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Some Applications of the Poincaré–Bendixson Theorem

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Abstract

We consider a $\mathcal{C}^1$ vector field X defined on an open subset U of the plane with compact closure. If X has no singular points and if U is simply connected, a weak version of the Poincaré–Bendixson theorem says that the limit sets of X in U are empty but that one can define non empty extended limit sets contained in the boundary of U. We give an elementary proof of this result, independent of the classical Poincaré–Bendixson theorem. A trapping triangle $\mathcal{T}$ based at p, for a $\mathcal{C}^1$ vector field X defined on an open subset $\mathcal{U}$ of the plane, is a topological triangle with a corner at a point p located on the boundary $\partial\mathcal{U}$ and a good control of the tranversality of X along the sides. The principal application of the weak Poincaré–Bendixson theorem is that a trapping triangle at p contains a separatrix converging toward the point p. This does not depend on the properties of X along $\partial\mathcal{U}$. For instance, X could be non differentiable at p, as in the example presented in the last section.

Keywords Weak Poincaré–Bendixson theorem · Extended limit sets · Trapping triangles · Separatrix

Mathematics Subject Classification Primary 34C05; Secondary 34A26

1 Introduction

We consider a $\mathcal{C}^1$ vector field X on an open subset $U \subset \mathbb{R}^2$. This vector field is integrable with a $\mathcal{C}^1$ flow $\varphi(t, m)$: for each $m \in U$, the map $t \mapsto \varphi(t, m)$ is the maximal trajectory with initial condition $\varphi(0, m) = m$ and is defined for $t \in (\tau_-(m), \tau_+(m))$, interval whose end points satisfy $-\infty \leq \tau_-(m) < 0 < \tau_+(m) \leq +\infty$.

Of primordial importance is to know the future of the trajectory when $t \to \tau_+(m)$ and its past when $t \to \tau_-(m)$. To this end, one introduces the limit sets $\omega(m)$, $\alpha(m)$ of X defined by:

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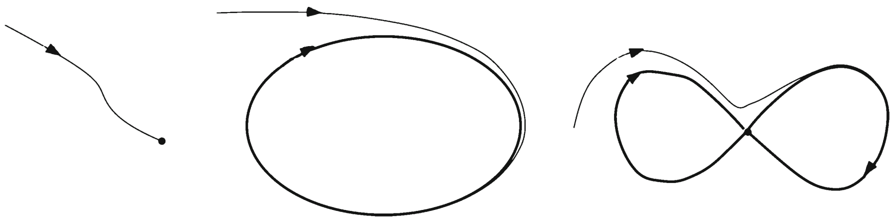


Fig. 1 Possible ω-limit sets

$$\omega(m) = \{p \in U \mid \exists (t_n) \to \tau_+(m), \text{ such that } (\varphi(t_n, m)) \to p\},$$

and

$$\alpha(m) = \{p \in U \mid \exists (t_n) \to \tau_-(m), \text{ such that } (\varphi(t_n, m)) \to p\},$$

The image of the trajectory: $\gamma = \gamma_m = \varphi((\tau_-(m), \tau_+(m)), m)$ is the orbit of m. The limit sets depend only on γ and we can denote them by: $\omega(\gamma), \alpha(\gamma)$. More precisely, if $\gamma_m^+ = \varphi([0, \tau_+(m)), m)$ is the positive half-orbit and $\gamma_m^- = \varphi((\tau_-(m)), 0], m)$ the negative half-orbit, $\omega(m)$ depends only on γ_m^+ and $\alpha(m)$ just of γ_m^-. See [4,6] for more information.

A description of the possible limit sets was given by Poincaré in [11] and this result was proved by Bendixson in [1]. More recent proofs of the so-called Poincaré–Bendixson Theorem may be found in [7,9]. The following version appeared in [6]:

Theorem 1.1 (Poincaré–Bendixson Theorem) *Let X be a $\mathcal{C}^1$ vector field X defined on an open set $U \subset \mathbb{R}^2$. Assume that the singular points of X are isolated. Consider a point $m \in U$ such that γ_m^+ is contained into a compact subset of U. Then $\omega(m)$ is either a singular point, a periodic orbit or a graphic: a topological immersion of the circle S^1, union of a finite number of regular orbits connecting a finite number of singular points (see Fig. 1). A similar result stands for the α-limit sets.*

Remark 1.2 Moreover for real analytic vector fields, it is possible to extend Theorem 1.1 to vector fields with non-isolated singularities (see [10]).

If γ_m^+ is not contained in a compact subset of U, it may happen that $\omega(m)$ is empty. But, if $\bar{U}$ is compact, it is always possible to extract from any sequence $(\varphi(t_n, m))$ defined as above, a subsequence converging toward a point of $\bar{U}$. Then, in the case that $\bar{U}$ is compact, it is natural to extend the definition of the limit sets by taking the limit of sequences $(\varphi(t_n, m))$ in $\bar{U}$. In this way, one defines what we call here the *extended limit sets* $\bar{\omega}(m)$ and $\bar{\alpha}(m)$ (see Definition 2.1). *These extended limit sets are always non empty.*

Using this notion of extended limit sets, Theorem 1.3 below gives conditions on U such that the limit sets in $\bar{U}$ are always contained in ∂U (or equivalently, such that the usual limit sets in U are empty). We call this result: *the weak Poincaré-Bendixson Theorem*, because it can be seen in some cases as a rather trivial particular case of the

Theorem 1.1. In fact we will give in Sect. 3 a simple proof of it, independent of the Theorem 1.1.

Theorem 1.3 (Weak Poincaré–Bendixson Theorem) *Let X be a $\mathcal{C}^1$ vector field defined on an open subset U of $\mathbb{R}^2$ with a compact closure $\bar{U}$ Assume that X has no singular points and that U is simply connected. Let m be a point in U. Then, the limit sets $\omega(m)$ and $\alpha(m)$ are empty or equivalently the extended limit sets $\bar{\omega}(m)$ and $\bar{\alpha}(m)$ are contained in the boundary $\partial U = \bar{U} \setminus U$ (see Definition 2.1).*

Remark 1.4 If one drops the condition that X has no singular points or that U is simply connected, it is very easy to find examples of extended limit sets not contained in ∂U.

Remark 1.5 In Theorem 1.3 we consider the closure $\bar{U}$ in $\mathbb{R}^2$. To say that $\bar{U}$ is compact is equivalent to say that U is bounded. We could consider the closure in any compactification of $\mathbb{R}^2$ and prove the same result as Theorem 1.3, with exactly the same proof. Taking the closure in a compactification of $\mathbb{R}^2$ would allow for instance to study trajectories in U with a limit set at infinity. It may be observed that there are different possible compactifications. The more usual compactifications in geometry are the Alexandroff compactification where $\mathbb{R}^2$ is identified with $\mathbb{C} \setminus \{\infty\}$ and the Lyapunov–Poincaré compactification where $\mathbb{R}^2$ is identified with the interior of the trigonometric disk (see [6]). In this paper, we are just interested in limit sets at finite distance. Then, Theorem 1.3 is just stated for bounded open sets.

