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Ergodicities of Infinite Dimensional Nonlinear Stochastic Operators

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Abstract

In the present paper, we introduce two classes $\mathcal{L}^+$ and $\mathcal{L}^-$ of nonlinear stochastic operators acting on the simplex of ℓ^1-space. For each operator V from these classes, we study omega limiting sets ω_V and $\omega_V^{(w)}$ with respect to ℓ^1-norm and pointwise convergence, respectively. As a consequence of the investigation, we establish that every operator from the introduced classes is weak ergodic. However, if V belongs to $\mathcal{L}^-$, then it is not ergodic (w.r.t ℓ^1-norm) while V is weak ergodic.

Keywords Stochastic operator · Infinite dimensional · Ergodic · Pointwise convergence

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1 Introduction

It is known [8,14] that nonlinear (in particular, quadratic) mappings appear in various branches of mathematics and their applications: the theory of differential equations,

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probability theory, the theory of dynamical systems, mathematical economics, mathematical biology, statistical physics, etc.

Recently nonlinear Markov chains become an interesting subject in many areas of applied mathematics. These chains are discrete time stochastic processes whose transitions, which are defined by stochastic hypermatrix $\mathcal{P} = (P_{i_1,\ldots,i_m,k})_{i_1,\ldots,i_m,k\in I}$, where $I \subset \mathbb{N}$ depend on both the current state and the current distribution of the process [11]. These processes were introduced by McKean [15] and have been extensively studied in the context of the nonlinear Chapman- Kolmogorov equation [7,19] as well as the nonlinear Fokker-Planck equation. On the other hand, we stress that such types of chains are generated by tensors (hypermatrices), therefore, this topic is closely related to the geometric and algebraic structures of tensors which have been systematically studied and has wide applications in scientific and engineering communities [12,13, 24].

Let us denote

$$S = \left\{ \mathbf{x} = (x_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \ : \ x_i \geq 0, \quad \sum_{i\in\mathbb{N}} x_i = 1 \right\}.$$

By means of $\mathcal{P}$ one defines an operator $V : S \to S$ by

$$(\mathfrak{B}(\mathbf{x}))_k = \sum_{i_1,i_2,\ldots,i_m\in\mathbb{N}} P_{i_1 i_2 \ldots i_m,k} x_{i_1} x_{i_2} \ldots x_{i_m}, \quad k \in \mathbb{N}.$$

This operator is called *m-ordered polynomial stochastic operator* (in short m-ordered PSO). Recently, in [18,20] we have studied surjectivity and bijectivity properties of $\mathfrak{B}$. However, limiting properties of the operator $\mathfrak{B}$ is not investigated (over infinite dimensional state space) at all. We stress that if the state space of $\mathfrak{B}$ is finite then such kind of operators have been examined in [24,26].

On the other hand, if one looks at the interacting populations which can be modeled by the Kolmogorov system

$$\begin{aligned} \frac{dx_k}{dt} &= x_k f_k(x_1, \ldots, x_n), \\ x_k(0) &\geq 0, \quad k = 1, \ldots, n, \end{aligned}$$

where $f_k(x_1, \ldots, x_n)$ are continuous differentiable functions. Such kind of models arise in biology, e.g., as food chain models [5] which lead to the investigation of Lotka–Volterra dynamical systems [27]. It is important to investigate dynamics of the associated system when the spices in the system is huge [3,23]. Roughly speaking, what happens if the game involves a large number of players? This naturally leads our attention to the following problem: what is the dynamical behavior of infinite dimensional operators acting on an infinite dimensional simplex? These investigations have essential applications in the game theory, evolutionary and dynamical aspects of population [1,8].

It is known that in statistical mechanics an ergodic hypothesis proposes a connection between dynamics and statistics. In the classical theory the assumption was made that the average time spent in any region of phase space is proportional to the volume of the region in terms of the invariant measure, or, more generally, that time averages may be replaced by space averages. On the basis of numerical calculations, Ulam [28] conjectured that an ergodic theorem holds for any 2-odered PSO acting on finite dimensional simplex. Afterwards, Zakharevich [29] proved that in general this conjecture is false. There are many results have been appeared to extend this result to Lotka–Volterra systems [10,21,25]. In the present paper, we are going to investigate this ergodicity question for general infinite dimensional stochastic operators.

