\}$ is bounded in E and $u_n \to 0$ in $L^\alpha_{\text{loc}}(\mathbb{R}^N)$ for $\alpha \in [2, 2^*)$, one has*

$$A_{n1} := \int_{\mathbb{R}^N}\Big(V(x) - V_0(x)\Big)|F^{-1}(u_n)|^2dx = o_n(1). \tag{2.2}$$

$$A_{n2} := \int_{\mathbb{R}^N}\Big[G(x, F^{-1}(u_n)) - G_0(x, F^{-1}(u_n))\Big]dx = o_n(1). \tag{2.3}$$

Proof Firstly, we give some useful inequalities which can be deduced by conditions (g_1), (g_2), (g_4) directly. For any $\delta > 0$, there exist $r_\delta > 0$, $C_\delta > 0$ and $\alpha \in (2, 2^*)$ such that

$$0 \leq g_0(x,t) \leq g(x,t) \leq \delta|t|, \quad \forall (x,t) \in \mathbb{R}^N \times [-r_\delta, r_\delta], \tag{2.4}$$

$$0 \leq g_0(x,t) \leq g(x,t) \leq \delta|t| + C_\delta|t|^{2^*-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{2.5}$$

$$0 \leq g_0(x,t) \leq g(x,t) \leq C_\delta|t| + \delta|t|^{2^*-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{2.6}$$

$$0 \leq g_0(x,t) \leq g(x,t) \leq \delta(|t| + |t|^{2^*-1}) + C_\delta|t|^{\alpha-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{2.7}$$

(i) The proof of (2.2).

When $k(x)\in\mathscr{F}_0$, for any $\epsilon>0$, there exists $R_\epsilon>0$ such that

$$\int_{|k(x)|\geq\epsilon}u^2dx\leq C_0\int_{B_{R_\epsilon+1}(0)}u^2dx+C_1\epsilon^{2/N}\|u\|_H^2,\quad\forall u\in E, \tag{2.8}$$

where C_0, C_1 are positive constants and independent on ϵ. Inequality (2.8) has already been proved in [13], we omit it here.

Let $k(x):=V(x)-V_0(x)\in\mathscr{F}_0$, then, $|k(x)|\leq 2|V_0(x)|\leq 2\|V_0\|_\infty$, by using Lemma 2.1-(3) and (2.8), we have

$$\begin{aligned}|A_{n1}|&\leq\int_{\mathbb{R}^N}|k(x)||F^{-1}(u_n)|^2dx\leq\int_{\mathbb{R}^N}|k(x)u_n^2|dx\\&=\int_{|k(x)|\geq\epsilon}|k(x)u_n^2|dx+\int_{|k(x)|<\epsilon}|k(x)u_n^2|dx\\&\leq 2\|V_0\|_\infty\left[C_0\int_{B_{R_\epsilon+1}(0)}u_n^2dx+C_1\epsilon^{\frac{2}{N}}\|u_n\|_H^2\right]+\epsilon\int_{\mathbb{R}^N}|u_n|^2dx\\&=o_n(1)+C_2\epsilon^{\frac{2}{N}}+C_3\epsilon.\end{aligned}$$

Let $\epsilon\to 0$, (2.2) is proved.

(ii)The proof of (2.3).

Set $h(x,s):=g(x,s)-g_0(x,s)\in\mathscr{F}$. For any $\epsilon>0$, there is $R_\epsilon>0$ such that

$$\text{meas}\{x\in B_1(y):|h(x,s)|\geq\epsilon\}<\epsilon,\quad\forall|y|\geq R_\epsilon,\quad|s|\leq 1/\epsilon.$$

Covering $\mathbb{R}^N$ by balls $B_1(y_i)$, $i\in\mathbb{N}$, in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls (see [24]). Without loss of generality, we suppose that $|y_i|<R_\epsilon$, $i=1,2,\cdots,n_\epsilon$ and $|y_i|\geq R_\epsilon$, $i=n_\epsilon+1,n_\epsilon+2,\cdots,+\infty$. By the mean value theorem, there exists $t_n\in[0,1]$ such that

$$G(x,F^{-1}(u_n))-G_0(x,F^{-1}(u_n))=[g(x,t_nF^{-1}(u_n))-g_0(x,t_nF^{-1}(u_n))]F^{-1}(u_n).$$

Set

$$\begin{aligned}\Omega^1&:=\{x\in B_1(y_i):|h(x,t_nF^{-1}(u_n))|<\epsilon\},\\\Omega^2&:=\{x\in B_1(y_i):|t_nF^{-1}(u_n)|\leq 1/\epsilon,\quad|h(x,t_nF^{-1}(u_n))|\geq\epsilon\},\\\Omega^3&:=\{x\in B_1(y_i):|t_nF^{-1}(u_n)|>1/\epsilon,\quad|h(x,t_nF^{-1}(u_n))|\geq\epsilon\}.\end{aligned}$$

Then we have

$$\begin{aligned}|A_{n2}|&\leq\int_{\mathbb{R}^N}|[g(x,t_nF^{-1}(u_n))-g_0(x,t_nF^{-1}(u_n))]F^{-1}(u_n)|dx\\&=\int_{\mathbb{R}^N}|h(x,t_nF^{-1}(u_n))F^{-1}(u_n)|dx\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n_\epsilon} \int_{B_1(y_i)} |h(x, t_n F^{-1}(u_n)) F^{-1}(u_n)| dx \\
&\quad + \sum_{i=n_\epsilon+1}^{+\infty} \int_{B_1(y_i)} |h(x, t_n F^{-1}(u_n)) F^{-1}(u_n)| dx \\
&= \sum_{i=1}^{n_\epsilon} \int_{B_1(y_i)} |h(x, t_n F^{-1}(u_n)) F^{-1}(u_n)| dx \\
&\quad + \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^1} |h(x, t_n F^{-1}(u_n)) F^{-1}(u_n)| dx \\
&\quad + \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^2} |h(x, t_n F^{-1}(u_n)) F^{-1}(u_n)| dx \\
&\quad + \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^3} |h(x, t_n F^{-1}(u_n)) F^{-1}(u_n)| dx \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