The principal interest of Theorem 1.3 is that there is no assumption on the vector field X along ∂U. For instance, X could be non differentiable at some points of ∂U and the classical Poincaré-Bendixson Theorem could not be applied in neighborhoods of such points. In Section 5, we will present an example of a vector field with a non-differentiable singular point, which stems from combustion theory (see [2]). Theorem 1.3 can be applied by putting the singular point at the boundary of the domain of study.

The extended limit sets are defined in Sect. 2, where some properties are given and where they are compared with the usual limit sets. In Sect. 3, we give the proof of Theorem 1.3. This very simple proof is based on the non-recurrence property satisfied by a vector field without singular points, defined on a simply connected open set U. In Sect. 4 we present some applications of Theorem 1.3. The most important one is the notion of *trapping triangle* which gives conditions to have a trajectory tending toward a point of the boundary.

This is illustrated in some detail in Sect. 5 where we recall how trapping triangles can be used to obtain interesting properties for a non-differentiable vector field introduced in [2]. The appendix is devoted to a sketch of proof of the Jordan-Schoenflies Theorem for $\mathcal{C}^1$ vector fields. This theorem enters as a key argument in the proof of Theorem 1.3.

2 Extended Limit Sets

In the whole section we assume that X is a $\mathcal{C}^1$ vector field, defined on an open set U of $\mathbb{R}^2$ with a compact closure $\bar{U}$.

Definition 2.1 (Extended limit sets) The $\bar{\omega}$-limit set of m (in $\bar{U}$) is the compact subset of $\bar{U}$ defined as:

$$\bar{\omega}(m) = \{p \in \bar{U} \mid \exists (t_n) \to \tau_+(m) \text{ such that } \varphi(t_n, m) \to p\}.$$

The $\bar{\alpha}$-limit set $\alpha(m)$ (in $\bar{U}$) is the $\bar{\omega}$-set of m, (in $\bar{U}$), for the field $-X$ (we have just to replace $\tau_+(m)$ by $\tau_-(m)$ in the above definition).

Remark 2.2 Extended limit sets $\bar{\omega}(m), \bar{\alpha}(m)$ are different from the usual ones, since we consider the limit values in $\bar{U}$ and not in U. It is the reason why we call them extended limit sets. We will write $\omega(m), \alpha(m)$ for the usual limit sets in U. Clearly, one has that $\omega(m) = \bar{\omega}(m) \cap U$ and $\alpha(m) = \bar{\alpha}(m) \cap U$. In fact, in Theorem 1.3 we are interested in a situation where the usual limit sets are empty, or equivalently where the extended limit sets are contained in ∂U.

It is easy to see that the extended limit sets are non-empty compact subsets of $\bar{U}$. The more important property is that they are limits of the trajectory of m in the Haudorff sense, for positive or negative time. We recall that the Hausdorff distance between a point p and a non-empty compact subset A of $\mathbb{R}^2$ is given by:

$$\mathrm{dist}_H(p, A) = \mathrm{Inf}\{||m - p|| \mid m \in A\},$$

where $|| \cdot ||$ is the Euclidean norm of $\mathbb{R}^2$. One has the following result:

Lemma 2.3 *For $m \in U$: $\mathrm{dist}_H(\varphi(t, m), \bar{\omega}(m)) \to 0$ when $t \to \tau_+(m)$. There is a similar result for $\bar{\alpha}(m)$ when $t \to \tau_-(m)$.*

Proof We have just to consider the case of $\bar{\omega}(m)$. Assume that $\mathrm{dist}(\varphi(t, m), \bar{\omega}(m)) \nrightarrow 0$. This means that there exist $\varepsilon_0 > 0$ and a sequence $(t_n) \to \tau_+(m)$ such that $\forall n$ one has that $\varphi(t_n, m) \in K(\varepsilon_0) = \{p \in \bar{U} \mid \mathrm{dist}(p, \bar{\omega}(m)) \geq \varepsilon_0\}$. As $K(\varepsilon_0)$ is compact, we can extract a subsequence $(t'_i) = (t_{n_i})$ such that the sequence $(\varphi(t'_i, m))$ converges toward some point $p_0 \in K(\varepsilon_0)$, while $(t'_i) \to \tau_+(m)$. This point p_0 belongs to $\bar{\omega}(m)$. We arrive to a contradiction, since $\mathrm{dist}_H(p_0, \bar{\omega}(m)) \geq \varepsilon_0 > 0$ by passing to the limit. □

Remark 2.4 We will use the following particular case of Lemma 2.3: to say that $\bar{\omega}(m)$ is reduced to a single point p is equivalent to say that $\varphi(t, m) \to p$ when $t \to \tau_+(m)$.

With a similar proof as in Lemma 2.3, one has the following:

Lemma 2.5 *The extended limit sets are connected subsets of $\bar{U}$.*

3 Proof of the weak Poincaré–Bendixson Theorem

We want to present a direct and simple proof, independent of the classical Poincaré-Bendixson Theorem. In fact this proof will just use an easy form of the non-existence of non-trivial recurrence property, adapted to the context. It is given in Lemma 3.1.

Of course, the non-trivial recurrence property is also a key ingredient in the proof of the classical Poincaré-Bendixson Theorem (see [7] or [9] for instance). One has the following:

Lemma 3.1 *Let X be a $\mathcal{C}^1$ vector field defined on a simply connected open set U, without singular points. Then, an orbit of X has at most one intersection point with an open transverse section to X, contained in U.*

Proof Let γ be an orbit of X and $\Sigma \subset U$ be an open transverse section to X. Assume that $\gamma \cap \Sigma$ contains at least two points. Let p, q be two such points, consecutive on γ. We denote by $\gamma(p, q)$ the closed segment of the orbit between p, q and by $\Sigma(p, q)$ the closed segment on Σ between p, q. Since the points p, q are consecutive on γ, one has that $\gamma(p, q) \cap \Sigma(p, q)$ is the set with two points $\{p, q\}$. This means that $\Gamma = \gamma(p, q) \cup \Sigma(p, q)$ is a $\mathcal{C}^1$-piecewise simple curve in U. It is easy to smoothen Γ in order to obtain a $\mathcal{C}^1$ curve $\tilde{\Gamma}$, transverse to X. This curve can be chosen $\mathcal{C}^0$ arbitrarily near Γ, and then contained in U (see Fig. 2: the segment of orbit $\gamma(p, q)$ is replaced by a transverse arc of curve $\tilde{\gamma}(p', q)$ inside a thin flow box T along $\gamma(p, q)$; this flow box is a curved rectangle with corners the points p, q, q', p'; a detailled proof can be found in [12]). It follows from the $\mathcal{C}^1$ Jordan-Schoenflies Theorem that this curve bounds a topological disk $\tilde{D} \subset U$. As X is transverse to $\partial \tilde{D} = \tilde{\Gamma}$, the disk $\tilde{D}$ must contain a singular point of X, as it follows for instance from the Brouwer Theorem. We have thus arrived to a contradiction. □

