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Topological Average Shadowing Property on Uniform Spaces

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Abstract
We introduce topological definition of average shadowing property. We prove that topological average shadowing property implies topological chain transitivity. In particular it is proved that for a dynamical system with dense minimal points, the topological average shadowing property implies topological strong ergodicity.

Keywords Topological average shadowing · Topologically chain transitive · Topologically ergodic · Topologically strongly ergodic · Uniform space

Mathematics Subject Classification 37B65 · 37B20

1 Introduction

The pseudo-orbit tracing property is one of the most important notions in dynamical systems, which is closely related to stability and chaos of systems [1–5]. This concept is motivated by computer simulations. More precisely, let X be a set and $f : X \to X$ be a map. Then in the computation of f with initial value $x_0 \in X$, computer approximates $f(x_0)$ by some point x_1. To continue the process, it computes the value x_2 as an

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approximation of $f(x_1)$ and so on. For formulating this concept we have to use the ‘distance’ between points to control approximation errors. In a metric space (X, d) one can approximates points using metric d and define a pseudo-orbit with error δ as a sequence $x_0, x_1, \ldots$ with $d(x_{j+1}, f(x_j)) < \delta$ for all $j \geq 0$. However, for general topological spaces such a distance cannot be found unless we have somewhat more structure than what the topology itself provides. This issue will be solved if we consider a completely regular topological spaces which is equipped with an structure, called uniformity, enabling us to control the distance between points in these spaces. Using this structure, Das et al. [6,7] generalized the usual definitions of shadowing, and chain recurrence for homeomorphisms to topological spaces. Then, the second author [8] proved that a dynamical system with ergodic shadowing is topologically chain transitive. The third author [9] introduced the topological concepts of weak uniformity, uniform rigidity, and multi-sensitivity and obtained some equivalent characterizations of uniform rigidity. Then, we [10] proved that a point transitive dynamical system in a Hausdorff uniform space is either almost (Banach) mean equicontinuous or (Banach) mean sensitive. Recently, we [11] generalized concepts of entropy points, expansivity and shadowing property for dynamical systems to uniform spaces and obtained a relation between topological shadowing property and positive uniform entropy. Good and Macías [12] obtained some equivalent characterizations and iteration invariance of various definitions of shadowing in the compact uniform spaces. For more results on shadowing properties on non-metrizable spaces, one is referred to [13–17] and references therein.

Nevertheless when calculating approximate trajectories, it makes sense to consider errors small on average, since controlling them in each iteration may be impossible. The notion of average pseudo-orbit introduced by Blank [18]. In a metric space (X, d) an average pseudo-orbit with error δ is a sequence $x_0, x_1, \ldots$ for which there is $N \in \mathbb{N}$ such that $\frac{1}{n}\sum_{j=0}^{n-1} d(x_{j+k+1}, f(x_{j+k})) < \delta$ for any $n \geq N$ and $k \in \mathbb{N}$. The average shadowing property is related to finding an averagely close real orbit for any average pseudo-orbit [19–24]. But in a general topological space we need some method to control the average of errors in a pseudo-orbit . Motivated by mentioned ideas, we show that average shadowing property can be defined in a natural way on uniform spaces. In order to do this, we control the average of errors of a pseudo-orbit in a non-metrizable topological space via infinite sequences of neighborhoods of diagonal. This paper introduce and studies topological definition of average shadowing property and obtains that topological average shadowing implies topological chain mixing. Then we show that for a dynamical system with dense minimal points, the topological average shadowing property implies topological strong ergodicity.

2 Basic Definitions and Preliminaries

A uniform structure on X is a *filter* $\mathscr{U}$ of subsets of $X \times X$ with the following properties.

(U1) every set $U \in \mathscr{U}$ contains the diagonal $\Delta_X = \{(x, x) : x \in X\}$;
(U2) if $U \in \mathscr{U}$, then $U^{-1} = \{(y, x) : (x, y) \in U\} \in \mathscr{U}$;

(U3) for any $U \in \mathscr{U}$ there exists $V \in \mathscr{U}$ such that $V \circ V \subset U$, where $V \circ V = \{(x, y) : \exists z \in X \text{ with } (x, z) \in V, (z, y) \in V\}$.

The set X with a uniformity $\mathscr{U}$ on it is called *uniform space* and denoted by $(X, \mathscr{U})$. For example if (X, ρ) is a pseudo-metric space, then the family $\{V_\epsilon^\rho \mid \epsilon > 0\}$ is a base for a uniformity on X, where $V_\epsilon^\rho = \{(x, y) | \rho(x, y) < \epsilon\}$. Each element of $\mathscr{U}$ is called *entourage* of X. An entourage E is said to be *symmetric* if $E = E^{-1}$. If $x \in X$ and $E \in \mathscr{U}$, then the set $E[x] = \{y \in X : (x, y) \in E\}$ is called the *cross-section* of E at a point x. If $\tau_{\mathscr{U}} = \{A \subset X : \forall a \in A, \exists E \in \mathscr{U}, \ such that\ E[a] \subset A\}$, then $\tau_{\mathscr{U}}$ is a topology on X which is called the *uniform topology* on X. A map $f : X \to X$ is called *uniformly continuous* if for any $E \in \mathscr{U}$ we conclude that $f^{-1}(E) \in \mathscr{U}$. In this paper, by a *dynamical system*, mean a pair (X, f), where X is a uniform space and $f : X \longrightarrow X$ is a uniformly continuous map (Note that any continuous map on a compact uniform space is uniformly continuous, so in this case it is enough to assume that f is continuous). Let $D \in \mathscr{U}$. A *D-chain* of length n is a sequence $\xi = \{x_i\}_{i=0}^n$ such that $(f(x_i), x_{i+1}) \in D$ for $i = 0, \ldots, n-1$. An infinite D-chain is called a *D-pseudo-orbit*. A D-pseudo-orbit $\xi = \{x_i\}$ is *E-shadowed* by a point $z \in X$ if $(f^n(z), x_n) \in E$ for all $n \in \mathbb{N}_0$. A dynamical system (X, f) has the *topological shadowing property TSP* [6] if for every entourage E of X, there exists an entourage D such that every D-pseudo-orbit is E-shadowed by some point in X. For $x \in X$ and $U, V \subset X$, let

$$N_f(x, U) = \{n \in \mathbb{N} \mid f^n(x) \in U\} \text{ and } N_f(U, V) = \{n \in \mathbb{N} \mid U \cap f^{-n}(V) \neq \emptyset\}.$$