It follows from (2.6) and Lemma 2.1-(3) that

$$
\begin{aligned}
I_1 &\leq (N+1) \int_{B_{R_\epsilon+1}(0)} |h(x, t_n F^{-1}(u_n)) F^{-1}(u_n)| dx \\
&\leq (N+1) \int_{B_{R_\epsilon+1}(0)} 2\big[C_\delta |t_n F^{-1}(u_n)| + \delta |t_n F^{-1}(u_n)|^{2^*-1}\big] |F^{-1}(u_n)| dx \\
&\leq 2(N+1) C_\delta \int_{B_{R_\epsilon+1}(0)} |u_n|^2 dx + 2(N+1)\delta \int_{B_{R_\epsilon+1}(0)} |u_n|^{2^*} dx \\
&= o_n(1) + C_4 \delta
\end{aligned}
$$

Let

$$
\begin{aligned}
\Omega^{11} &:= \{x \in B_1(y_i) : |h(x, t_n F^{-1}(u_n))| < \epsilon,\ |t_n F^{-1}(u_n)| \leq r_\delta\}, \\
\Omega^{12} &:= \{x \in B_1(y_i) : |h(x, t_n F^{-1}(u_n))| < \epsilon,\ |t_n F^{-1}(u_n)| > r_\delta\}.
\end{aligned}
$$

By using (2.4) and Lemma 2.1-(3), we obtain

$$
\begin{aligned}
I_2 &= \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^{11}} |h(x, t_n F^{-1}(u_n)) F^{-1}(u_n)| dx \\
&\quad + \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^{12}} |h(x, t_n F^{-1}(u_n)) F^{-1}(u_n)| dx
\end{aligned}
$$

$$
\begin{aligned}
&\le \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^{11}} 2\delta |t_n F^{-1}(u_n) F^{-1}(u_n)| dx + \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^{12}} \frac{\epsilon}{r_\delta} |F^{-1}(u_n)|^2 dx \\
&\le 2\delta \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^{11}} |u_n|^2 dx + \frac{\epsilon}{r_\delta} \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^{12}} |u_n|^2 dx \\
&\le 2(N+1)\delta \int_{\mathbb{R}^N} |u_n|^2 dx + \frac{(N+1)\epsilon}{r_\delta} \int_{\mathbb{R}^N} |u_n|^2 dx \\
&\le C_5\delta + C_6\epsilon.
\end{aligned}
$$

It follows from (2.6), Lemma 2.1-(3), the Hölder and Sobolev inequalities that

$$
\begin{aligned}
I_3 &\le \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^2} 2\Big[C_\delta |F^{-1}(u_n)|^2 + \delta |F^{-1}(u_n)|^{2^*} \Big] dx \\
&\le \sum_{i=n_\epsilon+1}^{+\infty} \Big[2C_\delta \int_{\Omega^2} |u_n|^2 dx + 2\delta \int_{\Omega^2} |u_n|^{2^*} dx \Big] \\
&\le 2C_\delta \sum_{i=n_\epsilon+1}^{+\infty} |\Omega^2|^{\frac{2}{N}} \Big(\int_{\Omega^2} |u_n|^{2^*} dx \Big)^{\frac{N-2}{N}} + 2(N+1)\delta \int_{\mathbb{R}^N} |u_n|^{2^*} dx \\
&\le 2C_\delta \epsilon^{\frac{2}{N}} \sum_{i=n_\epsilon+1}^{+\infty} C \int_{\Omega^2} (|\nabla u_n|^2 + |u_n|^2) dx + C_7\delta \\
&\le 2C_\delta \epsilon^{\frac{2}{N}} (N+1) C \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |u_n|^2) dx + C_7\delta \\
&= C_8 \epsilon^{\frac{2}{N}} + C_7\delta.
\end{aligned}
$$

Thanks to (2.7) and Lemma 2.1-(3) that

$$
\begin{aligned}
I_4 &\le \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^3} 2\Big[\delta |F^{-1}(u_n)|^2 + \delta |F^{-1}(u_n)|^{2^*} + C_\delta |F^{-1}(u_n)|^\alpha \Big] dx \\
&\le \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^3} 2\Big[\delta |u_n|^2 + \delta |u_n|^{2^*} + C_\delta |u_n|^\alpha \Big] dx \\
&\le 2\delta(N+1) \int_{\mathbb{R}^N} (|u_n|^2 + |u_n|^{2^*}) dx + 2C_\delta \epsilon^{2^*-\alpha} \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^3} |u_n|^{2^*} dx \\
&\le C_9\delta + C_{10}\epsilon^{2^*-\alpha}.
\end{aligned}
$$

Hence we have

$$
|A_{n2}| \le o_n(1) + C_4\delta + C_5\delta + C_6\epsilon + C_8\epsilon^{\frac{2}{N}} + C_7\delta + C_9\delta + C_{10}\epsilon^{2^*-\alpha}.
$$

Let $\epsilon \to 0$ and then $\delta \to 0$, we complete the proof of (2.3). □

Lemma 2.5 *Assume that (f), (V), (g_1), (g_2) and (1) of (g_4) hold, $\{u_n\} \subset E$ is bounded, $|z_n| \to +\infty$. Then for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, one has*

$$B_{n1} := \int_{\mathbb{R}^N} (V(x) - V_0(x)) \frac{F^{-1}(u_n)}{f(F^{-1}(u_n))} \varphi(x - z_n) dx = o_n(1) \tag{2.9}$$

$$B_{n2} := \int_{\mathbb{R}^N} [g(x, F^{-1}(u_n)) - g_0(x, F^{-1}(u_n))] \frac{\varphi(x - z_n)}{f(F^{-1}(u_n))} dx = o_n(1) \tag{2.10}$$

Proof (i) The proof of (2.9).