Remark 3.2 The Jordan-Schoenfies Theorem is true for any topological closed curve embedded in $\mathbb{R}^2$, also called a Jordan curve. This theorem says that there is a homeomorphism of $\mathbb{R}^2$ to itself sending the trigonometric circle Γ_0 onto Γ. As a consequence, Γ bounds a topological disk contained in a neighborhood U of Γ. The proof in this general $\mathcal{C}^0$ context is rather delicate (nevertheless, one can see [3] for an elementary proof). The proof is much easier for a $\mathcal{C}^1$ closed curve Γ. Since it is the only case used in this paper, a sketch of proof is given in Appendix.

Remark 3.3 One can find in [8] an easy topological proof of the Poincaré-Hopf formula, whose Brouwer Theorem is easily deduced, since the Euler characteristic of the disk is equal to 1.

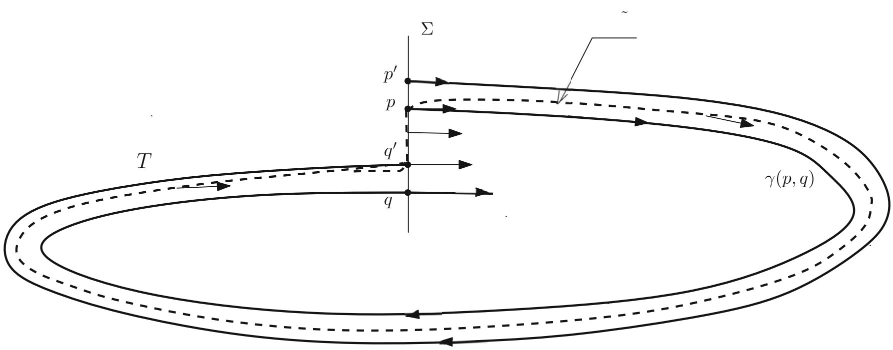


Fig. 2 Construction of $\tilde{\Gamma}$

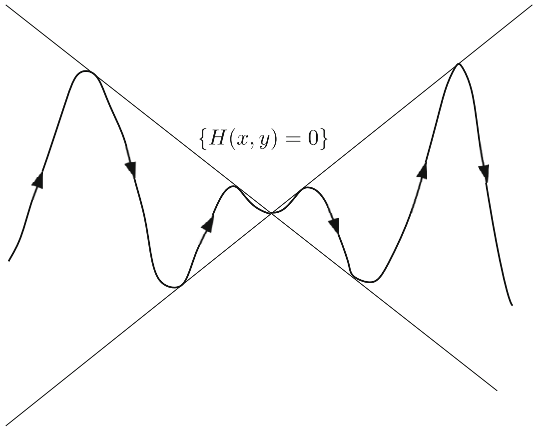


Fig. 3 The orbit $\{H(x, y) = 0\}$

Lemma 3.1 easily implies that:

Proof of Theorem 1.3 Consider a point $m \in U$. We have just to prove that $\bar{\omega}(m) \subset \partial U$: the proof for $\bar{\alpha}(m)$ is the same, changing X by $-X$.

We prove this result by contradiction. Let us assume that a point $p \in \bar{\omega}(m)$ belongs to U. There exists a sequence $(t_n) \to \tau_+$ such that $\varphi(t_n, m) \to p$. By hypothesis this point is regular: $(X(p) \neq 0)$. Take a flow-box W in U, diffeomorphic to a closed rectangle $\Sigma \times I$, centered at $(0, 0) \in \mathbb{R}^2$, where $p = (0, 0)$ and $\Sigma = \Sigma \times \{0\}$ is a transverse section to X. For n large enough, it is easy to change sligthly t_n in order that $\varphi(t_n, m) \in \Sigma$. As a consequence the half-orbit $\gamma_+(m)$ cuts Σ in infinitely many points. This contradicts Lemma 3.1. Then, one has that $\omega(m) = \bar{\omega}(m) \cap U = \emptyset$, i.e. that $\bar{\omega}(m) \subset \partial U$. □

4 Applications of the weak Poincaré–Bendixson Theorem

Applications of the weak Poincaré-Bendixson Theorem depend on the properties that one assumes for the vector field X on the boundary of U. It follows from Lemma 2.5 that an extended limit set is a compact connected subset of ∂U. Then, if ∂U is a topological curve, an extended limit set is either an isolated point or it is homeomorphic to a closed interval. This last possibility may occur when the properties of X are rather wild near the boundary. For instance, consider the Hamiltonian vector field X_H of Hamiltonian function $H(x, y) = y - x \sin x$. This Hamiltonian vector field has no singular points in the whole plane. Each trajectory oscillates indefinitely between a pair of lines $\{y \pm x = \text{Const.}\}$ (see Fig. 3). If we take the direct image of X_H by a smooth diffeomorphism of $\mathbb{R}^2$ onto the open disk U of radius 1, preserving the radial directions, we obtain a smooth vector field on U, whose limit sets are one of the intervals $\{-\frac{\pi}{2} \leq \theta \leq +\frac{\pi}{2}\}$ or $\{\frac{3\pi}{2} \leq \theta \leq +\frac{5\pi}{2}\}$ on the trigonometric circle.

The above example is of course rather pathological. We are more interested in finding conditions such that a trajectory in U tends toward a point of the boundary ∂U. There is a circumstance when this occurs rather trivially:

Lemma 4.1 *Let X be a $\mathcal{C}^1$ vector field defined on an open subset U of $\mathbb{R}^2$ with a compact closure. Let p be a point in $\bar{\omega}(m) \cap \partial U$, for some $m \in U$. Assume that X can be extended in a neighbohood W of p in $\mathbb{R}^2$ as a $\mathcal{C}^1$ vector field, that will still be called X. Also assume that ∂U is a regular $\mathcal{C}^1$ curve in a neighborhood of p and that $X(p) \neq 0$ is transverse to it. Then $\bar{\omega}(m) = \{p\}$ and the trajectory of the extended vector field X arrives to p at the time $\tau_+(m)$, which is the finite positive limit time of the trajectory in U.*

