Qualitative Theory of Dynamical Systems (2021) 20:88
https://doi.org/10.1007/s12346-021-00526-1

Reiterative Distributional Chaos in Non-autonomous Discrete Systems

Zongbin Yin[1] · **Qiaomin Xiang**[2] · **Xinxing Wu**[3]

Received: 13 September 2020 / Accepted: 13 September 2021 / Published online: 23 September 2021

Abstract
In this paper, several types of reiterative distributional chaos are concerned in discrete dynamical systems. Some implications between distributional chaos and reiterative distributional chaos are obtained. It is further shown that an equicontinuous non-autonomous system $(X, f_{1,\infty})$, where $f_{1,\infty} = \{f_i\}_{i\geq 1}$ is a sequence of self-maps of a metric space X, exhibits reiterative distributional chaos of type i ($i \in \{1, 1^+, 2, 2\frac{1}{2}, 2\frac{1}{2}-\}$) if and only if its kth iteration $f_{1,\infty}^{[k]}$ exhibits reiterative distributional chaos of type i for any $k \geq 2$.

Keywords Reiterative distributional chaos · Distributional chaos · Invariant · Non-autonomous systems

Mathematics Subject Classification MSC 54H20 · 37B55 · 37D45

1 Introduction

The chaotic dynamics of discrete systems has been extensively concerned over the past decades. The first description of the term "chaos" in discrete systems with strict mathematical language was proposed by Li and Yorke [17], where the asymptotic behavior of pairs was investigated. In [22], Schweizer and Smítal introduced the notion of distributional chaos for interval maps, as a natural strengthening of Li-Yorke chaos. They proved that distributional chaos is equivalent to positive topological entropy for a self-map of a compact interval, by considering the dynamics of pairs with certain

✉ Zongbin Yin
yinzb_math@163.com

1 School of Mathematics and Systems Science, Guangdong Polytechnic Normal University, Guangzhou 510665, People's Republic of China

2 School of Mathematics and Big Data, Foshan University, Foshan 528000, People's Republic of China

3 School of Sciences, Southwest Petroleum University, Chengdu 610500, People's Republic of China

statistical properties. Thereafter, distributional chaos was divided into several types, DC1, DC2, DC2$\frac{1}{2}$ and DC3, see [3,12]. Following this way, several other concepts of chaos were developed to characterize the complexity of dynamical systems, such as distributional chaos in a sequence [31], dense chaos [26], Li-Yorke sensitivity [1], $(\mathcal{F}_1, \mathcal{F}_2)$-chaos [27], etc.. For the relations between Li-Yorke chaos, distributional chaos and positive topological entropy of compact dynamical systems, we refer to [9,16,20] and the references therein.

As a young branch of topological dynamics, infinite-dimensional linear dynamics has turned into an active research area. Particularly, the hypercyclicity and chaos of continuous linear operators on Banach spaces or Fréchet spaces has been widely investigated and many beautiful and interesting results on this topic have been well-developed [11]. Due to the absence of compact structure, some dynamical results of linear operators are different from that of classical compact systems. For instance, a topologically transitive operator must be Li-Yorke chaotic; the full space could be a distributionally scrambled set of some operators [18]; the dynamics of an operator with infinite topological entropy could be very trivial [38] and so on. Among others, Bernardes et al. [6,7] showed that DC1 and DC2 are equivalent for linear operators on Banach spaces and there exist DC2 operators which are not mean Li-Yorke chaotic. For more results on Li-Yorke and distributionally chaotic operators, see [4,5,33,35–37]. Recently, Bonilla and Kostić [8] observed that the orbits of a Li-Yorke chaotic operator on a Banach space has additional statistical properties. To further investigate the relations between Li-Yorke chaos and distributional chaos, they proposed the concepts of reiterative distributional chaos of types 1, 1^+ and 2 for linear operators on Banach spaces and showed that a topologically mixing and RDC1^+ linear operator may not be RDC2, that there exists a RDC1 and RDC2 operator which is not RDC1^+, see also [14]. It is therefore meaningful to find the difference of the dynamical properties of reiterative distributional chaos within the setting of infinite-dimensional linear systems and of compact dynamical systems.

On the other hand, the classical results of autonomous discrete systems has been extended to the non-autonomous case, since non-autonomous discrete systems are more flexible than the autonomous ones for the investigation of real world problems and such systems have been widely applied in physics, engineering, mathematical biology, economics, etc.. Some results on topological entropy, sensitivity, mixing properties, chaos of non-autonomous discrete systems can be seen [2,10,13,19,21,23–25,32]. In [2], Balibrea and Oprocha showed that the weak mixing is stronger than Li-Yorke chaos, but positive topological entropy is not sufficient to imply Li-Yorke chaos for non-autonomous systems. Shao et al. [24] proved that Li-Yorke δ-chaos and distributional δ'-chaos in a sequence are equivalent for non-autonomous discrete systems on compact spaces, they also provided sufficient conditions for non-autonomous discrete systems to be distributionally chaotic. It is known that Li-Yorke chaos, DC1, DC2, DC2$\frac{1}{2}$ are iteration invariants for autonomous systems (see [10,12,30]), nevertheless, the existence of a DC3 pair is not preserved under iteration. For the non-autonomous case, it was further shown [23] that the properties of DC1, DC2 and DC2$\frac{1}{2}$ can be preserved under iteration if the family $\{f_i\}_{i\geq 1}$ is equicontinuous, which weakens the condition provided in [28,29,34]. Motivated by [23], we would like to further investigate the iteration invariance of reiterative distributional chaos in non-autonomous discrete systems.

