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Perturbed Nonlocal Stochastic Functional Differential Equations

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Abstract
This paper discusses the asymptotic behavior of the solution for a class of perturbed nonlocal stochastic functional differential equations (SFDEs, in short). By comparing it with the solution of the corresponding unperturbed one, we derive the conditions under which their solutions are close. Firstly, the results are established on finite time-intervals. Then, we also show the results hold when the length of time-interval tends to infinity as small perturbations tend to zero.

Keywords Nonlocal stochastic functional differential equation · Small perturbation · Closeness

Mathematics Subject Classification 60H10 · 60H20 · 34K50

1 Introduction

For the practical applications in mechanics, medicine biology, ecology and so on, stochastic functional differential equations (SFDEs, in short) attracted researchers' more attention. One can see [7,9–11,13] and the references therein. Moreover, nonlocal stochastic differential equations have potential application in finance market, one can see [1,8,12,14] for the details. Especially, Wu and Hu [15] introduced the following nonlocal SFDEs with infinite delay with the form

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$$\mathrm{d}y(t) = g\left(t, y_t, ||y_t||_p\right) \mathrm{d}t + \sigma\left(t, y_t, ||y_t||_p\right) \mathrm{d}B(t),\ t \geq 0, \tag{1}$$

where $y_t = y_t(\theta) =: \{y(t+\theta) : \theta \in (-\infty, 0]\}$, g and σ are two Borel measurable functions defined on the space $\mathbb{R}_+ \times BC((-\infty, 0]; \mathbb{R}^d) \times \mathbb{R}_+$. For $p \geq 2$, $||\cdot||_p$ is a norm in the space $L^p((-\infty, 0] \times \Omega; \mathbb{R}^d)$ with the form

$$||y_t||_p = \left[\int_{-\infty}^{0} \mathbb{E}|y(t+\theta)|^p \,\mathrm{d}\eta(\theta)\right]^{1/p},$$

where η is a probability measure and $BC((-\infty, 0]; \mathbb{R}^d)$ is the family of bounded continuous functions from $(-\infty, 0]$ to $\mathbb{R}^d$ with the norm $||\varphi|| = \sup_{-\infty<\theta\leq 0} |\varphi(\theta)|$. In [15], by virtue of the fixed point theorem, the authors proved the existence and uniqueness of the solution for the nonlocal SFDE (1) with the coefficients g and σ satisfying the Lipschitz condition and the linear growth condition.

In addition, SDEs with perturbations are very important not only from the theoretical point of view but also from the point of view of various application. One can see the related works by Janković and Jovanović [2–6] for discussing different class of perturbed stochastic differential equations.

If A is a vector, we denote its transpose as A^T. If A is a matrix, we denote its Frobenius norm as $|A| = \sqrt{\mathrm{trace}(A^T A)}$. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^d$ and $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space with its filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions. In the sequel, we assume that $B(t)$ is an d-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. For given constants $r, p > 0$, let $L^p([-r, 0]; \mathbb{R}^d)$ denote the family of $\mathbb{R}^d$-valued, Borel measurable functions $\psi(s)$, $-r \leq s \leq 0$, which is equipped with the following norm

$$||\psi||_{L^p} = \left(\int_{-r}^{0} |\psi(s)|^p \,\mathrm{d}s\right)^{1/p} < \infty.$$

Let $\mathscr{BC}_{\mathscr{F}_0}([-r, 0]; \mathbb{R}^d)$ be the family of continuous bounded $\mathbb{R}^d$-valued stochastic process $\phi = \{\phi(s), -r \leq s \leq 0\}$ such that $\phi(s)$ is $\mathscr{F}_0$-measurable for every s, here, we require that $\mathscr{F}_s = \mathscr{F}_0$ for $-r \leq s \leq 0$.

In this paper, we consider the following nonlocal SFDE with delay with the form

$$\begin{cases} \mathrm{d}y(t) = g(t, y_t, ||y_t||_2)\,\mathrm{d}t + \sigma(t, y_t, ||y_t||_2)\,\mathrm{d}B(t),\ t \geq 0, \\ y(t) = \phi(t),\ -r \leq t \leq 0, \end{cases} \tag{2}$$

where $y_t = y_t(\theta) = \{y(t+\theta) : -r \leq \theta \leq 0\}$ is an $L^2([-r, 0]; \mathbb{R}^d)$-valued stochastic process, $g : \mathbb{R}_+ \times L^2([-r, 0]; \mathbb{R}^d) \times \mathbb{R}_+ \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times L^2([-r, 0]; \mathbb{R}^d) \times \mathbb{R}_+ \to \mathbb{R}^{d\times n}$ are two Borel measurable functions, and $||\cdot||_2$ is a norm in the space $L^2([-r, 0] \times \Omega; \mathbb{R}^d)$ defined by

$$||y_t||_2 = \left(\int_{-r}^{0} \mathbb{E}|y(t+\theta)|^2 \,\mathrm{d}\mu(\theta)\right)^{1/2},$$

where μ is a probability measure.