Proof We choose W to be a flow box of the extended vector field X, diffeomorphic to $[-\varepsilon_0, \varepsilon_0] \times [-\delta_0, \delta_0]$, where $p = (0, 0)$. The intervals $[-\varepsilon_0, \varepsilon_0] \times \{\delta\}$ are segments of orbits and the intervals $\{\varepsilon\} \times [-\delta_0, \delta_0]$ are transverse sections, for all $(\varepsilon, \delta) \in [-\varepsilon_0, \varepsilon_0] \times [-\delta_0, \delta_0]$. As $p \in \bar{\omega}(m)$ there is a time t_0 such that $\varphi(t_0, m) \in W$, and more precisely $\varphi(t_0, m) = (\varepsilon_{t_0}, 0)$ for some $\varepsilon_{t_0} \in [\varepsilon_0, 0)$. Then, for all $t \geq t_0$, $\varphi(t, m)$ must be also a point of the same type $(\varepsilon_t, 0)$ for some $\varepsilon_t \in [\varepsilon_{t_0}, 0)$. The conclusions of the lemma clearly follow. □

4.1 Existence of Flow Boxes

Let X be a $\mathcal{C}^k$ vector field, with $k \geq 1$, defined on an open set U. The usual Flow-Box Theorem gives a normal form for X in a neighborhood of any regular point: if $m \in U$ is such that $X(m) \neq 0$, there exists a neighborhood W of $(0, 0) \in \mathbb{R}^2$ (with coordinates (x, y)) and a $\mathcal{C}^k$ diffeomorphism Φ of W into U, sending the vector field $\frac{\partial}{\partial x}$ on the vector field X. $T = \Phi(W) \subset U$ is called a *flow box* of X. Using Theorem 1.3, we can prove the following:

Proposition 4.2 *Let X be a $\mathcal{C}^k$ vector field, with $k \geq 1$, defined on an open set U, without singular points. Let $T \subset U$ be a $\mathcal{C}^k$-piecewise rectangle. Assume that T has two sides $[A, B]$, $[DC]$ which are segments of orbits and the two sides $[AD]$, $[BC]$ which are transverse sections such that X is pointing inward T along $[AD]$ and outward T along $[BC]$ (see Fig. 4). Then the trajectory starting at a point $m \in [AD]$ arrives at a point of $[BC]$ in a finite time $t(m)$. The function $t(m)$ is $\mathcal{C}^k$.*

Proof We can apply Theorem 1.3 to the interior $\mathrm{Int}(T)$ of T. If $m \in [AD]$, its trajectory passes through a nearby point m' located in $\mathrm{Int}(T)$. It follows from Theorem 1.3 that $\bar{\omega}(m') \in \partial T$. Since $\bar{\omega}(m')$ cannot contain points of the open arcs of trajectory $]AB[$ and $]DC[$, nor points of $[AB]$ because X is entering along this side, we have that $\bar{\omega}(m') \subset [BC]$. It follows from Lemma 4.1 that $\bar{\omega}(m')$ is just a point of $[BC]$ and as a consequence, there is a finite time $t(m)$ such that $\varphi(t(m), m) \in [BC]$. As the trajectory of m arrives transversally on $[BC]$, we can use the Cauchy Theorem in class $\mathcal{C}^k$ (which states that the flow is a $\mathcal{C}^k$ map) and the Inverse Function Theorem to show that $t(m)$ is a function of class $\mathcal{C}^k$. □

In fact a rectangle as in Proposition 4.2 is a flow box. More precisely one has the following:

Corollary 4.3 *Let X, U and T and $t(m)$ as in Proposition 4.2. Parametrize $[AD]$ by $y \in [0, 1]$. Let W be the curved rectangle of $\mathbb{R}^2$ defined by $W = \{(x, y) \in \mathbb{R}^2 \mid y \in$*

Fig. 4 A flow box

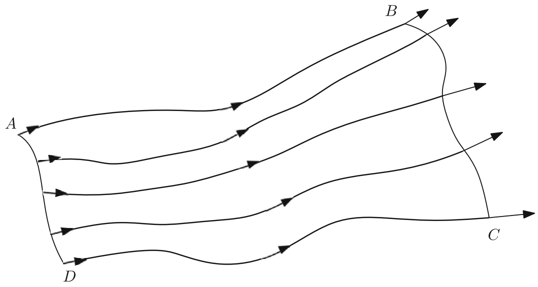
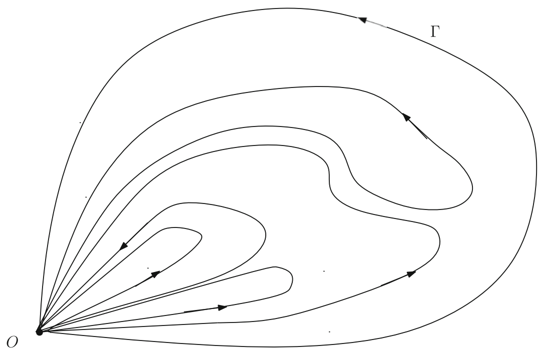


Fig. 5 Closed nodal region



$[0, 1], x \in [0, t(y)]\}$. *Then, the map* $(x, y) \mapsto \varphi(x, y)$ *is a* $\mathcal{C}^k$ *diffeomorphism from* W *onto* T*, sending* $\frac{\partial}{\partial x}$ *on* X.

Proof We identify $[AD]$ with the interval $[0, 1]$ parametrized by y. For any $y \in (01)$ the arc of orbit starting at y is disjoint from the sides $[AB]$ and $[DC]$. The same argument than the one used in Proposition 4.2 shows that if $y \neq y'$ are two points on $[AD]$, then the arcs of trajectories in T, starting at y and y' are disjoint. As the flow is injective on each arc, one has that the map $(x, y) \mapsto \varphi(x, y)$ is one-to-one from W onto T. As a consequence of the Cauchy Theorem, this map is $\mathcal{C}^k$. Since $D\varphi(x, y)[\frac{\partial}{\partial x}] = X(\varphi(x, y))$ and $D\varphi(x, y)[\frac{\partial}{\partial y}]$ is a vector tranverse to $X(\varphi(x, y))$, the map $(x, y) \to \varphi(x, y)$ has a maximal rank at each $(x, y) \in W$. Then, this map is a $\mathcal{C}^k$ diffeomorphism from W onto T. Finally, since $D\varphi(x, y)[\frac{\partial}{\partial x}] = X(\varphi(x, y))$, this map sends the vector field $\frac{\partial}{\partial x}$ to the vector field X. □

4.2 Closed Nodal Region

We consider a vector field X on an open set U. We assume that X has a singular point O in U and that X is $\mathcal{C}^1$ in $U \setminus \{O\}$. The following notion was introduced by Bendixson in [1]:

Definition 4.4 (Bendixson) A closed nodal region at O for X is a topological disk $D(\Gamma)$ pinched at a singular point O, bounded by an orbit Γ such that $\omega(\Gamma) = \alpha(\Gamma) = \{O\}$ and consisting of orbits with the same limit property (see Fig. 5).