The present paper is organized as follows. In Sect. 2, some basic concepts of discrete dynamical systems are given. In order to adapt the setting of infinite-dimensional linear systems and of compact dynamical systems, we unify the notations of Li-Yorke chaos, distributional chaos and reiterative distributional chaos of types 1, 1^+, 2, $2\frac{1}{2}$ and $2\frac{1}{2}-$ in a very general framework. It is shown that reiterative distributional chaos of type $2\frac{1}{2}$ and type 2 are equivalent for linear operators on Banach spaces; that reiterative distributional chaos of type 1 and type $2\frac{1}{2}-$ are equivalent for linear operators on Banach spaces. Moreover, there are no implications between reiterative distributional chaos of type 1 and type 2 for linear operators. Nevertheless, reiterative distributional chaos of type 1 and type $2\frac{1}{2}-$ are not equivalent for continuous maps on compact spaces, and the equivalence between reiterative distributional chaos of type $2\frac{1}{2}$ and type 2 does not hold for general continuous self-maps of a metric space. The obtained results complement the ones of [14].

Section 3 deals with the iterative invariance of various types of reiterative distributional chaos in non-autonomous discrete systems. It is shown that for an equicontinuous family $f_{1,\infty} = (f_i)_{i\geq 1}$ on a metric space X, the non-autonomous system $(X, f_{1,\infty})$ is reiteratively distributionally chaotic of type i ($i \in \{1, 1^+, 2, 2\frac{1}{2}, 2\frac{1}{2}-\}$) if and only if its kth iteration $(X, f_{1,\infty}^{[k]})$ is reiteratively distributionally chaotic of type i for any $k \geq 2$. It is worth noting that there exists a DC3 equicontinuous non-autonomous system $(X, f_{1,\infty})$ such that $f_{1,\infty}^{[2]}$ has no DC3 pairs. This situation is different from the case of autonomous systems [10].

2 Preliminaries and Reiterative Distributional Chaos

Let $(X, f_{1,\infty})$ be a non-autonomous discrete system, where $f_{1,\infty} = \{f_i\}_{i\geq 1}$ is a sequence of self-maps of a metric space (X, d). In the case that $f_i = f, \forall i \geq 1$ for some $f : X \to X$, it reduces to an autonomous discrete system (X, f). $(X, f_{1,\infty})$ is called *equicontinuous* if the sequence $\{f\}_{i\geq 1}$ is equicontinuous on X, that is, for any $t > 0$, there is $t' > 0$ such that $d(f_1^n x, f_1^n y) < t, \forall n \in \mathbb{N}$ for any $x, y \in X$ with $d(x, y) < t'$. Moreover, we say that $f_{1,\infty}$ is *uniformly Lipschitz continuous* if there exists $L > 0$ such that $d(f_i x, f_i y) < Ld(x, y)$ for any $x, y \in X$ and any $i \geq 1$. Clearly, uniform Lipschitz continuity is stronger than equicontinuity.

Given $x_0 \in X$, the orbit of x_0 under $f_{1,\infty}$ is denoted by $\text{orb}(x_0, f_{1,\infty}) = \{f_1^i(x_0)\}_{i\geq 0}$, where $f_1^i := f_i \circ f_{i-1} \circ \cdots \circ f_1$ for any $i \geq 1$ and $f_1^0 = id_X$ is the identify map on X. For the sake of convenience, let $f_n^i := f_{n+i-1} \circ f_{n+i-2} \circ \cdots \circ f_n$ and $f_n^0 = id_X$ for any $n, i \geq 1$. For each $k \in \mathbb{N}$, the kth iteration system of $(X, f_{1,\infty})$ is given by $(X, f_{1,\infty}^{[k]})$, where $f_{1,\infty}^{[k]} = \{f_{i(n-1)+1}^k\}_{i\geq 1}$. It is easy to see that the orbit $\text{orb}(x_0, f_{1,\infty}^{[k]})$ is contained in $\text{orb}(x_0, f_{1,\infty})$.

A pair of points $(x, y) \in X \times X$ is said to be *asymptotic* if $d(f_1^n x, f_1^n y)$ tends to zero as n tends to infinity; *proximal* if $\liminf_{n\to\infty} d(f_1^n x, f_1^n y) = 0$ and *syndetically* *proximal* if the set

$$N_{x,y}(f_{1,\infty},t) := \{i \geq 0 : d(f_1^i(x), f_1^i(y)) < t\}$$

is syndetic for any $t > 0$. Recall that a set $A \subset \mathbb{N}$ is syndetic if there is $k \in \mathbb{N}$ so that $A \cap [i, i+k] \neq \emptyset$ for any $i \in \mathbb{N}$. (x, y) is called a *scrambled pair* of $(X, f_{1,\infty})$ if it is proximal but not asymptotic. The system $(X, f_{1,\infty})$ is called *Li-Yorke chaotic* if it has an uncountable scrambled set $D \subset X$, namely, any distinct points x, y of D form a scrambled pair.

Given $A \subset \mathbb{N}$, its lower and upper densities are given by

$$\underline{\mathrm{d}}(A) := \liminf_{n\to\infty} \frac{|A \cap [1,n]|}{n}, \quad \overline{\mathrm{d}}(A) := \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n},$$

respectively, where $|M|$ denotes the cardinality of the set M. Moreover, the lower and upper Banach densities of A are given by

$$\underline{\mathrm{Bd}}(A) := \lim_{n\to\infty} \liminf_{m\to\infty} \frac{|A \cap [m+1, m+n]|}{n}$$

and

$$\overline{\mathrm{Bd}}(A) := \lim_{n\to\infty} \limsup_{m\to\infty} \frac{|A \cap [m+1, m+n]|}{n},$$

respectively. Denote by

$$F_{x,y}(f_{1,\infty},t) = \underline{\mathrm{d}}(N_{x,y}(f_{1,\infty},t)), \quad F^*_{x,y}(f_{1,\infty},t) = \overline{\mathrm{d}}(N_{x,y}(f_{1,\infty},t)), \ \forall t > 0,$$

the lower and upper distributional functions of (x, y), respectively.