Since $\varphi \in C_0^\infty(\mathbb{R}^N)$, we get that

$$\int_{B_{R_\epsilon+1}(0)} |\varphi(x - z_n)|^2 dx = o_n(1). \tag{2.11}$$

Let $k(x) := V(x) - V_0(x) \in \mathscr{F}_0$, by using Lemma 2.1-(3), (2.8), (2.11) and the Hölder inequality, we have

$$\begin{aligned}
|B_{n1}| &\le \int_{|k|\ge\epsilon} \left|\frac{k(x)F^{-1}(u_n)}{f(F^{-1}(u_n))}\varphi(x - z_n)\right| dx + \int_{|k|<\epsilon} \left|\frac{k(x)F^{-1}(u_n)}{f(F^{-1}(u_n))}\varphi(x - z_n)\right| dx \\
&\le 2\|V_0\|_\infty \int_{|k|\ge\epsilon} |u_n\varphi(x - z_n)| dx + \epsilon \int_{|k|<\epsilon} |u_n\varphi(x - z_n)| dx \\
&\le 2\|V_0\|_\infty \|u_n\|_2 \left(\int_{|k|\ge\epsilon} |\varphi(x - z_n)|^2 dx \right)^{1/2} + \epsilon \|u_n\|_2 \|\varphi\|_2 \\
&\le C_{11} \left(C_0 \int_{B_{R_\epsilon+1}(0)} |\varphi(x - z_n)|^2 dx + C_1 \epsilon^{2/N} \|\varphi\|_H^2 \right)^{1/2} + C_{12}\epsilon \\
&= o_n(1) + C_{13}\epsilon^{1/N} + C_{12}\epsilon.
\end{aligned}$$

Let $\epsilon \to 0$, (2.9) is proved.

(ii)The proof of (2.10).

Set $h(x, s) := g(x, s) - g_0(x, s) \in \mathscr{F}$. As the proof of Lemma 2.4, we can cover $\mathbb{R}^N$ by balls $B_1(y_i)$. Let

$$\begin{aligned}
\Omega^4 &:= \{x \in B_1(y_i) : |h(x, F^{-1}(u_n))| < \epsilon\}, \\
\Omega^5 &:= \{x \in B_1(y_i) : |F^{-1}(u_n)| \le 1/\epsilon, \ \ |h(x, F^{-1}(u_n))| \ge \epsilon\}, \\
\Omega^6 &:= \{x \in B_1(y_i) : |F^{-1}(u_n)| > 1/\epsilon, \ \ |h(x, F^{-1}(u_n))| \ge \epsilon\}.
\end{aligned}$$

Then, one has

$$|B_{n2}| \le \int_{\mathbb{R}^N} \left| h(x, F^{-1}(u_n)) \frac{\varphi(x - z_n)}{f(F^{-1}(u_n))} \right| dx$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n_\epsilon} \int_{B_1(y_i)} |h(x, F^{-1}(u_n)) \frac{\varphi(x-z_n)}{f(F^{-1}(u_n))}|dx \\
&\quad + \sum_{i=n_\epsilon+1}^{+\infty} \int_{B_1(y_i)} |h(x, F^{-1}(u_n)) \frac{\varphi(x-z_n)}{f(F^{-1}(u_n))}|dx \\
&= \sum_{i=1}^{n_\epsilon} \int_{B_1(y_i)} |h(x, F^{-1}(u_n)) \frac{\varphi(x-z_n)}{f(F^{-1}(u_n))}|dx \\
&\quad + \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^4} |h(x, F^{-1}(u_n)) \frac{\varphi(x-z_n)}{f(F^{-1}(u_n))}|dx \\
&\quad + \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^5} |h(x, F^{-1}(u_n)) \frac{\varphi(x-z_n)}{f(F^{-1}(u_n))}|dx \\
&\quad + \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^6} |h(x, F^{-1}(u_n)) \frac{\varphi(x-z_n)}{f(F^{-1}(u_n))}|dx \\
&:= I_5 + I_6 + I_7 + I_8.
\end{aligned}
$$

It follows from (2.6), Lemma 2.1-(3), (2.11) and the Hölder inequality that

$$
\begin{aligned}
I_5 &\leq (N+1) \int_{B_{R_\epsilon+1}(0)} |h(x, F^{-1}(u_n)) \frac{\varphi(x-z_n)}{f(F^{-1}(u_n))}|dx \\
&\leq (N+1) \int_{B_{R_\epsilon+1}(0)} 2\Big[C_\delta |F^{-1}(u_n)| + \delta |F^{-1}(u_n)|^{2^*-1}\Big] |\frac{\varphi(x-z_n)}{f(F^{-1}(u_n))}|dx \\
&\leq 2(N+1)\Big[C_\delta \int_{B_{R_\epsilon+1}(0)} |u_n \varphi(x-z_n)|dx + \delta \int_{B_{R_\epsilon+1}(0)} |u_n|^{2^*-1} |\varphi(x-z_n)|dx\Big] \\
&\leq 2(N+1)\Big[C_\delta \|u_n\|_2 \Big(\int_{B_{R_\epsilon+1}(0)} |\varphi(x-z_n)|^2 dx\Big)^{1/2} + \delta \|u_n\|_{2^*}^{2^*-1} \|\varphi\|_{2^*}\Big] \\
&= o_n(1) + C_{14}\delta.
\end{aligned}
$$

Thanks to Lemma 2.1-(3), we obtain

$$
\begin{aligned}
I_6 &= \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^4} |h(x, F^{-1}(u_n)) \frac{\varphi(x-z_n)}{f(F^{-1}(u_n))}|dx \\
&\leq \epsilon(N+1) \int_{\mathbb{R}^N} |\varphi(x-z_n)|dx \\
&= C_{15}\epsilon.
\end{aligned}
$$

