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Hyperbolicity, Shadowing, and Bounded Orbits

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Abstract

We show that a linear homeomorphism with the shadowing property of a Banach space is hyperbolic if and only if the set of points with bounded orbit is closed. The proof is based on an auxiliary type of shadowing called *bounded shadowing property*. We give examples where our result can be applied.

Keywords Linear homeomorphism · Banach space · Shadowing property

Mathematics Subject Classification Primary 54G99; Secondary 37B05

1 Introduction

The relationship between hyperbolicity and shadowing for linear homeomorphisms on Banach spaces has been explored in the literature. For instance, every hyperbolic linear homeomorphism has the shadowing property (and conversely in finite dimension [12]). Examples of non-hyperbolic linear homeomorphism with the shadowing property were found recently in [2]. Also recently, it was proved a fundamental equivalence between the following two properties for linear homeomorphisms L (Theorem 11 in [1]):

(a) L is hyperbolic
(b) L is expansive and has the shadowing property.

The goal of this paper is to further explore this relationship. Indeed, we prove that a linear homeomorphism with the shadowing property of a Banach space L is hyperbolic

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if and only if the set of points with bounded orbit is closed. The proof is based on an auxiliary type of shadowing called *bounded shadowing property*. Some examples will be given. Let us state our result in a precise way.

Hereafter X will denote a (complex) Banach space. $GL(X)$ denotes the set of linear homeomorphisms $L : X \to X$. The spectral radius operation is denoted by $r(\cdot)$. All Banach spaces under consideration will be *nontrivial*, i.e. different from 0.

Definition 1 We say that $L \in GL(X)$ is *hyperbolic* if its spectrum does not intersect the unitary complex circle. Equivalently, if there is a direct sum decomposition $X = S \oplus U$ formed by closed subspaces S and U such that

$$L(S) = S, \quad L^{-1}(U) = U, \quad r(L|_S) < 1 \quad \text{and} \quad r(L^{-1}|_U) < 1. \tag{1}$$

Definition 2 We say that L has the *shadowing property* if for every $\epsilon > 0$ there is $\delta > 0$ such that for every sequence $(x_n)_{n\in\mathbb{Z}}$ satisfying $\|L(x_n) - x_{n+1}\| \leq \delta$ $(\forall n \in \mathbb{Z})$, there is $x \in X$ such that $\|L^n(x) - x_n\| \leq \epsilon$ $(\forall n \in \mathbb{Z})$.

These definitions are very important in the theory of dynamical systems [9]. We will relate them through the following result. Denote by $E^c = E^c(L)$ the set of vectors with bounded orbits under L namely

$$E^c = \left\{ x \in X : \sup_{n\in\mathbb{Z}} \|L^n(x)\| < \infty \right\}.$$

Clearly E^c is a subspace of X which is *invariant*, i.e. $L(E^c) = E^c$. The topological version of E^c will be considered in (2).

With these definitions we can state our result.

Theorem 3 *A linear homeomorphism with the shadowing property of a Banach space $L : X \to X$ is hyperbolic if and only if E^c is closed.*

This paper is organized as follows. In Sect. 2 we introduce an useful variation of shadowing called *bounded shadowing property*. Some properties of this kind of shadowing will be proved. As a consequence we obtain Theorem 3. In Sect. 3 we give some short examples where this result can be applied. Some problems posed by anonymous referees will be presented.

2 Bounded Shadowing Property: Proof of Theorem 3

It is very important in computer simulations that approximate trajectories be approximated by true ones. When this happen we say that the system has the *shadowing property*. The shadowing property was discovered by Bowen in his study of the omega-limit sets for Axiom A diffeomorphisms [3]. Its precise definition is as follows: We say that a homeomorphism of a metric space $g : Y \to Y$ has the *shadowing property* if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-*pseudo orbit* (i.e. sequence $(x_n)_{n\in\mathbb{Z}}$ with $d(g(x_n), x_{n+1}) \leq \delta$ for $n \in \mathbb{Z}$) can be ϵ-*shadowed* (i.e. there is $x \in Y$ such that $d(g^n(x), x_n) \leq \epsilon$ for $n \in \mathbb{Z}$).