In this present paper, we introduce two classes $\mathcal{L}^+$ and $\mathcal{L}^-$ of nonlinear stochastic operators acting on the simplex of ℓ^1-space. For each operator V from these classes, we study omega limiting sets ω_V and $\omega_V^{(w)}$ with respect to ℓ^1-norm and pointwise convergence, respectively. As a consequence of the investigation, we establish that every operator from the introduced classes is weak ergodic. However, if V belongs to $\mathcal{L}^-$, then it is not ergodic (w.r.t ℓ^1-norm) while V is weak ergodic. Besides, at the final section, we described all linear stochastic operators belonging to the classes $\mathcal{L}^+$ and $\mathcal{L}^-$.

2 Nonlinear Stochastic Operators

By ℓ^1 we denote the usual sequence space with the norm:

$$\|\mathbf{x}\| = \sum_{k=1}^{\infty} |x_k|.$$

As usual we denote

$$c_0 = \left\{ \mathbf{x} = (x_1, x_2, x_3, \dots) : \lim_{n\to\infty} x_n = 0 \right\}.$$

For a given $r > 0$ we denote

$$\mathbf{B}_r^+ = \{\mathbf{x} \in \ell^1 : x_k \geq 0 \text{ for all } k \in \mathbb{N}, \ \|\mathbf{x}\| \leq r\}$$

and

$$S_r = \{\mathbf{x} \in \mathbf{B}_r^+ : \|\mathbf{x}\| = r\}.$$

By S we denote S_1, i.e. $S := S_1$. This set S is called *simplex* in ℓ^1. By *support* of $\mathbf{x} = (x_1, x_2, x_3, \dots) \in S$ we mean a set $supp(\mathbf{x}) = \{i \in \mathbb{N} : x_i \neq 0\}$. In what follows, by $\mathbf{e}_i$ we denote the standard basis in S, i.e. $\mathbf{e}_i = (\delta_{i1}, \delta_{i2}, \delta_{i3}, \dots)$ $(i \in \mathbb{N})$, where δ_{ij} is the Kroneker delta.

Every mapping $V : S \to S$ is called *stochastic*. In present paper, we deal with such kind of operators.

Now, let us provide some concrete examples of stochastic operators.

Example 2.1 Let $f : \mathbf{x} \in S \mapsto f(\mathbf{x}) \in (0, 1]$ be a continuous functional (in ℓ^1-norm). Let us consider a nonlinear map defined by

$$(V(\mathbf{x}))_k = x_k \left(1 + \sum_{i \in \mathbb{N}} a_{ki} x_i f(\mathbf{x})\right), \quad \forall k \in \mathbb{N}, \ \forall \mathbf{x} = (x_1, x_2, x_3, \ldots), \tag{2.1}$$

where $A = (a_{ij})$ is a skew-symmetric matrix with

$$a_{ki} = -a_{ik}, \ |a_{ki}| \leq 1 \ \text{ for every } k, i \in \mathbb{N}. \tag{2.2}$$

One can check that the operator (2.1) is stochastic. Such kind of mapping is a particular Lotka–Volterra operator one [22,27]. Moreover, if $f(\mathbf{x}) \equiv 1$, then such operator reduces to the Volterra one (see [16]). Hence, for each f, by $\mathcal{LV}_f$ we denote the set of all Lotka–Volterra operators given by (2.1). The symbol $\mathbb{A}$ denotes the set of all skew-symmetric matrices with (2.2). The representation (2.1) establishes a one-to-one correspondence $\mathfrak{g} : \mathcal{LV}_f \to \mathbb{A}$ by $\mathfrak{g}(V) = (a_{ki})$. It is clear that $\mathfrak{g}$ is affine, hence $\mathcal{LV}_f$ is convex, and moreover, this correspondence allows to investigate certain geometric properties of $\mathcal{LV}_f$ by means of structure of the set $\mathbb{A}$ (see [16] for more details).

Remark 2.2 We notice that if the population which obeys the Volterra rule and at the same time a number of species is huge, then it is convenient to investigate dynamical behavior of Volterra operators on an infinite dimensional simplex [23].

In statistical mechanics an ergodic hypothesis proposes a connection between dynamics and statistics. In the classical theory the assumption was made that the average time spent in any region of phase space is proportional to the volume of the region in terms of the invariant measure, or, more generally, that time averages may be replaced by space averages. Therefore, in the present paper, our main aim is to study ergodicities of infinite dimensional stochastic operators.