An infinite subset $A \subset \mathbb{N}$ is *relatively dense* (or *syndetic*) if there exists $k > 0$ such that $\{n, n+1 \ldots, n+k\} \cap A \neq \emptyset$ for all $n \in \mathbb{N}_0$. Let (X, f) be a dynamical system. A point $x \in X$ is *minimal* (or *almost periodic*) if $N_f(x, U)$ is syndetic for every neighborhood U of x. We denote by $\mathcal{M}(f)$ the set of all minimal points.

A dynamical system (X, f) is

(**TCR**) *topologically chain recurrent* if, for any entourage D of X and any point $x \in X$, there exists an D-chain from x to itself;
(**TCT**) *topologically chain transitive* if, for any entourage D of X and any two points $x, y \in X$, there exists an D-chain from x to y;
(**TCM**) *topologically chain mixing* if, for any two points $x, y \in X$ and any entourage D of X, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, there exists a D-chain from x to y of length n;
(**TT**) *topologically transitive* if $N_f(U, V)$ is a non-empty set for any pair of nonempty open subsets U, V of X;
(**TTT**) *topologically totally transitive* if f^n is topologically transitive for any $n \in \mathbb{N}$;
(**TE**) *topologically ergodic* if $\overline{d}(N_f(U, V)) = \limsup_{n\to\infty} \frac{N_f(U,V)\cap\{1,2,\ldots,n\}}{n} > 0$ for any pair of nonempty open subsets U, V of X;
(**TSE**) *topologically strongly ergodic* if $N_f(U, V)$ is a syndetic set for any pair of nonempty open subsets U, V of X;
(**TTSE**) *topologically totally strongly ergodic* if f^n is topologically ergodic for any $n \in \mathbb{N}$.

It is clear that

$$\mathbf{TTSE} \Rightarrow \mathbf{TSE} \Rightarrow \mathbf{TE} \Rightarrow \mathbf{TT} \Rightarrow \mathbf{TCT} \Rightarrow \mathbf{TCR}.$$

3 Topological Average Shadowing Property

Denote by $\Sigma_{\mathscr{U}}$ the family of all sequences $\mathcal{E} = \{E_i\}_{i=0}^{\infty}$ of entourages in $\mathscr{U}$ with $E_0 = X \times X$, such that $E_{i+1} \subset E_i$ for all $i \in \mathbb{N}_0$. For a sequence $\mathcal{E} = \{E_i\}_{i=0}^{\infty} \in \Sigma_{\mathscr{U}}$, a map $f : X \to X$ and a sequence $\xi = \{x_0, x_1, \dots\}$ in X we define

$$\begin{aligned}\mathcal{A}_n(\xi, f, \mathcal{E}) &= \mathcal{A}_n(\xi, f, \{E_i\}_{i=0}^{\infty}) \\ &= \inf\left\{\sum_{j=0}^{n} \frac{1}{2^{\sigma(j)}} \mid \ (x_{j+1}, f(x_j)) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\right\}; \quad n \in \mathbb{N} \\ &= \sum_{j=0}^{n} \inf\left\{\frac{1}{2^{\sigma(j)}} \mid \ (x_{j+1}, f(x_j)) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^j\right\}; \quad n \in \mathbb{N},\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}_n(\xi, z, f, \mathcal{E}) &= \mathcal{A}_n(\xi, z, f, \{E_i\}_{i=0}^{\infty}) \\ &= \inf\left\{\sum_{j=0}^{n} \frac{1}{2^{\sigma(j)}} \mid \ (x_j, f^j(z)) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\right\}; \quad n \in \mathbb{N},\end{aligned}$$

where $\mathbb{N}_0^n$ is the set of all maps from $\{0, 1, \dots, n\}$ to $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Remark 1 Let $(X, \mathscr{U})$ be a compact uniform space and $f : X \to X$ be a continuous map. If $\mathcal{E} = \{E_i\}_{i=0}^{\infty} \in \Sigma_{\mathscr{U}}$ and $\xi = \{x_i\}_{i=0}^{\infty}$ be a sequence in X, then for any $m, n, k \in \mathbb{N}$ we have

1. $0 \le \mathcal{A}_n(\xi, f, \mathcal{E}) \le n$;
2. $\mathcal{A}_n(\xi, f, \mathcal{E}) \le \mathcal{A}_{n+1}(\xi, f, \mathcal{E}) \le \mathcal{A}_n(\xi, f, \mathcal{E}) + 1$;
3. $\mathcal{A}_n(\xi, f, \{E_i\}_{i=0}^{\infty}) = \frac{1}{2^k}\mathcal{A}_n(\xi, f, \{E_i'\}_{i=0}^{\infty})$, where $E_i' = E_{i+k}$ for $i \ge 1$ and $E_i' = X \times X$.
4. $\mathcal{A}_n(\xi, f, \{E_i'\}_{i=0}^{\infty}) \le \mathcal{A}_n(\xi, f, \{E_i\}_{i=0}^{\infty})$, where $E_i' = E_{ik}$;
5. $\mathcal{A}_{m+n}(\xi, f, \mathcal{E}) = \mathcal{A}_m(\xi, f, \mathcal{E}) + \mathcal{A}_n(T^m(\xi), f, \mathcal{E})$, where $T : X^{\mathbb{N}_0} \to X^{\mathbb{N}_0}$ is the shift map.