Several variations of this property have been proposed in the literature. Here we will consider a variation for which not every pseudo orbit but the bounded ones can be shadowed.

Definition 4 A homeomorphism $g : Y \to Y$ has the *bounded shadowing property* if for every $\epsilon > 0$ there is $\delta > 0$ such that every *bounded* δ-pseudo orbit can be ϵ-shadowed.

Clearly, every homeomorphism with the shadowing property has the bounded shadowing property. These properties are equivalent on bounded spaces (like the compact ones). We do not know however if they are equivalent in general. Nevertheless, we will prove that these properties are equivalent for linear homeomorphisms on finite dimensional Banach spaces (Proposition 11 below).

The main motivation for this definition is the elementary lemma below. Given $g : Y \to Y$ as above we define

$$E^c = \{y \in Y : \text{ the orbit } (g^n(y))_{n\in\mathbb{Z}} \text{ of } y \text{ is bounded}\}. \tag{2}$$

Clearly E^c is an invariant set, i.e. $g(E^c) = E^c$.

Lemma 5 *If a homeomorphism of a metric space $g : Y \to Y$ has the bounded shadowing property, then the restriction $g|_{E^c} : E^c \to E^c$ has the bounded shadowing property too.*

Proof Let $\epsilon > 0$ be given and δ be from the bounded shadowing property of g. Let $(x_n^c)_{n\in\mathbb{Z}}$ be a bounded sequence in E^c such that $d(g(x_n^c), x_{n+1}^c) \le \delta$ for every $n \in \mathbb{Z}$. Then, the bounded shadowing property of g provides $y \in Y$ such that

$$d(g^n(y), x_n^c) \le \epsilon, \qquad \forall n \in \mathbb{Z}.$$

Since $(x_n^c)_{n\in\mathbb{Z}}$ is bounded, there is $\Delta > 0$ such that

$$d(x_1^c, x_n^c) \le \Delta, \qquad \forall n \in \mathbb{Z}.$$

Then,

$$d(g^n(y), x_1^c) \le d(g^n(y), x_n^c) + d(x_n^c, x_1^c) \le \epsilon + \Delta, \qquad \forall n \in \mathbb{Z},$$

so $(g^n(y))_{n\in\mathbb{Z}}$ is bounded. Then, $y \in E^c$ and so $g|_{E^c}$ has the bounded shadowing property. This completes the proof. □

To apply this lemma we will need two more lemmas. The first is the invariance of the bounded shadowing property under certain topological conjugacies.

Lemma 6 *Let Y, Z be metric spaces and $H : Z \to Y$ be a Lipeomorphism (i.e. Lipschitz homeomorphism with Lipschitz inverse). If $P : Z \to Z$ is a homeomorphism with the bounded shadowing property, then so does $H \circ P \circ H^{-1} : Y \to Y$.*

Proof Take $\epsilon > 0$. Since every Lipschitz map is uniformly continuous, there is $\epsilon' > 0$ such that

$$a, b \in Z \text{ and } d(a, b) \leq \epsilon' \qquad \text{implies} \qquad d(H(a), H(b)) \leq \epsilon.$$

For this ϵ' we take δ' from the bounded shadowing property of P and, for this δ', we take $\delta > 0$ such that

$$c, d \in Y \text{ and } d(c, d) \leq \delta \qquad \text{implies} \qquad d(H^{-1}(c), H^{-1}(d)) \leq \delta'.$$

Now, let $(y_n)_{n\in\mathbb{Z}}$ be a bounded sequence in Y such that

$$d(H \circ P \circ H^{-1}(y_n), y_{n+1}) \leq \delta, \qquad \forall n \in \mathbb{Z}.$$