For a given operator V on S, by $\{V^n(\mathbf{x}_0)\}_{n=0}^\infty$ we denote the trajectory of a point $\mathbf{x}_0 \in S$ under V. By $\omega_V(\mathbf{x}_0)$ (respectively, $\omega_V^{(w)}(\mathbf{x}_0)$) we denote the set of limit points of $\{V^n(\mathbf{x}_0)\}_{n=0}^\infty$ with respect to ℓ^1-norm (respectively, pointwise convergence). Namely, one has

$$\omega_V(\mathbf{x}_0) := \bigcap_{n \geq 0} \overline{\bigcup_{k \geq n} V^k(\mathbf{x}_0)}^{\|\cdot\|}.$$

Equivalently, $\mathbf{x}^* \in \omega_V(\mathbf{x}_0)$ means that there exists a subsequence $\{n_k\}$ such that

$$V^{n_k}(\mathbf{x}_0) \xrightarrow{\|\cdot\|} \mathbf{x}^*, \quad n_k \to \infty.$$

Similarly, we have

$$\omega_V^{(w)}(\mathbf{x}_0) := \bigcap_{n \geq 0} \overline{\bigcup_{k \geq n} V^k(\mathbf{x}_0)}^{\rho},$$

here ρ is the metric given by (A.1).

So, $\mathbf{x}^* \in \omega_V^{(w)}(\mathbf{x}_0)$ means that there exists a subsequence $\{n_k\}$ such that

$$V^{n_k}(\mathbf{x}_0) \xrightarrow{\text{p.w.}} \mathbf{x}^*, \quad n_k \to \infty.$$

In what follows, by a *fixed point* of V we mean a vector $\mathbf{x} \in S$ such that $V(\mathbf{x}) = \mathbf{x}$. By $Fix(V)$ we denote the set of all fixed points of V.

Obviously, if $\omega_V(\mathbf{x}_0)$ consists of a single point, i.e. $\omega_V(\mathbf{x}_0) = \{\mathbf{x}^*\}$, then the trajectory $\{V^n(\mathbf{x}_0)\}_{n=0}^{\infty}$ converges to $\mathbf{x}^*$. Moreover, $\mathbf{x}^*$ is a fixed point of V. However, looking ahead, we remark that convergence of trajectories is not a typical case for the dynamical systems (2.1). Therefore, it is of particular interest to obtain an upper bound for $\mathbf{x}_0 \in S$, i.e., to determine a sufficiently "small" set containing limiting point $\mathbf{x}^*$ under trajectory of Volterra operators.

Remark 2.3 If we consider stochastic operator V on finite dimensional simplex S^{d-1} the compactness of S^{d-1} implies $\omega_V(\mathbf{x}_0) \neq \emptyset$ for any $\mathbf{x}_0 \in S^{d-1}$. It turns out that this property is violated in the infinite dimensional setting (see Example 2.5). If we consider a stochastic operator V on infinite dimensional simplex, then according to Lemma A.4 one has $\omega_V^{(w)}(\mathbf{x}_0) \neq \emptyset$ for any $\mathbf{x}_0 \in S$. That fact gives a motivation in studying relationship between the sets $\omega_V(\mathbf{x}_0)$ and $\omega_V^{(w)}(\mathbf{x}_0)$

Definition 2.4 A stochastic operator $V : S \to S$ is called

(i) ergodic, if for every $\mathbf{x}_0 \in S$ the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} V^k(\mathbf{x}_0)$$

exists in ℓ^1-norm.

(ii) weak ergodic, if for every $\mathbf{x}_0 \in S$ the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} V^k(\mathbf{x}_0)$$

exists in pointwise convergence.

On the basis of numerical calculations, Ulam [28] conjectured that an ergodic theorem holds for any QSO V on finite dimensional simplex. Afterwards, Zakharevich [29] proved that in general this conjecture is false. In the present paper, we investigate this ergodic property for general infinite dimensional stochastic operators.

Let us consider one interesting example.

Example 2.5 Let $(a_{ij})_{i,j\geq 1}$ be an infinite dimensional skew-symmetric matrix such that $a_{ki} = a \in [-1, 0)$ for all $i > k$. Then the corresponding Volterra operator has the following form

$$(V(\mathbf{x}))_k = x_k(1+a-ax_k) - 2ax_k\sum_{i<k} x_i, \qquad \forall k \in \mathbb{N}, \ \ \forall \mathbf{x} \in S.$$