Definition 3.1 For $\mathcal{D} \in \Sigma_{\mathscr{U}}$, a *topological average $\mathcal{D}$-pseudo-orbit* of f is a sequence $\{x_i\}$ in X such that $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\xi, f, \mathcal{D}) = 0$. Let $\mathcal{E} = \{E_i\} \in \Sigma_{\mathscr{U}}$. We say that the sequence $\{x_i\}$ is *$\mathcal{E}$ -shadowed on average by some point $z \in X$*, if $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\xi, z, f, \mathcal{E}) = 0$. We say that the map f has the *topological average shadowing property TASP*, if for every $\mathcal{E} = \{E_i\} \in \Sigma_{\mathscr{U}}$, there exists $\mathcal{D} = \{D_i\}_{i=0}^{\infty} \in \Sigma_{\mathscr{U}}$ such that every topological average $\mathcal{D}$-pseudo-orbit is $\mathcal{E}$-shadowed on average by some point of X.

Remark 2 If (X, d) is a compact metric space, then for any neighborhood U of Δ_X, we can find $\delta > 0$ such that $V^d_\delta \subset U$. On the other hand, every V^d_δ is a neighborhood of Δ_X. Moreover if $\{x_i\}$ is a topological average $\{V^d_{\frac{\delta}{2^i}}\}$-pseudo-orbit, then $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\xi, f, \{V^d_{\frac{\delta}{2^i}}\}) = 0$ and there exists $N \in \mathbb{N}$ such that $\frac{1}{n}\mathcal{A}_n(\xi, f, \{V^d_{\frac{\delta}{2^i}}\}) < 1$ for $n \geq N$. One can easily check that $\frac{1}{n}\sum_{i=0}^{n-1} d(x_{i+k+1}, f(x_{k+i})) < \delta$ for all $n \geq N$ and $k \in \mathbb{N}$. This shows that for any $\delta > 0$, we can find a $\mathcal{D} = \{D_i\}_{i=0}^{\infty} \in \Sigma_{\mathscr{U}}$ such that every topological average $\mathcal{D}$-pseudo orbit is an average δ-pseudo orbit. But the converse is not true, for example, if we consider the identity map on S^1 with the usual topology, then for $\mathcal{D} = \{V_{\frac{1}{i}}\}_{i=0}^{\infty} \in \Sigma_{\mathscr{U}_{S^1}}$ and any $\delta > 0$, the sequence $x_{j+1} = x_j + \delta/2$ with $x_0 = 0$ is an average δ-pseudo orbit which is not a topological average $\mathcal{D}$-pseudo orbit. That is, this remark does not implies that the topological average shadowing property is equivalent to the usual average shadowing property when the uniform structure is came from a metric d.

Recall that two dynamical systems $f : X \to X$ and $g : Y \to Y$ are *topologically semi-conjugate*, if there exists a continuous map h from X onto Y, such that $h \circ f = g \circ h$. In this case h is called a *semi-conjugacy* between f and g.

Theorem 3.1 *Let $(X, \mathscr{U}_X)$ and $(Y, \mathscr{U}_Y)$ be two compact uniform spaces. Let $h : X \to Y$ be an homeomorphism. If (X, f) has the topological average shadowing property, then $(Y, h \circ f \circ h^{-1})$ has topological average shadowing property.*

Proof Consider $H : X \times X \to Y \times Y$ given by $H(x, y) = (h(x), h(y))$. For any $\mathcal{E} = \{E_i\} \in \Sigma_{\mathscr{U}_Y}$, observe that $H^{-1}(\mathcal{E}) = \{(h^{-1} \times h^{-1})E_i\} \in \Sigma_{\mathscr{U}_X}$ is a sequence of entourages in X. Let $\mathcal{D} = \{D_i\} \in \Sigma_{\mathscr{U}_X}$ be an $H^{-1}(\mathcal{E})$-modulus of topological average shadowing property of f. Suppose that $\{y_i\}_{i=0}^{\infty}$ is a topological average $H(\mathcal{D})$-pseudo-orbit for $h \circ f \circ h^{-1}$, that is

$$\lim_{n\to\infty} \frac{1}{n} \inf \left\{ \sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad (h \circ f \circ h^{-1}(y_{j-1}), y_j) \in (h \times h)D_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} = 0.$$

hence $\{h^{-1}(y_i)\}_{i=0}^{\infty}$ is a topological average $\mathcal{D}$-pseudo-orbit for f. Then by topological average shadowing property of f there exists a point $z \in X$ such that

$$\lim_{n\to\infty} \frac{1}{n} \inf \left\{ \sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad (f^j(z), h^{-1}(y_j)) \in (h \times h)^{-1}E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} = 0$$

which implies that

$$\lim_{n\to\infty} \frac{1}{n} \inf \left\{ \sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad ((h \circ f \circ h^{-1})^j(h(z)), y_j) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} = 0.$$

Thus, $h \circ f \circ h^{-1}$ has the topological average shadowing property in Y. □

Proposition 3.1 *Let $(X, \mathscr{U})$ be a compact uniform space. Let f be a continuous map from X onto itself. If (X, f) has the topological average shadowing property, then so does (X, f^k) for every $k \in \mathbb{N}$.*

Proof Fix $k > 1$ and let $\mathcal{E} = \{E_i\} \in \Sigma_{\mathscr{U}}$. Since f has the topological average shadowing property, there exists $\mathcal{D} = \{D_i\} \in \Sigma_{\mathscr{U}}$ such that every topological average $\mathcal{D}$-pseudo-orbit is $\mathcal{E}$-shadowed on average by some point in X. Let $\xi = \{x_i\}_{i=0}^{\infty}$ be a topological average $\mathcal{D}$-pseudo-orbit of f^k, that is

$$\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\xi, f^k, \mathcal{D}) = 0.$$