Then, $d(P(H^{-1}(y_n)), H^{-1}(y_{n+1})) \leq \delta'$ for $n \in \mathbb{Z}$. Since

$$d(H^{-1}(y_n), H^{-1}(y_0)) \leq Lip(H^{-1})d(y_n, y_0)$$

(where $Lip(H^{-1})$ is the Lipschitz constant of H^{-1}) and $(y_n)_{n\in\mathbb{Z}}$ is bounded, we also have that $(H^{-1}(y_n))_{n\in\mathbb{Z}}$ is bounded. Then, the bounded shadowing property provides $z \in Z$ such that

$$d(P^n(z), H^{-1}(y_n)) \leq \epsilon', \qquad \forall n \in \mathbb{Z}.$$

It follows that $d(H(P^n(z)), y_n) \leq \epsilon$ so $y = H(z)$ satisfies

$$d((H \circ P \circ H^{-1})^n(y), y_n) = d(H(P^n(z)), y_n) \leq \epsilon, \qquad \forall n \in \mathbb{Z},$$

proving the result. □

To state the next lemma we need the following concept.

Definition 7 A *linear isometry* of a Banach space X is a linear homeomorphism $L : X \to X$ such that $\|L(x)\| = \|x\|$ for every $x \in X$.

It is known that a linear isometry of a Banach space does not have the shadowing property (Lemma 2 in [10]). Replacing shadowing by bounded shadowing in this assertion we obtain the following one.

Lemma 8 *A linear isometry of a Banach space does not have the bounded shadowing property.*

Proof Suppose that there is a Banach space X and $L \in GL(X)$ with the bounded shadowing property. Take $\delta > 0$ from this property for $\epsilon = 1$, and let $x \in X$ be an arbitrary point. Choose a sequence $0 = q_0, q_1, \cdots, q_r = x$ such that $\|q_{i+1} - q_i\| \leq \delta$ for $0 \leq i \leq r-1$. Define $(p_i)_{i\in\mathbb{Z}}$ by

$$p_i = \begin{cases} L^i(q_0), \text{ if } i < 0 \\ L^i(q_i), \text{ if } 0 \leq i \leq r \\ L^i(q_r), \text{ if } r < i. \end{cases}$$

Since

$$\sup_{i<0} \|L^i(q_0)\| = \|q_0\| < \infty \quad \text{and} \quad \sup_{i>r} \|L^i(q_r)\| = \|q_r\| < \infty,$$

the sequence $(p_i)_{i\in\mathbb{Z}}$ is bounded. Moreover,

$$\|L(p_i) - p_{i+1}\| = \|L^{i+1}(q_i) - L^{i+1}(q_{i+1})\| = \|q_{i+1} - q_i\| \leq \delta, \quad \forall 0 \leq i \leq r-1,$$

so $\|L(p_i) - p_{i+1}\| \leq \delta$ for $i \in \mathbb{Z}$ thus there is $z \in X$ such that

$$\|L^i(z) - p_i\| \leq 1, \qquad \forall i \in \mathbb{Z}.$$

Since L is a linear isometry,

$$\|z - q_i\| = \|z - L^{-i}(p_i)\| = \|L^i(z) - p_i\| \leq 1, \qquad \forall 0 \leq i \leq r.$$

In particular, $\|z\| = \|z - 0\| = \|z - q_0\| \leq 1$ and $\|z - x\| = \|z - q_r\| \leq 1$ so

$$\|x\| \leq \|z\| + \|z - x\| \leq 2, \qquad \forall x \in X$$

hence $X = \{0\}$ a contradiction. This completes the proof. □

The key to prove Theorem 3 is given below. To state it we will use the following definition.

Definition 9 A linear homeomorphism of a Banach space $L : X \to X$ is *expansive* if for every $x \in X$ with $\|x\| = 1$ there is $n \in \mathbb{Z}$ such that $\|L^n(x)\| \geq 2$.

With this definition we can state the following proposition.

Proposition 10 *If $L \in GL(X)$ has the bounded shadowing property and E^c is closed, then L is expansive.*

Proof By Item (c) of Proposition 19 in [2] it suffices to prove $E^c = \{0\}$. Suppose by contradiction that $E^c \neq \{0\}$. Clearly E^c is a subspace and so E^c is a closed subspace with $L(E^c) = E^c$. Then, the restriction $L|_{E^c} \in GL(E^c)$ is well defined. Define the Banach space $Z = (E^c, \|\cdot\|)$ where $\|\cdot\|$ is the induced norm from X and $P \in GL(Z)$ is just the restriction $P = L|_{E^c}$.

