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Existence and Stability of Solutions to Neutral Conformable Stochastic Functional Differential Equations

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Abstract
This paper studies conformable stochastic functional differential equations of neutral type. Firstly, the existence and uniqueness theorem of a solution is established. Secondly, the moment estimation and exponential stability results are given. Thirdly, the Ulam type stability in mean square is discussed. Finally, two examples are given to illustrate our results.

Keywords Conformable · Neutral type stochastic functional differential equations · Exponential stability · Ulam's type stability

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1 Introduction

Various derivatives are used in the literature to study properties in physics, chemistry, biology, engineering and economics; see [1,2]. With applications in mind one is usually

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faced with challenges for derivatives in theoretical analysis and computer simulation. As a result it is of interest to use a simple and well-behaved derivative to describe practical problems in engineering.

The conformable derivative is an extension of the classical limit definition of the derivative of a function and was proposed in Khalil et al. [3]. Its physical interpretation, Leibniz rule, Chain rule, exponential functions, Gronwall's inequality, integration by parts and Taylor power series were discussed in [4–9]. Ma et al. [10] applied the conformable derivative to a grey system model and showed that the conformable derivative is suitable and well-behaved. Moreover, Abel's formula, Sturms theorems, Lotka-Volterra model, Ulam's stability, variational iteration method have been studied extensively in [11–18]. Recently the authors in [19,20] applied the conformable derivative to stochastic differential equations and studied conformable Itô stochastic differential equations, existence results for solutions, Lyapunov stability, almost surely exponential stability and Ulam type stability.

Neutral stochastic functional differential equation (NSFDEs) is a special kind of stochastic equation, depending on the past and present values but also involves derivatives with delays as well as the function itself. Such equations are more difficult to motivate but often arise in the study of two or more simple oscillatory systems with some interconnections between them. The study of NSFDEs is now a hot topic. Existence, stability, and almost surely asymptotic estimations of the solution and random periodic solutions for NSFDEs was studied extensively in [21–25]. Approximate controllability and optimal control of NSFDEs with time lag in control was reported in [26,27]. Ahmadova et al. [28,29] studied the existence and Ulam–Hyers stability of Caputo-type fractional NSFDEs. The authors in [30] studied the Ulam–Hyers stability of Caputo-type fractional stochastic differential equations with time delays. For more details on the averaging principle and large deviations, we refer the reader to [31–35]. Zhu et al.'s recent work on stochastic functional (delay) differential equations provide effective theoretical support for potential applications in artificial intelligence, electrical and electronic engineering and robust control and related work can be found in [36–39].

Motivated by [19,20], we study neutral conformable stochastic functional differential equations

$$\begin{aligned} &\mathfrak{D}_0^\alpha[X(t)-D(X_t)] = f(t,X_t)+g(t,X_t)\frac{dW(t)}{dt},\\ &X(0)=X_0,\quad \alpha\in(0,1],\ t\in[0,T], \end{aligned} \tag{1}$$

where $\mathfrak{D}_0^\alpha$ is the conformable derivative, $W(\cdot)$ is a m-dimensional standard scalar Brownian motion defined on a complete probability space $(\Omega,\mathcal{F},P)$ with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let $\tau\geq 0$ and $X_t:=\{X(t+\theta),-\tau\leq\theta\leq 0\}$ is the past history of the state. Now $\mathbb{C}([-\tau,0];\mathbb{R}^n)$ denotes the family of continuous functions $\varphi:[-\tau,0]\to\mathbb{R}^n$ with the norm $||\varphi||=\sup_{-\tau\leq\theta\leq 0}|\varphi(\theta)|$, $L^2_{\mathcal{F}_{t_0}}([-\tau,0],\mathbb{R}^n)$ denotes the family of all $\mathcal{F}_{t_0}$-measurable $\mathbb{C}([-\tau,0];\mathbb{R}^n)$-random variables ϕ such that $E||\phi||^2<\infty$. Also $f:[0,T]\times\mathbb{C}([-\tau,0];\mathbb{R}^n)\to\mathbb{R}^n$, $g:[0,T]\times\mathbb{C}([-\tau,0];\mathbb{R}^n)\to\mathbb{R}^{m\times n}$ and

$D : \mathbb{C}([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ are Borel measurable. Next $X_0 = \xi = \{\xi(\theta), -\tau \le \theta \le 0\} \in L^2_{\mathcal{F}_{t_0}}([-\tau, 0], \mathbb{R}^n)$ and $||\cdot||$ is the norm of $\mathbb{R}^n$.

In this paper, we present neutral conformable stochastic functional differential equations. In Sect. 3, existence and uniqueness of the solution for Eq. (1) is discussed. In Sect. 4, the results on moment estimation are given and exponential stability is proved by the Razumikhin argument. In Sect. 5, we discuss Ulam type stability in mean square via Gronwall's inequality. Examples are given to illustrate our results in Sect. 6. Some concluding remarks are provided in the final section.

2 Preliminaries

Definition 2.1 (see [3, Definition 2.1]) The conformable derivative with low index 0 of a function $f : [0, \infty) \to \mathbb{R}$ is defined as

$$\mathfrak{D}_0^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + t^{1-\alpha}\varepsilon) - f(t)}{\varepsilon}, \quad t > 0, \quad 0 < \alpha \le 1.$$

while $\mathfrak{D}_0^\alpha f(0) = \lim_{t \to 0^+} \mathfrak{D}_0^\alpha f(t)$. Note for $t > 0$, f has a conformable derivative $\mathfrak{D}_0^\alpha f(t)$ iff f is differentiable at t and $\mathfrak{D}_0^\alpha f(t) = t^{1-\alpha} f'(t)$ holds.

Definition 2.2 (see [3, Definition 3.1]) The conformable integral with low index 0 of a function $f : [0, \infty) \to \mathbb{R}$ is defined as

$$\mathrm{I}_0^\alpha f(t) = \int_0^t f(s) d\frac{s^\alpha}{\alpha} = \int_0^t f(s) s^{\alpha-1} ds, \quad s > 0, \quad 0 < \alpha \le 1.$$

Let $Y \in \mathbb{C}^{2,1}(\mathbb{R} \times \mathbb{R}^+, \mathbb{R})$ denote the family of all real-valued functions $Y(X(\cdot), \cdot)$ defined on $\mathbb{R} \times \mathbb{R}^+$ such that they are continuously twice differentiable in X and once in t. Now, we introduce the following Itô formula in a conformable sense.