Let the coefficients g and σ satisfy the following Lipschitz and linear growth conditions, that is, there exists a positive constant $K > 0$, for $t \geq 0$, $y, y' \in \mathbb{R}^d$, $\varphi, \varphi' \in L^2([-r,0];\mathbb{R}^d)$, such that

$$\begin{aligned} &|g(t,\varphi,y)-g(t,\varphi',y')| \vee |\sigma(t,\varphi,y)-\sigma(t,\varphi',y')| \leq k(||\varphi-\varphi'||_{L^2}+|y-y'|),\\ &|g(t,\varphi,y)| \vee |\sigma(t,\varphi,y)| \leq k(||\varphi||_{L^2}+|y|). \end{aligned} \tag{3}$$

Now, we propose the perturbed nonlocal SFDE with delay, that is, for a small parameter $\varepsilon \in (0,1)$ with the form

$$\begin{cases} \mathrm{d}y^\varepsilon(t) = \widetilde{g}(t, y_t^\varepsilon, ||y_t^\varepsilon||_2, \varepsilon)\,\mathrm{d}t + \widetilde{\sigma}(t, y_t^\varepsilon, ||y_t^\varepsilon||_2, \varepsilon)\,\mathrm{d}B(t),\ t \geq 0, \\ y^\varepsilon(t) = \phi^\varepsilon(t),\ -r \leq t \leq 0, \end{cases} \tag{4}$$

where $\widetilde{g}, \widetilde{\sigma}, \phi^\varepsilon$ have the following form

$$\widetilde{g}(t,\varphi,y,\varepsilon)=g(t,\varphi,y)+\alpha(t,\varphi,y,\varepsilon),\ \ \widetilde{\sigma}(t,\varphi,y,\varepsilon)=\sigma(t,\varphi,y)+\beta(t,\varphi,y,\varepsilon), \tag{5}$$

where α, β are the perturbed parameters defined as g, σ respectively. In what way, (4) could be regarded as *the perturbed equation* with respect to *the unperturbed equation* (2).

Motivated by the above works, the aim of this paper is to establish the relation between $y(t)$, the solution of (2), and $y^\varepsilon(t)$, the solution of (4) and show their closeness in the sense of $(2m)$-th moment for $m \in \mathbb{N}$. In doing so, we introduce the following assumptions.

(H1) For $m \geq 1$ and a non-random function $\delta(\varepsilon)$, it holds that

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[-r,0]} |\phi(t)|^{2m}\right] < \infty, \mathbb{E}\left[\sup_{t\in[-r,0]} |\phi^\varepsilon(t)|^{2m}\right] < \infty,\\ &\mathbb{E}\left[\sup_{t\in[-r,0]} |\phi^\varepsilon(t)-\phi(t)|^{2m}\right] \leq \delta(\varepsilon). \end{aligned} \tag{6}$$

(H2) There exist two non-negative bounded functions $\overline{\alpha}(\cdot)$ and $\overline{\beta}(\cdot)$, defined on $[0,T]$ and dependent on ε such that

$$\begin{aligned} &\sup_{\varphi\in L^2([-r,0];\mathbb{R}^d),\, y\in\mathbb{R}^d} |\alpha(t,\varphi,y,\varepsilon)| \leq \overline{\alpha}(t,\varepsilon),\\ &\sup_{\varphi\in L^2([-r,0];\mathbb{R}^d),\, y\in\mathbb{R}^d} |\beta(t,\varphi,y,\varepsilon)| \leq \overline{\beta}(t,\varepsilon). \end{aligned} \tag{7}$$

(H3) We assume that the functions g, σ, α, β satisfy the Lipschitz and linear growth conditions proposed previously.

Then, under the above conditions and by the same procedures as Wu and Hu [15], we can easily show that (2) has a unique solution. Moreover, it holds that $\mathbb{E}\left[\sup_{-r\le t\le T}|y(t;\phi)|^{2m}\right]<\infty$ and $\mathbb{E}\left[\sup_{-r\le t\le T}|y^{\varepsilon}(t;\phi^{\varepsilon})|^{2m}\right]<\infty$.

The paper is organized as follows. In Sect. 2, we introduce some preliminaries. Section 3 is devoted to the main result. In Sect. 4, an example is given to illustrate the obtained result. In the last Section, concluding remarks are given.

2 Preliminaries

Let's first prove an independent result, which is crucial for the next part.

Lemma 1 ([6], Gronwall-Bellman inequality) *For three non-negative and continuous functions defined on* $[0,T]$, $\nu(t)$, $b(t)$ *and* $c(t)$ *satisfying that*

$$\nu(t)\le C+\int_0^t b(s)\nu(s)\mathrm{d}s+\int_0^t c(s)\nu^{\alpha}(s)\mathrm{d}s,\ t\in[0,T],$$

where $C>0,\ 0\le\alpha<1$ *are constants. Then, it holds the following relation*

$$\nu(t)\le\left[C^{1-\alpha}e^{(1-\alpha)\int_0^t b(s)\mathrm{d}s}+(1-\alpha)\int_0^t c(s)e^{(1-\alpha)\int_s^t b(r)\mathrm{d}r}\mathrm{d}s\right]^{1/(1-\alpha)},\ t\in[0,T]. \tag{8}$$

Theorem 1 *Let* $y^{\varepsilon}(t)$ *and* $y(t)$ *be the solutions of the* (2) *and* (4)*, respectively, defined on a finite interval* $[0,T]$ *, and let the assumption* (H1)– (H3) *be satisfied. Then, for* $t\in[0,T]$ *and* $m>1$*,*

$$\begin{aligned}&\mathbb{E}\left[\sup_{s\in[-r,t]}|y^{\epsilon}(s)-y(s)|^{2m}\right]\\&\le\left[(5\delta(\epsilon))^{\frac{m-1}{m}}e^{8m(m-1)\int_0^t(a_m+b_m+c_m)\mathrm{d}s}\right.\\&\left.+8m(m-1)\int_0^t(b_m+c_m)e^{8m(m-1)\int_s^t(a_m+b_m+c_m)\mathrm{d}\tau}\mathrm{d}s\right]^{\frac{m}{m-1}},\end{aligned}\tag{9}$$

where

$$\begin{aligned}a_m&=Tk^2r+8k^2r+8(m-1)^2Tk^4r^2,\\b_m&=Tk^2Ae^{\varphi s}+8k^2Ae^{\varphi s}+8(m-1)^2Tk^4(Ae^{\varphi s})^2,\\c_m&=T\bar{\alpha}^2+4\bar{\beta}^2+(m-1)^2T\bar{\beta}^4,\\A&=A(\phi,T,\varepsilon),\\\varphi&=\varphi(T).\end{aligned}$$