Now we show that $V^n(\mathbf{x}) \xrightarrow{\text{p.w.}} \mathbf{0}$ for any initial point $\mathbf{x} \in S$ with $|supp(\mathbf{x})| = \infty$. Let us suppose that $|supp(\mathbf{x})| = \infty$. For any $m \geq 1$ we define a point $x_{(m)} = \sum_{i\leq m} x_i$. One can see that $x_{(m)} \in [0, 1)$. We consider a function $f_a(x_{(m)}) = x_{(m)}\left(1 + a - ax_{(m)}\right)$, where $a \in [-1, 0)$. From $0 \leq x_{(m)} < 1$ we can easily verify that $0 \leq f_a(x_{(m)}) < x_{(m)}$. Hence, a sequence $\{f_a^n(x_{(m)})\}_{n=1}^{\infty}$ is a decreasing and bounded. From these and $Fix(f_a) \cap [0, 1) = \{0\}$ we conclude that $f_a^n(x_{(m)}) \to 0$ as $n \to \infty$. Consequently, we obtain $\sum_{k\leq m}(V^n(\mathbf{x}))_k \to 0$ as $n \to \infty$. This yields that for any $k \leq m$ it holds $(V^n(\mathbf{x}))_k \to 0$ as $n \to \infty$. From arbitrariness of $m \geq 1$ we have $V^n(\mathbf{x}) \xrightarrow{\text{p.w.}} \mathbf{0}$. Due to $\mathbf{0} \notin S$ and Lemma A.5 one concludes $\omega_V(\mathbf{x}) = \emptyset$.

Now let us suppose that $|supp(\mathbf{x})| = m_0$. Then we have $\sum_{k=1}^{m_0} x_k = 1$ and $x_k = 0$ for any $k > m_0$. So, one has $(V^n(\mathbf{x}))_k = 0$ for any $n \in \mathbb{N}$ and $k > m_0$.

On the other hand, from $\sum_{k=1}^{m_0-1} x_k < 1$ it follows that $f_a(x_{(m_0-1)}) \to 0$ as $n \to \infty$. Consequently, one gets

$$\sum_{k=1}^{m_0}(V^n(\mathbf{x}))_k \to 0, \quad \text{as } n \to \infty$$

It yields that

$$(V^n(\mathbf{x}))_k \to 0, \quad \text{for any } k < m_0 \tag{2.3}$$

Finally, the equality

$$\sum_{k=1}^{m_0}(V^n(\mathbf{x}))_k = 1$$

together with (2.3) implies $(V^n(\mathbf{x}))_{m_0} \to 1$, which means that $V^n(\mathbf{x}) \xrightarrow{\text{p.w.}} \mathbf{e}_{m_0}$. Hence, by Lemma A.5 we obtain $V^n(\mathbf{x}) \xrightarrow{\|\cdot\|} \mathbf{e}_{m_0}$.

Consequently, for any $\mathbf{x} \in S$ we find that

$$\omega_V(\mathbf{x}) = \begin{cases} \mathbf{e}_{m_0}, & \text{if } \max\{supp(\mathbf{x})\} = m_0, \\ \emptyset, & \text{if } |supp(\mathbf{x})| = \infty. \end{cases} \tag{2.4}$$

$$\omega_V^{(w)}(\mathbf{x}) = \begin{cases} \mathbf{e}_{m_0}, & \text{if } \max\{supp(\mathbf{x})\} = m_0, \\ \mathbf{0}, & \text{if } |supp(\mathbf{x})| = \infty. \end{cases} \tag{2.5}$$

From (2.4) one can see that V is ergodic at $\mathbf{x}_0 \in S$ if $|supp(\mathbf{x}_0)| < \infty$.

Now we consider a case $|supp(\mathbf{x}_0)| = \infty$ and we show that V is not ergodic at $\mathbf{x}_0$.

Assume that V is ergodic at $\mathbf{x}_0 \in S$ ($|supp(\mathbf{x}_0)| = \infty$). Then there exists $\hat{\mathbf{x}} \in S$ such that

$$\frac{1}{n}\sum_{k=0}^{n} V^k(\mathbf{x}_0) \xrightarrow{\|\cdot\|} \hat{\mathbf{x}}, \quad \text{as } n \to \infty. \tag{2.6}$$

Then by Lemma A.5 we obtain

$$\frac{1}{n}\sum_{k=0}^{n} V^k(\mathbf{x}_0) \xrightarrow{\text{p.w.}} \hat{\mathbf{x}}, \quad \text{as } n \to \infty.$$

On the other hand, from (2.5) it follows that

$$\frac{1}{n}\sum_{k=0}^{n} V^k(\mathbf{x}_0) \xrightarrow{\text{p.w.}} \mathbf{0}, \quad \text{as } n \to \infty. \tag{2.7}$$

Hence, (2.6),(2.7) implies $\hat{\mathbf{x}} = \mathbf{0}$, which contradicts to $\hat{\mathbf{x}} \in S$. So, we infer that V is not ergodic at point $\mathbf{x}_0 \in S$.

In the present paper, we extend an idea of this example for more general stochastic operators. To do so, we need to introduce several notions.