Putting $f(x_i), f^2(x_i), \ldots, f^{k-1}(x_i)$ between x_i and x_{i+1} for all $i \geq 0$, we get a topological average $\mathcal{D}$-pseudo-orbit ξ' for f. Let $y_{lk+j} = f^j(x_l)$, for all $0 \leq j \leq k$ and all $l \geq 0$. Then

$$\lim_{n\to\infty} \frac{1}{n} \inf \left\{ \sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad (f(y_{j-1}), y_j) \in D_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} = 0$$

that is, the sequence $\{y_i\}_{i=0}^{\infty}$ is topological average $\mathcal{D}$-pseudo-orbit of f. So there is a point $z \in X$ such that

$$\lim_{n\to\infty} \frac{1}{nk} \inf \left\{ \sum_{j=1}^{nk} \frac{1}{2^{\sigma(j)}} : \quad (f^j(z), y_j) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} = 0.$$

Therefore

$$\lim_{n\to\infty} \frac{1}{n} \inf \left\{ \sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad (f^{jk}(z), x_j) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} = 0.$$

hence f^k has topological average shadowing property. □

The following theorem is the topological version of [20, Theorem 3.1], done for metric spaces.

Theorem 3.2 *Let $(X, \mathscr{U})$ be a compact uniform space. Let f be a continuous map from X onto itself. If f has the topological average shadowing property, then f is topologically chain transitive.*

Proof Suppose x and y are two distinct points in X and $D \in \mathscr{U}$. Let $\mathcal{E} = \{E_i\} \in \Sigma_{\mathscr{U}}$ be such that $E_1 \subset D$ and $(f \times f)E_1 \subset D$. Since f has topological average shadowing property. there exists $\mathcal{D} = \{D_i\} \in \Sigma_{\mathscr{U}}$ such that every topological average $\mathcal{D}$-pseudo-orbit, can be $\mathcal{E}$-shadowed on average by some point in X.

Now, we construct a sequence $\{w_i\}_{i=0}^{\infty}$ as follows.

$$
\begin{aligned}
&w_0 = x,\ w_1 = y\\
&w_2 = x,\ w_3 = y\\
&w_4 = x,\ w_5 = f(x), w_6 = y_{-1}\ , w_7 = y\\
&\vdots\\
&w_{2^k} = x,\ w_{2^k+1} = f(x), \ldots, w_{2^k+2^{k-1}-1} = f^{2^{k-1}-1}(x), \quad w_{2^k+2^{k-1}} = y_{-2^{k-1}+1}, \ldots, w_{2^{k+1}-1} = y\\
&\vdots
\end{aligned}
$$

where $f(y_{-j}) = y_{-j+1}$ for every $j > 0$ and $y_0 = y$.

For $n \geq 2$ we obtain that

$$
\inf \left\{ \sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad (f(w_{j-1}), w_j) \in D_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} \leq 2 \log_2 n,
$$

thus $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\{w_i\}, f, \mathcal{D}) = 0$. That is, $\{w_i\}$ is a topological average $\mathcal{D}$-pseudo-orbit of f. Hence there exists a point $z \in X$ such that $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\{w_i\}, z, f, \mathcal{E}) = 0$. Now, we prove the following claims.

Claim 1 There exist infinitely many positive integers j such that

$$
w_{n_j} \in \{x, f(x), \ldots, f^{2^j-1}(x)\} \qquad \text{and} \qquad (f^{n_j}(z), w_{n_j}) \in E_1;
$$

Claim 2 There exist infinitely many positive integers l such that

$$
w_{n_l} \in \{y_{-2^l+1}, \ldots, y_{-1}, y\} \qquad \text{and} \qquad (f^{n_l}(z), w_{n_l}) \in E_1.
$$

Proof of claim 1. It is enough to prove the condition (1).

Suppose on the contrary that there is a positive integer N such that for all integer $k > N$, whenever

$$
w_i \in \{x, f(x), \ldots, f^{2^k-1}(x)\},
$$

it is obtained that $(f^i(z), w_i) \notin E_1$, hence $(f^i(z), w_i) \notin E_j$ for all $j = 1, 2, \ldots, n$. Therefore

$$
\inf \left\{ \sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad (f^j(z), w_j) \notin E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} \geq \frac{n}{2}
$$

which implies that $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\{w_i\}, f, \mathcal{E}) \neq 0$, a contradiction. Hence the claim (1) holds.

By the claim, we can pick two positive integers j_0 and l_0 such that $n_{j_0} < n_{l_0}$ and

$$w_{n_{j_0}} \in \{x, f(x), \ldots, f^{2^{j_0}-1}(x)\} \text{ and } (f^{n_{j_0}}(z), w_{n_{j_0}}) \in E_1;$$
$$w_{n_{l_0}} \in \{y_{2^{l_0}+1}, \ldots, y_{-1}, y\} \text{ and } (f^{n_{l_0}}(z), w_{n_{l_0}}) \in E_1.$$

Assume that $w_{n_{j_0}} = f^{j_1}(x)$ and $w_{n_{l_0}} = y_{-l_1}$ for some $j_1, l_1 > 0$. Hence we have a D-chain from x to y as follows.

$$x, f(x), \ldots, f^{j_1}(x) = w_{n_{j_0}}, f^{n_{j_0}+1}(z), f^{n_{j_0}+2}(z), \ldots, f^{n_{l_0}-1}(z),$$
$$w_{n_{l_0}} = y_{-l_1}, y_{-l_1+1}, \ldots, y.$$

Therefore, f is topological chain transitive. □

Corollary 3.1 *Let $(X, \mathscr{U})$ be a compact uniform space. Let f be a continuous map from X onto itself. If (X, f) has the topological average shadowing property, then it is topologically chain mixing.*