Proof Let us take

$$z^{\varepsilon}(t) = y^{\varepsilon}(t) - y(t),\ \ \Delta^{\varepsilon}(t) = \mathbb{E}\left[\sup_{s\in[-r,t]} |z^{\varepsilon}(s)|^{2m}\right].$$

To estimate $\Delta^{\varepsilon}(t)$, applying the Itô formula to $|z^{\varepsilon}(t)|^m$ and taking expectations, we have

$$|z^{\varepsilon}(t)|^m = |z^{\varepsilon}(0)|^m + m\left(I_1(t) + I_2(t) + \frac{1}{2}(m-1)I_3(t)\right), \tag{10}$$

where

$$\begin{aligned}
I_1(t) &= \int_0^t \left[\widetilde{g}\left(s, y_s^{\varepsilon}, ||y_s^{\varepsilon}||_2, \varepsilon\right) - g(s, y_s, ||y_s||_2)\right] |z^{\varepsilon}(s)|^{m-1}\mathrm{d}s,\\
I_2(t) &= \int_0^t \left[\widetilde{\sigma}\left(s, y_s^{\varepsilon}, ||y_s^{\varepsilon}||_2, \varepsilon\right) - \sigma(s, y_s, ||y_s||_2)\right] |z^{\varepsilon}(s)|^{m-1}\mathrm{d}B(s),\\
I_3(t) &= \int_0^t \left[\widetilde{\sigma}\left(s, y_s^{\varepsilon}, ||y_s^{\varepsilon}||_2, \varepsilon\right) - \sigma(s, y_s, ||y_s||_2)\right]^2 |z^{\varepsilon}(s)|^{m-2}\mathrm{d}s.
\end{aligned}$$

Then, we get

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]} |z^{\varepsilon}(s)|^{2m}\right] &= \mathbb{E}\left[\sup_{s\in[0,t]} \left(|z^{\varepsilon}(s)|^m\right)^2\right]\\
&\le 4\delta(\varepsilon) + 4m^2\mathbb{E}\left[\sup_{s\in[0,t]} |I_1(s)|^2\right] + 4m^2\mathbb{E}\left[\sup_{s\in[0,t]} |I_2(s)|^2\right]\\
&\quad + m^2(m-1)^2\mathbb{E}\left[\sup_{s\in[0,t]} |I_3(s)|^2\right].
\end{aligned}\tag{11}$$

Step 1. From (5) and (10), for $t \in [0, T]$, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]} |I_1(s)|^2\right]\\
&= \mathbb{E}\left[\sup_{u\in[0,t]} \left(\int_0^u [\widetilde{g}(s, y_s^{\varepsilon}, ||y_s^{\varepsilon}||_2, \varepsilon) - g(s, y_s, ||y_s||_2)]|z^{\varepsilon}(s)|^{m-1}\mathrm{d}s\right)^2\right]\\
&= \mathbb{E}\left[\sup_{u\in[0,t]} \left(\int_0^u [g(s, y_s^{\varepsilon}, ||y_s^{\varepsilon}||_2) - g(s, y_s, ||y_s||_2)\right.\right.\\
&\quad \left.\left. + \alpha(s, y_s^{\varepsilon}, ||y_s^{\varepsilon}||_2, \varepsilon)]|z^{\varepsilon}(s)|^{m-1}\mathrm{d}s\right)^2\right].
\end{aligned}$$

By applying Holder's inequality to $\mathbb{E}\left[\sup_{s\in[0,t]}|I_1(s)|^2\right]$, we get

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]}|I_1(s)|^2\right]\\
&\leq t\mathbb{E}\left[\sup_{u\in[0,t]}\int_0^u\left[g\left(s,y_s^\varepsilon,||y_s^\varepsilon||_2\right)-g(s,y_s,||y_s||_2)\right.\right.\\
&\quad\left.\left.+\alpha\left(s,y_s^\varepsilon,||y_s^\varepsilon||_2,\varepsilon\right)\right]^2|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\right]\\
&\leq 2t\mathbb{E}\left[\sup_{u\in[0,t]}\int_0^u\left([g\left(s,y_s^\varepsilon,||y_s^\varepsilon||_2\right)-g(s,y_s,||y_s||_2)]^2\right.\right.\\
&\quad\left.\left.+\alpha\left(s,y_s^\varepsilon,||y_s^\varepsilon||_2,\varepsilon\right)^2]\right)|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\right].
\end{aligned}$$

In view of (3), we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]}|I_1(s)|^2\right]\\
&\leq 2tk^2\mathbb{E}\left[\sup_{u\in[0,t]}\int_0^u\left(||y_s^\varepsilon-y_s||_{L^2}+\left|||y_s^\varepsilon||_2-||y_s||_2\right|\right)^2|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\right]\\
&\quad+2t\mathbb{E}\left[\sup_{u\in[0,t]}\int_0^u\alpha^2(s,y_s^\varepsilon,||y_s^\varepsilon||_2,\varepsilon)|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\right].
\end{aligned}\tag{12}$$