It follows from (2.6), Lemma 2.1-(3) and the Hölder, Young and Sobolev inequalities that

$$
\begin{aligned}
I_7 &\leq \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^5} 2\Big[C_\delta|F^{-1}(u_n)| + \delta|F^{-1}(u_n)|^{2^*-1}\Big]|\frac{\varphi(x-z_n)}{f(F^{-1}(u_n))}|dx \\
&\leq \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^5} 2C_\delta|u_n\varphi(x-z_n)|dx + \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^5} 2\delta|u_n|^{2^*-1}|\varphi(x-z_n)|dx \\
&\leq 2C_\delta \sum_{i=n_\epsilon+1}^{+\infty} |\Omega^5|^{\frac{2}{N}} \left(\int_{\Omega^5} |u_n\varphi(x-z_n)|^{\frac{N}{N-2}}dx\right)^{\frac{N-2}{N}} + 2(N+1)\delta\|u_n\|_{2^*}^{2^*-1}\|\varphi\|_{2^*} \\
&\leq 2C_\delta\epsilon^{\frac{2}{N}} \sum_{i=n_\epsilon+1}^{+\infty} \left(\int_{\Omega^5} (\frac{|u_n|^{2^*}}{2} + \frac{|\varphi(x-z_n)|^{2^*}}{2})dx\right)^{\frac{N-2}{N}} + C_{16}\delta \\
&\leq 2C_\delta\epsilon^{\frac{2}{N}} \sum_{i=n_\epsilon+1}^{+\infty} 2^{\frac{N-2}{N}}\Bigg[\left(\frac{1}{2}\int_{\Omega^5} |u_n|^{2^*}dx\right)^{\frac{N-2}{N}} \\
&\quad + \left(\frac{1}{2}\int_{\Omega^5} |\varphi(x-z_n)|^{2^*}dx\right)^{\frac{N-2}{N}}\Bigg] + C_{16}\delta \\
&\leq 2C_\delta\epsilon^{\frac{2}{N}}(N+1)C\Bigg[\int_{\mathbb{R}^N} (|\nabla u_n|^2 + |u_n|^2)dx \\
&\quad + \int_{\mathbb{R}^N} (|\nabla\varphi(x-z_n)|^2 + |\varphi(x-z_n)|^2)dx\Bigg] + C_{16}\delta \\
&= C_{17}\epsilon^{\frac{2}{N}} + C_{16}\delta.
\end{aligned}
$$

By using (2.7), Lemma 2.1-(3) and the Hölder inequality, one has

$$
\begin{aligned}
I_8 &\leq \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^6} 2\Big[\delta|F^{-1}(u_n)| + \delta|F^{-1}(u_n)|^{2^*-1} + C_\delta|F^{-1}(u_n)|^{\alpha-1}\Big]|\frac{\varphi(x-z_n)}{f(F^{-1}(u_n))}|dx \\
&\leq \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^6} 2\Big[\delta|u_n| + \delta|u_n|^{2^*-1} + C_\delta|u_n|^{\alpha-1}\Big]|\varphi(x-z_n)|dx \\
&\leq 2\delta(N+1)\left(\int_{\mathbb{R}^N} |u_n\varphi(x-z_n)|dx + \int_{\mathbb{R}^N} |u_n|^{2^*}|\varphi(x-z_n)|dx\right) \\
&\quad + 2C_\delta\epsilon^{2^*-\alpha} \sum_{i=n_\epsilon+1}^{+\infty} \int_{\Omega^6} |u_n|^{2^*-1}|\varphi(x-z_n)|dx \\
&\leq 2\delta(N+1)\left(\|u_n\|_2\|\varphi\|_2 + \|u_n\|_{2^*}^{2^*-1}\|\varphi\|_{2^*}\right) + 2C_\delta\epsilon^{2^*-\alpha}\|u_n\|_{2^*}^{2^*-1}\|\varphi\|_{2^*} \\
&= C_{18}\delta + C_{19}\epsilon^{2^*-\alpha}.
\end{aligned}
$$

Hence we obtain

$$|B_{n2}| \le o_n(1) + C_{14}\delta + C_{15}\epsilon + C_{17}\epsilon^{\frac{2}{N}} + C_{16}\delta + C_{18}\delta + C_{19}\epsilon^{2^*-\alpha}.$$

Let $\epsilon \to 0$ and then $\delta \to 0$, we complete the proof. □

3 Proof of Theorem 1.1

Define

$$\mathcal{N} = \{u \in E : \langle I'(u), u\rangle = 0, u \neq 0\},\ \mathcal{N}_0 = \{u \in E : \langle I_0'(u), u\rangle = 0, u \neq 0\},$$
$$c = \inf_{u\in\mathcal{N}} I(u),\ c_0 = \inf_{u\in\mathcal{N}_0} I_0(u),$$

where

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left[|\nabla u|^2 + V(x)|F^{-1}(u)|^2\right]dx - \int_{\mathbb{R}^N} G(x, F^{-1}(u))dx,$$
$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left[|\nabla u|^2 + V_0(x)|F^{-1}(u)|^2\right]dx - \int_{\mathbb{R}^N} G_0(x, F^{-1}(u))dx.$$

Lemma 3.1 *Suppose that conditions (f), (V) and $(g_1) - (g_4)$ hold, then for each $u \in E$, $u \neq 0$, there is a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$. Moreover, the maximum of $I(tu)$ for $t \ge 0$ is achieved at t_u.*

Proof By the inequality (2.5), Lemma 2.1-(3), one has

$$G(x, F^{-1}(tu)) \le \frac{\delta}{2}|F^{-1}(tu)|^2 + \frac{C_\delta}{2^*}|F^{-1}(tu)|^{2^*} \le \frac{\delta}{2}t^2u^2 + \frac{C_\delta}{2^*}t^{2^*}u^{2^*}. \quad (3.1)$$