Since L has the bounded shadowing property, P has the bounded shadowing property by Lemma 5. We now construct a second Banach space Y as follows. By the definition of E^c one has

$$\sup_{n\in\mathbb{Z}} \|P^n(z)\| < \infty, \qquad \forall z \in Z.$$

Then,

$$\sup_{n\in\mathbb{Z}} \|P^n\| < \infty \tag{3}$$

by the Banach-Steinhouse Theorem. Now define the new norm $\|\cdot\|'$ in Z by

$$\|z\|' = \sup_{n\in\mathbb{Z}} \|L^n(z)\|, \qquad \forall z \in Z.$$

Clearly $\|z\| \leq \|z\|'$ for $z \in Z$. Define

$$M = \sup_{n\in\mathbb{Z}} \|P^n\|.$$

Then, (3) implies $0 < M < \infty$. Moreover, $\|z\|' \leq M\|z\|$ for all $z \in Z$ namely

$$\|z\| \leq \|z\|' \leq M\|z\|, \qquad \forall z \in Z. \tag{4}$$

Put $Y = (Z, \|\cdot\|')$. It follows from (4) that Y is a Banach space.

Next we define $H : Z \to Y$ as the identity of Z. It follows from (4) that H is a Lipeomorphism. Since P has the bounded shadowing property, so does $H \circ P \circ H^{-1}$ by Lemma 6. Moreover,

$$\|H \circ P \circ H^{-1}(z)\|' = \|L(z)\|' = \sup_{n\in\mathbb{Z}} \|L^n(L(z))\| = \sup_{n\in\mathbb{Z}} \|L^n(z)\| = \|z\|',$$

for every $z \in Y = E$ so $H \circ P \circ H^{-1}$ is a linear isometry of Y. This contradicts Lemma 8 completing the proof. □

Now we prove our result.

Proof of Theorem 3 Let X be a Banach space and $L \in GL(X)$ with the shadowing property such that E^c is closed. In particular, L has the bounded shadowing property. So, L is expansive by Proposition 10. Thus, L is hyperbolic by [1]. This completes the proof. □

Now, we present three results about linear homeomorphisms with the bounded shadowing property having their own interest. The first is the equivalence between the bounded shadowing and shadowing properties for linear homeomorphisms on finite dimensional Banach spaces. More precisely, we have the following result.

Proposition 11 *A linear homeomorphism of a finite dimensional Banach space has the bounded shadowing property if and only if it has the shadowing property.*

Proof We just need to prove the necessity. Let $L \in GL(X)$ where X is a finite dimensional Banach space be with the bounded shadowing property. Since X is finite, E^c is closed and so L is expansive by Proposition 10. Since hyperbolicity and expansivity are equivalent for finite dimensional linear homeomorphisms [12], we have that L is hyperbolic. Since hyperbolic operators have the shadowing property, we are done. □

Proposition 10 suggests the study of expansive linear homeomorphisms with the bounded shadowing property on Banach spaces. By [1] it is reasonable is to believe that they are hyperbolic. If this were true, then such homeomorphisms would be uniformly expansive. Recall that a linear homeomorphism $L \in GL(X)$ is *uniformly expansive* if there is $N \in \mathbb{N}$ such that $\|L^N(x)\| \geq 2\|x\|$ or $\|L^{-N}(x)\| \geq 2\|x\|$ for every $x \in X$. It is well known that a linear homeomorphism is uniformly expansive if and only if its approximate point spectrum does not intersect the unit circle [8]. As in Theorem 27 of [2] we obtain the following result.