The following result was proved by Bendixson in [1]:

Lemma 4.5 *Let X be a vector field on a simply connected open set U. Assume that X has a unique singular point O in U and that X is $\mathcal{C}^1$ in $U \setminus \{O\}$. One also assumes that X is a has an orbit Γ such that $\omega(\Gamma) = \alpha(\Gamma) = \{O\}$. Then $\Gamma \cup \{O\}$ is the boundary in U of a topological disk which is a closed nodal region at O for X.*

Proof It follows from the Jordan–Schoenflies Theorem that $\Gamma \cup \{O\}$ is the boundary of a topological disk D in U. Since the interior of this disk D is simply connected and contains no singular point, we can apply Theorem 1.3 to it: the extended limit sets of any point of the interior of D are contained in $\partial D = \Gamma \cup \{O\}$. Since these limit sets cannot contain any point of Γ, they are reduced to $\{O\}$. □

Remark 4.6 Elliptic sectors, used in the classification of the phase portrait of the vector fields near an isolated singular point (see [5,12]), are simple examples of closed nodal regions. We can introduce an order in the set of orbits contained in $D(\Gamma)$, stating that $\tilde{\Gamma}_1$ is less than $\tilde{\Gamma}_2$ if and only if $\tilde{\Gamma}_1 \subset D(\tilde{\Gamma}_2)$. This order is total for an elliptic sector but is just partial in general. As a consequence, the phase portrait inside a general closed nodal region may be much more complicated than the simple 1-parameter family of orbits that one finds inside an elliptic sector. See Fig. 5 for an example of a closed nodal region which is not an elliptic sector.

4.3 Trapping Triangles

We consider a vector field X defined on an open subset U of $\mathbb{R}^2$, with a not necessarily compact closure $\bar{U}$. We assume that X is $\mathcal{C}^1$ on U, but nothing is said about a possible extention of X along the boundary $\partial U = \bar{U} \setminus U$. We look for conditions which could ensure that a trajectory in $\mathcal{U}$ converges toward some point $p \in \partial U$. We will use the following:

Definition 4.7 A trapping triangle $\mathcal{T} = [pqr]$ at $p \in \partial U$, for the vector field X, is a topological triangle contained in $\bar{U}$, with corners p, q, r such that $\mathcal{T} \cap \partial U = \{p\}$ (or equivalently $\mathcal{T} \setminus \{p\} \subset U$). This triangle has three sides $[pq]$, $[p, r]$ and $[qr]$. The arcs $(pq]$, $(p, r]$ and $[qr]$ are $\mathcal{C}^1$ regular arcs (i.e. contained into regular open curves of class $\mathcal{C}^1$) . We assume that X has no singular point in the interior of $\mathcal{T}$ and that X is tranverse to $(p, q]$, $[r, p)$ and $[qr]$. Moreover, we assume that X points outside $\mathcal{T}$ along (p, q), (p, r) and inside $\mathcal{T}$ along (q, r) (see Fig. 6).

As a consequence of Theorem 1.3, one has the following result:

Lemma 4.8 *Let $\mathcal{T} = [pqr]$ be a trapping triangle as in Definition 4.7. There exist points $m \in (q, r)$ whose trajectories $\varphi(t, m)$ remain in $\mathcal{T}$ for all times and tends toward p for $t \to \tau^+(m)$. Trajectories starting at other points of (q, r) cut the side $(p, q]$ or the side $(p, r]$ after a finite time.*

Proof We begin by extending slightly the triangle $\mathcal{T}$ into a new triangle $\mathcal{T}' = [pq'r']$, with the same properties than $\mathcal{T}$, by taking a $\mathcal{C}^1$ regular arc $[q', r']$ near $[q, r]$ (see Fig. 6). We call $\mathcal{U}$ the interior of $\mathcal{T}'$. It is clear that the pair $(\mathcal{U}, X)$ satisfies the statements of Theorem 1.3 and one considers X restricted to $\mathcal{U}$ in the rest of the proof. Let m be a point of (q, r). As $(q, r) \subset \mathcal{U}$, by Theorem 1.3 one has that $\bar{\omega}(m) \subset [p, q'] \cup$

Fig. 6 Trapping Triangle

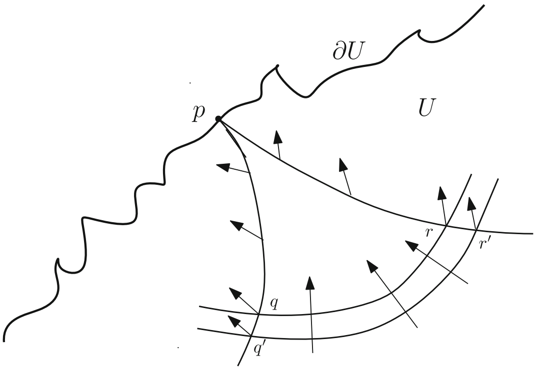
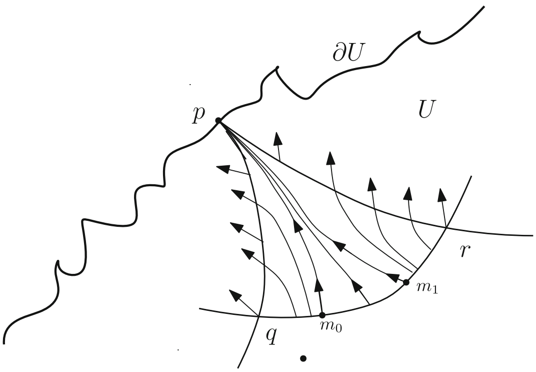


Fig. 7 Dynamics inside the trapping triangle



$[p, r'] \cup [q', r']$. As a consequence of the direction of X along (q', r'), no point of $\bar{\omega}(m)$ can belong to (q', r'). Then, $\bar{\omega}(m) \subset [q', p] \cup [p, r']$.

By Lemma 4.1, we know that, if $\bar{\omega}(m)$ contains a point a of $[q', p] \cup [p, r'] \backslash \{p\}$ then, $\bar{\omega}(m) = \{a\}$. For the same reason, if $\bar{\omega}(m)$ contains the point p, it cannot contain any other point in $[q', p] \cup [p, r']$ and then its is reduced to $\{p\}$. It follows that one has just three possibilities: $\bar{\omega}(m)$ is a point in $(p, q']$, a point of $(p, r']$ or the point p. The set $\mathcal{O}_q$ of the points of (q, r) whose $\bar{\omega}$-set belongs to $(p, q']$ is a non empty open set of (q, r). The reason is that, if a is such a point, one has that $X(a) \neq 0$. Then $\tau_+(m)$ is finite and is a value attained by the flow of X on U. Moreover, as the trajectory is transverse at a to the regular curve $(p, q']$, these properties remain true for the points m' near m on (q, r) : the trajectory through m' attains also $(p, q']$ after a finite time. For the same reason, the set $\mathcal{O}_r$ of the points of (q, r) whose $\bar{\omega}$-set belongs to $(p, r']$ is a non empty open set of (q, r). Since (q, r) is connected $F_p = (q, r) \backslash \mathcal{O}_q \cup \mathcal{O}_r$ is non-empty. A point $m \in F_p$ is such that $\bar{\omega}(m) = \{p\}$. By Lemma 2.3 (see also Remark 2.4) this means that the trajectory of m tends toward p when $t \to \tau_+(m)$. It is clear that this trajectory remains in U. □