It follows from Lemma 2.1-(3), (3.1) and the Sobolev inequality and Lemma 2.2 that

$$\begin{aligned} h(t) = I(tu) &= \frac{1}{2}\int_{\mathbb{R}^N}\left[|\nabla(tu)|^2 + V(x)|F^{-1}(tu)|^2\right]dx - \int_{\mathbb{R}^N} G(x, F^{-1}(tu))dx \\ &\ge \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx + \frac{t^2}{2a^2}\int_{\mathbb{R}^N}V(x)u^2dx - \frac{t^2\delta}{2}\int_{\mathbb{R}^N}u^2dx - \frac{t^{2^*}C_\delta}{2^*}\int_{\mathbb{R}^N}u^{2^*}dx \\ &\ge \frac{t^2}{2a^2}\|u\|^2 - \frac{t^2\delta}{2}C_1\|u\|^2 - t^{2^*}C_2\|u\|^{2^*}, \end{aligned}$$

for some positive constants C_1, C_2. We choose $\delta > 0$ small enough, such that $\frac{1}{2a^2} - \frac{\delta}{2}C_1 > 0$. Therefore, we can get $h(t) > 0$ whenever $t > 0$ is small enough.

By Lemma 2.1-(3) and $G(x, s) \ge G_0(x, s)$, we have

$$\frac{h(t)}{t^2} \le \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2dx - \int_{\mathbb{R}^N}G_0(x, F^{-1}(tu))dx$$

$$\leq \frac{1}{2}\|u\|^2 - \int_{u\neq 0} \frac{G_0(x, F^{-1}(tu))}{|F^{-1}(tu)|^2} \cdot \frac{|F^{-1}(tu)|^2}{(tu)^2} \cdot u^2 dx$$

Thanks to (4) of (g_4) and Lemma 2.1-(5), we can deduce that the last integral on the right-hand side above tends to infinity with t. Hence, $h(t) \to -\infty$ as $t \to \infty$ and h has a positive maximum.

The condition $h'(t) = 0$ is equivalent to

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = \int_{u\neq 0} \left[\frac{g(x, F^{-1}(tu))}{tuf(F^{-1}(tu))} - \frac{V(x)F^{-1}(tu)}{f(F^{-1}(tu))tu} \right] u^2 dx.$$

Let

$$Z(s) := \frac{g(x, s)}{f(s)F(s)} - \frac{V(x)s}{f(s)F(s)}.$$

By (g_3) and Lemma 2.1-(7), $s \mapsto Z(s)$ is strictly increasing for $s > 0$, so there is a unique $t_u > 0$ such that $h'(t_u) = 0$. The conclusion is true since $h'(t) = t^{-1}\langle I'(tu), tu\rangle$. □

As the argument in [24] (Theorem 4.2), we obtain the following lemma.

Lemma 3.2 *Suppose that (f), (V) hold, g satisfies* $(g_1) - (g_4)$, *then*

$$c = \inf_{u\in\mathcal{N}} I(u) = \inf_{u\in E} \max_{t>0} I(tu) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, I(\gamma(t)) < 0\}$.

Remark 3.3 The conclusions of Lemmas 3.1 and 3.2 are also suitable for the periodic functional I_0.

Next, we will give the boundedness of the Cerami sequences.

Lemma 3.4 *Suppose that (f), (V) and* $(g_1) - (g_4)$ *hold. Let* $\{u_n\} \subset E$ *be a* $(C)_c$ *sequence for the functional* I. *Then* $\{u_n\}$ *is bounded in* E.

Proof Suppose by contradiction that $\{u_n\} \subset E$ be a sequence such that $\|u_n\| \to \infty$, $I(u_n) \to c$ and $(1 + \|u_n\|)\|I'(u_n)\| \to 0$. Set $v_n := \frac{u_n}{\|u_n\|}$, then, there is a $v \in E$ such that $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$. If $v \neq 0$, let $\Omega_* = \{x \in \mathbb{R}^N : v(x) > 0\}$, then $|\Omega_*| > 0$. For a.e. $x \in \Omega_*$, one has

$$u_n(x) \to +\infty \quad \text{as} \quad \|u_n\| \to +\infty,$$

since $v_n(x) = \frac{u_n(x)}{\|u_n\|} \to v(x) > 0$ for a.e. $x \in \Omega_*$, from Lemma 2.1-(5) and the fact that $F^{-1}(t)$ is strictly increasing, we can deduce that for a.e. $x \in \Omega_*$,

$$F^{-1}(u_n) \to +\infty \quad \text{as} \quad \|u_n\| \to +\infty.$$

It follows from Lemma 2.1-(3)(5) and $(g_4)-(1)(4)$ that

$$
\begin{aligned}
0 &= \limsup_{n\to\infty} \frac{I(u_n)}{\|u_n\|^2} \\
&\le \limsup_{n\to\infty} \frac{\frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^N} G_0(x, F^{-1}(u_n))dx}{\|u_n\|^2} \\
&= \frac{1}{2} - \liminf_{n\to\infty} \frac{\int_{\Omega_*} G_0(x, F^{-1}(u_n))dx}{\|u_n\|^2} \\
&= \frac{1}{2} - \liminf_{n\to\infty} \int_{\Omega_*} \frac{G_0(x, F^{-1}(u_n))}{|F^{-1}(u_n)|^2} \cdot \frac{|F^{-1}(u_n)|^2}{u_n^2} \cdot v_n^2 dx \\
&= -\infty.
\end{aligned}
$$

A contradiction, thus $v = 0$. Define

$$
\beta := \limsup_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B_1(z)} v_n^2 dx.
$$