A pair (x, y) is called *DC1* of $f_{1,\infty}$ if $F^*_{x,y}(f_{1,\infty},t) \equiv 1$ and $F_{x,y}(f_{1,\infty},\delta) = 0$ for some $\delta > 0$; *DC2* if $F^*_{x,y}(f_{1,\infty},t) \equiv 1$ and $F_{x,y}(f_{1,\infty},\delta) < 1$ for some $\delta > 0$; *DC*$2\frac{1}{2}$ if $F_{x,y}(f_{1,\infty},t) < c < F^*_{x,y}(f_{1,\infty},t), \forall t \in (0,\delta)$, for some $\delta, c > 0$; *DC*3 if $F_{x,y}(f_{1,\infty},t) < F^*_{x,y}(f_{1,\infty},t)$ for all t in an interval. For $i \in \{1, 2, 2\frac{1}{2}, 3\}$, a set $S \subset X$ is called a *distributionally scrambled set of type* i of $f_{1,\infty}$ (simply, a DCi set) if any distinct point $x, y \in S$ forms a DCi pair. Furthermore, S is called a *strong DC1 (DC2, DC*$2\frac{1}{2}$*) set* of $f_{1,\infty}$ if there is $\delta > 0$ so that any $x, y \in S (x \neq y)$ forms a DC1 (DC2, DC$2\frac{1}{2}$, respectively) pair with respect to the same δ.

Definition 1 For $i \in \{1, 2, 2\frac{1}{2}, 3\}$, a system $(X, f_{1,\infty})$ is called *distributionally chaotic of type* i (or simply, DCi) if it has an uncountable DCi set S. In additional, $(X, f_{1,\infty})$ is called *uniformly DC*i ($i \in \{1, 2, 2\frac{1}{2}\}$) if S could be a strong DCi set.

Inspired by the concepts of reiterative distributional chaos of linear operators introduced in [14], we use the following versions of reiterative distributional chaos of types $1, 1^+, 2, 2\frac{1}{2}$ and $2\frac{1}{2}-$ for general non-autonomous discrete systems by modifying the upper (or lower) density in the definitions of different types of distributional chaos (DC1, DC2, DC$2\frac{1}{2}$) with upper (or lower) Banach density.

Let

$$BF_{x,y}(f_{1,\infty},t)=\underline{\mathrm{Bd}}(N_{x,y}(f_{1,\infty},t)),\ \forall t>0,$$
$$BF^*_{x,y}(f_{1,\infty},t)=\overline{\mathrm{Bd}}(N_{x,y}(f_{1,\infty},t)),\ \forall t>0.$$

A pair $(x,y)\in X\times X$ is called *RDC*1^+ of $f_{1,\infty}$ if $F^*_{x,y}(f_{1,\infty},t)=1$, $\forall t>0$ and $BF_{x,y}(f_{1,\infty},\delta)=0$ for some $\delta>0$; *RDC1* if $F^*_{x,y}(f_{1,\infty},t)\geq c$, $\forall t>0$ and $BF_{x,y}(f_{1,\infty},\delta)=0$ for some $c,\delta>0$; *RDC2* if $BF^*_{x,y}(f_{1,\infty},t)=1$, $\forall t>0$ and $F_{x,y}(f_{1,\infty},\delta)<1$ for some $\delta>0$; *RDC*$2\frac{1}{2}$ if $F_{x,y}(f_{1,\infty},t)<c<BF^*_{x,y}(f_{1,\infty},t)$, $\forall t\in(0,\delta)$, for some $\delta,c>0$; *RDC*$2\frac{1}{2}-$ if $BF_{x,y}(f_{1,\infty},t)<c<F^*_{x,y}(f_{1,\infty},t)$, $\forall t\in(0,\delta)$, for some $\delta,c>0$.

A set $S\subset X$ is called a reiterative distributionally scrambled set of type i ($i\in\{1,1^+,2,2\frac{1}{2},2\frac{1}{2}-\}$) of $f_{1,\infty}$ (or simply, a RDCi set) if any distinct point $x,y\in S$ forms a RDCi pair. Moreover, S is called a strong RDCi set of $f_{1,\infty}$ if any $x,y\in S(x\neq y)$ forms a RDCi pair with respect to a constant $\delta>0$ which is independent on x and y.

Definition 2 Given $i\in\{1,1^+,2,2\frac{1}{2},2\frac{1}{2}-\}$, a system $(X,f_{1,\infty})$ is called *reiteratively distributionally chaotic of type i* (or simply, RDCi) if it has an uncountable DCi set S. In additional, $(X,f_{1,\infty})$ is called uniformly RDCi if S is a strong RDCi set.

According to Definition 1 and 2, it is easily seen that DC1 implies DC2, DC2 implies DC$2\frac{1}{2}$ and DC$2\frac{1}{2}$ implies Li-Yorke chaos; that a RDCi system is Li-Yorke chaotic for any $i\in\{1,1^+,2,2\frac{1}{2},2\frac{1}{2}-\}$. Also, the following implications hold: DC1 $\Rightarrow$ RDC1^+ $\Rightarrow$ RDC1 $\Rightarrow$ RDC$2\frac{1}{2}-$; DC2 $\Rightarrow$ RDC2 $\Rightarrow$ RDC$2\frac{1}{2}$; DC$2\frac{1}{2}$ $\Rightarrow$ RDC$2\frac{1}{2}$ and RDC$2\frac{1}{2}-$. Generally, these types of distributional chaos and reiterative distributional chaos given in Definition 1 and 2 are not equivalent to each other even for the autonomous case. In the context of infinite-dimensional linear systems, some interesting implications between different notions of chaos were obtained recently. For instance, DC1 and DC2 were proved to be equivalent for a linear operator acting on a Banach space, and there exists DC2 operators which are not mean Li-Yorke chaotic [7]. There is a topologically mixing and RDC1^+ linear operator which is not RDC2; a RDC1 and RDC2 operator which is not RDC1^+ [8].