We recall that an ℓ^1-continuous function $\varphi : S \to \mathbb{R}$ is called a *Lyapunov function* for stochastic operator V if the limit $\lim_{n\to\infty} \varphi(V^n(\mathbf{x}))$ exists for any initial point $\mathbf{x} \in S$.

Obviously, if φ is Lyapunov function for V and $\lim_{n\to\infty} \varphi(V^n(\mathbf{x}_0)) = \mathbf{x}^*$, then $\omega_V(\mathbf{x}_0) \subset \varphi^{-1}(\mathbf{x}^*)$. Consequently, to determine more precisely of $\omega_V(\mathbf{x}_0)$ we should construct as much as possible Lyapunov functions.

However, to investigate $\omega_V^{(w)}(\mathbf{x}_0)$ of stochastic operator usual Lyapunov functions may not be applicable. Therefore, we want to introduce a notion *quasi Lyapunov function* which is pointwise continuous rather than ℓ^1-norm continuity.

First, we denote by $\mathbf{b}_\downarrow$ a non-increasing sequence $\{b_k\}$, i.e.

$$\mathbf{b}_\downarrow = (b_1, b_2, b_3 \dots,), \quad \text{such that } b_1 \geq b_2 \geq b_3 \geq \cdots$$

A pointwise continuous function $\varphi : \mathbf{B}_1^+ \to \mathbb{R}$ is called a *quasi Lyapunov function* for V if the limit $\lim\limits_{n\to\infty} \varphi(V^n(\mathbf{x}))$ exists for any initial point $\mathbf{x} \in S$.

We note that in Lemma A.6 it has been described all pointwise continuous linear functionals defined on $\mathbf{B}_1^+$. Based on that result, we introduce the following class of stochastic operators. Given a sequence $\mathbf{b}_\downarrow \in c_0$ let us denote

$$\varphi_{\mathbf{b}_\downarrow}(\mathbf{x}) = \sum_{k=1}^{\infty} b_k x_k. \tag{2.8}$$

Definition 2.6 We say a stochastic operator V belongs to the class $\mathcal{L}^+$ (resp. $\mathcal{L}^-$) if for every $\mathbf{b}_\downarrow \in c_0$ and for any $\mathbf{x} \in S$ the sequence $\left\{\varphi_{\mathbf{b}_\downarrow}(V^n(\mathbf{x}))\right\}_{n\geq 0}^\infty$ is increasing (resp. decreasing).

We point out that, due to $0 \leq b_n \leq b_1$ for any $n \geq 1$, one has $0 \leq \varphi_{\mathbf{b}_\downarrow}(\mathbf{y}) \leq b_1$ for every $\mathbf{y} \in \mathbf{B}_1^+$. Therefore,

$$0 \leq \varphi_{\mathbf{b}_\downarrow}(V^n(\mathbf{x})) \leq b_1, \quad \forall n \in \mathbb{N}, \ \forall \mathbf{x} \in S.$$

Hence, from Definition 2.6 it immediately follows that for every $V \in \mathcal{L}^+ \cup \mathcal{L}^-$ the functional $\varphi_{\mathbf{b}_\downarrow}$ is a quasi Lyapunov function for V.

Remark 2.7 We notice that if $V \in \mathcal{L}^+$ if and only if

$$\varphi_{\mathbf{b}_\downarrow}(V(\mathbf{x})) \geq \varphi_{\mathbf{b}_\downarrow}(\mathbf{x}), \quad \mathbf{x} \in \mathbf{B}_1^+. \tag{2.9}$$

Similarly, if $V \in \mathcal{L}^-$ if and only if

$$\varphi_{\mathbf{b}_\downarrow}(V(\mathbf{x})) \leq \varphi_{\mathbf{b}_\downarrow}(\mathbf{x}), \quad \mathbf{x} \in \mathbf{B}_1^+. \tag{2.10}$$

The following result described some properties of the classes $\mathcal{L}^+$, $\mathcal{L}^-$.

Proposition 2.8 *The following statements hold true:*

(i) the sets $\mathcal{L}^+$ and $\mathcal{L}^-$ are convex;
(ii) for every $V_1, V_2 \in \mathcal{L}^+$ one has $V_1 \circ V_2 \in \mathcal{L}^+$;
(iii) for every $V_1, V_2 \in \mathcal{L}^-$ one has $V_1 \circ V_2 \in \mathcal{L}^-$;
(iv) one has $\mathcal{L}^+ \cap \mathcal{L}^- = \{Id\}$, here Id is an identity operator;
(v) For every $V \in \mathcal{L}^+$ (resp. $V \in \mathcal{L}^-$) and every $\mathbf{x} \in S$ the trajectory $\{V^n(\mathbf{x})\}$ pointwise converges.