It is possible to give more information about the phase portrait of X in $\mathcal{T}$:

Lemma 4.9 *Assume that X is $\mathcal{C}^k$ for $1 \leq k \leq +\infty$. Let $\mathcal{T} = [pqr]$ be a trapping triangle as in Definition 4.7. The set of points of (q, r) whose trajectories tend toward* *p is a closed interval* $[m_0, m_1] \subset (q, r)$*, maybe reduced to a single point. The trajectory from* $m \in (q, m_0)$ *reaches* (q, p) *after a finite time* $t(m)$*. Similarly, the trajectory from* $m \in (m_1, r)$ *reaches* (r, p) *after a finite time* $t(m)$ *These functions* $t(m)$ *are* $\mathcal{C}^k$ *(see Fig. 7).*

Proof We use the notations introduced in the proof of Lemma 4.8: $\mathcal{U}$ is the interior of a larger triangle $[pq'r']$ and $\mathcal{O}_q$, $\mathcal{O}_r$, F_p the subsets of (q, r) defined as above in the proof of Lemma 4.8. Let m_0 be the upper bound of points $m \in (q, r)$ such that $(q, m) \subset \mathcal{O}_q$. As $\mathcal{O}_q$ and $\mathcal{O}_r$ are open subsets of (q, r) the point m_0 cannot belong to any of them. Then $m_0 \in F_p$. In a similar way, we can find a point $m_1 \in F_p$ associated to $\mathcal{O}_r$. Clearly we have that $m_0 \leq m_1$ for the orientation going from q to r. If $m_0 \neq m_1$, we can apply Theorem 1.3 to the triangle $[pm_0m_1]$ with sides the positive half orbits $\gamma^+(m_0)$, $\gamma^+(m_1)$ union their $\bar{\omega}$-limit p and the subarc $[m_0, m_1]$ on (q, r): for any point $m \in (m_0, m_1)$ the limit set $\bar{\omega}(m) = \{p\}$, since this limit set cannot contains points of the regular orbits $\gamma^+(m_0)$, $\gamma^+(m_1)$. Then, we have that $[m_0, m_1]$ is the set F_p of points on (q, r) whose trajectory tends toward p and that $\mathcal{O}_q = (q, m_0)$, $\mathcal{O}_r = (m_1, r)$. The fact that the function $t(m)$ is $\mathcal{C}^k$ follows from the Cauchy Theorem in class $\mathcal{C}^k$, which states that the flow map $(t, m) \mapsto \varphi(t, m)$ is $\mathcal{C}^k$, and from the Inverse Function Theorem used to define implicitly the functions $t(m)$. □

5 How to Use Trapping Triangles?

Trapping triangles can be used in order to obtain the existence of a separatrix tending toward a singular point p. To this end one places this point at the boundary of an open set U, shows that there exists a trapping triangle at p and applies Lemma 4.8. Next, by chosing suitable other trapping triangles, it may be possible to obtain more precise information about the detected separatrix. This method uses principally qualitative arguments with a minimum of computations which are in general rather direct. Moreover, one can apply the method to a vector field not differentiable for instance at the point p. In such a case, it is not possible to apply the classical Poincaré-Bendixson Theorem in a neighborhood of p.

We would like to illustrate this method with an example presented in a recent paper (see [2]), which deals with a free interface problem in combustion theory. More specifically, one considers a system of two reaction-diffusion equations that models diffusional-thermal combustion with ignition-temperature kinetics and fractional order α. Looking for special solutions, namely one-dimensional traveling waves, turns out to be equivalent to finding a trajectory tending towards the origin for the vector field X_c with differential equation :

$$\begin{cases} x' = y, \\ y' = \dfrac{1}{\Lambda}\left(cy + x^\alpha\right). \end{cases} \tag{5.1}$$

Moreover, this trajectory must satisfy the initial conditions: $x(0) = v_0$ and $y(0) = -\frac{c}{\Lambda}(1 - v_0)$. Here, on the one hand, $c > 0$ is the speed of the traveling wave (to be determined), $v_0 \in [0, 1]$ is fixed at this phase of the study; on the other hand, $\Lambda > 0$ (the inverse of the Lewis number) and $\alpha \in [0, 1]$ are physical parameters. For instance, the parameter α can change with the ratio of the two reactants and may take non integer values. For such values of α the vector field X_c is just defined for $x \geq 0$ and is not differentiable along the axis $\{x = 0\}$.

For physical reasons, the vector field X_c is considered in the quadrant $Q = \{x \geq 0, y \leq 0\}$. For $c = 0$, the vector field X_0 is Hamiltonian with Hamiltonian function

$$H(x, y) = \frac{1}{2}y^2 - \frac{1}{\Lambda(1+\alpha)}x^{1+\alpha}.$$

In Q, this vector field has a stable separatrix L_0 at the origin O :

$$L_0 := \left\{ y = y_0(x) = -\left(\frac{2}{\Lambda(1+\alpha)}\right)^{1/2} x^{\frac{1+\alpha}{2}} \right\}. \tag{5.2}$$

For any $c \geq 0$, we have that:

$$X_c \cdot H(x, y) = -\frac{1}{\Lambda}yx^\alpha + \frac{1}{\Lambda}(cy + x^\alpha)y = \frac{c}{\Lambda}y^2.$$

This implies that, for $c > 0$, the vector field X_c is transverse to L_0 and directed downwards all along L_0, outside O.

Now, for any $v_0 > 0$, we consider in Q a trapping triangle $\mathcal{T}_{v_0}$. This triangle has three corners: O, $A_{v_0} = (v_0, 0)$ and $B_{v_0} = (v_0, y_0(v_0))$; and three sides denoted as follows: $[OA_{v_0}]$ on the $0x$-axis, $[OB_{v_0}]$ on the curve L_0 and $[A_{v_0}B_{v_0}]$. Since the vector field X_c is vertical at the point A_{v_0}, one chooses for side $[A_{v_0}B_{v_0}]$ a vertical segment in $\{x = v_0\}$ modified by a small bump near A_{v_0}, in order that X_c be transverse along this side with a left direction. The vector field X_c is transverse and has an upward direction along $(OA_{v_0}] = [OA_{v_0}]\backslash\{O\}$. As already mentioned, X_c is transverse and has a downward direction along $(OB_{v_0}] = [OB_{v_0}]\backslash\{O\}$. (see Fig. 8).