Proposition 12 *An expansive linear homeomorphism with the bounded shadowing property of a Banach space is uniformly expansive.*

Proof Let $L \in GL(X)$ be expansive with the bounded shadowing property for some Banach space X. If L were not uniformly expansive, then its approximate point spectrum would intersect the unit complex circle at some λ (see [8]). We take δ from the bounded shadowing property for $\epsilon = \frac{1}{2}$. Since λ is an approximate eigenvalue, there is $x \in X$ with $\|x\| = 1$ such that $\|L(x) - \lambda x\| < \delta$. By taking $x_n = \lambda^n x$ for $n \in \mathbb{Z}$ one has $\|L(x_n) - x_{n+1}\| = |\lambda^n|\|L(x) - \lambda x\| < \delta$ for every $n \in \mathbb{Z}$. Since $\|x_n\| = \|\lambda^n x\| = \|x\| = 1$ for $n \in \mathbb{Z}$ we also have that $(x_n)_{n\in\mathbb{Z}}$ is bounded. Then, the bounded shadowing property provides $y \in X$ such that $\|L^n(y) - x_n\| < \frac{1}{2}$ for $n \in \mathbb{Z}$. Therefore, $\frac{1}{2} < \|L^n(y)\| < \frac{3}{2}$ for all $n \in \mathbb{Z}$ contradicting that L is expansive (c.f. [2]). This completes the proof. □

The following is a corollary of the proof of Proposition 10. Recall that a homeomorphism of a metric space $g : Y \to Y$ is *equicontinuous* if the family $\{g^n : n \in \mathbb{Z}\}$ is equicontinuous i.e. for every $\epsilon > 0$ there is $\delta > 0$ such that if $y, y' \in Y$ and $d(y, y') \leq \delta$, then $d(g^n(y), g^n(y')) < \epsilon$ for every $n \in \mathbb{Z}$.

It was proved in [10] that an equicontinuous linear homeomorphism of a Banach space does not have the shadowing property. A direct corollary of the methods above is that the same result is true for linear homeomorphisms with the bounded shadowing property. More precisely, we have the following result.

Corollary 13 *An equicontinuous linear homeomorphism of a Banach space does not have the bounded shadowing property.*

Proof Let L be the operator in the statement. Since L is equicontinuous, we can see that

$$M = \sup_{n\in\mathbb{Z}} \|L^n\| < \infty.$$

This allows to reproduce the proof of Proposition 10 to obtain a Lipeomorphism conjugating L to a linear isometry. Now we apply Lemmas 6 and 8 to conclude the proof. □

3 Examples and Questions

First we present some short examples where our result can be applied. The first is precisely the equivalence between shadowing and hyperbolicity in finite dimension [12]: This follows form our theorem and the fact that every finite dimensional subspace is closed.

Example 14 A linear homeomorphism of a finite dimensional Banach space is hyperbolic if and only if it has the shadowing property.

Another consequence is as follows. Given a Banach space X and $L \in GL(X)$ we define

$$E^{cs} = \left\{ x \in X : \sup_{n \geq 0} \|L^n(x)\| < \infty \right\}, E^{cu} = \left\{ x \in X : \sup_{n \geq 0} \|L^{-n}(x)\| < \infty \right\}.$$

Clearly E^{cs} and E^{cu} are subspaces which are invariant, i.e. $L(E^{cs}) = E^{cs}$ and $L(E^{cu}) = E^{cu}$. With these definitions we can state simple proposition below.

Proposition 15 *If L is hyperbolic, then L has the shadowing property and both E^{cs} and E^{cu} are closed.*

Proof Since L is hyperbolic, L has the shadowing property and is uniformly expansive. From this expansivity and the characterization of E^{cs} and E^{cu} given in Proposition 2 of [1] we obtain the result. □

As a corollary of our result we will obtain the converse of the above proposition.

Example 16 A linear homeomorphism of a Banach space $L : X \to X$ is hyperbolic if and only if it has the shadowing property and both E^{cs} and E^{cu} are closed.

Proof The necessity follows from the previous proposition. For the sufficiency we note that if E^{cu} and E^{cu} are closed, then $E^c = E^{cs} \cap E^{cu}$ also is and then L is hyperbolic by Theorem 3. □

A third example is related to the new form of hyperbolicity for linear homeomorphisms defined as follows.