Lemma 2.3 (see [19, Theorem 2.8]) *Let* $0 < T < +\infty$, $X(t), t \in [0, T]$ *be an Itô process for*

$$\mathfrak{D}_0^\alpha X(t) = f(t) + g(t)\frac{dW(t)}{dt}, \ \alpha \in (0, 1],$$

$Y(\cdot) := Y(X(\cdot), \cdot) \in \mathbb{C}^{2,1}(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$. *Then for* $Y(t), t \in [0, T]$,

$$\begin{aligned} dY(t) = &\frac{\partial Y(X(t), t)}{\partial t} dt + \frac{\partial Y(X(t), t)}{\partial X} f(t) t^{\alpha-1} dt \\ &+ \frac{\partial Y(X(t), t)}{\partial X} g(t) t^{\alpha-1} dW(t) + \frac{1}{2}\frac{\partial Y^2(X(t), t)}{\partial X^2} g^2(t) t^{2\alpha-2} dt. \end{aligned}$$

Lemma 2.4 (see [21, p. 204, Lemma 2.3]) *Let* $a, b \ge 0$ *and* $0 < \lambda < 1$. *Then*

$$|a + b|^2 \le \frac{a^2}{\lambda} + \frac{b^2}{1 - \lambda}.$$

Lemma 2.5 (see [21, p. 40, Theorem 7.3]) *Let* $g \in \mathbb{L}^2(\mathbb{R}^+, \mathbb{R}^{n\times n})$. *Denote*

$$x(t) = \int_0^t g(s)dW(s), \quad A(t) = \int_0^t |g(s)|^2 ds, \ t \geq 0.$$

Then, for every $p > 0$*, there exists two positive constants* c_p, C_p *(depending only on* p*), such that*

$$c_p E||A(t)||^{\frac{p}{2}} \leq E\left(\sup_{0\leq s\leq t} ||x(s)||^p\right) \leq C_p E||A(t)||^{\frac{p}{2}}.$$

for all $t \geq 0$*. In particular, one may take*

$$\begin{array}{ccc} c_p = \left(\frac{p}{2}\right)^p, & C_p = \left(\frac{32}{p}\right)^{\frac{p}{2}}, & if\ 0 < p < 2; \\ c_p = 1, & C_p = 4, & if\ p = 2; \\ c_p = (2p)^{-\frac{p}{2}}, & C_p = \left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{p-1}, & if\ p > 2. \end{array}$$

Let $\mathcal{M}^2([a, b]; \mathbb{R})$ denote the space of all real-valued measurable $\{\mathcal{F}_t\}$-adapted processes $f = \{f(t)_{a\leq t\leq b}\}$ such that

$$E\int_a^b |f(t)|^2 dt < \infty.$$

Lemma 2.6 (see [21, Lemma 5.4]) *If* $f \in \mathcal{M}^2([a, b]; \mathbb{R})$*, then*

$$E\left(\int_a^b f(t)dW(t)\right) = 0,$$
$$E\left(\left|\int_a^b f(t)dW(t)\right|^2\right) = E\left(\int_a^b |f(t)|^2 dt\right).$$

Lemma 2.7 (see [40, Theorem 1]) *Let* $x(\cdot)$, $g(\cdot)$ *be real continuous functions on* $[t_0, t_1]$, $f(\cdot) \geq 0$ *is an integrable function over interval* $[t_0, t_1]$ *and* $g(\cdot) \geq 0$ *is nondecreasing. If*

$$x(t) \leq g(t) + \int_{t_0}^t f(\tau)x(\tau)d\tau, \quad t \in [t_0, t_1],$$

then

$$x(t) \leq g(t) \exp\left(\int_{t_0}^t f(\tau)d\tau\right), \quad t \in [t_0, t_1].$$

Lemma 2.8 (see [21, Theorem 3.8]) *Let* $\{M_t\}_{t\geq a}$ *be an* $\mathbb{R}^n$*-valued martingale and* $[a, b]$ *an interval in* $\mathbb{R}^+$. *If* $p \geq 1$ *and* $M_t \in \mathbb{L}^p(\Omega, \mathbb{R}^n)$, *then*

$$P\left\{\omega : \sup_{a\leq t\leq b} |M_t(\omega)| \geq c\right\} \leq \frac{1}{c^p} E|M_b|^p, \quad c > 0.$$

Lemma 2.9 (see [21, Lemma 2.4]) (Borel–Cantelli's lemma) *Let* $\{A_k\} \subset \mathcal{F}$ *and* $\sum_{k=1}^{\infty} P(A_k) < \infty$. *Then,* $P\{\lim_{k\to\infty} \sup A_k \ \ i.o.\} = 0$, *where, i.o. means infinitely often.*

3 Existence and Uniqueness Result

In this part, we study the existence and uniqueness of the solution of Eq. (1). Let $\mathcal{L}^p([a, b]; \mathbb{R}^n)$ denote the family of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a\leq t\leq b}$ such that $\int_a^b |f(t)|^p dt < \infty$ a.s. Now $E(X) = \int_\Omega X(\omega) dP(\omega)$ is the expectation of X (with respect to P). Also $M^p([-\tau, T], \mathbb{R}^n)$ denotes the family of process $\{f(t)\}_{-\tau\leq t\leq T} \in \mathcal{L}^p([-\tau, T], \mathbb{R}^n)$ such that $E(\int_{-\tau}^t ||f(s)||^p ds) < \infty$. Similar to [21, p. 203, Definition 2.1], for some $\{f(s, X_s)\} \in \mathcal{L}^1([0, T]; \mathbb{R}^n)$, $\{g(s, X_s)\} \in \mathcal{L}^2([0, T]; \mathbb{R}^n)$, we introduce the following definition.

Definition 3.1 A $\mathbb{R}^n$-valued stochastic process $X(\cdot)$ is a solution of (1), if $X(t)$ is continuous and $\mathbb{F}_t$-adapted and satisfies

$$\begin{aligned} X(t) - D(X_t) = X(0) - D(X_0) &+ \int_0^t f(s, X_s) s^{\alpha-1} ds \\ &+ \int_0^t g(s, X_s) s^{\alpha-1} dW(s), \quad t \in [0, T]. \end{aligned} \tag{2}$$

Let $a \vee b$ denote the maximum of a and b, we introduce the following assumptions.

(H1) There exists a constant $L > 0$ such that for all $X, \hat{X} \in \mathbb{C}([-\tau, 0]; \mathbb{R}^n)$, $0 \leq t \leq T$

$$||f(t, X) - f(t, \hat{X})||^2 \vee ||g(t, X) - g(t, \hat{X})||^2 \leq L||X - \hat{X}||^2.$$

(H2) There exists a constant $L > 0$ such that for all $(t, X) \in [0, T] \times \mathbb{C}([-\tau, 0]; \mathbb{R}^n)$

$$||f(t, X)||^2 \vee ||g(t, X)||^2 \leq L(1 + ||X||^2).$$

(H3) There exists a constant $\lambda \in (0, 1)$ such that for all $X, Y \in \mathbb{C}([-\tau, 0]; \mathbb{R}^n)$

$$||D(X) - D(Y)|| \leq \lambda||X - Y||.$$

Lemma 3.2 *Assume* (H2) *and* (H3) *hold, and* $X(\cdot)$ *is a solution of* (1). *Then*

$$E\left(\sup_{-\tau\le t\le T}||X(t)||^2\right)\le\left[1+\frac{4+\lambda\sqrt{\lambda}}{(1-\lambda)(1-\sqrt{\lambda})}E(||\xi||^2)\right]e^{\frac{3L(1+T)T^{2\alpha-1}}{(1-\lambda)(1-\sqrt{\lambda})(2\alpha-1)}},$$

holds for $\frac{1}{2}<\alpha\le 1$.