Since

$$\begin{aligned}
&\mathbb{E}\left[\sup_{u\in[0,t]}\int_0^u\left(||y_s^\varepsilon-y_s||_{L^2}+|||y_s^\varepsilon||_2-||y_s||_2|\right)^2|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\right]\\
&\leq 2\mathbb{E}\left[\sup_{u\in[0,t]}\int_0^u(||y_s^\varepsilon-y_s||_{L^2}^2)|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\right]\\
&\quad+2\mathbb{E}\left[\sup_{u\in[0,t]}\int_0^u\left|||y_s^\varepsilon||_2-||y_s||_2\right|^2|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\right],
\end{aligned}\tag{13}$$

we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{u\in[0,t]}\int_0^u(||y_s^\varepsilon-y_s||_{L^2}^2)|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\right]\\
&=\mathbb{E}\left[\sup_{u\in[0,t]}\int_0^u\left(\int_{-r}^0|y^\varepsilon(s+\tau)-y(s+\tau)|^2\mathrm{d}\tau\right)|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\right]
\end{aligned}$$

$$\leq r\int_0^t \Delta^\varepsilon(s)\mathrm{d}s, \tag{14}$$

$$\mathbb{E}\left[\sup_{u\in[0,t]}\int_0^u \left|||y_s^\varepsilon||_2 - ||y_s||_2\right|^2 |z^\varepsilon(s)|^{2m-2}\mathrm{d}s\right]$$
$$\leq \mathbb{E}\left[\sup_{u\in[0,t]}\int_0^u \left(\int_{-r}^0 E|z(s+\theta)|^2\mathrm{d}\mu(\theta)\right)|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\right]$$
$$\leq \mathbb{E}\left[\sup_{u\in[0,t]}\int_0^u \left(\sup_{-r<\tau\leq s}|z^\varepsilon(\tau)|^2\right)|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\right]$$
$$\leq \mathbb{E}\left[\sup_{u\in[0,t]}\int_0^u Ae^{\varphi s}|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\right]$$
$$= \int_0^t Ae^{\varphi s}(\Delta^\varepsilon(s))^{\frac{m-1}{m}}\mathrm{d}s, \tag{15}$$

where

$$A = A(\phi, T, \varepsilon),\ \varphi = \varphi(T).$$
$$\mathbb{E}\left[\sup_{u\in[0,t]}\int_0^u \alpha^2\left(s, y_s^\varepsilon, ||y_s^\varepsilon||_2, \varepsilon\right)|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\right]$$
$$\leq \int_0^t \bar{\alpha}^2(s,\varepsilon)\left(\Delta^\varepsilon(s)\right)^{\frac{m-1}{m}}\mathrm{d}s, \tag{16}$$

Therefore, from (12)–(15) and (16), we get

$$\mathbb{E}\left[\sup_{s\in[0,t]}|I_1(s)|^2\right] \leq 2tk^2r\int_0^t \Delta^\varepsilon(s)\mathrm{d}s$$
$$+2t\int_0^t \left[k^2Ae^{\varphi s} + \bar{\alpha}^2(s,\varepsilon)\right](\Delta^\varepsilon(s))^{\frac{m-1}{m}}\mathrm{d}s. \tag{17}$$

By the following elementary inequality $|a|^{r_1} \leq |a|^{r_2} + |a|$, $0 < r_2 \leq r_1 < 1$ and putting $a = \Delta^\varepsilon(s), r_1 = \frac{m-1}{m}, r_2 = \frac{1}{m}$, we get $(\Delta^\varepsilon(s))^{\frac{m-1}{m}} \leq (\Delta^\varepsilon(s))^{\frac{1}{m}} + \Delta^\varepsilon(s)$. Thus, the relation (17) becomes

$$\mathbb{E}\left[\sup_{s\in[0,t]}|I_1(s)|^2\right] \leq \int_0^t \left[2tk^2r + 2tk^2Ae^{\varphi s} + 2t\bar{\alpha}^2(s,\varepsilon)\right](\Delta^\varepsilon(s))\mathrm{d}s$$
$$+\int_0^t 2t\left[k^2Ae^{\varphi s} + \bar{\alpha}^2(s,\varepsilon)\right](\Delta^\varepsilon(s))^{\frac{1}{m}}\mathrm{d}s. \tag{18}$$

Step 2. In order to estimate $\mathbb{E}\Big[\sup_{s\in[0,t]}|I_2(s)|^2\Big]$, by the Burkholder–Davis–Gundy inequality, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[0,t]}|I_2(s)|^2\Big]\\
&=\mathbb{E}\Big[\sup_{u\in[0,t]}\Big(\int_0^u[\widetilde{\sigma}(s,y_s^\varepsilon,||y_s^\varepsilon||_2,\varepsilon)-\sigma(s,y_s,||y_s||_2)]|z^\varepsilon(s)|^{m-1}\mathrm{d}B(s)\Big)^2\Big]\\
&\le 4\mathbb{E}\int_0^t\left[\widetilde{\sigma}\left(s,y_s^\varepsilon,||y_s^\varepsilon||_2,\varepsilon\right)-\sigma(s,y_s,||y_s||_2)\right]^2|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\\
&=4\mathbb{E}\int_0^t\left[\sigma\left(s,y_s^\varepsilon,||y_s^\varepsilon||_2\right)+\beta(s,y_s^\varepsilon,||y_s^\varepsilon||_2,\varepsilon)\right.\\
&\quad\left.-\sigma(s,y_s,||y_s||_2)\right]^2|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\\
&\le 8\mathbb{E}\int_0^t\Big(\sigma\left[\left(s,y_s^\varepsilon,||y_s^\varepsilon||_2\right)-\sigma(s,y_s,||y_s||_2)\right]^2\\
&\quad+\beta^2(s,y_s^\varepsilon,||y_s^\varepsilon||_2,\varepsilon)\Big)|z^\varepsilon(s)|^{2m-2}\mathrm{d}s.
\end{aligned}\tag{19}$$