In the following, we focus on some implications of RDC$2\frac{1}{2}$ and RDC$2\frac{1}{2}-$ within the framework of infinite-dimensional linear systems and of compact dynamical systems, respectively. Let us begin with an example of RDC$2\frac{1}{2}$ linear operator which is not RDC1.

Example 1 Let $v=(v_i)_{i\geq 1}$ be a sequence of positive numbers given by:

$$v_i=\begin{cases}2^{i-P_j-j}, & i\in[P_j,Q_j),\\ 1, & i\in[Q_j,P_{j+1}),\end{cases} \tag{1}$$

where $P_1 = 1$, $Q_1 = 4$ and $P_{k+1} = (k+1)^2 + \sum_{j=1}^{k} j!$, $Q_{k+1} = P_{k+1} + 2k + 3$ for any $k \geq 1$. Let $l_v^1(\mathbb{N}) = \{(x_i)_{i\geq 1} : \sum_{i\geq 1} |x_i| v_i < \infty\}$ be a weighted l^1-space with the norm $\|(x_i)_{i\geq 1}\| = \sum_{i\geq 1} |x_i| v_i$. Consider the forward shift F acting on $l_v^1(\mathbb{N})$, namely $F : (x_1, x_2, x_3, \cdots) \mapsto (0, x_1, x_2, x_3, \cdots)$. It is easy to see that F is a continuous linear operator. We show that F is RDC2$\frac{1}{2}$, but it is not RDC1.

Let $e_i, i \geq 1$ be the usual standard basis on $l_v^1(\mathbb{N})$. Then $\|F^n e_i\| = \|e_{i+n}\| = v_{i+n}$ for any $n, i \geq 1$. It follows from (1) that $\liminf_{n\to\infty} \|F^n e_i\| = 0$ and the lower density of the set $\{n \geq 1 : \|F^n e_i\| \geq s\}$ equals to one for any $s \in (0, 1)$. So $(e_1, 0)$ is a RDC2$\frac{1}{2}$ pair of F and span$\{e_1\}$ is a RDC2$\frac{1}{2}$ set. However, for any nonzero vector $y = (y_i)_i \in l_v^1(\mathbb{N})$ (assume $y_r \neq 0$ for some $r \geq 1$), it is easy to see $\|F^n y\| \geq \|F^n e_r\|$ for any $n \geq 1$. Therefore $F^*_{y,0}(F, s) = 0$ for any $s \in (0, 1)$. This shows that F has no RDC1 pairs.

The next result obtains a basic relation between RDC1 operators and RDC2$\frac{1}{2}$ operators, whose proof follows from the argument of [14, Theorem 3.16].

Theorem 1 *Let $f : X \to X$ be a continuous linear operator on a Banach space X. Then f is RDC1 if and only if it is RDC2$\frac{1}{2}-$.*

Proof It suffices to show that RDC2$\frac{1}{2}-$ implies RDC1. Suppose that f is RDC2$\frac{1}{2}-$ and (x, y) is a RDC2$\frac{1}{2}-$ pair of f. Then there exist $\delta > 0, c > 0$ so that $BF_{x,y}(f_{1,\infty}, t) < c < F^*_{x,y}(f_{1,\infty}, t)$ for all $t \in (0, \delta)$. Let $z = x - y$ and $Y = \overline{\text{span}}\{\text{orb(z, f)}\}$. Clearly Y is invariant under f and the limitation $g := f|_Y$ on Y is also RDC2$\frac{1}{2}-$ and thus Li-Yorke chaotic. According to [14, Lemma 3.15], there exists a residual set of vectors u on Y for which there is $B \subset \mathbb{N}$ with $\overline{d}(B) > 0$ such that $\lim_{n\in B} \|g^n u\| = 0$. By [5, Proposition 5], g has a residual set of vectors $w \in Y$ satisfying $\limsup_{n\to\infty} \|g^n w\| = \infty$. Hence, there exists $v \in Y$ so that $\lim_{n\in B} \|f^n v\| = 0$ and $\limsup_{n\to\infty} \|f^n v\| = \infty$. It follows from [14, Theorem 3.16] that f is RDC1. □

Theorem 2 *Let $f_{1,\infty} = \{f_i\}_{i\geq 1}$ be uniformly Lipschitz continuous on a metric space (X, d). Then $(X, f_{1,\infty})$ is RDC2 if and only if it is RDC2$\frac{1}{2}$. In particular, a continuous linear operator f acting on a Banach space X is RDC2 if and only if it is RDC2$\frac{1}{2}$.*

Proof Since $\{f_i\}_{i\geq 1}$ is uniformly Lipschitz continuous, there exists $L > 1$ such that $d(f_i x, f_i y) < Ld(x, y)$ for any $x, y \in X$ and any $i \geq 1$. It suffices to show that any RDC2$\frac{1}{2}$ pair of $f_{1,\infty}$ is also a RDC2 pair. Suppose that (x, y) is a RDC2$\frac{1}{2}$ pair of $f_{1,\infty}$, that is, there are $\delta, c > 0$ such that $F_{x,y}(f_{1,\infty}, t) < c < BF^*_{x,y}(f_{1,\infty}, t)$ for all $t \in (0, \delta)$. Pick an increasing sequence $(n_k)_k$ satisfying $d(f_1^{n_k} x, f_1^{n_k} y) < L^{-2k}$. Denote $A = \bigcup_{k\geq 1} [n_k, n_k + k] \bigcap \mathbb{N}$. Then $Bd^*(A) = 1$ and $d(f_1^n x, f_1^n y) < L^k d(f_1^{n_k} x, f_1^{n_k} y) < L^{-k}$ for each $n \in [n_k, n_k + k]$. Therefore $\lim_{n\to\infty, n\in A} d(f_1^n x, f_1^n y) = 0$. This indicates that $BF^*_{x,y}(f_{1,\infty}, t) = 1$ for any $t > 0$. □

It is worth mentioning that contrary to what happens in Theorem 1, the notions of RDC1 and RDC2$\frac{1}{2}-$ for compact dynamical systems (namely, a continuous self-map of a compact metric space) are not equivalent to each other, as showed in the next example.