Proof By Theorem 3.1 if f has the average shadowing property, then so does f^n for every $n > 0$. Hence by Theorem 3.2, f^n is chain transitive for every $n > 0$, then by Proposition 3.3 in [25], f is chain mixing. □

A sequence $\xi = \{x_i\}_{i=0}^{\infty}$ is called an ergodic D-pseudo-orbit provided that the set $\Lambda^c(\xi, f, D) = \{i \in \mathbb{N}_0 | \ (x_{i+1}, f(x_i)) \notin D\}$ has density zero, that is $\lim_{n\to\infty} 1/n|\Lambda_n^c(\xi, f, D)| = 0$, where $\Lambda_n^c(\xi, f, D) = \Lambda^c(\xi, f, D) \cap \{0, 1, \ldots, n-1\}$. For an entourage E of X, an ergodic D-pseudo-orbit ξ is said to be ergodically E-shadowed by a point $z \in X$ provided that the set $\Lambda^c(\xi, z, f, E) = \{i \in \mathbb{N}_0 | \ (x_i, f^i(z)) \notin E\}$ has density zero, that is

$$\lim_{n\to\infty} 1/n|\Lambda_n^c(\xi, z, f, E)| = 0.$$

A dynamical system (X, f) has the topological ergodic shadowing property provided that for any entourage E of X there exists an entourage D of X such that any ergodic D-pseudo-orbit can be ergodically E-shadowed by some point in X [8,17]. A dynamical system (X, f) has the topological $\underline{d}$-shadowing if for every entourage E there is an entourage D such that every ergodic D-pseudo-orbit is E-shadowed by some point z in such a way that the set $\Lambda(\xi, z, f, E) = \{i \in \mathbb{N}_0 | \ (x_i, f^i(z)) \in E\}$ has positive lower density, that is $\liminf_{n\to\infty} 1/n|\Lambda_n(\xi, z, f, E)| > 0$.

A dynamical system (X, f) has the topological pseudo orbital specification if for every entourage E there exists an entourage D and a positive integer M such that for any non-negative integers $a_1 \le b_1 < a_2 \le b_2 < \cdots < a_n \le b_n$ with $a_{j+1} - b_j > M$ for $1 \le j \le (n-1)$ and any D-chain $\xi_1, \xi_2, \ldots, \xi_n$ where $\xi_j = \{x_{(i,j)}\}_{a_j \le i \le b_j}$ for $1 \le j \le n$, there is $z \in X$ such that $(f^i(z), x_{(i,j)}) \in E$ for $a_j \le i \le b_j$, $1 \le j \le n$. By [26, Main Theorem] and Corollary 3.1 we obtain the following corollary.

Corollary 3.2 *Let $(X, \mathscr{U})$ be a compact uniform space and f be a continuous map from X onto itself. If (X, f) has topological average shadowing property and topological shadowing property, then (X, f) has*

1. *Topological pseudo orbital specification property;*
2. *Topological ergodic shadowing property;*
3. *Topological $\underline{d}$-shadowing property.*

The following theorem generalizes [22, Theorem 3.1] from metric spaces to uniform spaces.

Theorem 3.3 *Let $(X, \mathscr{U})$ be a compact uniform space. Let f be a continuous map from X onto itself. If f has the topological average shadowing property and the minimal points of f are dense in X, then f is topologically totally strongly ergodic.*

Proof Let U, V be two non-empty open subsets of X. By density of minimal points of f in X, we can choose $x \in U \cap \mathcal{M}(f)$, $y \in V \cap \mathcal{M}(f)$ and $E \in \mathscr{U}$ such that $E[x] \subset U$, and $E[y] \subset V$. There is a symmetric entourage $\hat{E}$ such that $\hat{E} \circ \hat{E} \subset E$. Since $x, y \in \mathcal{M}(f)$, hence $J_x = \{n \in \mathbb{Z}_+ : \quad f^n(x) \in \hat{E}[x]\}$ and $J_y = \{n \in \mathbb{Z}_+ : f^n(y) \in \hat{E}[y]\}$ are syndetic. Thus, there are $N_1, N_2 \in \mathbb{N}$ such that $[n, n+N_1] \cap J_x \neq \emptyset$ and $[n, n+N_2] \cap J_y \neq \emptyset$ for each $n \in \mathbb{Z}_+$. Let $N = \max\{N_1, N_2\}$ then by uniform continuity of f there exists $D \in \mathscr{U}$ such that $(f \times f)^i D \subset \hat{E}$ for $i = 1, 2, \ldots, N$. Choose $\mathcal{E} = \{E_i\} \in \Sigma_{\mathscr{U}}$ such that $E_i \subset D$ for $i \geq 1$. Since f has topological average shadowing property then there exists $\mathcal{D} = \{D_i\} \in \Sigma_{\mathscr{U}}$ such that every topological average $\mathcal{D}$-pseudo-orbit, $\mathcal{E}$-shadowed on average by some point of X. Now, we construct a sequence $\{w_i\}_{i=0}^{\infty}$ as follows.

$$
\begin{aligned}
& w_0 = x, \; w_1 = y \\
& w_2 = x, \; w_3 = y \\
& w_4 = x, \; w_5 = f(x), w_6 = y \;, w_7 = f(y) \\
& w_8 = x, \; w_9 = f(x), \; w_{10} = f^2(x) \;, w_{11} = f^3(x), w_{12} = y, \; w_{13} = f(y), \; w_{14} = f^2(y), \; w_{15} = f^3(y) \\
& \vdots \\
& w_{2^k} = x, \ldots, w_{2^k + 2^{k-1} - 1} = f^{2^{k-1}-1}(x), \; w_{2^k + 2^{k-1}} = y, \ldots, w_{2^{k+1}-1} = f^{2^{k-1}-1}(y) \\
& \ddots
\end{aligned}
$$

For $n \geq 2$ we obtain that

$$
\inf \left\{ \sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad (f(w_{j-1}), w_j) \in D_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} \leq 2 \log_2 n,
$$

therefore $\lim_{n\to\infty} \frac{1}{n} \mathcal{A}_n(\{w_i\}, f, \mathcal{D}) = 0$. That is, $\{w_i\}$ is a topological average $\mathcal{D}$-pseudo-orbit of f. Thus there exists a point $w \in X$ such that $\lim_{n\to\infty} \frac{1}{n} \mathcal{A}_n(\{w_i\}, w, f, \mathcal{E}) = 0$. We have the following claims.