Proof (i). Let $V_1, V_2 \in \mathcal{L}^+$ and $\lambda \in (0, 1)$. Then it is clear that $V = \lambda V_1 + (1 - \lambda)V_2$ is stochastic operator. Take any $\mathbf{b}_\downarrow \in c_0$. Then, due to (2.9), one gets

$$\begin{aligned}
\varphi_{\mathbf{b}_\downarrow}(V(\mathbf{x})) &= \lambda\varphi_{\mathbf{b}_\downarrow}(V_1(\mathbf{x})) + (1-\lambda)\varphi_{\mathbf{b}_\downarrow}(V_2(\mathbf{x})) \\
&\geq \lambda\varphi_{\mathbf{b}_\downarrow}(\mathbf{x}) + (1-\lambda)\varphi_{\mathbf{b}_\downarrow}(\mathbf{x}) \\
&\geq \varphi_{\mathbf{b}_\downarrow}(\mathbf{x})
\end{aligned}$$

which, due Remark 2.7, yields $V \in \mathcal{L}^+$.

By the same argument, one can prove that $\mathcal{L}^-$ is convex.

The statements (ii) and (iii) obviously follow from Remark 2.7.

(iv). Let $V \in \mathcal{L}^+ \cap \mathcal{L}^-$. Then for all $\mathbf{b}_\downarrow \in c_0$ from Remark 2.7 it follows that

$$\varphi_{\mathbf{b}_\downarrow}(V(\mathbf{x})) = \varphi_{\mathbf{b}_\downarrow}(\mathbf{x}), \qquad \forall \mathbf{x} \in S. \tag{2.11}$$

For any integer $k \geq 0$, let us define linear functionals by

$$\varphi_k(\mathbf{x}) = \begin{cases} \sum_{j=1}^k x_j, & \text{if } k \geq 1, \\ 0, & \text{if } k = 0, \end{cases} \qquad \forall \mathbf{x} \in S.$$

It is clear that φ_k is a quasi Lyapunov function for V. From the equality

$$\varphi_{k+1}(\mathbf{x}) - \varphi_k(\mathbf{x}) = x_{k+1},$$

and (2.11) we obtain

$$(V(\mathbf{x}))_k = x_k, \quad \forall k \in \mathbb{N},$$

which implies the statement.

(v). Let $V \in \mathcal{L}^+ \cup \mathcal{L}^-$. Then the convergence $\{\varphi_k(V^n(\mathbf{x}))\}$ yields that the sequence $\{V^n(\mathbf{x})_k\}$ converges for every $k \in \mathbb{N}$ which implies the required assertion. □

Now let us show that the introduced classes are not empty.

Example 2.9 Denote

$$\mathbb{A}^+ = \{(a_{ki}) \in \mathbb{A} : \ a_{ki} \geq 0 \text{ for all } k < i\},$$
$$\mathbb{A}^- = \{(a_{ki}) \in \mathbb{A} : \ a_{ki} \leq 0 \text{ for all } k < i\}.$$

We define two subclasses of Lotka–Volterra operators (see (2.1)) corresponding to these sets by

$$\mathcal{LV}_f^+ = \{V \in \mathcal{LV}_f \ : \ \mathfrak{g}(V) \in \mathbb{A}^+\}, \quad \mathcal{LV}_f^- = \{V \in \mathcal{LV}_f \ : \ \mathfrak{g}(V) \in \mathbb{A}^-\}.$$

Let us show that $\mathcal{LV}_f^+ \subset \mathcal{L}^+$ and $\mathcal{LV}_f^- \subset \mathcal{L}^-$. Indeed, take any $\mathbf{b}_\downarrow \in c_0$. One can check that $b_k \geq 0$ for any $k \in \mathbb{N}$, since $\mathbf{b}_\downarrow$ is a decreasing sequence with $b_n \to 0$. Then, from (2.1), we obtain

$$\varphi_{\mathbf{b}_\downarrow}(V(\mathbf{x})) - \varphi_{\mathbf{b}_\downarrow}(\mathbf{x}) = f(\mathbf{x}) \sum_{k=1}^{\infty} \sum_{i=k+1}^{\infty} a_{ki}(b_k - b_i) x_k x_i.$$

Since the sequence $\{b_n\}$ is decreasing, then one gets

$$\varphi_{\mathbf{b}_\downarrow}(V(\mathbf{x})) - \varphi_{\mathbf{b}_\downarrow}(\mathbf{x}) \begin{cases} \geq 0, \text{ if } V \in \mathcal{LV}_f^+, \\ \leq 0, \text{ if } V \in \mathcal{LV}_f^-. \end{cases}$$

This yields that for any $\mathbf{x} \in S$ the sequence $\{\varphi_{\mathbf{b}_\downarrow}(V^n(\mathbf{x}))\}$ is increasing if $V \in \mathcal{LV}_f^+$, and it is decreasing if $V \in \mathcal{LV}_f^-$.