Example 2 Let $\Omega_2 = \{0, 1\}^{\mathbb{N}}$ be a symbol space equipped with the metric $\rho(\omega, \gamma) = 2^{-\min\{n\in\mathbb{N}:\omega_n\neq\gamma_n\}}$, for any distinct $\omega = (\omega_k)_k, \gamma = (\gamma_k)_k \in \Omega_2$. The shift map $\sigma : \Omega_2 \to \Omega_2$ is given by $(\sigma(\omega))_i = \omega_{i+1}, \omega \in \Omega_2$. A nonempty closed set $X \subset \Omega_2$ is called a subshift provided $\sigma(X) \subset X$. Consider a hereditary shift $X \subset \Omega_2$ (namely, if $\omega' \in \Omega_2$ satisfies $\omega'_n \leq \omega_n, \forall n \in \mathbb{N}$ for some $\omega \in X$, then $\omega' \in X$). Kwietniak [15] showed that X is DC1 if and only if X is not proximal, and that X is DC2 if and only if it is DC3, if and only if it has positive topological entropy, if and only if there exists $\omega = (\omega_k)_k \in X$ such that $\{i \in \mathbb{N} : \omega_i = 1\}$ has positive upper Banach density.

By [20, Theorem 14], if X is proximal, then any $(x, y) \in X\times X$ forms a syndetically proximal pair (i.e., for any $\varepsilon > 0$, the set $\{j \in \mathbb{N} : \rho(\sigma^j x, \sigma^j y) < \varepsilon\}$ is syndetic), particularly, $BF_{x,y}(\sigma, \varepsilon) > 0$ for any $\varepsilon > 0$. Therefore, every RDC1 hereditary shift X is not proximal and so it is DC1 and RDC1$^+$. On the other hand, if there exist $x = (x_i)_i, y = (y_i)_i \in X$ such that $BF_{x,y}(\sigma, s) < 1$ for some $s > 0$, then it is easy to show that the set $\{j \in \mathbb{N} : x_j \neq y_j\}$ has positive upper Banach density. Thus, either $\overline{\mathrm{Bd}}\{i \in \mathbb{N} : x_i = 1\} > 0$ or $\overline{\mathrm{Bd}}\{i \in \mathbb{N} : y_i = 1\} > 0$. In this case, X is DC2. Therefore it follows that both RDC2$\frac{1}{2}$ and RDC2$\frac{1}{2}-$ are equivalent to DC2 for hereditary shifts. By [15, Theorem 5.6], there exists a DC2 hereditary shift which is not DC1 and hence not RDC1.

Unfortunately, we do not know whether or not RDC2$\frac{1}{2}$ and RDC2 are equivalent for compact dynamical systems. We end this section with a RDC2$\frac{1}{2}$ continuous map on a noncompact metric space, which is not RDC2.

Example 3 Let H be an infinite-dimensional real Hilbert space with a basis $\{e_i\}_{i\geq 0}$. Denote $A_k = \{k! + 2, k! + 4, \cdots, k! + 2k\}$ for any $k \geq 3$ and $A := \cup_{k\geq 3} A_k$. Let

$$X := \left(\bigcup_{k\geq 3} \{\lambda e_i : \lambda \geq k^{-1}, i \in A_k\} \right) \cup \{\lambda e_i : \lambda \geq 1, i \geq 0, i \notin A\}$$

be equipped with the following metric,

$$d(\lambda e_i, \mu e_j) = \begin{cases} |\lambda - \mu|, & \text{if } i = j, \\ |\lambda| + |\mu|, & \text{if } i \neq j. \end{cases} \tag{2}$$

Define the map $f : X \to X$ by: $f(\lambda e_i) = (w_{i+1}\lambda)e_{i+1}, \ \forall \lambda e_i \in X$, where the sequence $\{w_j\}_{j\geq 1}$ of positive numbers is given by

$$w_j = \begin{cases} \frac{1}{k}, & \text{if } j = k! + 2i \text{ for some } k \geq 3 \text{ and } 1 \leq i \leq k, \\ k, & \text{if } j = k! + 2i + 1 \text{ for some } k \geq 3 \text{ and } 1 \leq i \leq k, \\ 1, & \text{otherwise.} \end{cases} \tag{3}$$

It is not hard to see that f is continuous. We check that $\{ae_0 : a \geq 1\}$ is a RDC2$\frac{1}{2}$ set of $f_{1,\infty}$. Indeed, given $a > b \geq 1$. For any $n \geq 1$,

$$d(f^n(ae_0), f^n(be_0)) = d(w_1 \cdots w_n a e_n, w_1 \cdots w_n a e_n) = w_1 \cdots w_n(a - b).$$

It follows from (3) that $d(f^n(ae_0), f^n(be_0)) = \frac{a-b}{k}$, if $n = k! + 2i$ for $k \geq 3$ and $1 \leq i \leq k$; otherwise, $d(f^n(ae_0), f^n(be_0)) = a - b$. Thus $BF^*_{ae_0,be_0}(f, t) = \frac{1}{2}$ and $F_{ae_0,be_0}(f, t) = 0$ for any $t \in (0, a - b)$.