As a consequence of Lemma 4.8, there exists an orbit L_c of X_c in Q which tends toward O and which is a stable separatrix at this singular point. Much more is obtained in [2] about L_c. This orbit is the unique one tending toward O. Essentially, it is a graph of a smooth function $y_c(x)$ defined for $x \in (0, +\infty)$ and which extends continuously by $y_c(0) = 0$.

It is possible to use a finer trapping triangle $\mathcal{T}'_{v_0}$ in order to obtain the following expression:

$$y_c(x) = -\left(\frac{2}{\Lambda(1+\alpha)}\right)^{1/2} x^{\frac{1+\alpha}{2}} + o\left(x^{\frac{1+\alpha}{2}}\right). \tag{5.3}$$

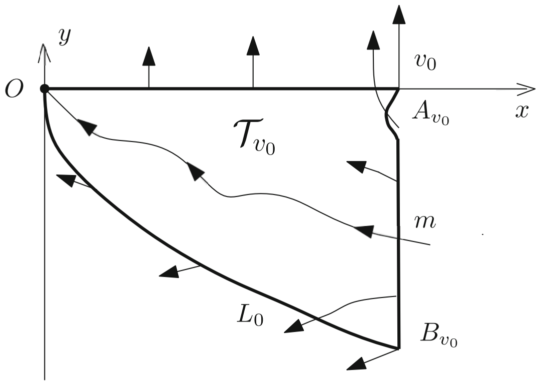


Fig. 8 Trapping triangle $\mathcal{T}_{v_0}$

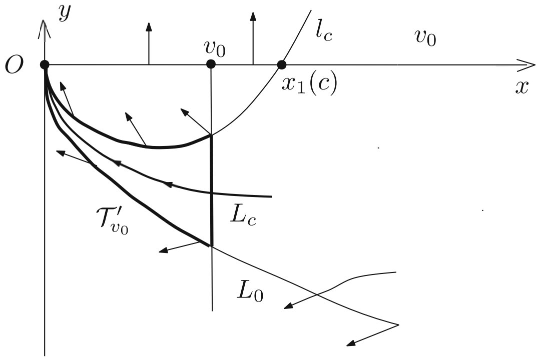


Fig. 9 Trapping triangle $\mathcal{T}'_{v_0}$

To this end, we consider the curve:

$$l_c := \left\{ y = \tilde{y}_c(x) = -\left(\frac{2}{(1+\alpha)\Lambda}\right)^{1/2} x^{\frac{1+\alpha}{2}} + \frac{c}{\Lambda}x \right\}. \tag{5.4}$$

It is easy to see that the vector field X_c is transverse along l_c, with an upward direction (see [2] for the computation). One can observe that the curve l_c cuts the Ox-axis at the value $x_1(c) = \left(\frac{2\Lambda}{(1+\alpha)c^2}\right)^{\frac{1}{1-\alpha}} > 0$ and remains in the quadrant Q only for $x \in [0, x_1(c)]$. Nevertheless, we can construct a new trapping triangle $\mathcal{T}'_{v_0}$, using the curves L_0 and l_c, with a vertical side in $\{x = v_0\}$ when $0 < v_0 < x_1(c)$ (see Fig. 9; since $v_0 < x_1(c)$, one does not need now to modify the vertical side by a bump). The graph of L_c is trapped inside $\mathcal{T}'_{v_0}$. It follows that $y_0(x) < y_c(x) < \tilde{y}_c(x)$ for $0 < x < v_0$ and these inequalities imply the asymptotic relation (5.3).

In [2], it is also shown that for any $x > 0$, the function $c \to y_c(x)$ is continuous and increasing. This allows to find a value $c(v_0)$ in order to fulfill the above initial conditions: $c(v_0)$ is the unique solution of the equation $y_c(v_0) = -\frac{c}{\Lambda}(1 - v_0)$. These results are obtained using a new well-chosen trapping triangle.

It follows from (5.3) that the time $R(v_0)$ "to arrive at O from the initial condition $(v_0, -\frac{c}{\Lambda}(1 - v_0))$" is finite if $\alpha < 1$ (stated more precisely, $R(v_0)$ is the finite limit time $\tau_+(v_0, -\frac{c}{\Lambda}(1-v_0))$ for the vector field $X_{c(v_0)}$). Moreover, using estimations on $c(v_0)$ given in [2] (which may be also obtained by a qualitative argument) and a new trapping triangle where the curve l_c is replaced by a curve $d_k := \{y = -kx^{\frac{1+\alpha}{2}}\}$ for a well-chosen $k < 0$, one obtains the following inequalities:

$$\frac{(2(1+\alpha)\Lambda)^{1/2}}{1-\alpha} v_0^{\frac{1-\alpha}{2}} < R(v_0) < \frac{4\Lambda^{1/2}}{(1-\alpha)(1+\alpha)^{1/2}} \frac{v_0^{\frac{1-\alpha}{2}}}{1-v_0}, \tag{5.5}$$

for $0 < v_0 < 1$ and $0 \leq \alpha < 1$.

The value of $R(v_0)$ is directly related to the *trailing interface* at which the deficient reactant is completely comsumed in the combustion model. As explained in [2], the result that $R(v_0)$ is finite for $0 \leq \alpha < 1$ and the inequalities (5.5) are highly significant for this problem.

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Data availability statement The author confirm that the data supporting the findings of this study are available within this article (and the articles referred in bibliography).

Appendix: The Jordan–Schoenflies Theorem in class $\mathcal{C}^1$

We want to give a sketch of the proof for the following restricted version of the Jordan-Schoenflies Theorem, the only version used in the present article. The support of a diffeomorphism H of $\mathbb{R}^2$ is the closure of the set: $\{m \in \mathbb{R}^2 | \; m \neq H(m)\}$.

Theorem 5.1 (Jordan-Schoenflies Theorem in class $\mathcal{C}^1$) *Let Γ be a $\mathcal{C}^1$ closed regular curve in $\mathbb{R}^2$. There exists a $\mathcal{C}^1$ diffeomorphism H of $\mathbb{R}^2$, with compact support, sending the trigonometric circle onto Γ.*

Proof Let (x, y) be Cartesian coordinates in the plane $\mathbb{R}^2$. We choose a tubular neighborhood T of Γ with a $\mathcal{C}^1$ trivialization $T \cong S^1 \times [-1, +1]$ such that $S^1 \times \{0\}$ corresponds to Γ. The segments $\{\theta\} \times [-1, +1]$ give a $\mathcal{C}^1$ normal fibration $\mathcal{N}$ along Γ. A first step is to approach Γ by a $\mathcal{C}^\infty$ closed curve Γ_1 in the $\mathcal{C}^1$ topology. In a second step Γ_1 can be approached in the $\mathcal{C}^\infty$ topology by a smooth closed Γ_2 in generic position in relation with the foliation $\mathcal{F}$ by the horizontal lines $\{y = \text{Const.}\}$, meaning that all contact points of Γ_2 with $\mathcal{F}$ are quadratic and located on different leaves (see Fig. 10). We can choose Γ_2 sufficiently near Γ in order that Γ_2 is inside the interior of T and transverse to $\mathcal{N}$. Then, Γ_2 is given in the trivialization, by the graph of a map from S^1 to $]-1, 1[$ and it is easy to construct a $\mathcal{C}^1$ diffeomorphism of $\mathbb{R}^2$, with support in T, sending each fiber of $\mathcal{N}$ onto itself and sending Γ onto Γ_2 (see Fig. 11).