Proof For all $0\le t\le T$, let

$$N^*(t)=\xi(0)+\int_0^t f(s,X_s)s^{\alpha-1}ds+\int_0^t g(s,X_s)s^{\alpha-1}dW(s),\ t\in[0,T],$$

and we obtain

$$X(t)=D(X_t)-D(X_0)+N^*(t).$$

From (2) and applying Lemma 2.4, for all $0\le t\le T$, we get

$$\begin{aligned}||X(t)||^2&\le\frac{1}{\lambda}||D(X_t)-D(X_0)||^2+\frac{1}{1-\lambda}||N^*(t)||^2\\&\le\lambda||X_t-\xi||^2+\frac{1}{1-\lambda}||N^*(t)||^2\\&\le\sqrt{\lambda}||X_t||^2+\frac{\lambda}{1-\sqrt{\lambda}}||\xi||^2+\frac{1}{1-\lambda}||N^*(t)||^2.\end{aligned}$$

Noting that $\sup_{-\tau\le s\le t}||X(s)||^2\le||\xi||^2+\sup_{0\le s\le t}||X(s)||^2$, one obtains

$$\begin{aligned}E\left(\sup_{-\tau\le s\le t}||X(t)||^2\right)&\le\sqrt{\lambda}E\left(\sup_{-\tau\le s\le t}||X(t)||^2\right)+\frac{1+\lambda-\sqrt{\lambda}}{1-\sqrt{\lambda}}E||\xi||^2\\&\quad+\frac{1}{1-\lambda}E\left(\sup_{-\tau\le s\le t}||N^*(s)||^2\right).\end{aligned}$$

Hence

$$\begin{aligned}E\left(\sup_{-\tau\le s\le t}||X(t)||^2\right)&\le\frac{1+\lambda-\sqrt{\lambda}}{(1-\sqrt{\lambda})^2}E||\xi||^2\\&\quad+\frac{1}{(1-\lambda)(1-\sqrt{\lambda})}E\left(\sup_{-\tau\le s\le t}||N^*(s)||^2\right).\end{aligned}\tag{3}$$

On the other hand, from (H1), one can show that

$$\begin{aligned}E\left(\sup_{-\tau\le s\le t}||N^*(s)||^2\right)&\le 3E||\xi||^2+3L(1+T)\\&\int_0^t\left[1+E\left(\sup_{-\tau\le s\le t}||X(s)||^2\right)\right]|s^{\alpha-1}|^2ds.\end{aligned}$$

Substituting this into (3), we have

$$1 + E\left(\sup_{-\tau\le t\le T} ||X(t)||^2\right) \le 1 + \frac{4+\lambda\sqrt{\lambda}}{(1-\lambda)(1-\sqrt{\lambda})}E||\xi||^2 + \frac{3L(1+T)}{(1-\lambda)(1-\sqrt{\lambda})}\int_0^t \left[1 + E\left(\sup_{-\tau\le s\le t} ||X(s)||^2\right)\right] |s^{\alpha-1}|^2 ds.$$

Since $E||\xi||^2 < \infty$, using Lemma 2.7, we have

$$E\left(\sup_{-\tau\le t\le T} ||X(t)||^2\right) \le \left[1 + \frac{4+\lambda\sqrt{\lambda}}{(1-\lambda)(1-\sqrt{\lambda})}E(||\xi||^2)\right] e^{\frac{3L(1+T)T^{2\alpha-1}}{(1-\lambda)(1-\sqrt{\lambda})(2\alpha-1)}}.$$

The proof is complete. □

Theorem 3.3 *Suppose that* (H1), (H2) *and* (H3) *hold. Then* (1) *has a unique solution* $X(\cdot) \in M^2([-\tau, T]; \mathbb{R}^n)$ *given by* (2) *provided that* $\alpha \in (\frac{1}{2}, 1]$.

Proof *Existence* We first show the local existence of a solution. Let $\bar{T}$ be sufficiently small such that

$$\kappa := \lambda + \frac{2L(1+\bar{T})\bar{T}^{2\alpha-1}}{(1-\lambda)(2\alpha-1)} < 1. \tag{4}$$

Define $X_0^0 = \xi$ and $X^0(t) = \xi(0)$ for $t \in [0, \bar{T}]$. For each $n = 1, 2, \ldots$, consider the Picard iteration

$$X^n(t) - D(X_t^{n-1}) = \xi(0) - D(\xi) + \int_0^t f(s, X_s^{n-1})s^{\alpha-1}ds + \int_0^t g(s, X_s^{n-1})s^{\alpha-1}dW(s). \tag{5}$$

From Lemma 3.2, $X^n(\cdot) \in M^2([-\tau, \bar{T}]; \mathbb{R}^n)$. Then, for all $0 \le t \le \bar{T}$, we have

$$X^1(t) - X^0(t) = X^1(t) - \xi(0) = D(X_t^0) - D(\xi) + \int_0^t f(s, X_s^0)s^{\alpha-1}ds + \int_0^t g(s, X_s^0)s^{\alpha-1}dW(s).$$

Thus

$$E\left(\sup_{0\le s\le t} ||X^1(t) - X^0(t)||^2\right) \le \lambda E\left(\sup_{0\le s\le t} ||X_t^0 - \xi||^2\right) + \frac{2L(1+\bar{T})}{1-\lambda}E\int_0^t (1 + ||X_t^0||^2)s^{2\alpha-2}ds \le 2\lambda E||\xi||^2 + \frac{2L(1+\bar{T})\bar{T}^{2\alpha-1}}{(1-\lambda)(2\alpha-1)}E(1 + ||X_t^0||^2) := C. \tag{6}$$

Note also that for any $n \geq 1, 0 \leq t \leq \bar{T}$,

$$\begin{aligned} X^{n+1}(t) - X^n(t) &= D(X_t^n) - D(X_t^{n-1}) + \int_0^t [f(s, X_s^n) - f(s, X_s^{n-1})]s^{\alpha-1}ds \\ &\quad + \int_0^t [g(s, X_s^n) - g(s, X_s^{n-1})]s^{\alpha-1}dW(s). \end{aligned}$$

One has

$$\begin{aligned} E\left(\sup_{0\leq t\leq \bar{T}} ||X^{n+1}(t) - X^n(t)||^2\right) &\leq \lambda E\left(\sup_{0\leq t\leq \bar{T}} ||X_t^n - X_t^{n-1}||^2\right) \\ &\quad + \frac{2L(1+\bar{T})}{1-\lambda}\int_0^t E\left(\sup_{0\leq t\leq \bar{T}} ||X_s^n - X_s^{n-1}||^2\right)s^{2\alpha-2}ds \\ &\leq \kappa E\left(\sup_{0\leq t\leq \bar{T}} ||X_s^n - X_s^{n-1}||^2\right) \\ &\leq \kappa^n E\left(\sup_{0\leq t\leq \bar{T}} ||X_s^1 - X_s^0||^2\right) \\ &\leq C\kappa^n. \end{aligned} \tag{7}$$

Combine with (6) and condition (4), we get

$$E\left(\sup_{0\leq t\leq \bar{T}} ||X^{n+1}(t) - X^n(t)||^2\right) \leq C\kappa^n \to 0, \quad n \to \infty.$$

From above

$$X^n(t) = X^0(t) + \sum_{k=1}^{n-1}\left(X^{k+1}(t) - X^k(t)\right), \tag{8}$$

converges uniformly on the interval $[0, \bar{T}]$. Denote the limit of $X^n(\cdot)$ by $X(\cdot)$. Clearly, $X(\cdot)$ is continuous and $\mathbb{F}_t$-adapted. From (7), $\{X^n(\cdot)\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{L}^2[0, \bar{T}]$.

Hence, let $n \to \infty$ in (5), we obtain

$$\begin{aligned} X(t) - D(X_t) = X(0) - D(X_0) &+ \int_0^t f(s, X_s)s^{\alpha-1}ds \\ &+ \int_0^t g(s, X_s)s^{\alpha-1}dW(s), \quad t \in [0, \bar{T}]. \end{aligned}$$

From the idea of continuation of a solution and Lemma 3.2, repeating the above procedures, we obtain that the Eq. (1) has a solution in the intervals $[\bar{T}, 2\bar{T}]$, $[2\bar{T}, 3\bar{T}]\ldots$, and thus, (1) has a solution on the entire interval $[0, T]$ since there exists a positive integer k such that $k\bar{T} > T$.