We further show that $f_{1,\infty}$ has no RDC2 pairs. Given distinct $ae_p, be_q \in X$. In the case of $p = q$ and $a \neq b$, it follows that

$$BF^*_{ae_p,be_p}(f, t) = BF^*_{ae_0,be_0}(f, t) = \frac{1}{2}, \ \forall t \in \left(0, \frac{|a-b|}{p}\right).$$

In the case of $p \neq q$ (assume $p > q$), then

$$d(f^n(ae_p), f^n(be_q)) > aw_{p+1} \cdots w_{p+n} = d(f^n(ae_p), f^n(2ae_p))$$

for any $n \geq 1$. Therefore $BF^*_{ae_p,be_q}(f, t) \leq BF^*_{ae_p,2ae_p}(f, t) = \frac{1}{2}$ for any $t \in \left(0, \frac{a}{p}\right)$. In both cases, $BF^*_{ae_p,be_q}(f, t) < 1$ for some $t > 0$, so (ae_p, be_q) is not a RDC2 pair of f.

3 Iterative Property of Reiterative Distributional Chaos

This section deals with the iteration invariance of reiterative distributional chaos. In [23], the properties of DC1,DC2,and DC2$\frac{1}{2}$ were showed to be invariants under iterations for an equicontinuous system $(X, f_{1,\infty})$. In particular, they obtained the following lemma.

Lemma 1 ([23]) *Let $(X, f_{1,\infty})$ be equicontinuous and $x, y \in X$. Let $\hat{f}_{1,\infty}$ be the Nth iteration of $f_{1,\infty}$. Then for any $s > 0$, there is $t_s > 0$ such that for any $t \in (0, t_s]$,*

(i) $F_{x,y}(\hat{f}_{1,\infty}, s) \geq F_{x,y}(f_{1,\infty}, t)$;
(ii) $F^*_{x,y}(\hat{f}_{1,\infty}, s) \geq F^*_{x,y}(f_{1,\infty}, t)$;
(iii) $F^*_{x,y}(\hat{f}_{1,\infty}, t) \leq F^*_{x,y}(f_{1,\infty}, s)$;
(iv) $F_{x,y}(\hat{f}_{1,\infty}, t) \leq F_{x,y}(f_{1,\infty}, s)$.

For convenience of notations, denote $\xi^0_{x,y}(f_{1,\infty}, t) = \eta^0_{x,y}(f_{1,\infty}, t) = 0$,

$$\xi^n_{x,y}(f_{1,\infty}, t) := \left|\{0 \leq i < n : d(f_1^i(x), f_1^i(y)) < t\}\right|,$$

and

$$\eta^n_{x,y}(f_{1,\infty}, t) := \left|\{0 \leq i < n : d(f_1^i(x), f_1^i(y)) \geq t\}\right|,$$

for any $t > 0$ and any $n \geq 1$. It is not hard to check the following properties.

Lemma 2 *Let $f_{1,\infty} = \{f_i\}_{i \geq 1}$ be equicontinuous on X. Given $k \in \mathbb{N}$ and $t > 0$. Then there is $t' > 0$ such that for any $x, y \in X$, the following relations hold:*

(a) $BF^*_{x,y}(f^{[k]}_{1,\infty}, t') \le BF^*_{x,y}(f_{1,\infty}, t)$;
(b) $BF_{x,y}(f_{1,\infty}, t') \le BF_{x,y}(f^{[k]}_{1,\infty}, t)$;
(c) $BF^*_{x,y}(f_{1,\infty}, t') \le BF^*_{x,y}(f^{[k]}_{1,\infty}, t)$;
(d) $BF_{x,y}(f^{[k]}_{1,\infty}, t') \le BF_{x,y}(f_{1,\infty}, t)$.

Proof We only show the relation (a) and leave the others to the reader, since these proofs are similar.

Since $f_{1,\infty} = \{f_i\}_{i\ge 1}$ is equicontinuous, for $t > 0$, there exists $t' > 0$ such that for any $x, y \in X$ and any $i \in \mathbb{N}$,

$$d(f_1^{ki}x, f_1^{ki}y) < t' \implies d(f_1^{ki+j}x, f_1^{ki+j}y) < t, \; j = 0, 1, \cdots, k-1. \tag{4}$$

For any $n, m \in \mathbb{N}$, it follows from (4) that

$$\begin{aligned} k\left(\xi^{n+m}_{x,y}(f^{[k]}_{1,\infty}, t') - \xi^{m}_{x,y}(f^{[k]}_{1,\infty}, t')\right) &\le \xi^{k(n+m)}_{x,y}(f_{1,\infty}, t) - \xi^{km}_{x,y}(f_{1,\infty}, t) \\ &\le \sup_{l\ge 0}\left\{\xi^{kn+l}_{x,y}(f_{1,\infty}, t) - \xi^{l}_{x,y}(f_{1,\infty}, t)\right\}. \end{aligned}$$

Then

$$\frac{1}{n}\sup_{m\ge 0}\left\{\xi^{n+m}_{x,y}(f^{[k]}_{1,\infty}, t') - \xi^{m}_{x,y}(f^{[k]}_{1,\infty}, t')\right\} \le \frac{1}{kn}\sup_{l\ge 0}\left\{\xi^{kn+l}_{x,y}(f_{1,\infty}, t) - \xi^{l}_{x,y}(f_{1,\infty}, t)\right\},$$

which further implies $BF^*_{x,y}(f^{[k]}_{1,\infty}, t') \le BF^*_{x,y}(f_{1,\infty}, t)$. □

Theorem 3 *Let $f_{1,\infty} = \{f_i\}_{i\ge 1}$ be equicontinuous on X and $k \in \mathbb{N}$. Then*

(i) *$(X, f_{1,\infty})$ is RDC1 (RDC1^+) if and only if $(X, f^{[k]}_{1,\infty})$ is RDC1 (RDC1^+).*
(ii) *$(X, f_{1,\infty})$ is RDC2 if and only if $(X, f^{[k]}_{1,\infty})$ is RDC2.*
(iii) *$(X, f_{1,\infty})$ is RDC$2\frac{1}{2}$ (RDC$2\frac{1}{2}-$) if and only if $(X, f^{[k]}_{1,\infty})$ is RDC$2\frac{1}{2}$ (respectively, RDC$2\frac{1}{2}-$).*

Proof We only prove the case (i) of RDC1. The rest cases can be followed similarly.