Claim 1 There exist infinitely many $k \in \mathbb{N}$ such that there exists $n_k \in \{2^k, 2^k + 1, \ldots, 2^k + 2^{k-1} - 1\}$ with

$$
w_{n_k} \in \{x, f(x), \ldots, f^{2^{k-1}-1}(x)\} \quad \text{and} \quad (f^{n_k}(w), w_{n_k}) \in D;
$$

Claim 2 There exist infinitely many $l \in \mathbb{N}$ such that there exists $m_l \in \{2^l, 2^{l-1}, \ldots, 2^{l+1}-1\}$ with

$$w_{m_l} \in \{y, f(y), \ldots, f^{2^{l-1}-1}(y)\} \quad \text{and} \quad (f^{m_l}(w), w_{m_l}) \in D.$$

Proof of claim 1. It is enough to prove (1). Assume on the contrary that there is $M \in \mathbb{N}$ such that for all integers $k > M$, whenever $i \in \{2^k, 2^k+1, \ldots, 2^k+2^{k-1}-1\}$, which implies $w_i \in \{x, f(x), \ldots, f^{2^{k-1}-1}(x)$, it is obtained that $(f^i(w), w_i) \notin E_j$ for all $j = 1, 2, \ldots, n$. Therefore,

$$\inf\left\{\sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad (f^j(w), w_j) \notin E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\right\} \geq \frac{n}{2}$$

thus,

$$\lim_{n\to\infty} \frac{1}{n} \inf\left\{\sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad (f^j(w), w_j) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\right\} \neq 0$$

which is a contradiction. Hence the claim 1 holds.
By the claim, we can choose n_{k_0}, m_{l_0}, r, t with $m_{l_0} > n_{k_0} + N$ and $w_{n_{k_0}} = f^t(x)$, $w_{m_{l_0}} = f^r(y)$ such that

$$(f^{n_{k_0}}(w), w_{n_{k_0}}) = (f^{n_{k_0}}(w), f^t(x)) \in D \tag{1}$$

$$(f^{m_{l_0}}(w), w_{m_{l_0}}) = (f^{m_{l_0}}(w), f^r(y)) \in D \tag{2}$$

since $[t, t+N] \cap J_x \neq \emptyset$, $[r, r+N] \cap J_y \neq \emptyset$, there are $0 \leq k, l \leq N$ such that $f^{t+k}(x) \in \hat{E}[x]$, $f^{r+l}(y) \in \hat{E}[y]$. By 1 and 2, we obtain $(f^{n_{k_0}+k}(w), f^{t+k}(x)) \in \hat{E}$ and $(f^{m_{l_0}+l}(w), f^{r+l}(y)) \in \hat{E}$. Thus $f^{n_{k_0}+k}(w) \in E[x] \subset U$ and $f^{m_{l_0}+l}(w) \in E[y] \subset V$. Let $n_0 = m_{l_0} - n_{k_0} + l - k > 0$, then it follows that $U \cap f^{-n_0}(V) \neq \emptyset$. Choose $W = U \cap f^{-n_0}(V) \neq \emptyset$. Hence there is $s \in \mathcal{M}(f) \cap W$. Let $J = \{n \in \mathbb{Z}_+ : \quad f^n(s) \in W\}$, then J is syndetic. For $m \in J$, since $\emptyset \neq U \cap f^{-n_0}(V) \cap f^{-m}(U \cap f^{-n_0}(V)) \subset U \cap f^{-(n_0+m)}(V)$, $N(U, V) \neq \emptyset$. thus $N(U, V)$ is syndetic. Since U, V are arbitrary, f is topological strongly ergodic. For any $k \in \mathbb{N}$, since f has the topological average shadowing property, f^k has the topological average shadowing property by proposition 3.1, so f^k is topological strongly ergodic. Hence f is topological totally strongly ergodic. □

4 Examples

The following example shows that **TSP** $\nRightarrow$ **TASP**.

Example 4.1 Let $X = \{a, b, c\}$. Consider following subsets of $X \times X$;

$U_0 = \Delta_X \cup \{(a, b), (b, a)\}$
$U_1 = \Delta_X \cup \{(a, b), (b, a), (a, c)\}$
$U_2 = \Delta_X \cup \{(a, b), (b, a), (a, c), (c, a)\}$
$U_3 = \Delta_X \cup \{(a, b), (b, a), (a, c), (b, c)\}$
$U_4 = \Delta_X \cup \{(a, b), (b, a), (a, c), (c, b)\}$
$U_5 = \Delta_X \cup \{(a, b), (b, a), (c, a)\}$
$U_6 = \Delta_X \cup \{(a, b), (b, a), (c, a), (b, c)\}$
$U_7 = \Delta_X \cup \{(a, b), (b, a), (c, a), (c, b)\}$
$U_8 = \Delta_X \cup \{(a, b), (b, a), (b, c)\}$
$U_9 = \Delta_X \cup \{(a, b), (b, a), (c, b)\}$
$U_{10} = \Delta_X \cup \{(a, b), (b, a), (b, c), (c, b)\}$
$U_{11} = \Delta_X \cup \{(a, b), (b, a), (b, c), (c, b), (a, c).\}$
$U_{12} = \Delta_X \cup \{(a, b), (b, a), (b, c), (c, b), (c, a)\}$
$U_{13} = \Delta_X \cup \{(a, b), (b, a), (b, c), (a, c).(c, a)\}$
$U_{14} = \Delta_X \cup \{(a, b), (b, a), (c, b), (a, c).(c, a)\}$.