Uniqueness Let $X(\cdot)$, $\tilde{X}(\cdot)$ be two solutions of (1), and from Lemma 3.2, both of them belong to $M^2([-\tau, T]; \mathbb{R}^n)$. Note that

$$X(t) - \tilde{X}(t) = D(X_t) - D(\tilde{X}_t) + N(t),$$

where

$$N(t) = \int_0^t [f(s, X_s) - f(s, \tilde{X}_s)]s^{\alpha-1}ds + \int_0^t [g(s, X_s) - g(s, \tilde{X}_s)]s^{\alpha-1}dW(s).$$

From Lemma 2.4, we get

$$||X(t) - \tilde{X}(t)||^2 \leq \lambda||X_t - \tilde{X}_t||^2 + \frac{1}{1-\lambda}||N(t)||^2.$$

Therefore

$$E\left(\sup_{-\tau\leq s\leq t} ||X(s) - \tilde{X}(s)||^2\right)$$
$$\leq \lambda E\left(\sup_{-\tau\leq s\leq t} ||X(s) - \tilde{X}(s)||^2\right) + \frac{1}{1-\lambda}E\left(\sup_{-\tau\leq s\leq t} ||N(t)||^2\right).$$

This implies

$$E\left(\sup_{-\tau\leq s\leq t} ||X(s) - \tilde{X}(s)||^2\right) \leq \frac{1}{(1-\lambda)^2}E\left(\sup_{-\tau\leq s\leq t} ||N(t)||^2\right).$$

Note that

$$E\left(\sup_{-\tau\leq s\leq t} ||N(t)||^2\right) \leq 2L(1+T)\int_0^t ||X_s - \tilde{X}_s||^2 s^{2\alpha-2}ds.$$

Thus

$$E\left(\sup_{-\tau\leq s\leq t} ||X(s) - \tilde{X}(s)||^2\right) \leq \frac{2L(1+T)}{(1-\lambda)^2}\int_0^t ||X_s - \tilde{X}_s||^2 s^{2\alpha-2}ds.$$

Using Lemma 2.7, we have

$$E\left(\sup_{-\tau\leq s\leq t} ||X(s) - \tilde{X}(s)||^2\right) = 0,$$

which implies

$$P\left(\sup_{-\tau\leq s\leq t} ||X(s) - \tilde{X}(s)|| > 0\right) = 0.$$

Thus, we almost surely have $X(t) = \tilde{X}(t)$, which ends the proof. □

Remark 3.4 Consider (1) on $[0, \infty)$, and f, g are the mappings from $[0, \infty) \times \mathbb{C}([-\tau, 0]; \mathbb{R}^n)$ to $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, respectively. If (H1), (H2) and (H3) hold on $[0, T]$, then, (1) has a unique global solution $X(\cdot, \xi)$ on the entire interval $[-\tau, \infty)$.

Remark 3.5 In [28] the authors investigated the existence and uniqueness of mild solutions to stochastic neutral differential equations involving Caputo fractional time derivative operator with Lipschitz coefficients and under some Caratheodory-type conditions on the coefficients through the Picard approximation technique.

4 Moment Estimates and Exponential Stability

Now we establish the moment estimates and exponential stability theory for the global solution of (1) on $[0, \infty)$. We impose a linear growth condition for the function $D(\cdot)$. Assume that there exists a constant $\lambda \in (0, 1)$ such that for all $\varphi \in \mathbb{C}([-\tau, 0]; \mathbb{R}^n)$

$$||D(\varphi)|| \leq \lambda ||\varphi||. \tag{9}$$

Note that (9) follows from (H3) if in addition $D(0) = \mathbf{0}$, where $\mathbf{0}$ is an n-dimensional zero vector.

Lemma 4.1 (see [21, p. 213, Theorem 4.5]) *Let $p \geq 2$, $E||\xi||^p < \infty$, (H2) and (9) hold. Then*

$$||X(s) - D(X_s)||^{p-1} \cdot ||f(s, X_s)|| \leq \sqrt{2L}(1+\lambda)^{p-1}(1 + ||X_s||^p),$$

and

$$||X(s) - D(X_s)||^{p-2} \cdot ||g(s, X_s)||^2 \leq 2L(1+\lambda)^{p-2}(1 + ||X_s||^p),$$

hold for $0 \leq s \leq t \leq T$.

Lemma 4.2 (see [21, p. 212, Lemma 4.3]) *Let $p \geq 1$ and (9) holds. Then*

$$||\varphi(0) - D(\varphi)||^p \leq (1+\lambda)^p ||\varphi||^p,$$

for all $\varphi \in \mathbb{C}([-\tau, 0]; \mathbb{R}^n)$.

Lemma 4.3 (see [21, p. 212, Lemma 4.4]) *Let $p > 1$ and (9) holds. Then*

$$\sup_{0\leq s\leq t} ||X(s)||^p \leq \frac{\lambda}{1-\lambda}||\xi||^p + \frac{1}{(1-\lambda)^p} \sup_{0\leq s\leq t} ||X(s) - D(X_s)||^p.$$

Theorem 4.4 *Let* $p \geq 2$, $E||\xi||^p < \infty$, (H2) *and* (9) *hold. Then*

$$E\left(\sup_{-\tau\leq s\leq t}||X(s)||^p\right) \leq (1+C_4E||\xi||^p)\exp\left[\frac{2C_1t^{\alpha}}{\alpha(1-\lambda)^p}+\frac{2(C_2+C_3)}{(1-\lambda)^p}\frac{t^{2\alpha-1}}{2\alpha-1}\right], \tag{10}$$

hold for $\frac{1}{2} < \alpha \leq 1$, *where*

$$C_1 = p\sqrt{2L}(1+\lambda)^{p-1},\quad C_2 = p(p-1)L(1+\lambda)^{p-2},$$
$$C_3 = 32Lp^2(1+\lambda)^{p-2},\quad C_4 = 1+\frac{\lambda}{1-\lambda}+\frac{2(1+\lambda)^p}{(1-\lambda)^p}.$$

Proof Applying the Itô formula in the conformable sense (i.e. Lemma 2.3), one sees that

$$||X(t)-D(X_t)||^p \leq ||\xi(0)-D(\xi)||^p+p\int_0^t||X(s)-D(X_s)||^{p-1}||f(s,X_s)||s^{\alpha-1}ds$$
$$+\frac{p(p-1)}{2}\int_0^t||X(s)-D(X_s)||^{p-2}||g(s,X_s)||^2s^{2\alpha-2}ds$$
$$+p\int_0^t||X(s)-D(X_s)||^{p-1}||g(s,X_s)||s^{\alpha-1}dW(s).$$

Next, using Lemma 4.1 and 4.2, we get

$$E\left(\sup_{0\leq s\leq t}||X(s)-D(X_s)||^p\right) \leq (1+\lambda)^pE||\xi||^p$$
$$+C_1\int_0^t(1+E||X_s||^p)s^{\alpha-1}ds+C_2\int_0^t(1+E||X_s||^p)s^{2\alpha-2}ds$$
$$+p\int_0^t||X(s)-D(X_s)||^{p-1}||g(s,X_s)||s^{\alpha-1}dW(s),$$

where $C_1 = p\sqrt{2L}(1+\lambda)^{p-1}$, $C_2 = p(p-1)L(1+\lambda)^{p-2}$. From Lemma 2.5, we have

$$p\int_0^t||X(s)-D(X_s)||^{p-1}\cdot||g(s,X_s)||s^{\alpha-1}dW(s)$$
$$\leq \frac{1}{2}E\left(\sup_{0\leq s\leq t}||X(s)||^p\right)+32Lp^2(1+\lambda)^{p-2}\int_0^t(1+E||X_s||^p)s^{2\alpha-2}ds.$$

This implies

$$E\left(\sup_{0\le s\le t}||X(s)-D(X_s)||^p\right)\le 2(1+\lambda)^p E||\xi||^p$$
$$+2C_1\int_0^t(1+E||X_s||^p)s^{\alpha-1}ds+2(C_2+C_3)\int_0^t(1+E||X_s||^p)s^{2\alpha-2}ds,$$

where $C_3=32Lp^2(1+\lambda)^{p-2}$.