If $\beta = 0$, by the Lions lemma [24] (Lemma 1.21), we get $v_n \to 0$ in $L^\alpha(\mathbb{R}^N)$ for $\alpha \in (2, 2^*)$. It follows from (2.7) and Lemma 2.1-(3) that

$$
\begin{aligned}
\int_{\mathbb{R}^N} G(x, F^{-1}(tv_n))dx &\le \frac{\delta}{2}\int_{\mathbb{R}^N} |F^{-1}(tv_n)|^2 dx + \frac{\delta}{2^*}\int_{\mathbb{R}^N} |F^{-1}(tv_n)|^{2^*} dx \\
&\quad + \frac{C_\delta}{\alpha}\int_{\mathbb{R}^N} |F^{-1}(tv_n)|^\alpha dx \le \frac{\delta}{2}t^2 \int_{\mathbb{R}^N} |v_n|^2 dx \\
&\quad + \frac{\delta}{2^*}t^{2^*}\int_{\mathbb{R}^N} |v_n|^{2^*} dx + \frac{C_\delta}{\alpha}t^\alpha \int_{\mathbb{R}^N} |v_n|^\alpha dx \\
&= o_n(1)(\delta \to 0).
\end{aligned}
$$

Especially, set $t = 4\sqrt{c}$, we obtain

$$
\int_{\mathbb{R}^N} G(x, F^{-1}(4\sqrt{c}v_n))dx = o_n(1). \tag{3.2}
$$

By Lemma 2.1-(4), one has $F^{-1}(4\sqrt{c}v_n) \to 4\sqrt{c}v_n$, since $4\sqrt{c}v_n \to 0$ a.e. in $\mathbb{R}^N$. Then, we can deduce that

$$
\int_{\mathbb{R}^N} V(x)\left[(4\sqrt{c}v_n)^2 - F^{-1}(4\sqrt{c}v_n)^2\right]dx = o_n(1). \tag{3.3}
$$

Setting

$$
k(x, s) = \frac{g(x, F^{-1}(s))}{f(F^{-1}(s))} - V(x)\frac{F^{-1}(s)}{f(F^{-1}(s))} + V(x)s,
$$

and

$$K(x,s) := \int_0^s k(x,t)dt = G(x, F^{-1}(s)) - \frac{1}{2}V(x)|F^{-1}(s)|^2 + \frac{1}{2}V(x)s^2.$$

Then, thanks to (3.2) and (3.3), we can obtain that

$$\int_{\mathbb{R}^N} K(x, 4\sqrt{c}v_n)dx = \int_{\mathbb{R}^N} G(x, F^{-1}(4\sqrt{c}v_n))dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)\left[(4\sqrt{c}v_n)^2 - F^{-1}(4\sqrt{c}v_n)^2\right]dx = o_n(1).$$

By the continuity of I, there exists $t_n \in [0,1]$ such that $I(t_n u_n) = \max_{0\leq t\leq 1} I(tu_n)$. Since $\|u_n\| \to \infty$, we have $\frac{4\sqrt{c}}{\|u_n\|} \leq 1$ when n is large enough. Hence, one has

$$\begin{aligned}
I(t_n u_n) + o_n(1) &\geq I\left(\frac{4\sqrt{c}}{\|u_n\|}u_n\right) + o_n(1) = I(4\sqrt{c}v_n) + o_n(1) \\
&= 8c\|v_n\|^2 - \int_{\mathbb{R}^N} K(x, 4\sqrt{c}v_n)dx + o_n(1) \\
&= 8c.
\end{aligned}$$

Note that $I(u_n) \to c$, so $0 < t_n < 1$ and $\langle I'(t_n u_n), t_n u_n\rangle = 0$ when n is large enough. By (g_3) and Lemma 2.1-(7), the function

$$\frac{k(x,s)}{s} = \frac{g(x, F^{-1}(s))}{f(F^{-1}(s))s} - V(x)\frac{F^{-1}(s)}{f(F^{-1}(s))s} + V(x)$$

is strictly increasing for $s > 0$. Since $\{u_n\}$ is a Cerami sequence of I and the Monotonicity of $\frac{k(x,s)}{s}$, we can conclude

$$\begin{aligned}
c &= I(u_n) + o_n(1) \\
&= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle + o_n(1) \\
&= \int_{\mathbb{R}^N} \left(\frac{1}{2}k(x,u_n)u_n - K(x,u_n)\right)dx + o_n(1) \\
&\geq \int_{\mathbb{R}^N} \left(\frac{1}{2}k(x,t_n u_n)t_n u_n - K(x,t_n u_n)\right)dx + o_n(1) \\
&= I(t_n u_n) - \frac{1}{2}\langle I'(t_n u_n), t_n u_n\rangle + o_n(1) \\
&= I(t_n u_n) + o_n(1) \\
&\geq 8c,
\end{aligned}$$

which is a contradiction.

If $\beta > 0$, by the definition of β, there is $z_n \in \mathbb{R}^N$ such that

$$\frac{\beta}{2} < \int_{B_1(z_n)} v_n^2 dx.$$

If z_n is bounded, there exists $R > 0$ such that

$$\frac{\beta}{2} < \int_{B_R(0)} v_n^2 dx,$$

which is a contradiction with $v_n \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$.