Then $\mathscr{U} = \{U_0, U_1, U_2, \ldots, U_{14}, X \times X\}$ is a uniformity on X for which $\tau_{\mathscr{U}} = \{\emptyset, \{a, b\}, \{c\}, X\}$. Let $f : X \to X$ be a permutation defined by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then f is uniformly continuous. We show that f does not have the average shadowing property. Let $E_0 = X \times X$ and $E_i = U_0$ for all $i \geq 1$. Then $\mathcal{E} = \{E_i\} \in \Sigma_{\mathscr{U}}$. Put $x_0 = a$ and for each $i \in \mathbb{N}$ and $k \geq 0$ define

$$x_i = \begin{cases} f^i(a) & \text{if } 2^{2k} \leq i < 2^{2k+1}, \\ c & \text{if } 2^{2k+1} \leq i < 2^{2k+2} \cdot \end{cases}$$

In other word

$$x_0, x_1, x_2, \cdots = \underbrace{a, b}_{2}, \underbrace{c, c}_{2}, \underbrace{a, b, a, b}_{4}, \underbrace{c, c, \ldots, c}_{8}, \underbrace{a, b, a, b, \ldots, b}_{16}, \underbrace{c, c, \ldots, c, c}_{32}, \ldots.$$

Let $\xi = \{x_i\}$. Since any element of $\mathscr{U}$ contains (a, b) and (b, a), we obtain

$$\frac{1}{n} \inf \left\{ \sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad (f(x_{j-1}), x_j) \in D_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} \leq \frac{\log_2 n}{n},$$

for $n \geq 2$ and arbitrary $\mathcal{D} = \{D_i\} \in \Sigma_{\mathscr{U}}$. That is ξ is an average $\mathcal{D}$-pseudo-orbit. For $2^k \leq n < 2^{k+1}$, we obtain

$$\frac{1}{n} \mathcal{A}_n(\xi, a, f, \mathscr{E}) = \frac{1}{n} \mathcal{A}_n(\xi, b, f, \mathscr{E}) \geq \frac{\sum_{i=0}^{k} 2^{2i-1}}{\sum_{i=0}^{2k} 2^i},$$
$$\frac{1}{n} \mathcal{A}_n(\xi, c, f, \mathscr{E}) \geq \frac{\sum_{i=0}^{k} 2^{2i}}{\sum_{i=0}^{2k} 2^i}.$$

hence

$$\lim_{n\to\infty} \frac{1}{n} \mathcal{A}_n(\xi, a, f, \mathscr{E}) = \lim_{n\to\infty} \frac{1}{n} \mathcal{A}_n(\xi, b, f, \mathscr{E}) \geq \frac{1}{3},$$
$$\lim_{n\to\infty} \frac{1}{n} \mathcal{A}_n(\xi, c, f, \mathscr{E}) \geq \frac{2}{3}.$$

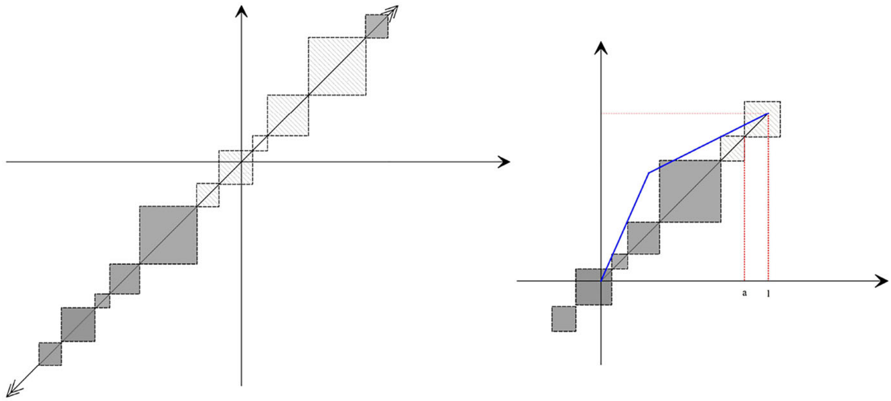

Fig. 1 Filter base for Michael line uniformity and the map f

Therefore ξ could not be $\mathcal{E}$-shadowed in average by any point in X. This implies that f does not have the topological average shadowing property. It is easy to show that f has the topological shadowing property. Indeed U_0 is an E-modulus of shadowing property for any $E \in \mathscr{U}$. Also f is not topologically transitive, since $f^n(\{c\}) \cap \{a, b\} = \emptyset$ for all $n \geq 0$.

The following example shows that the condition of surjectivity in the Theorem 3.2 is necessary.

Example 4.2 Let $(X, \mathscr{U})$ be a compact uniform space with more than one element. Let $a \in X$ be arbitrary. Consider the constant map $f : X \to X$, by $f(x) = a$ for all $x \in X$. Let $\mathcal{E} = \{E_i\} \in \Sigma_{\mathscr{U}}$ be arbitrary and $\mathcal{D} = \mathcal{E}$.Then for any $\xi = \{x_i\}_{i=0}^{\infty}$ and $n \in \mathbb{N}$,

$$\mathcal{A}_n(\xi, f, \mathcal{D}) = \mathcal{A}_n(\xi, x_0, f, \mathcal{E}).$$

This implies that f has the topological average shadowing property. But it is easy to check that f is not chain transitive.

By applying the method in [20, Example 5.4] we construct a dynamical system with **TASP** in a non-metrizable topological space.