Applying Lemma 4.3, we obtain

$$E\left(\sup_{0\le s\le t}||X(s)||^p\right)\le\left(\frac{\lambda}{1-\lambda}+\frac{2(1+\lambda)^p}{(1-\lambda)^p}\right)E||\xi||^p$$
$$+\frac{2C_1}{(1-\lambda)^p}\int_0^t(1+E||X_s||^p)s^{\alpha-1}ds$$
$$+\frac{2(C_2+C_3)}{(1-\lambda)^p}\int_0^t(1+E||X_s||^p)s^{2\alpha-2}ds.$$

Consequently

$$\begin{aligned}
&1+E\left(\sup_{-\tau\le s\le t}||X(s)||^p\right)\\
&\le 1+E||\xi||^p+E\left(\sup_{0\le\varsigma\le s}||X(\varsigma)||^p\right)\\
&\le 1+C_4E||\xi||^p+\frac{2C_1}{(1-\lambda)^p}\int_0^t(1+E(\sup_{-\tau\le\varsigma\le s}||X(\varsigma)||^p))s^{\alpha-1}ds\\
&\quad+\frac{2(C_2+C_3)}{(1-\lambda)^p}\int_0^t(1+E(\sup_{-\tau\le\varsigma\le s}||X(\varsigma)||^p))s^{2\alpha-2}ds\\
&=1+C_4E||\xi||^p+\\
&\quad\int_0^t(1+E(\sup_{-\tau\le\varsigma\le s}||X(\varsigma)||^p))\left[\frac{2C_1}{(1-\lambda)^p}s^{\alpha-1}+\frac{2(C_2+C_3)}{(1-\lambda)^p}s^{2\alpha-2}\right]ds,
\end{aligned}$$

where $C_4=(1+\frac{\lambda}{1-\lambda}+\frac{2(1+\lambda)^p}{(1-\lambda)^p})$. Finally, using Lemma 2.7, we obtain that

$$1+E\left(\sup_{-\tau\le s\le t}||X(s)||^p\right)\le(1+C_4E||\xi||^p)\exp\left[\frac{2C_1t^\alpha}{\alpha(1-\lambda)^p}+\frac{2(C_2+C_3)}{(1-\lambda)^p}\frac{t^{2\alpha-1}}{2\alpha-1}\right],$$

which gives (10). The proof is finished. □

Remark 4.5 When $\alpha=1$ in (10), the result of Theorem 4.4 is consistent with that of [21, p. 213, Theorem 4.5].

Now, we establish a result of exponential stability by the Razumikhin argument. Let $\mathbf{L}^2_{\mathcal{F}}([-\tau,0],\mathbb{R}^n)$ denote the family of all $\mathbb{C}([-\tau,0];\mathbb{R}^n)$ -valued random variable

ξ such that $E|\xi|^2 < \infty$. We furthermore assume that $f(0,t) = \mathbf{0}$, $g(0,t) = \mathbf{0}$ and $D(0) = \mathbf{0}$ and we introduce several assumptions.

(V1) There is a constant $\lambda \in (0,1)$ such that

$$E||D(\phi)||^2 \le \lambda^2 \sup_{-\tau\le\theta\le 0} E||\phi(\theta)||^2, \quad \phi \in \mathbf{L}^2_{\mathcal{F}}([-\tau,0],\mathbb{R}^n).$$

(V2) Let $q > (1-\lambda)^{-2}$. There is a $\eta > 0$ such that for all $t \ge 0$,

$$E[2(\phi(0) - D(\phi))^T f(\phi,t)t^{\alpha-1} + ||g(\phi,t)t^{\alpha-1}||^2] \le -\eta E||\phi(0) - D(\phi)||^2.$$

(V3) For any $\phi \in \mathbf{L}^2_{\mathcal{F}}([-\tau,0],\mathbb{R}^n)$,

$$E||\phi(\theta)||^2 < qE||\phi(0) - D(\phi)||^2, \ -\tau \le \theta \le 0.$$

Lemma 4.6 (see [21, p. 222 Theorem 6.2]) *Let* (V1) *hold for some* $\lambda \in (0,1)$. *Then*

$$E||\phi(0) - D(\phi)||^2 \le (1+\lambda)^2 \sup_{-\tau\le\theta\le 0} E||\phi(\theta)||^2,$$

hold for all $\phi \in \mathbf{L}^2_{\mathcal{F}}([-\tau,0],\mathbb{R}^n)$.

Lemma 4.7 (see [21, p. 223, Theorem 6.1]) *Let* (V1), (V2), (V3) *hold,* $\alpha = 1$. *Then, for all* $\xi \in \mathbf{L}^2_{\mathcal{F}}([-\tau,0],\mathbb{R}^n)$,

$$E||X(t,\xi)||^2 \le q(1+\lambda)^2 e^{-\tilde{\beta}t} \sup_{-\tau\le\theta\le 0} E||\phi(\theta)||^2, \ t \ge 0,$$

where $\tilde{\beta} = \min\{\eta, \frac{1}{\tau}\ln[\frac{q}{(1+\lambda\sqrt{q})^2}]\} > 0$. *In other words, the trivial solution of* (1) *is exponential stable in mean square.*

Lemma 4.8 (see [21, p. 222 Theorem 6.3]) *Let* (V1) *hold for some* $\lambda \in (0,1)$, $\rho \ge 0$ *and* $0 < \beta < \tau^{-1}\ln[\frac{1}{\lambda^2}]$, $X(\cdot)$ *be a solution of* (1). *If*

$$e^{\tilde{\beta}t} E||X(t) - D(X_t)||^2 \le (1+\lambda)^2 \sup_{-\tau\le\theta\le 0} E||\phi(\theta)||^2, \ \ 0 \le t \le \rho, \qquad (11)$$

then

$$e^{\tilde{\beta}t} E||X(t)||^2 \le \frac{(1+\lambda)^2}{(1-\lambda e^{\beta\tau/2})^2} \sup_{-\tau\le\theta\le 0} E||\phi(\theta)||^2, \ \ -\tau \le t \le \rho.$$

Lemma 4.9 ([21, p. 227, Corollary 6.6]) *Let* (V1) *hold and assume that* $\beta_1, \beta_2 > 0$ *such that*

$$\begin{aligned} &E[2(\phi(0) - D(\phi))^T f(\phi,t) + ||g(\phi,t)||^2] \\ &\quad \le -\beta_1 E||\phi(0)||^2 + \beta_2 \sup_{-\tau\le\theta\le 0} E||\phi(\theta)||^2, \ t \ge 0, \end{aligned} \qquad (12)$$

for all $\xi \in \mathbf{L}^2_{\mathcal{F}}([-\tau, 0], \mathbb{R}^n)$. *If*

$$0 < \lambda < \frac{1}{2}, \quad and \quad \beta_1 > \frac{\beta_2}{(1-2\lambda)^2}, \tag{13}$$

then, the trivial solution of (1) *when* $\alpha = 1$ *is exponential stable in mean square (also almost surely exponentially stable).*