Definition 17 We say that $L \in GL(X)$ is *generalized hyperbolic* if there is a direct sum decomposition $X = S \oplus U$ formed by closed subspaces S and U such that

$$L(S) \subset S, \quad L^{-1}(U) \subset U, \quad r(L|_S) < 1 \quad \text{and} \quad r(L^{-1}|_U) < 1.$$

Notice that this definition is obtained by replacing the equalities in (1) by inclusions. In particular, every hyperbolic linear homeomorphism is generalized hyperbolic. Further remarks are as follows.

Remark 18 (a) This concept was implicitly introduced in 2018 ([2]). It reappeared with the name "generalized hyperbolic" in [4]. It is known that every generalized hyperbolic linear homemorphism has the shadowing property [2]. Also, a number of interesting properties were obtained in [4]. All known examples of linear homeomorphisms with the shadowing property are generalized hyperbolic. This motivates the question if, conversely, every linear homeomorphism with the shadowing property is generalized hyperbolic [6].

(b) The question was answered positively for certain operators. These include the weighted shifts [1] (to be defined below) and also a large class of L^p-operators [6, 7].

About these homeomorphism we obtain the following example.

Example 19 A generalized hyperbolic linear homeomorphism L is hyperbolic if and only if E^c is closed.

Theorem 3 motivates the analysis of E^c in specific examples. The natural ones are the weighted shifts.

Given a Banach space X we define $X^{\mathbb{Z}}$ as the set of maps $x : \mathbb{Z} \to X$. If $1 \leq p < \infty$, we define

$$l^p(X) = \left\{ x \in X^{\mathbb{Z}} : \sum_{n\in\mathbb{Z}} \|x(n)\|^p < \infty \right\}.$$

It follows that $l^p(X)$ is a Banach space if endowed with the norm

$$\|x\| = \left(\sum_{n\in\mathbb{Z}} \|x(n)\|^p \right)^{\frac{1}{p}}.$$

Fix $\omega \in \mathbb{C}^{\mathbb{Z}}$ (the set of maps $\omega : \mathbb{Z} \to \mathbb{C}$) with

$$0 < \inf_{n\in\mathbb{Z}} |\omega(n)| \leq \sup_{n\in\mathbb{Z}} |\omega(n)| < \infty. \tag{5}$$

Define the linear homeomorphisms $B_\omega, F_\omega : l^p(X) \to l^p(X)$ by

$$B_\omega(x)(n) = \omega(n+1)x(n+1), \quad F_\omega(x)(n) = \omega(n-1)x(n-1)$$

These operators are called *backward* (resp. *forward*) *weighted shifts*. They are conjugated each other so we concentrate on B_ω only. We have the following example for $X = \mathbb{C}$.

Example 20 If B_w has the shadowing property, then either $E^c = \{0\}$ or E^c is dense in $l^p(X)$.

Proof If $E^c \neq \{0\}$, B_w is not hyperbolic so

$$\lim_{n\to\infty}\left(\sup_{k\in\mathbb{N}}|\omega(-k)\omega(-k-1)\cdots\omega(-k-n)|\right)^{\frac{1}{n}} < 1$$

and

$$\lim_{n\to\infty}\left(\inf_{k\in\mathbb{N}}|\omega(k)\omega(k+1)\cdots\omega(k+n)|\right)^{\frac{1}{n}} > 1$$

by Theorem 18 in [1]. This implies that E^c contains those $x \in l^p(X)$ such that $x(n) = 0$ for all but finitely many $n \in \mathbb{Z}$. Therefore, E^c is dense in $l^p(X)$. □

Anonymous referees asked us about the dynamics of non-hyperbolic linear homeomorphism with the shadowing property on Banach spaces $L : X \to X$. More precisely, they asked if $L|_{\overline{E^c}}$ is topologically transitive or chaotic or frequently hypercyclic or if $\overline{E^c} = X$ where $\overline{E^c}$ is the closure of E^c.

Recall that a linear homeomorphism of a Banach space $T : Y \to Y$ is *frequently hypercyclic* provided there exists a vector y such that for every nonempty open subset U of Y, the set of integers n such that $T^n(y)$ belongs to U has positive lower density. All such homeomorphisms are *topologically transitive*, i.e. for every pair (U, V) of nonempty open subsets of Y there exists an integer n such that $T^n(U) \cap V \neq \emptyset$.