If z_n is unbounded, up to a subsequence, $|z_n| \to \infty$. Let $w_n(x) := v_n(x + z_n) = \frac{u_n(x+z_n)}{\|u_n\|}$, we have

$$\frac{\beta}{2} < \int_{B_1(0)} w_n^2 dx. \tag{3.4}$$

There is a function $w \in E$ such that $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ and $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$. Moreover, by (3.4), one has $w(x) \neq 0$. Define $\Omega_{**} = \{x \in \mathbb{R}^N : w(x) > 0\}$, then $|\Omega_{**}| > 0$ and for a.e. $x \in \Omega_{**}$, we have

$$u_n(x) \to +\infty \quad \text{as} \quad \|u_n\| \to +\infty.$$

Since $F^{-1}(t)$ is strictly increasing for $t \geq 0$, by Lemma 2.1-(5), we can conclude that for a.e. $x \in \Omega_{**}$,

$$F^{-1}(u_n) \to +\infty \quad \text{as} \quad \|u_n\| \to +\infty.$$

Then, one has

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} G(x, F^{-1}(u_n))dx}{\|u_n\|^2} \\
&\quad \geq \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} G_0(x + z_n, F^{-1}(u_n(x + z_n)))dx}{\|u_n\|^2} \\
&\quad \geq \liminf_{n\to\infty} \int_{\Omega_{**}} \frac{G_0(x + z_n, F^{-1}(u_n(x + z_n)))}{|F^{-1}(u_n(x + z_n))|^2} \frac{|F^{-1}(u_n(x + z_n))|^2}{(u_n(x + z_n))^2} w_n^2 dx \\
&\quad = +\infty.
\end{aligned}$$

Hence

$$\begin{aligned}
0 &= \limsup_{n\to\infty} \frac{I(u_n)}{\|u_n\|^2} \\
&\leq \frac{1}{2} - \liminf_{n\to\infty} \frac{1}{\|u_n\|^2} \int_{\mathbb{R}^N} G(x, F^{-1}(u_n))dx
\end{aligned}$$

$$= -\infty,$$

this contradiction finished the proof. □

Lemma 3.5 *Suppose that conditions (f), (V) and* $(g_1)-(g_4)$ *are satisfied. If* $u \in \mathcal{N}$ *and* $I(u) = c$*, then* u *is a ground state solution of problem (1.1) (see* [13, 16]*).*

Proof of Theorem 1.1. From Lemma 3.1, we see that I satisfies the mountain pass geometry. Then, we can get a Cerami sequence on level c, where $c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t))$. We invoke Lemma 3.2 to get $c = \inf_{u\in\mathcal{N}} I(u)$. Applying Lemma 3.4, the $(C)_c$ sequence is bounded. Then, we may get, up to a subsequence, $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. By using the Lebesgue dominated convergence theorem, through the standard discussion, we can get that

$$0 = \langle I'(u_n), \phi\rangle + o_n(1) = \langle I'(u), \phi\rangle,$$

for any $\phi \in C_0^\infty(\mathbb{R}^N)$, i.e. u is a weak solution of problem (1.1).

(i) The case $u \neq 0$. Since u is a weak solution of problem (1.1), $I(u) \geq c$. By Lemma 2.1-(6), (2.1) and the Fatou lemma, one has

$$\begin{aligned}
c &= \liminf_{n\to\infty}\left(I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle\right)\\
&= \liminf_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^N} V(x)\left[|F^{-1}(u_n)|^2 - \frac{F^{-1}(u_n)}{f(F^{-1}(u_n))}u_n\right]dx\right.\\
&\quad\left.+\int_{\mathbb{R}^N}\left(\frac{g(x, F^{-1}(u_n))}{2f(F^{-1}(u_n))}u_n - G(x, F^{-1}(u_n))\right)dx\right]\\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} V(x)\left[|F^{-1}(u)|^2 - \frac{F^{-1}(u)}{f(F^{-1}(u))}u\right]dx\\
&\quad+\int_{\mathbb{R}^N}\left(\frac{g(x, F^{-1}(u))}{2f(F^{-1}(u))}u - G(x, F^{-1}(u))\right)dx\\
&= I(u) - \frac{1}{2}\langle I'(u), u\rangle\\
&= I(u).
\end{aligned}$$

Hence, $I(u) = c$ and $I'(u) = 0$, which implies that u is a ground state solution of problem (1.1).

(ii) The case $u = 0$. Define

$$\beta := \limsup_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_1(z)} u_n^2 dx.$$

If $\beta = 0$, by the Lions lemma [24] (Lemma 1.21), we get $u_n \to 0$ in $L^\alpha(\mathbb{R}^N)$ for $\alpha \in (2, 2^*)$. It is implied by (2.1) and condition (f) that

$$0 \le t^2 - \frac{tF(t)}{f(t)} \to 0(t \to 0). \tag{3.5}$$

Combining (3.5) with (2.7) and Lemma 2.1-(3), we obtain

$$\begin{aligned} c &= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle + o_n(1) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} V(x)\left[|F^{-1}(u_n)|^2 - \frac{F^{-1}(u_n)}{f(F^{-1}(u_n))}u_n\right]dx + \int_{\mathbb{R}^N} \frac{1}{2}\frac{g(x, F^{-1}(u_n))}{f(F^{-1}(u_n))}u_n dx \\ &\quad - \int_{\mathbb{R}^N} G(x, F^{-1}(u_n))dx \\ &\le o_n(1) + \frac{1}{2}\int_{\mathbb{R}^N}\left[\delta|F^{-1}(u_n)| + \delta|F^{-1}(u_n)|^{2^*-1} + C_\delta|F^{-1}(u_n)|^{\alpha-1}\right]u_n dx \\ &\le o_n(1) + \frac{\delta}{2}\int_{\mathbb{R}^N}|u_n|^2dx + \frac{\delta}{2}\int_{\mathbb{R}^N}|u_n|^{2^*}dx + \frac{C_\delta}{2}\int_{\mathbb{R}^N}|u_n|^\alpha dx \\ &= o_n(1) + C_{20}\delta \\ &\to 0(\delta \to 0). \end{aligned}$$

A contradiction, thus $\beta > 0$. By the definition of β, up to a subsequence, there exist $R > 0$ and $z_n \in \mathbb{Z}^N$ such that

$$\int_{B_R(0)} u_n^2(x + z_n)dx = \int_{B_R(z_n)} u_n^2(x)dx > \frac{\beta}{2}.$$

If z_n is bounded, there is $R' > 0$ such that

$$\int_{B_{R'}(0)} u_n^2 dx \ge \int_{B_R(z_n)} u_n^2 dx > \frac{\beta}{2},$$

which contradicts with $u_n \to u = 0$ in $L^2_{loc}(\mathbb{R}^N)$. Thus, z_n is unbounded, going if necessary to a subsequence, $|z_n| \to \infty$. Let $w_n(x) := u_n(x + z_n)$, then there exists a function $w \in E\backslash\{0\}$ such that $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^2_{loc}(\mathbb{R}^N)$ and $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$.