Example 4.3 Let $\mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$. Let $\mathbf{a} = \{a_i\}_{i \in \mathbb{Z}} \subset \mathbb{P}$ be an increasing bi-sequence for which there exists a positive integer k such that $a_i + 1 = a_{i+k}$ for all $i \in \mathbb{Z}$. Put

$$U_a = \bigcup_{i \in \mathbb{Z}} \{\{(a_i, a_i)\} \cup (a_i, a_{i+1}) \times (a_i, a_{i+1})\} \subset \mathbb{R} \times \mathbb{R}.$$

(See Fig. 1). Then the family $\mathcal{B} = \{U_{\mathbf{a}}\}$ is a filter base and generate a uniformity $\mathscr{U}$. Consider this uniformity on S^1 (just taking the projection modulo 1). Then the topology generated by this uniformity is the Michael line topology

$\tau_M = \{U \cup F : U$ is open in usual topology and $F \subset \mathbb{P}\}$ which is not metrizable [27, b-13]. Consider the map $f : S^1 \to S^1$ by

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{3}, \\ \frac{1}{2}x + \frac{1}{2} & \text{if } \frac{1}{3} \le x \le 1 \end{cases}.$$

It is easy to see that f is uniformly continuous, in fact, $x \in \mathbb{P}$ iff $f^{-1}(x) \in \mathbb{P}$ and for any $U_a \in \mathcal{B}$,

$$f^{-1}(U_a) = \bigcup_{i \in \mathbb{Z}} \{\{(f^{-1}(a_i), f^{-1}(a_i))\} \cup (f^{-1}(a_i), f^{-1}(a_{i+1})) \times (f^{-1}(a_i), f^{-1}(a_{i+1}))\} \in \mathcal{B}.$$

Let $\mathcal{E} = \{E_i\} \in \Sigma_{\mathscr{U}}$ be arbitrary. Let $\mathcal{D} = \{D_i\}$, where $D_0 = S^1 \times S^1$ and $D_i \subset E_i \cap V_{\frac{1}{2^i}}$, for all $i \ge 1$. We show that any average $\mathcal{D}$-pseudo orbit can be $\mathcal{E}$-shadowed by the fixed point 1. Let $k \in \mathbb{N}$ be arbitrary. By the structure of the uniformity there exists $a, b \in \mathbb{P}$ such that $(1, 1) \in (a, b) \times (a, b) \subset E_k$. Put $\epsilon = 1 - a > 0$. Let $\{x_j\}$ be an average $\mathcal{D}$-pseudo orbit, that is, $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\{x_j\}, f, \mathcal{D}) = 0$. Since $\lim_{n\to\infty} f^n(x) = 1$ for all $x \in [\epsilon, 1]$, there exists $N \in \mathbb{N}$ such that

$$f^n(x) \in (1 - \frac{\epsilon}{2}, 1] forall\ n \ge Nandall\ x \in [\epsilon, 1]. \tag{3}$$

By [28, Theorem 1.20], there is a set $J_0 \subset \mathbb{Z}^+$ of zero density such that

$$\lim_{j\to\infty} \inf\{\frac{1}{2^{\sigma(j)}} | \quad (x_{j+1}, f(x_j)) \in D_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\} = 0$$

provided $n \notin J_0$. By uniform continuity of f there exists $\delta \in (0, \epsilon)$ such that

$$(f^i \times f^i)(V_\delta) \subset V_{\frac{\epsilon}{2N}} \text{ for all } 0 \le i \le N. \tag{4}$$

Choose $m > k$ such that $D_m \subset V_\delta$. Let $J_N = \{j : \{jN, jN + 1, \ldots, jN + N - 1\} \cap J_0 \ne \emptyset\}$ and $J'_N = \bigcup_{j \in J_N}\{jN, jN + 1, \ldots, jN + N - 1\}$. Then J'_N has density zero and $\lim_{n\to\infty} \inf\{\frac{1}{2^{\sigma(j)}} | \quad (x_{j+1}, f(x_j)) \in D_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\} = 0$ provided $n \notin J'_N$. Therefore there exists $N_1 \ge N$ such that

$$(x_{j+1}, f(x_j)) \in D_m \text{ for all } j \ge N_1 \text{ and } j \notin J'_N \tag{5}$$

hence it follows from (4) that

$$(f^N(x_{jN}), x_{(j+1)N}) \in \underbrace{V_{\frac{\epsilon}{2N}} \circ V_{\frac{\epsilon}{2N}} \circ \cdots \circ V_{\frac{\epsilon}{2N}}}_{\text{N times}} = V_{\frac{\epsilon}{2}} \text{ for all } \ j > N_1\, and\ j \notin J'_N. \tag{6}$$

Let $\hat{J}_N = \bigcup_{j \in J_N} \{(j-1)N, (j-1)N+1, \ldots, jN, jN+1, \ldots, jN+N-1\}$ and let $\hat{J}_N^c = \mathbb{N} \setminus \hat{J}_N$. Then $\hat{J}_N$ has density zero. It follows from (3) and (6) that

$$x_{(j+1)N+k} \in (a, 1], \quad 0 \le k \le N-1$$

for all $j \in \hat{J}_N^c \cap [N_1, \infty)$. That is $(x_{(j+1)N+k}, 1) \in E_k$ for all $j \in \hat{J}_N^c \cap [N_1, \infty)$. Since k is arbitrary we conclude that,

$$\lim_{j\to\infty} \inf \left\{ \frac{1}{2^{\sigma(j)}} : \quad (f^j(1), x_j) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} = 0.$$

hence it follows from [28, Theorem 1.20] that

$$\lim_{n\to\infty} \frac{1}{n} \inf \left\{ \sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad (f^j(1), x_j) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n \right\} = 0.$$

This shows that the fixed point 1 topologically $\mathcal{E}$-shadows $\{x_i\}_{i=0}^{\infty}$ in average. Therefore, f has the topological average shadowing property.

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