We do not have answers for such questions yet. However, we can exhibit examples where positive answers hold.

Example 21 By Example 20 and Theorem B in [2] there is Banach space X and an open set of frequently hypercyclic (hence topologically transitive) linear homeomorphisms with the shadowing property $\mathcal{O}$ of X such that E^c is dense in X for every $L \in \mathcal{O}$.

Next we consider the *multiplication operators*. Let (X, μ) be a measure space, and $g \in L^\infty(X, \mu)$. For any $1 \leq p < \infty$ we have the linear map $M_g : L^p(X, \mu) \to L^p(X, \mu)$ defined by $M_g(f) = g \cdot f$. Since $g \in L^\infty(X, \mu)$, M_g is bounded with $\|M_g\|_p \leq \|g\|_\infty$. Also $M_g \in GL(L^p(X, \mu))$ if and only if $1/g \in L^\infty(X, \mu)$ in whose case $(M_g)^{-1} = M_{1/g}$. This will be assumed in what follows. In the following lemma we will compute E^c for the multiplication operators M_g.

Lemma 22

$$E^c = \{f \in L^p(X, \mu) : \mu(\{x \in X : f(x) \neq 0 \textit{ and } |g(x)| \neq 1\}) = 0\}.$$

Proof Given $f \in L^p(X, \mu)$ and $k \in \mathbb{N}$ we define

$$C_k^+(f) = \left\{x \in X : |f(x)| \geq \frac{1}{k} \text{ and } |g(x)| \geq 1 + \frac{1}{k}\right\} \quad \text{and}$$

$$C_k^-(f) = \left\{x \in X : |f(x)| \geq \frac{1}{k} \text{ and } |g(x)| \leq 1 - \frac{1}{k}\right\}.$$

These are disjoint sets and if $C_k = C_k^+ \cup C_k^-$ then

$$\{x \in X : f(x) \neq 0 \text{ and } |g(x)| \neq 1\} = \bigcup_{k=1}^{\infty} C_k. \tag{6}$$

Now we have

$$\mu(C_k^+)\frac{1}{k^p}\left(1+\frac{1}{k}\right)^{pn} \leq \int_{C_k^+} |g(x)|^{pn}|f(x)|^p d\mu(x) \leq \|M_g^n(f)\|_p^p.$$

Likewise,

$$\mu(C_k^-)\frac{1}{k^p}\left(1-\frac{1}{k}\right)^{-pn} \leq \|M_g^n(f)\|_p^p.$$

Then, if $f \in E^c$ we get

$$\mu(C_k^+) \leq \left(\sup_{n\in\mathbb{Z}} \|M_g^n(f)\|\right)^p k^p \left(1+\frac{1}{k}\right)^{-pn} \to 0 \quad \text{as} \quad n \to \infty$$

so $\mu(C_k^+) = 0$. Likewise $\mu(C_k^-) = 0$ for every $k \in \mathbb{N}$. Then, $\mu(C_k) = 0$ for every $k \in \mathbb{N}$ so $\mu(\{x \in X : f(x) \neq 0 \text{ and } |g(x)| \neq 1\}) = 0$ by (6).

Conversely, suppose that $f \in L^p(X,\mu)$ satisfies $\mu(\{x \in X : f(x) \neq 0 \text{ and } |g(x)| \neq 1\}) = 0$. It follows that $f(x) \neq 0 \Rightarrow |g(x)| = 1$. So,

$$\|M_g^n(f)\|_p^p = \int_X |g(x)|^{pn}|f(x)|^p d\mu(x) = \int_X |f(x)|^p d\mu(x) = \|f\|_p^p, \ \forall n \in \mathbb{Z},$$

proving $f \in E^c$. This completes the proof. □

With this lemma we obtain the following one yet about M_g.