(i) (Necessity). Suppose that S is an uncountable RDC1 set of $f_{1,\infty}$. Given $(x, y) \in S^2$ $(x \ne y)$. There are $c, \delta > 0$ so that $F^*_{x,y}(f_{1,\infty}, t) \ge c$ and $BF_{x,y}(f_{1,\infty}, \delta) = 0$ for any $t > 0$. For $\delta > 0$, it follows from Lemma 2 (d) that there is $\delta' > 0$ such that $BF_{x,y}(f^{[k]}_{1,\infty}, \delta') \le BF_{x,y}(f_{1,\infty}, \delta) = 0$. Given $\varepsilon > 0$, there exists $\varepsilon' > 0$ satisfying $F^*_{x,y}(f^{[k]}_{1,\infty}, \varepsilon) \ge F^*_{x,y}(f_{1,\infty}, \varepsilon') \ge c$. Therefore (x, y) is a RDC1 pair of $f^{[k]}_{1,\infty}$ and S is a RDC1 set of $f^{[k]}_{1,\infty}$.

(Sufficiency). Assume that (u, v) is a RDC1 pair of $f^{[k]}_{1,\infty}$, that is, there exist $c, \delta > 0$ so that $F^*_{u,v}(f^{[k]}_{1,\infty}, t) \ge c$, $BF_{u,v}(f^{[k]}_{1,\infty}, \delta) = 0$ for any $t > 0$. By Lemma 2 (b), for $\delta > 0$, there is $\delta' > 0$ such that $BF_{u,v}(f_{1,\infty}, \delta') \le BF_{u,v}(f^{[k]}_{1,\infty}, \delta) = 0$. Given $\varepsilon > 0$, there exists $\varepsilon' > 0$ satisfying $F^*_{u,v}(f_{1,\infty}, \varepsilon) \ge F^*_{u,v}(f^{[k]}_{1,\infty}, \varepsilon') \ge c$. So (u, v) is a RDC1 pair of $f_{1,\infty}$. This indicates that any RDC1 set S of $f^{[k]}_{1,\infty}$ is also RDC1 of $f_{1,\infty}$. Therefore $f_{1,\infty}$ is RDC1. □

The results of Theorem 3 extend those of distributional chaos obtained in [29] and [23]. From the proof of Theorem 3, it indicates that $S \subset X$ is strong RDCi for $f_{1,\infty}$ if and only if it is strong RDCi of $f_{1,\infty}^{[k]}$, under the equicontinuity of $\{f_i\}_{i\geq 1}$.

Corollary 1 *Let $f_{1,\infty} = \{f_i\}_{i\geq 1}$ be equicontinuous on X and $k \in \mathbb{N}$. Let $i \in \{1, 1^+, 2, 2\frac{1}{2}, 2\frac{1}{2}-\}$. Then $(X, f_{1,\infty})$ is uniformly RDC i if and only if $(X, f_{1,\infty}^{[k]})$ is uniformly RDC i.*

In [10], Dvořáková showed that if a continuous self-map f of a compact space has a DC3 pair, then so does its k-th iteration f^k, for any $k \geq 2$. Actually, this result is true for continuous maps acting on general metric spaces. However, this property does not hold for non-autonomous discrete systems.

Example 4 Consider the maps $f_i(x) = a_i x$ on $\mathbb{R}$, where $a_i = 1$ if $i = 2$ or $i \in [(2k+1)!+1, (2k+2)!]$ for $k \geq 1$; $a_i = 2$ if $i = 1$ or i is even in $[(2k)!+1, (2k+1)!]$ for $k \geq 1$; otherwise, let $a_i = \frac{1}{2}$. Then for any $x \in \mathbb{R}$, $f_1^i(x) = x$ for any odd integer $i \in [(2k)!+1, (2k+1)!]$, $k \geq 1$, and $f_1^i(x) = 2x$ for other $i > 3$. Given $x, y \in \mathbb{R}$ with $x \neq y$, let $a = |x - y|$. It is easy to see that $F_{x,y}(f_{1,\infty}, t) = 0 < F_{x,y}^*(f_{1,\infty}, t) = \frac{1}{2}$ for any $t \in (a, 2a]$. Thus, $\mathbb{R}$ is a DC3 set of $f_{1,\infty}$. However, $f_{1,\infty}^{[2]}$ has no DC3 pairs, since $f_1^{2i}(x) = 2x$ for any $i \geq 1$ and any $x \in \mathbb{R}$.

Acknowledgements This work was supported by the National Natural Science Foundation of China (No. 11801096, 11901091), Natural Science Foundation of Guangdong Province (No. 2020A1515010339), Natural Science Research Project of Guangdong Province (No. 2017KQNCX122), the Science and Technology Innovation Team of Education Department of Sichuan for Dynamical System and its Applications (No. 18TD0013) and Youth Science and Technology Innovation Team of Southwest Petroleum University for Nonlinear Systems (No. 2017CXTD02). The authors would like to express their gratitude to the anonymous referees for their valuable comments and suggestions.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Akin, E., Kolyada, S.: Li-Yorke sensitivity. Nonlinearity **16**, 1421–1433 (2003)
2. Balibrea, F., Oprocha, P.: Weak mixing and chaos in non-autonomous discrete systems. Appl. Math. Lett. **25**, 1135–1141 (2012)
3. Balibrea, F., Smítal, J., Štefánková, M.: The three versions of distributional chaos. Chaos Solit. Fract. **23**, 1581–1583 (2005)
4. Bernardes, N.C., Bonilla, A., Müller, V., Peris, A.: Distributional chaos for linear operators. J. Funct. Anal. **265**, 2143–2163 (2013)
5. Bernardes, N.C., Bonilla, A., Müller, V., Peris, A.: Li-Yorke chaos in linear dynamics. Ergod. Theory Dyn. Syst. **35**, 1723–1745 (2015)
6. Bernardes, N.C., Bonilla, A., Peris, A.: Mean Li-Yorke chaos in Banach spaces. J. Funct. Anal. **3**, 108343 (2020)
7. Bernardes, N.C., Bonilla, A., Peris, A., Wu, X.: Distributional chaos for operators on Banach spaces. J. Math. Anal. Appl. **459**, 797–821 (2018)
8. Bonilla, A., Kostić, M.: Reiterative distributional chaos on banach spaces. Int. J. Bifurc. Chaos **29**, 1950201 (2019)