Example 2.10 Let T be a right shift operator on ℓ^1, i.e. $T(\mathbf{x}) = (0, x_1, x_2, x_3, \ldots)$ for every $\mathbf{x} \in \ell^1$. One can see that T is a stochastic. Moreover, for any $\mathbf{b}_\downarrow \in c_0$ one has $b_{i+1}x_i \leq b_i x_i$, $\forall i \in \mathbb{N}$. Consequently, we have $\varphi_{\mathbf{b}_\downarrow}(T(\mathbf{x})) \leq \varphi_{\mathbf{b}_\downarrow}(\mathbf{x})$ for every $\mathbf{x} \in S$. By definition this means that $T \in \mathcal{L}^-$.

3 Main Results

In this section, we prove main results of this paper. First we study the limiting sets $\omega_V(\mathbf{x})$, $\omega_V^{(w)}(\mathbf{x})$ for operators from the classes $\mathcal{L}^+$ and $\mathcal{L}^-$.

Theorem 3.1 *Let $V \in \mathcal{L}^+ \cup \mathcal{L}^-$ and $\mathbf{x}_0 \in S$. Then the following statements hold:*

(i) if $V \in \mathcal{L}^+$ then $\omega_V^{(w)}(\mathbf{x}_0) \subset S$;
(ii) if $V \in \mathcal{L}^-$ then $\omega_V^{(w)}(\mathbf{x}_0) \subset S_r$ for some $r \le 1$.

Proof First, we note that if $\mathbf{x}_0 \in Fix(V)$ then the statements are obvious. Now, let us assume $\mathbf{x}_0 \in S\backslash Fix(V)$. According to Proposition A.4 (see also Proposition 2.8) we infer that $\omega_V^{(w)}(\mathbf{x}_0) \neq \emptyset$ for any $V \in \mathcal{L}^+ \cup \mathcal{L}^-$. Moreover, $\omega_V^{(w)}(\mathbf{x}_0) \subset \mathbf{B}_1^+$.

(i). Let $V \in \mathcal{L}^+$, and pick a point $\mathbf{a} \in \omega_V^{(w)}(\mathbf{x}_0)$. Let us show that $\|\mathbf{a}\| = 1$.

Assume that $\|\mathbf{a}\| < 1$. Due to $\|\mathbf{x}_0\| = 1$, for a positive number $\varepsilon = \frac{1-\|\mathbf{a}\|}{2}$, there exists an integer $m \ge 1$ such that

$$\sum_{k=1}^{m} x_k^{(0)} > 1 - \varepsilon.$$

For a given m let us define a sequence $\mathbf{b}_{\downarrow}^{[m]} = (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \ldots)$ as follows

$$\tilde{b}_k = \begin{cases} 1, & k \le m; \\ \frac{1}{2^k}, & k > m. \end{cases} \tag{3.1}$$

It is clear that $\mathbf{b}_{\downarrow}^{[m]} \in c_0$. Since $V \in \mathcal{L}^+$, the functional $\varphi_{\mathbf{b}_{\downarrow}^{[m]}}$ is a quasi Lyapunov function for V. Hence,

$$\varphi_{\mathbf{b}_{\downarrow}^{[m]}}(\mathbf{x}_0) = \sum_{k=1}^{m} x_k^{(0)} + \sum_{k=m+1}^{\infty} \frac{x_k^{(0)}}{2^k} \ge \sum_{k=1}^{m} x_k^{(0)} > 1 - \varepsilon$$

and

$$\varphi_{\mathbf{b}_{\downarrow}^{[m]}}(\mathbf{a}) = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{\infty} \frac{a_k}{2^k} \le \sum_{k=1}^{\infty} a_k = \|\mathbf{a}\|.$$

From the definition of the class $\mathcal{L}^+$, we infer that the sequence $\{\varphi_{\mathbf{b}_{\downarrow}^{[m]}}(V^n(\mathbf{x}_0))\}$ is increasing. Therefore,

$$\begin{aligned} \varphi_{\mathbf{b}_{\downarrow}^{[m]}}(V^n(\mathbf{x}_0)) - \varphi_{\mathbf{b}_{\downarrow}^{[m]}}(\mathbf{a}) &\ge \varphi_{\mathbf{b}_{\downarrow}^{[m]}}(\mathbf{x}_0) - \varphi_{\mathbf{b}_{\downarrow}^{[m]}}(\mathbf{a}) \\ &> 1 - \varepsilon - \|\mathbf{a}\| \\ &= \frac{1 + \|\mathbf{a}\|}{2} > 0. \end{aligned}$$

This contradicts to the pointwise continuity of $\varphi_{\mathbf{b}_{\downarrow}^{[m]}}$ at point $\mathbf{a}$. So, we conclude $\|\mathbf{a}\| = 1$, which yields $\omega_V^{(w)}(\mathbf{x}_0) \subset S$.