Theorem 4.10 *Let* (V1)*,* (V2)*,* (V3) *hold. Then, for all* $\xi \in \mathbf{L}^2_{\mathcal{F}}([-\tau, 0], \mathbb{R}^n)$*,*

$$E||X(t,\xi)||^2 \leq q(1+\lambda)^2 e^{-\bar{\beta}t} \sup_{-\tau\leq\theta\leq 0} E||\phi(\theta)||^2, \; t \geq 0, \tag{14}$$

and

$$\lim_{t\to\infty} \sup \frac{1}{t} \ln||X(t,\xi)|| \leq -\frac{\bar{\beta}}{2}, \quad t \geq 0, \quad a.s.$$

where $\bar{\beta} = \min\{\eta, \frac{1}{\tau}\ln[\frac{q}{(1+\lambda\sqrt{q})^2}]\}$. *That is, the trivial solution of* (1) *is almost surely exponentially stable.*

Proof Note that $\beta > 0$, $\frac{q}{(1+\lambda\sqrt{q})^2} > 1$, and we have $q > (1-\lambda)^{-2}$. Fix any $\xi \in \mathbf{L}^2_{\mathcal{F}}([-\tau, 0], \mathbb{R}^n)$ and assume that $\sup_{-\tau\leq\theta\leq 0} E|\xi(\theta)|^2 > 0$, and $\beta \in (0, \bar{\beta})$ be arbitrary. It is easy to see that

$$0 < \beta < \bar{\beta} \leq \min\left\{\eta, \frac{1}{\tau}\ln\left(\frac{1}{\lambda^2}\right)\right\}, \quad q > \frac{e^{\beta\tau}}{(1-\lambda e^{\lambda\tau/2})^2} > \frac{1}{(1-\lambda e^{\lambda\tau/2})^2}. \tag{15}$$

We now claim that

$$e^{\beta t} E|X(t) - D(X_t)|^2 \leq (1+\lambda)^2 \sup_{-\tau\leq\theta\leq 0} E|\xi(\theta)|^2, \; t \geq 0. \tag{16}$$

If so, using Lemma 4.8 with (16) and combine with (15), one can show that

$$e^{\beta t} E|X(t)|^2 \leq q(1+\lambda)^2 \sup_{-\tau\leq\theta\leq 0} E|\xi(\theta)|^2, \; t \geq 0.$$

Then, the desired result (14) follows by letting $\beta \to \bar{\beta}$. Next we show (16) by contradiction. Suppose (16) is not true. Then, from Lemma 4.6, we can get that there is a constant $\rho \geq 0$ such that

$$\begin{aligned} e^{\beta t} E||X(t) - D(X_t)||^2 &\leq e^{\beta\rho} E||X(\rho) - D(X_\rho)||^2 \\ &= (1+\lambda)^2 \sup_{-\tau\leq\theta\leq 0} E||\xi(\theta)||^2, \; 0 \leq t \leq \rho. \end{aligned}$$

Moreover, there is a sequence of $\{t_k\}_{k\geq 0}$ such that $t_k \to \rho$ and

$$e^{\beta\cdot t_k}E||X(t_k) - D(X_{t_k})||^2 > e^{\beta\rho}E||X(\rho) - D(X_\rho)||^2. \tag{17}$$

Now, recalling $\beta < \eta$, using the conformable type Itô formula (Lemma 2.3), Lemma 4.8 and (V2), for all sufficiently small $h > 0$, we have

$$\begin{aligned}
&e^{\beta(\rho+h)}E||X(\rho+h) - D(X_{\rho+h})||^2 - e^{\beta(\rho)}E|X(\rho) - D(X_\rho)|^2 \\
&= \int_\rho^{\rho+h} e^{\beta t_k}[\beta E||X(t) - D(X_t)||^2]t^{\alpha-1}dt \\
&\quad + \int_\rho^{\rho+h} e^{\beta t_k}E[2(X(t) - D(X_t))^T f(t, X_t)t^{\alpha-1} + ||g(t, X_t)t^{\alpha-1}||^2]dt \\
&\leq 0.
\end{aligned}$$

This contradicts with (17), so (16) and (14) must hold.

Next, noting that $X(t, \xi)$ be a $\mathbb{R}^n$-valued martingale, let $\varepsilon > 0$, and applying Lemma 2.8 to (14), for all $t \geq 0$, $\omega \in \Omega$, we have

$$P\{\omega : ||X(t, \xi)||^2 > e^{-(\beta-\varepsilon)t}\} \leq Me^{-\varepsilon t},$$

where M is a normal number. From Lemma 2.9, we have

$$P\{\omega : ||X(t, \xi)||^2 > e^{-(\beta-\varepsilon)t},\ i.o.\} = 0.$$

Thus, we almost surely have $|X(t, \xi)|^2 \leq e^{-(\beta-\varepsilon)t}$. Further, we have

$$\lim_{t\to\infty} \sup \frac{1}{t} \ln ||X(t, \xi)|| \leq -\frac{\beta - \varepsilon}{2}, \quad t \geq 0.\ a.s.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\lim_{t\to\infty} \sup \frac{1}{t} \ln ||X(t, \xi)|| \leq -\frac{\beta}{2}, \quad t \geq 0.\ a.s.$$

which completes the proof. □

Corollary 4.11 *Let* (V1) *hold and assume that*

$$\begin{aligned}
&E[2(\phi(0) - D(\phi)^T f(\phi, t)t^{\alpha-1}) + ||g(\phi, t)t^{\alpha-1}||^2] \\
&\quad \leq -\beta_1 E||\phi(0)||^2 + \beta_2 \sup_{-\tau\leq\theta\leq 0} E||\phi(\theta)||^2,\ t \geq 0,
\end{aligned} \tag{18}$$

hold for all $\phi \in \mathbf{L}^2_{\mathcal{F}}([-\tau, 0], \mathbb{R}^n)$*, and* $\beta_1, \beta_2 > 0$*. If*

$$0 < \lambda < \frac{1}{2}, \quad and \quad \beta_1 > \frac{\beta_2}{(1 - 2\lambda)^2}, \tag{19}$$

then, the trivial solution of (1) *is exponential stable in mean square (also almost surely exponentially stable).*

Proof From Lemma 4.9 and using the Itô formula in the conformable sense (i.e. Lemma 2.3), one can complete the proof. □

5 Ulam Type Stability

In this part, we discuss the Ulam type stability of (1) in the one-dimensional case. Let $\mathbb{J} := [0, T]$, $Y_t := \{Y(t+\theta), -\tau \leq \theta \leq 0\}$ be the past history of the state, and for $\forall \varepsilon > 0$, $\varphi(\cdot) \in \mathbb{C}(\mathbb{J}, \mathbb{R}^+)$, we consider (1) and the following inequality

$$\left| \mathfrak{D}_0^\alpha [Y(t) - D(Y_t)] - f(t, Y_t) - g(t, Y_t)\frac{dW(t)}{dt} \right| \leq \varepsilon, \quad \frac{1}{2} < \alpha \leq 1, \; t \in \mathbb{J}, \tag{20}$$

and

$$\left| \mathfrak{D}_0^\alpha [Y(t) - D(Y_t)] - f(t, Y_t) - g(t, Y_t)\frac{dW(t)}{dt} \right| \leq \varepsilon\varphi(t), \quad \frac{1}{2} < \alpha \leq 1, \; t \in \mathbb{J}. \tag{21}$$

Definition 5.1 The solution $X(\cdot)$ of (1) is called almost surely Ulam–Hyers stable in mean square, if for $\forall \varepsilon > 0$, there exists a constant $N > 0$ such that for each process $Y(\cdot) \in \mathbb{L}_n^2(\mathbb{J})$ a solution of (20), then

$$E\left(\sup_{-\tau \leq t \leq T} |Y(t) - X(t)|^2 \right) \leq N\varepsilon, \quad t \in \mathbb{J}.$$

Remark 5.2 A process $Y(\cdot) \in \mathbb{L}_n^2(\mathbb{J})$ is a solution of (20) iff for $\forall \varepsilon > 0$, there exists a function $G(t) \in \mathbb{L}_n^2(\mathbb{J})$ such that (i) $|G(t)| < \sqrt{\varepsilon}$; (ii) $\mathfrak{D}_0^\alpha[Y(t) - D(Y_t)] = f(t, Y_t) + g(t, Y_t)\frac{dW(t)}{dt} + G(t), \; t \in \mathbb{J}$.