It follows from (2.9) and (2.10) that, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned} 0 &= \langle I'(u_n), \varphi(x - z_n)\rangle + o_n(1) \\ &= \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla\varphi(x - z_n)dx + \int_{\mathbb{R}^N} V(x)\frac{F^{-1}(u_n)}{f(F^{-1}(u_n))}\varphi(x - z_n)dx \\ &\quad - \int_{\mathbb{R}^N} \frac{g(x, F^{-1}(u_n))}{f(F^{-1}(u_n))}\varphi(x - z_n)dx + o_n(1) \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla \varphi(x - z_n)dx + \int_{\mathbb{R}^N} V_0(x)\frac{F^{-1}(u_n)}{f(F^{-1}(u_n))}\varphi(x - z_n)dx \\
&\quad - \int_{\mathbb{R}^N} \frac{g_0(x, F^{-1}(u_n))}{f(F^{-1}(u_n))}\varphi(x - z_n)dx + o_n(1) \\
&= \int_{\mathbb{R}^N} \nabla w_n \cdot \nabla \varphi dx + \int_{\mathbb{R}^N} V_0(x)\frac{F^{-1}(w_n)}{f(F^{-1}(w_n))}\varphi dx \\
&\quad - \int_{\mathbb{R}^N} \frac{g_0(x, F^{-1}(w_n))}{f(F^{-1}(w_n))}\varphi dx + o_n(1) \\
&= \int_{\mathbb{R}^N} \nabla w \cdot \nabla \varphi dx + \int_{\mathbb{R}^N} V_0(x)\frac{F^{-1}(w)}{f(F^{-1}(w))}\varphi dx - \int_{\mathbb{R}^N} \frac{g_0(x, F^{-1}(w))}{f(F^{-1}(w))}\varphi dx
\end{aligned}
$$

i.e. w is a weak solution of the periodic equation (1.9).

On the one hand, by Lemma 3.1, for any $u \in E\backslash\{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathscr{N}$. Moreover, the maximum of $I(tu)$ for $t \geq 0$ is achieved at t_u. Thanks to $V(x) \leq V_0(x)$ and $G(x, s) \geq G_0(x, s)$, we obtain

$$
c \leq I(t_u u) \leq I_0(t_u u) \leq \max_{t>0} I_0(tu),
$$

hence $c \leq \inf_{u\in E} \max_{t>0} I_0(tu)$. It follows from Remark 3.3 that $c \leq c_0$.

On the other hand, by (2.2), (2.3), $V(x) \leq V_0(x)$, $g(x, s) \geq g_0(x, s)$, Lemma 2.1-(6), (2.1) and the Fatou lemma, we get

$$
\begin{aligned}
c &= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle + o_n(1) \\
&= \frac{1}{2}\int_{\mathbb{R}^N} V(x)|F^{-1}(u_n)|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} V(x)\frac{F^{-1}(u_n)}{f(F^{-1}(u_n))}u_n dx \\
&\quad + \int_{\mathbb{R}^N} \frac{1}{2}\frac{g(x, F^{-1}(u_n))}{f(F^{-1}(u_n))}u_n dx - \int_{\mathbb{R}^N} G(x, F^{-1}(u_n))dx + o_n(1) \\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} V_0(x)|F^{-1}(u_n)|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} V_0(x)\frac{F^{-1}(u_n)}{f(F^{-1}(u_n))}u_n dx \\
&\quad + \int_{\mathbb{R}^N} \frac{1}{2}\frac{g_0(x, F^{-1}(u_n))}{f(F^{-1}(u_n))}u_n dx - \int_{\mathbb{R}^N} G_0(x, F^{-1}(u_n))dx + o_n(1) \\
&= \frac{1}{2}\int_{\mathbb{R}^N} V_0(x)\left[|F^{-1}(w_n)|^2 - \frac{F^{-1}(w_n)}{f(F^{-1}(w_n))}w_n\right]dx \\
&\quad + \int_{\mathbb{R}^N} \left[\frac{g_0(x, F^{-1}(w_n))}{2f(F^{-1}(w_n))}w_n - G_0(x, F^{-1}(w_n))\right]dx + o_n(1) \\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} V_0(x)\left[|F^{-1}(w)|^2 - \frac{F^{-1}(w)}{f(F^{-1}(w))}w\right]dx \\
&\quad + \int_{\mathbb{R}^N} \left[\frac{g_0(x, F^{-1}(w))}{2f(F^{-1}(w))}w - G_0(x, F^{-1}(w))\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&= I_0(w) - \frac{1}{2}\langle I_0'(w), w\rangle \\
&= I_0(w) \geq c_0.
\end{aligned}
$$

Hence $I_0(w) = c_0 = c$. Lemma 3.1 implies that there is a unique $t_w > 0$ such that $t_w w \in \mathscr{N}$. Then, we get

$$c \leq I(t_w w) \leq I_0(t_w w) \leq I_0(w) = c,$$

i.e. c is achieved by $t_w w$. It follows from Lemma 3.5 that $t_w w$ is a ground state solution of problem (1.1).

From (i), (ii), we can obtain that problem (1.1) has a nonnegative ground state solution $u \in E$. Furthermore, the maximum principle implies $u > 0$, this ends the proof. □

Data Availability No data, models, or code were generated or used during the study.

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