Similarly,

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]}|I_2(s)|^2\right]&\le 8k^2\mathbb{E}\int_0^t\left(||y_s^\varepsilon-y_s||_{L_2}+\left|||y_s^\varepsilon||_2-||y_s||_2\right|\right)^2|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\\
&\quad+8\mathbb{E}\int_0^t\beta^2(s,y_s^\varepsilon,||y_s^\varepsilon||_2,\varepsilon)|z^\varepsilon(s)|^{2m-2}\mathrm{d}s.
\end{aligned}\tag{20}$$

Furthermore, we have

$$\begin{aligned}
&\mathbb{E}\int_0^t\left(||y_s^\varepsilon-y_s||_{L_2}+\left|||y_s^\varepsilon||_2-||y_s||_2\right|\right)^2|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\\
&\le 2\mathbb{E}\int_0^t\left(||y_s^\varepsilon-y_s||_{L_2}^2+\left|||y_s^\varepsilon||_2-||y_s||_2\right|^2\right)|z^\varepsilon(s)|^{2m-2}\mathrm{d}s
\end{aligned}\tag{21}$$

and

$$\begin{aligned}
&\mathbb{E}\int_0^t(||y_s^\varepsilon-y_s||_{L^2}^2)|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\\
&=\mathbb{E}\int_0^t\left(\int_{-r}^0|y^\varepsilon(s+\tau)-y(s+\tau)|^2\mathrm{d}\tau\right)|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\\
&\le r\int_0^t\Delta^\varepsilon(s)\mathrm{d}s,
\end{aligned}\tag{22}$$

$$\mathbb{E}\int_0^t\left|||y_s^\varepsilon||_2-||y_s||_2\right|^2|z^\varepsilon(s)|^{2m-2}\mathrm{d}s$$

$$
\begin{aligned}
&\leq \mathbb{E}\int_0^t \left(\int_{-r}^0 E[\sup_{-r<\tau\leq s}|z^\varepsilon(\tau)|^2]\right)|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\\
&\leq \int_0^t Ae^{\varphi s}|z^\varepsilon(s)|^{2m-2}\mathrm{d}s\\
&\leq \int_0^t Ae^{\varphi s}(\Delta^\varepsilon(s))^{\frac{m-1}{m}}\mathrm{d}s, \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\int_0^t \beta^2(r, y_r^\varepsilon, ||y_r^\varepsilon||_2, \varepsilon)(z^\varepsilon(r))^{2m-2}\mathrm{d}r\\
&\leq \int_0^t \bar{\beta}^2(r,\varepsilon)(\Delta^\varepsilon(r))^{\frac{m-1}{m}}\mathrm{d}r. \qquad (24)
\end{aligned}
$$

Now, substituting (21)–(24) into (20) yields that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]}|I_2(s)|^2\right] &\leq 16k^2 r\int_0^t \Delta^\varepsilon(u)\mathrm{d}u\\
&+2\int_0^t [8k^2Ae^{\varphi u}+4\bar{\beta}^2(u,\varepsilon)](\Delta^\varepsilon(u))^{\frac{m-1}{m}}\mathrm{d}u. \qquad (25)
\end{aligned}
$$

Since $a=\Delta^\varepsilon(u)$, $r_1=\frac{m-1}{m}$, $r_2=\frac{1}{m}$, we obtain $(\Delta^\varepsilon(u))^{\frac{m-1}{m}} \leq (\Delta^\varepsilon(u))^{\frac{1}{m}}+\Delta^\varepsilon(u)$. (25) yields that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]}|I_2(s)|^2\right] &\leq \int_0^t \left[16k^2r+16k^2Ae^{\varphi u}+8\bar{\beta}^2(u,\varepsilon)\right](\Delta^\varepsilon(u))\mathrm{d}u\\
&+\int_0^t \left[16k^2Ae^{\varphi u}+8\bar{\beta}^2(u,\varepsilon)\right](\Delta^\varepsilon(u))^{\frac{1}{m}}\mathrm{d}u. \qquad (26)
\end{aligned}
$$

Step 3. Now, we give the estimate for $\mathbb{E}\left[\sup_{s\in[0,t]}|I_3(s)|^2\right]$.

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]}|I_3(s)|^2\right]\\
&=\mathbb{E}\left[\sup_{u\in[0,t]}\left(\int_0^u [\widetilde{\sigma}(s, y_s^\varepsilon, ||y_s^\varepsilon||_2, \varepsilon)-\sigma(s, y_s, ||y_s||_2)]^2|z^\varepsilon(s)|^{m-2}\mathrm{d}s\right)^2\right]\\
&\leq \mathbb{E}\left[\sup_{u\in[0,t]}\left(\int_0^u [\sigma(s, y_s^\varepsilon, ||y_s^\varepsilon||_2)-\sigma(s, y_s, ||y_s||_2)\right.\right.\\
&\left.\left.+\beta(s, y_s^\varepsilon, ||y_s^\varepsilon||_2, \varepsilon)]^2|z^\varepsilon(s)|^{m-2}\mathrm{d}s\right)^2\right]\\
&\leq t\mathbb{E}\left[\sup_{u\in[0,t]}\int_0^u [\sigma(s, y_s^\varepsilon, ||y_s^\varepsilon||_2)-\sigma(s, y_s, ||y_s||_2)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\beta(s, y_s^\varepsilon, ||y_s^\varepsilon||_2, \varepsilon)]^4 |z^\varepsilon(s)|^{2m-4} \mathrm{d}s\Bigg] \\
&\leq 8t\mathbb{E}\Bigg[\sup_{u\in[0,t]} \int_0^u [\sigma(s, y_s^\varepsilon, ||y_s^\varepsilon||_2) - \sigma(s, y_s, ||y_s||_2)]^4 |z^\varepsilon(s)|^{2m-4} \mathrm{d}s\Bigg] \\
&\quad +8t\mathbb{E}\left[\sup_{u\in[0,t]} \int_0^u \beta^4(s, y_s^\varepsilon, ||y_s^\varepsilon||_2, \varepsilon) |z^\varepsilon(s)|^{2m-4} \mathrm{d}s\right] \\
&\leq 64k^4 t\mathbb{E}\left[\sup_{u\in[0,t]} \int_0^u \left(||y_s^\varepsilon - y_s||_{L_2}^4 + \left|||y_s^\varepsilon||_2 - ||y_s||_2\right|^4\right) |z^\varepsilon(s)|^{2m-4} \mathrm{d}s\right] \\
&\quad +8t\mathbb{E}\left[\sup_{u\in[0,t]} \int_0^u \beta^4(s, y_s^\varepsilon, ||y_s^\varepsilon||_2, \varepsilon) |z^\varepsilon(s)|^{2m-4} \mathrm{d}s\right].
\end{aligned} \tag{27}
$$