Lemma 23 *E^c is closed.*

Proof Take a sequence $f_l \in E^c$ and $f \in L^p(X,\mu)$ such that $f_l \to f$ in L^p. If $f \notin E^c$, we can assume by Lemma 22 that $\mu(C) > 0$ where

$$C = \left\{x \in X : |f(x)| \geq \frac{1}{k} \text{ and } |g(x)| \geq 1 + \frac{1}{k}\right\}$$

for some $k \in \mathbb{N}$. Fix $0 < \epsilon < \frac{1}{k}$ and $k' > k$ such that $\frac{1}{k'} < \frac{1}{k} - \epsilon$.

Since $f_l \to f$ in L^p, $f_l \to f$ in measure too. Then, there is $l \in \mathbb{N}$ such that

$$\mu(C \cap \{|f_l - f| \geq \epsilon\}) < \frac{\mu(C)}{2}.$$

Now take $x \in C \cap \{|f_l - f| < \epsilon\}$. Then,

$$|f_l(x)| \geq |f(x)| - |f_l(x) - f(x)| > \frac{1}{k} - \epsilon > \frac{1}{k'}.$$

Since $k' > k$, we also have $|g(x)| \geq 1 + \frac{1}{k'}$ proving

$$C \cap \{|f_l - f| < \epsilon\} \subset \{x \in X : f_l(x) \neq 0 \text{ and } |g(x)| \neq 1\}.$$

This implies

$$\mu(\{x \in X : f_l(x) \neq 0 \text{ and } |g(x)| \neq 1\}) \geq \mu(C \cap \{|f_l - f| < \epsilon\}).$$

On the other hand,

$$\mu(C \cap \{|f_l - f| < \epsilon) = \mu(C) - \mu(C \cap \{|f_l - f| \geq \epsilon\}) > \frac{\mu(C)}{2} > 0$$

so $\mu(\{x \in X : f_l(x) \neq 0 \text{ and } |g(x)| \neq 1\}) > 0$ contradicting $f_l \in E^c$ by Lemma 22. This completes the proof. □

Therefore, we have the following characterization.

Example 24 M_g has the shadowing property if and only if M_g is hyperbolic.

Proof If M_g has the shadowing property, then M_g is hyperbolic by Theorem 3 and Lemma 23. Since every hyperbolic linear homeomorphism has the shadowing property, we are done. □

A particular case of Example 24 is as follows. Given a Banach space X, and $\omega \in \mathbb{C}^{\mathbb{Z}}$ as in (5), we define the *diagonal operator* $D_\omega : l^p(X) \to l^p(X)$ by

$$D_\omega(x)(n) = \omega(n)x(n), \qquad \forall x \in l_p(X), \forall n \in \mathbb{Z}.$$

Example 25 D_ω has the shadowing property if and only if D_ω is hyperbolic.

This example is a direct consequence of Theorem 27 in [2].

The multiplication version of the spectral theorem (c.f. Theorem 4.6 in [5]) implies that for every normal operator of a Hilbert space $L : H \to H$ there is a measure space (X, μ) such that L is unitarily equivalent to a multiplication operator of $L^2(X, \mu)$. Then, Example 24 also implies the following well known result by Mazur (Theorem 1 in [11], see also Corollary 28 in [2]).

Example 26 An invertible normal operator on a Hilbert space has the shadowing property if and only if it is hyperbolic.

It is worth noting that multiplication operators were studied with a different approach for $p = 2$ by Mazur [11].

An anonymous referee also asked if Theorem 3 holds in the non-invertible case. More precisely, is a non-invertible operator with the positively shadowing property of a Banach space $L : X \to X$ hyperbolic if and only if $\{x \in X : \sup_{n\in\mathbb{N}} \|L^n(x)\| < \infty\}$ is closed?

Recall that L has the *positively shadowing property* if for every $\epsilon > 0$ there is $\delta > 0$ such that for every sequence $(x_n)_{n\geq 0}$ with $\|L(x_n) - x_{n+1}\| < \delta$ for $n \in \mathbb{N}$, there is $x \in X$ such that $\|L^n(x) - x_n\| < \epsilon$ for $n \geq 0$. Some parts of the proof of Theorem 3 holds in this setting as well. This and the fact that positively expansive operators with the positively shadowing property are hyperbolic [1] looks to imply positive answer for this question.

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Data availability The data that support the findings of this study are available from the corresponding author upon reasonable request.

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