We now consider the smooth curve Γ_2. The position of Γ_2 with respect to the horizontal foliation may be rather complicated, with a lot of horizontal contact points, as it is suggested in Fig. 10. In the rest of the proof we explain how to simplify this by means of a diffeomorphism of $\mathbb{R}^2$ with compact support.

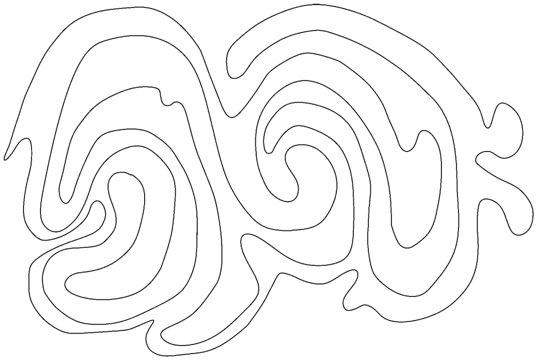

Fig. 10 A curve Γ_2

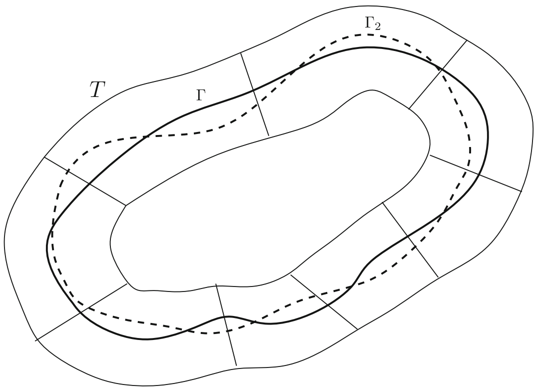

Fig. 11 Smoothing Γ

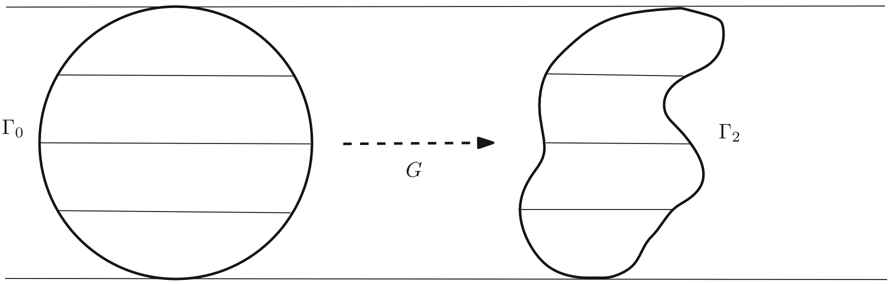

Fig. 12 Curve Γ_2 with two contact points

If there are just two such horizontal contact points, a maximum and a minimum for the y-function, we can displace Γ_2 by an affine map such that the maximum is on the line $\{y = 1\}$ and the minimum on the line $\{y = -1\}$. It is now very easy to construct a smooth diffeomorphism $G(x, y) = (g(x, y), y)$ sending Γ_0 onto Γ_2 (see Fig. 12)

If there are more than 3 contact points, it is possible to prove that there is at least a pair of two successive contact points on Γ_2, a minimum p and a maximum q, in the position illustrated in Fig. 13: there is a disk B such that ∂B is the union of an arc $\Gamma_2(p, p')$ on Γ_2 containing q in its interior (and no other contact point) and the horizontal segment $[pp']$. Moreover, at p, the complement $\Gamma_2 \setminus \Gamma_2(p, p')$ starts outside

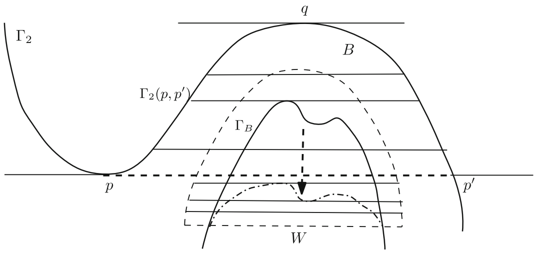

Fig. 13 Cleanning out a disk B

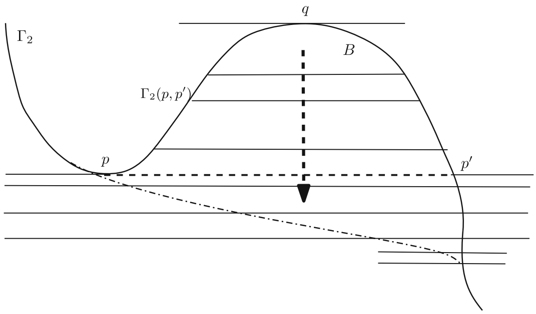

Fig. 14 Elimination of the pair (p, q)

B. The disk B may contain other parts of Γ_2 disjoint from $\Gamma_2(p, p')$. Let Γ_B be their union. The existence of such a pair (p, q) of contact points is the key point of the proof. It can be obtained by an easy recurrence argument on the number of contact points. We will not elaborate further on it.

We now consider such a pair (p, q) and explain how to eliminate it. We proceed in two sub-steps. First, we choose a disk W disjoint from $\Gamma_2(p, p')$, and such that Γ_B is inside $W \cap B$. Then, we push Γ_B outside B by a smooth diffeomorphism, with support in W which, in a neighborhood of Γ_B, sends horizontal intervals into horizontal intervals located outside B (see Fig. 13).

We obtain a new curve Γ_3, in generic position and coinciding with Γ_2 in a neighborhood of $\Gamma_2(p, p')$, with the same number of horizontal contact points as Γ_2. But now the same disk B is associated to Γ_3 and does not contain other parts of Γ_3 than $\Gamma_2(p, p')$. It is now easy to construct a diffeomorphism, with support in a compact neighborhood of B, which pushes the arc $\Gamma_2(p, p')$ downward outside B, in order to eliminate the pair (p, q) without modifying the other contact points nor creating new ones (see Fig. 14)

One can apply this argument by recurrence to finish with a curve which has just two contact points. Finally, we have obtained a sucession of diffeomorphisms of $\mathbb{R}^2$, with compact support and of class at least $\mathcal{C}^1$, whose composition (of them or their inverse) sends the trigomometric circle onto the given initial closed curve Γ. $\square$

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