While,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{u\in[0,t]} \int_0^u ||y_s^\varepsilon - y_s||_{L_2}^4 |z^\varepsilon(s)|^{2m-4} \mathrm{d}s\right] \\
&= \mathbb{E}\left[\sup_{u\in[0,t]} \int_0^u \left(\int_{-r}^0 |y^\varepsilon(s+\tau) - y(s+\tau)|^2 \mathrm{d}\tau\right)^2 |z^\varepsilon(s)|^{2m-4} \mathrm{d}s\right] \\
&\leq r^2 \int_0^t \Delta^\varepsilon(s) \mathrm{d}s,
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{u\in[0,t]} \int_0^u \left|||y_s^\varepsilon||_2 - ||y_s||_2\right|^4 |z^\varepsilon(s)|^{2m-4} \mathrm{d}s\right] \\
&\leq \mathbb{E}\left[\sup_{u\in[0,t]} \int_0^u \left(\int_{-r}^0 \mathbb{E}|z^\varepsilon(s+\theta)|^2 \mathrm{d}\mu\right)^2 |z^\varepsilon(s)|^{2m-4} \mathrm{d}s\right] \\
&\leq \mathbb{E}\left[\sup_{u\in[0,t]} \int_0^u (Ae^{\varphi s})^2 |z^\varepsilon(s)|^{2m-4} \mathrm{d}s\right] \\
&= \int_0^t (Ae^{\varphi s})^2 (\Delta^\varepsilon(s))^{\frac{m-2}{m}} \mathrm{d}s,
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{u\in[0,t]} \int_0^u \beta^4(s, y_s^\varepsilon, ||y_s^\varepsilon||_2, \varepsilon)] |z^\varepsilon(s)|^{2m-4} \mathrm{d}s\right] \\
&\leq \int_0^t \bar{\beta}^4(s, \varepsilon) (\Delta^\varepsilon(s))^{\frac{m-2}{m}} \mathrm{d}s.
\end{aligned} \tag{30}
$$

Then, it follows from (27) that

$$\mathbb{E}\left[\sup_{s\in[0,t]}|I_3(s)|^2\right]$$
$$\leq 64k^4tr^2\int_0^t \Delta^\varepsilon(u)\mathrm{d}u + 8t\left(\int_0^t [8k^4(Ae^{\varphi u})^2+\bar{\beta}^4(u,\varepsilon)](\Delta^\varepsilon(u))^{\frac{m-2}{m}}\mathrm{d}u\right)$$
$$\leq \int_0^t [64k^4tr^2+64k^4t(Ae^{\varphi u})^2+8t\bar{\beta}^4(u,\varepsilon)](\Delta^\varepsilon(u))\mathrm{d}u$$
$$+\int_0^t [64k^4t(Ae^{\varphi u})^2+8t\bar{\beta}^4(u,\varepsilon)](\Delta^\varepsilon(u))^{\frac{1}{m}}\mathrm{d}u. \tag{31}$$

Step 4. It follows from the definition of $\Delta^\varepsilon(t)$, we can derive

$$\Delta^\varepsilon(t) \leq \delta(\varepsilon) + E\left[\sup_{s\in[0,t]}|z^\epsilon(s)|^{2m}\right]. \tag{32}$$

Substituting (18), (26) and (31) into (32), we have

$$\Delta^\varepsilon(t) \leq 5\delta(\varepsilon) + 8m^2\int_0^t (a_m+b_m+c_m)(\Delta^\varepsilon(s))\mathrm{d}s$$
$$+8m^2\int_0^t (b_m+c_m)(\Delta^\varepsilon(s))^{\frac{1}{m}}\mathrm{d}s, \tag{33}$$

where a_m , b_m and c_m are determined by (9). In view of the Gronwall-Bellman inequality, the estimate (9) holds. □

3 Main Result

Since the magnitude of the perturbations of (2) is determined by the quantities $\delta(\cdot)$, $\alpha(\cdot)$, $\beta(\cdot)$, and $A(\cdot)$, it is natural to impose some conditions on these quantities and see how $\Delta_t^\varepsilon = \mathbb{E}\left[\sup_{s\in[-r,t]}|y^\epsilon(s)-y(s)|^{2m}\right] \to 0$ as $\varepsilon \to 0$ and on which intervals this convergence holds.