Definition 5.3 The solution $X(\cdot)$ of (1) is called almost surely Ulam–Hyers–Rassias stable in mean square, if there exists a constant $\tilde{N} > 0$ such that for $\forall \varepsilon > 0$, $\varphi(\cdot) \in \mathbb{C}(\mathbb{J}, \mathbb{R}^+)$ and for each process $Y(\cdot) \in \mathbb{L}_n^2(\mathbb{J})$ a solution of (21), then

$$E\left(\sup_{-\tau \leq t \leq T} |Y(t) - X(t)|^2 \right) \leq \tilde{N}\varepsilon\varphi(t), \quad t \in \mathbb{J}.$$

Remark 5.4 A process $Y(\cdot) \in \mathbb{L}_n^2(\mathbb{J})$ is a solution of (21) iff for $\forall \varepsilon > 0$, there exists a function $\bar{G}(t) \in \mathbb{L}_n^2(\mathbb{J})$ such that (i) $|\bar{G}(t)| < \sqrt{\varepsilon\varphi(t)}$; (ii) $\mathfrak{D}_0^\alpha[Y(t) - D(Y_t)] = f(t, Y_t) + g(t, Y_t)\frac{dW(t)}{dt} + \bar{G}(t), \; t \in \mathbb{J}$.

Let

$$\mathbf{N}^*(t) = \xi(0) + \int_0^t f(s, Y_s)s^{\alpha-1}ds + \int_0^t g(s, Y_s)s^{\alpha-1}dW(s),\ t \in \mathbb{J}.$$

Lemma 5.5 *Let $Y(\cdot)$ be a solution of Eq.* (20). *Then*

$$E\left(\left|Y(t) - D(Y_t) + D(Y_0) - \mathbf{N}^*(t)\right|^2\right) \le \frac{\varepsilon T^{2\alpha}}{2\alpha - 1}, \quad t \in \mathbb{J}. \tag{22}$$

Proof For all $t \in \mathbb{J}$, $\alpha \in (0, 1]$ note that,

$$\mathfrak{D}_0^\alpha[Y(t) - D(Y_t)] = f(t, Y_t) + g(t, Y_t)\frac{dW(t)}{dt} + G(t)$$

with initial value $Y(0) = Y_0 = X_0$. Then, the solution can be expressed as

$$\begin{aligned} Y(t) = D(Y_t) - D(Y_0) + \xi(0) &+ \int_0^t f(s, Y_s)s^{\alpha-1}ds \\ &+ \int_0^t g(s, Y_s)s^{\alpha-1}dW(s) + \int_0^t G(s)s^{\alpha-1}ds,\ t \in \mathbb{J}. \end{aligned}$$

By Hölder's inequality, we get

$$\begin{aligned} E(|Y(t) - D(Y_t) + D(Y_0) - \mathbf{N}^*(t)|^2) &= E\left(|\int_0^t G(s)s^{\alpha-1}ds|^2\right) \\ &\le \left|\int_0^t G(s)s^{\alpha-1}ds\right|^2 \\ &\le \int_0^t |G(s)|^2 ds \int_0^t s^{2(\alpha-1)}ds \\ &\le \varepsilon t \cdot \frac{t^{2\alpha-1}}{2\alpha - 1} \\ &\le \frac{\varepsilon T^{2\alpha}}{2\alpha - 1}, \quad t \in \mathbb{J}. \end{aligned}$$

This finishes the proof. □

Similar to Lemma 5.5, we have

Lemma 5.6 *Let $Y(\cdot)$ be a solution of Eq.* (21). *Then*

$$E\left(\left|Y(t) - D(Y_t) + D(Y_0) - \mathbf{N}^*(t)\right|^2\right) \le \frac{\varepsilon\varphi(t)T^{2\alpha}}{2\alpha - 1}, \quad t \in \mathbb{J}. \tag{23}$$

Theorem 5.7 *Suppose that* (H1), (H2), (H3) *hold and* $\lambda \in (0, \frac{1}{2})$, $\alpha \in (\frac{1}{2}, 1]$. *Then, the solution of* (1) *is almost surely Ulam–Hyers stable on* $\mathbb{J}$.

Proof Let $Y(\cdot) \in \mathbb{L}_n^2[0,T]$ be a solution of (20), and $X(\cdot)$ be a solution of (1) given by (2). Note that $Y_0 = X_0$, from Lemmas 2.6, 5.5, (H1) and (H3), for $0 \le t \le T$, we get

$$
\begin{aligned}
&E(|Y(t)-X(t)|^2) = E(|Y(t)-Y(t)+Y(t)-X(t)|^2)\\
&= E\left(\left|Y(t)-D(Y_t)+D(Y_0)-\mathbf{N}^*(t)+D(Y_t)-D(Y_0)-D(X_t)+D(X_0)\right.\right.\\
&\quad \left.\left.+\int_0^t (f(Y_s,s)-f(X_s,s))s^{\alpha-1}ds+\int_0^t (g(Y_s,s)-g(X_s,s))s^{\alpha-1}dW(s)\right|^2\right)\\
&\le 4E(|Y(t)-D(Y_t)+D(Y_0)-\mathbf{N}^*(t)|^2+4\lambda^2E(|Y_t-X_t|^2)\\
&\quad +4L^2t\int_0^t E(|Y_s-X_s|^2)s^{2(\alpha-1)}ds+4L^2\int_0^t E(|Y_s-X_s|^2)s^{2(\alpha-1)}ds\\
&\le \frac{4\varepsilon T^{2\alpha}}{2\alpha-1}+4\lambda^2E(|Y_t-X_t|^2)+4L^2(1+T)\int_0^t E(|Y_s-X_s|^2)s^{2(\alpha-1)}ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&E\left(\sup_{-\tau\le t\le T}|Y(t)-X(t)|^2\right)\\
&\le \frac{4\varepsilon T^{2\alpha}}{2\alpha-1}+4\lambda^2E\left(\sup_{-\tau\le t\le T}|Y(t)-X(t)|^2\right)\\
&\quad +4L^2(1+T)\int_0^t E\left(\sup_{-\tau\le t\le T}|Y(s)-X(s)|^2\right)s^{2(\alpha-1)}ds.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&E\left(\sup_{-\tau\le t\le T}|Y(t)-X(t)|^2\right)\\
&\le \frac{4\varepsilon T^{2\alpha}}{(2\alpha-1)(1-4\lambda^2)}+\frac{4L^2(1+T)}{1-4\lambda^2}\int_0^t E\left(\sup_{-\tau\le t\le T}|Y(s)-X(s)|^2\right)s^{2(\alpha-1)}ds.
\end{aligned}
$$

Next, using Lemma 2.7, we have

$$
\begin{aligned}
&E\left(\sup_{-\tau\le t\le T}|Y(t)-X(t)|^2\right)\\
&\le \frac{4\varepsilon T^{2\alpha}}{(2\alpha-1)(1-4\lambda^2)}\exp\left(\frac{4L^2(1+T)}{(1-4\lambda^2)}\int_0^t s^{2(\alpha-1)}ds\right)\\
&\le \frac{4\varepsilon T^{2\alpha}}{(2\alpha-1)(1-4\lambda^2)}\exp\left(\frac{4L^2(1+T)}{1-4\lambda^2}\frac{T^{2\alpha-1}}{2\alpha-1}\right)\\
&= N\varepsilon,
\end{aligned}
$$

where $N := N(\alpha, T, \lambda) = \frac{4T^{2\alpha}}{(2\alpha-1)(1-4\lambda^2)} \exp\left(\frac{4L^2(1+T)}{1-4\lambda^2}\frac{T^{2\alpha-1}}{2\alpha-1}\right)$.