9. Downarowicz, T., Lacroix, Y.: Measure-theoretic chaos. Ergod. Th. Dyn. Syst. **34**, 110–131 (2014)
10. Dvořáková, J.: On a problem of iteration invariants for distributional chaos. Commun. Nonlin. Sci. Numer. Simul. **17**, 785–787 (2012)
11. Grosse-Erdmann, K.G., Peris, A.: Linear Chaos. Springer, Berlin (2011)
12. Hantáková, J.: Iteration problem for distributional chaos. Int. J. Bifurc. Chaos **27**, 1750183 (2017)
13. Kolyada, S., Snoha, L.: Topological entropy of nonautonomous dynamical systems. Random Comput. Dyn. **4**, 205–233 (1996)
14. Kostić, M.: Reiterative m_n-Distributional Chaos of Type s in Fréchet Spaces. Bull. Malays. Math. Sci. Soc. **43**, 3963–4005 (2020)
15. Kwietniak, D.: Topological entropy and distributional chaos in hereditary shifts with applications to spacing shifts and Beta shifts. Discrete Contin. Dynam. Syst. **33**, 2451–2467 (2013)
16. Li, J., Ye, X.: Recent development of chaos theory in topological dynamics. Acta Mathematica Sinica **32**, 83–114 (2016)
17. Li, T., Yorke, J.: Period three implies chaos. Amer. Math. Monthly **82**, 985–992 (1975)
18. Martínez-Giménez, F., Oprocha, P., Peris, A.: Distributional chaos for operators with full scrambled sets. Math. Z. **274**, 603–612 (2013)
19. Miralles, A., Murillo-Arcila, M., Sanchis, M.: Sensitive dependence for nonautonomous discrete dynamical systems. J. Math. Anal. Appl. **463**, 268–275 (2018)
20. Oprocha, P.: Distributional chaos revisited. Trans. Amer. Math. Soc. **361**, 4901–4925 (2009)
21. Salman, M., Das, R.: Multi-transitivity in non-autonomous discrete systems. Topol. Appl. **278**, 107237 (2020)
22. Schweizer, B., Smítal, J.: Measures of chaos and a spectral decomposition of dynamical systems on the interval. Trans. Amer. Math. Soc. **344**(2), 737–754 (1994)
23. Shao, H., Shi, Y.: Some weak versions of distributional chaos in non-autonomous discrete systems. Commun. Nonlinear Sci. Numer. Simulat. **70**, 318–325 (2019)
24. Shao, H., Shi, Y., Zhu, H.: On distributional chaos in non-autonomous discrete systems. Chaos Soliton Fractals **107**, 234–243 (2018)
25. Shi, Y., Chen, G.: Chaos of time-varying discrete dynamical systems. J. Differ. Equ. Appl. **15**, 429–449 (2009)
26. Snoha, L.: Dense chaos. Comment. Math. Univ. Carolin. **33**(4), 747–752 (1992)
27. Tan, F., Xiong, J.: Chaos via Furstenberg family couple. Topol. Appl. **156**, 525–532 (2009)
28. Tang, X., Chen, G., Lu, T.: Some iterative properties of $(\mathcal{F}_1, \mathcal{F}_2)$-chaos in non-autonomous discrete systems. Entropy **20**, 188 (2018)
29. Wang, J.: On the iteration invariance of distributional chaos of type $2\frac{1}{2}$ in non-autonomous discrete systems. Qual. Theor. Dyn. Syst. **18**, 711–721 (2019)
30. Wang, L., Huan, S., Huang, G.: A note on Schweizer-Smital chaos. Nonlin. Anal. **68**, 1682–1686 (2008)
31. Wang, L., Huang, G., Huan, S.: Distributional chaos in a sequence. Nonlin. Anal. **67**(7), 2131–2136 (2007)
32. Wu, X., Chen, G.: Answering two open problems on Banks theorem for non-autonomous dynamical systems. J. Differ. Equ. Appl. **25**, 1790–1794 (2019)
33. Wu, X., Chen, G., Zhu, P.: Invariance of chaos from backward shift on the Köthe sequence space. Nonlinearity **27**, 271–288 (2014)
34. Wu, X., Zhu, P.: Chaos in a class of non-autonomous discrete systems. Appl. Math. Lett. **26**, 431–436 (2013)
35. Wu, X., Zhu, P.: Li-Yorke chaos of backward shift operators on Köthe sequence spaces. Topol. Appl. **160**, 924–929 (2013)
36. Yin, Z., He, S., Huang, Y.: On Li-Yorke and distributionally chaotic direct sum operators. Topol. Appl. **239**, 35–45 (2018)
37. Yin, Z., Huang, Y.: Remarks on multiples of distributionally chaotic operators. Studia Math. **243**, 25–52 (2018)
38. Yin, Z., Wei, Y.: Recurrence and topological entropy of translation operators. J. Math. Anal. Appl. **460**, 203–215 (2018)