Theorem 2 *Under the conditions of Theorem 1, let $\delta(\cdot)$, $\bar{\alpha}(\cdot)$, $\bar{\beta}(\cdot)$, and $\bar{A}(\cdot)$ tend to zero as ε tends to zero. Then, it holds that*

$$\mathbb{E}\left[\sup_{t\in[-r,T]}|y^\epsilon(t)-y(t)|^{2m}\right] \to 0 \text{ as } \varepsilon \to 0. \tag{34}$$

Proof In the sequel, let

$$\bar{\alpha}(\varepsilon) = \sup_{t\in[0,T]}\bar{\alpha}(t,\varepsilon), \quad \bar{\beta}(\varepsilon) = \sup_{t\in[0,T]}\bar{\beta}(t,\varepsilon), \quad \bar{A}(\varepsilon) = \sup_{t\in[0,T]}A(\phi,t,\varepsilon), \tag{35}$$
$$\xi(\varepsilon) = \max\{\,\delta(\varepsilon),\ \bar{\alpha}^2(\varepsilon),\ \bar{\beta}^2(\varepsilon),\ \bar{A}(\varepsilon)\}. \tag{36}$$

It is obvious that $\lim_{\varepsilon\to 0}\xi(\varepsilon)=0$. And from the b_m and c_m defined in Theorem 1, we can have two polynomials p_m and q_m satisfy

$$b_m \le \xi(\varepsilon)p_m, \quad c_m \le \xi(\varepsilon)q_m.$$

Then, one get

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[-r,T]}|y^{\epsilon}(t)-y(t)|^{2m}\right]\\
&\le\left[(5\xi(\varepsilon))^{\frac{m-1}{m}}e^{8m(m-1)\int_0^T(a_m+b_m+c_m)\mathrm{d}s}\right.\\
&\quad\left.+16m(m-1)\xi(\varepsilon)(p_m+q_m)\int_0^T e^{8m(m-1)\int_s^t(a_m+b_m+c_m)\mathrm{d}\tau}\mathrm{d}s\right]^{\frac{m}{m-1}}\\
&\le\xi(\varepsilon)\left[5^{\frac{m-1}{m}}e^{8m(m-1)\int_0^T(a_m+b_m+c_m)\mathrm{d}s}\right.\\
&\quad\left.+16m(m-1)(p_m+q_m)\int_0^T e^{8m(m-1)\int_s^t(a_m+b_m+c_m)\mathrm{d}\tau}\mathrm{d}s\right]^{\frac{m}{m-1}}\\
&=\xi(\varepsilon)e^{8m^2\int_0^T(a_m+b_m+c_m)\mathrm{d}s}\left[5^{\frac{m-1}{m}}+16m(m-1)(p_m+q_m)T\right]^{\frac{m}{m-1}}\\
&=\xi(\varepsilon)\eta(T)e^{8m^2\int_0^T(a_m+b_m+c_m)\mathrm{d}s},
\end{aligned}\tag{37}$$

where

$$\eta(T)=\left[5^{\frac{m-1}{m}}+16m(m-1)(p_m+q_m)T\right]^{\frac{m}{m-1}}.$$

For T is finite and $\lim\limits_{\varepsilon\to 0}\xi(\varepsilon)=0$, it yields that

$$\Delta^{\varepsilon}(T)\to 0 \quad \text{as} \quad \varepsilon\to 0.$$

In what follows, we show that when the finite time-intervals whose length tends to infinity as $\varepsilon\to 0$ such that

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{t\in[-r,T]}|y^{\epsilon}(t)-y(t)|^{2m}\right]=0$$

on these intervals. □

Theorem 3 *Under the conditions of Theorem 2, for $t\in[-r,\infty]$ and an arbitrary $\rho\in(0,1)$, there is a positive number $T(\varepsilon)>0$, which is determined by*

$$T(\varepsilon)=\frac{1}{3\varphi}\ln\frac{-\rho\ln\xi(\varepsilon)-j_m}{d_m+e_m+h_m},\tag{38}$$

where

$$\begin{aligned} d_m &= 64m^2k^2r + 32m^2\bar{\beta}^2(\varepsilon) + 64m^2k^2\varphi^{-1}A - 64m^2k^2A\varphi^{-2}, \\ e_m &= 4m^2k^2r + 32m^2(m-1)^2k^4r^2 + 4m^2\bar{\alpha}^2(\varepsilon) \\ &\quad +4m^2(m-1)^2\bar{\beta}^4(\varepsilon) - 16m^2(m-1)^2k^4\varphi^{-2}A^2 + 64m^2k^2\varphi^{-1}A, \\ h_m &= 32m^2(m-1)^2k^4\varphi^{-1}A^2, \\ j_m &= 64m^2k^2A\varphi^{-2} + 16m^2(m-1)^2k^4A^2\varphi^{-2}, \end{aligned} \tag{39}$$

such that

$$\mathbb{E}\left[\sup_{t\in[-r,T(\varepsilon)]} |y^{\epsilon}(t) - y(t)|^{2m}\right] \to 0\,, \quad \text{as } \varepsilon \to 0. \tag{40}$$

Proof For fixed $T > 0$, on the fixed time-interval $[-r, T]$, it holds that

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\sup_{t\in[-r,T]} |y^{\epsilon}(t) - y(t)|^{2m}\right] = 0.$$

Thus, we know that there is a positive constant $T = T(\varepsilon)$ and determine effectively $T(\varepsilon)$ such that (40) holds. We know from the given requirements, if we let

$$e^{8m^2\int_0^T (a_m+b_m+c_m)\mathrm{d}s} \to 0, \quad \text{as } \varepsilon \to 0,$$

the corresponding conclusion can be obtained.