From Definition 5.1, the solution of (1) is almost surely Ulam–Hyers stable. This completes the proof of the theorem. □

Theorem 5.8 *Suppose that* (H1), (H2), (H3) *hold,* $\lambda \in (0, \frac{1}{2})$, $\alpha \in (\frac{1}{2}, 1]$ *and* $\varphi(\cdot)$ *be nondecreasing. Then, the solution of* (1) *is almost surely Ulam–Hyers–Rassias stable on* $\mathbb{J}$.

Proof Let $Y(\cdot) \in \mathbb{L}_n^2[0, T]$ be a solution of (21), and $X(\cdot)$ be a solution of (1) given by (2). Note that $Y_0 = X_0$, from Lemma 2.6, Lemma 5.6, (H1) and (H3). Repeating the procedures in the proof of Theorem 5.7, we have

$$
\begin{aligned}
E\left(\sup_{-\tau\le t\le T}|Y(t)-X(t)|^2\right) &\le \frac{4\varepsilon^2 T^{2\alpha}\varphi^2(t)}{(2\alpha-1)(1-4\lambda^2)} \\
&+\frac{4L^2(1+T)}{1-4\lambda^2}\int_0^t E\left(\sup_{-\tau\le t\le T}|Y(s)-X(s)|^2\right)s^{2(\alpha-1)}ds.
\end{aligned}
$$

Noting $\varphi(\cdot)$ is nondecreasing, then, $(\varphi^2(t))' = 2\varphi(t)\varphi'(t) \ge 0$. Using Lemma 2.7, we obtain

$$
\begin{aligned}
&E\left(\sup_{-\tau\le t\le T}|Y(t)-X(t)|^2\right) \\
&\le \frac{4\varepsilon T^{2\alpha}\varphi(t)}{(2\alpha-1)(1-4\lambda^2)}\exp\left(\frac{4L^2(1+T)}{1-4\lambda^2}\int_0^t (t-s)^{2(\alpha-1)}ds\right) \\
&\le \frac{4\varepsilon T^{2\alpha}\varphi(t)}{(2\alpha-1)(1-4\lambda^2)}\exp\left(\frac{4L^2(1+T)}{1-4\lambda^2}\frac{T^{2\alpha-1}}{2\alpha-1}\right) \\
&= N\varphi(t)\varepsilon.
\end{aligned}
$$

From Definition 5.3, the solution of (1) is almost surely Ulam–Hyers–Rassias stable. The proof is now complete. □

Remark 5.9 Theorems 5.7 and 5.8 show two different Ulam stability, that is, the error norm is limited by $N\varepsilon$ and $N\varphi(\cdot)\varepsilon$, respectively. This property is very necessary in iterative learning control, tracking control, consensus control and synchronization of multi-agent systems.

Remark 5.10 In [29], the Ulam–Hyers stability of Caputo type fractional NSFDEs is studied (note the Ulam–Hyers–Rassias stability is not considered). A similar comment applies to [30].

6 Examples

Example 6.1 Consider the one-dimensional linear neutral conformable stochastic delay differential equations on $t > 0$.

$$\mathfrak{D}_0^\alpha[x(t) - \frac{1}{4}x(t-\tau)] = -6t^{1-\alpha}x(t) + t^{1-\alpha}x(t-\tau)\frac{dW(t)}{dt},$$
$$x(0) = x_0,\ \alpha \in (0, 1], \tag{24}$$

where $\tau > 0$, $W(\cdot)$ is a one-dimensional Brownian motion.

For $x, y \in \mathbb{R}$ and $t > 0$, one has

$$\begin{aligned}
&2\left(x - \frac{1}{4}y\right)\left(-6t^{1-\alpha}xt^{\alpha-1}\right) + \left(t^{1-\alpha}yt^{\alpha-1}\right)^2 \\
&= -12x^2 + 3xy + y^2 \\
&\le -\frac{21}{2}x^2 + \frac{5}{2}y^2,
\end{aligned}$$

where $2xy \le x^2 + y^2$ was used. Let $\beta_1 = \frac{21}{2}$ and $\beta_2 = \frac{5}{2}$. Let $\lambda = \frac{1}{4}$ and note $\beta_1 > \frac{\beta_2}{(1-2\lambda)^2}$. From Corollary 4.11, the trivial solution of Eq. (24) is almost surely exponentially stable.

Example 6.2 Consider the one-dimensional neutral conformable stochastic delay differential equations on $t \in [0, 10]$

$$\mathfrak{D}_0^\alpha[x(t) - \lambda x(t-\tau)] = ax(t) + bx(t-\tau)\frac{dW(t)}{dt},$$
$$x(0) = 1,\ \alpha \in \left(\frac{1}{2}, 1\right], \tag{25}$$

where $\tau > 0$, a, b is a constant, and $W(\cdot)$ is a one-dimensional Brownian motion.

Set $\varepsilon > 0$, $\varphi(t) = e^{\frac{t^\alpha}{\alpha}}$. For $t \in [0, 10]$, let $G(t) = \sqrt{\varepsilon} \cdot e^{\frac{t^\alpha}{\alpha}}$, $a = b = 1$, $\tau = 0.5$, $\lambda = \frac{1}{3}$ and

$$\mathfrak{D}_0^\alpha\left[y(t) - \frac{1}{2}y(t-\tau)\right] = ay(t) + by(t-\tau)\frac{dW(t)}{dt} + G(t),$$
$$y(0) = 1,\ \alpha \in \left(\frac{1}{2}, 1\right]. \tag{26}$$

From Theorem 3.3, the existence and uniqueness of a solution of Eqs. (25) and (26) can be guaranteed. From Theorem 5.7, we obtain

$$E\left(\sup_{-0.5 \le t \le 10} |y(t) - x(t)|^2\right) \le N(\alpha)\varepsilon\, e^{\frac{2t^\alpha}{\alpha}},\ N(\alpha) = \frac{7.2 \times 10^{2\alpha+2}}{2\alpha - 1} e^{\frac{7.92 \times 10^{2\alpha}}{2\alpha-1}}.$$

From Definition 5.3, the solution of (25) is almost surely Ulam–Hyers–Rassias stable on [0, 10]. Similarly, we can show the solution of (25) is also almost surely Ulam–Hyers stable on [0, 10].

7 Conclusion

In this paper, we discuss the neutral conformable stochastic functional differential equations. In detail, the existence and uniqueness theorem, moment estimation and exponential stability are given. Moreover, we discuss the Ulam type stability of the solution of the equation.

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