$$\begin{aligned} &8m^2\int_0^{T(\varepsilon)} (a_m + b_m + c_m)\mathrm{d}s \\ &\le \left[64m^2k^2r + 32m^2\bar{\beta}^2(\varepsilon) + 64m^2k^2\varphi^{-1}A - 64m^2k^2A\varphi^{-2}\right]e^{\varphi T(\varepsilon)} \\ &\quad +4m^2k^2r + 32m^2(m-1)^2k^4r^2 + 4m^2\bar{\alpha}^2(\varepsilon)e^{2\varphi T(\varepsilon)} \\ &\quad +32m^2(m-1)^2k^4\varphi^{-1}A^2e^{3\varphi T(\varepsilon)} \\ &\quad +32m^2(m-1)^2k^4\varphi^{-1}A^2\varphi^{-2} \\ &\le (d_m + e_m + h_m)e^{3\varphi T(\varepsilon)} + j_m \\ &= -\rho \ln \xi(\varepsilon), \end{aligned} \tag{41}$$

which yields that (38). □

4 An Example

Example 1 Let us discuss the following perturbed SFDE

$$dy(t)^{\varepsilon} = \left[\int_{-0.1}^{0} \frac{|y^{\varepsilon}(t+\theta)|}{1+|y^{\varepsilon}(t+\theta)|}\mathrm{d}\theta + \sin\frac{2^{-t}\varepsilon}{1+|y^{\varepsilon}(t+\theta)|}\right]\mathrm{d}t + \left[\ln(e^{-s})\left|\int_{-0.1}^{0}[y^{\varepsilon}(t+\theta)+(2+r)^{\frac{-1}{m}}]\mathrm{d}B_{\theta}\right|+1\right]\mathrm{d}B_t, \qquad (42)$$

while

$$dy(t) = \left(\int_{-0.1}^{0} \frac{|y(t+\theta)|}{1+|y(t+\theta)|}\mathrm{d}\theta\right)\mathrm{d}t + \ln\left(e^{-s}\left|\int_{-0.1}^{0} y(t+\theta)\mathrm{d}B_{\theta})\right|+1\right)\mathrm{d}B_t, \qquad (43)$$

is the corresponding unperturbed one. It is obvious that (42) and (43) satisfy the global Lipschitz condition and the linear growth condition and holds that $\mathbb{E}\left(\sup_{-r\le t\le T}|y(t;\phi)|^{2m}\right) < \infty$. Then, these equations has unique solutions. Moreover, all the conditions of Theorems 1, 2 and 3 are satisfied. Here,

$$\delta(\varepsilon) = \varepsilon^{2m},\ \ \bar{\alpha}(\varepsilon) = \sin\varepsilon,\ \ \bar{\beta}(\varepsilon) = 2^{\frac{-1}{\varepsilon}},$$
$$\xi(\varepsilon) = \max\{\ \xi^2,\ \ \sin\varepsilon,\ \ 2^{\frac{-2}{\varepsilon}}\} = \sin\varepsilon.$$

Therefore, by (38), it holds that

$$\mathbb{E}\left[\sup_{t\in[-r,T(\varepsilon)]}|y^{\epsilon}(t)-y(t)|^{2m}\right] \to 0,\ \ \text{as}\ \ \varepsilon \to 0.$$

5 Conclusions

In this paper, we study perturbed nonlocal SFDEs with delay. We establish the relation between $y(t)$ and $y^{\varepsilon}(t)$. What's more, for an interval $[0, T(\varepsilon)]$ whose length tends to infinity as $\varepsilon \to 0$, it holds that $\mathbb{E}\left[\sup_{t\in[0,T(\varepsilon)]}|y^{\epsilon}(t)-y(t)|^{2m}\right] \to 0$ as $\varepsilon \to 0$. An example is provided to illustrate the feasibility of the obtained result.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Hu, Y., Wu, F.: A class of stochastic differential equations with expectations in the coefficients. Nonlinear Anal. **81**, 190–199 (2013)
2. Janković, S., Jovanović, M.: Perturbed stochastic hereditary differential equations with integral contractors. Comput. Math. Appl. **42**, 871–881 (2001)
3. Janković, S., Jovanović, M.: Generalized stochastic perturbation-depending differential equations. Stoch. Anal. Appl. **20**, 1281–1307 (2002)
4. Janković, S., Jovanović, M.: On perturbed nonlinear Itô type stochastic integrodifferential equations. J. Math. Anal. Appl. **269**, 301–316 (2002)
5. Janković, S., Jovanović, M.: Functionally perturbed stochastic differential equations. Math. Nachr. **279**, 1808–1822 (2006)
6. Janković, S., Jovanović, M.: Neutral stochastic functional differential equations with additive perturbations. Appl. Math. Comput. **213**, 370–379 (2009)
7. Mao, X.: Stochastic Differential Equations and Applications. Horwood, Chichestic, UK (1997)
8. Peter, K., Thomas, L.: A Peano-like theorem for stochastic differential equations with nonlocal sample dependence. Stoch. Anal. Appl. **31**, 19–30 (2013)
9. Ren, Y., Lu, S., Xia, N.: Remarks on the existence and uniqueness of the solutions to stochastic functional differential equations with infinite delay. J. Comput. Appl. Math. **220**, 364–372 (2008)
10. Ren, Y., Xia, N.: Existence, uniqueness and stability of the solutions to neutral stochastic functional differential equations with infinite delay. Appl. Math. Comput. **210**, 72–79 (2009)
11. Ren, Y., Chen, L.: A note on the neutral stochastic functional differential equations with infinite delay and Poisson jumps in an abstract space. J. Math. Phys. **50**, 082704 (2009)
12. Sheinkman, J., LeBaron, B.: Nonlinear dynamics and stock returns. J. Busines **62**, 311–337 (1989)
13. Stoica, G.: A stochastic delay financial model. Proc. Am. Math. Soc. **133**, 1837–1841 (2005)
14. Thomas, L.: Nonlocal stochastic differential equations: existence and uniqueness of solutions. Bol. Soc. Esp. Mat. Apl. SeMA **51**, 99–107 (2010)
15. Wu, F., Hu, S.: On a class of nonlocal stochastic functional differential equations with infinite delay. Stoch. Anal. Appl. **29**, 713–721 (2011)