(ii). Let $V \in \mathcal{L}^-$. Pick any $\mathbf{x}, \mathbf{y} \in \omega_V^{(w)}(\mathbf{x}_0)$. Now, let us show that $\|\mathbf{x}\| = \|\mathbf{y}\|$.

Suppose that $\|\mathbf{x}\| > \|\mathbf{y}\|$. For a positive $\varepsilon = \frac{\|\mathbf{x}\|-\|\mathbf{y}\|}{2}$ there exists an integer $m \geq 1$ such that

$$\sum_{k=1}^{m} x_k > \|\mathbf{x}\| - \varepsilon.$$

Furthermore, for a given $m \geq 1$, let us consider $\mathbf{b}_{\downarrow}^{[m]} \in c_0$ given by (3.1). Then, due to $V \in \mathcal{L}^-$, the functional $\varphi_{\mathbf{b}_{\downarrow}^{[m]}}$ is a quasi Lyapunov function for V, and there exists $\xi \in [0, 1]$ such that

$$\lim_{n\to\infty} \varphi_{\mathbf{b}_{\downarrow}^{[m]}}(V^n(\mathbf{x}_0)) = \xi. \tag{3.2}$$

On the other hand, we have

$$\begin{aligned}
\varphi_{\mathbf{b}_{\downarrow}^{[m]}}(\mathbf{x}) &= \sum_{k=1}^{m} x_k + \sum_{k=m+1}^{\infty} \frac{x_k}{2^k} \\
&\geq \sum_{k=1}^{m} x_k > \|\mathbf{x}\| - \varepsilon \\
&= \frac{\|\mathbf{x}\| + \|\mathbf{y}\|}{2} > \|\mathbf{y}\| \\
&\geq \sum_{k=1}^{m} y_k + \sum_{k=m+1}^{\infty} \frac{y_k}{2^k} \\
&= \varphi_{\mathbf{b}_{\downarrow}^{[m]}}(\mathbf{y}).
\end{aligned}$$

Thus, $\varphi_{\mathbf{b}_{\downarrow}^{[m]}}(\mathbf{x}) > \varphi_{\mathbf{b}_{\downarrow}^{[m]}}(\mathbf{y})$ while $\|\mathbf{x}\| > \|\mathbf{y}\|$. However, it contradicts to (3.2). This means $\|\mathbf{x}\| = \|\mathbf{y}\|$ which yields $\omega_V^{(w)}(\mathbf{x}_0) \subset S_r$ for some $r \geq 0$. Finally, Proposition A.4 implies $r \leq 1$. □

Next result establishes relation between $\omega_V(\mathbf{x}_0)$ and $\omega_V^{(w)}(\mathbf{x}_0)$.

Theorem 3.2 *Let $V \in \mathcal{L}^+ \cup \mathcal{L}^-$ and $\mathbf{x}_0 \in S$. Then the following statements hold:*

(i) if $V \in \mathcal{L}^+$ then $\omega_V(\mathbf{x}_0) = \omega_V^{(w)}(\mathbf{x}_0)$;
(ii) if $V \in \mathcal{L}^-$, then $\omega_V(\mathbf{x}_0) = \omega_V^{(w)}(\mathbf{x}_0)$ iff $\omega_V(\mathbf{x}_0) \neq \emptyset$.

Proof (i). Let $V \in \mathcal{L}^+$. Then according to Theorem 3.1, we have $\omega_V^{(w)}(\mathbf{x}_0) \subset S$. By Lemma A.5 one gets $\omega_V(\mathbf{x}_0) = \omega_V^{(w)}(\mathbf{x}_0)$.

(ii) Let $V \in \mathcal{L}^-$. First, we assume that $\omega_V(\mathbf{x}_0) = \omega_V^{(w)}(\mathbf{x}_0)$. Then, Proposition A.4 yields $\omega_V^{(w)}(\mathbf{x}_0) \neq \emptyset$ which means $\omega_V(\mathbf{x}_0) \neq \emptyset$.

Now, suppose that $\omega_V(\mathbf{x}_0) \neq \emptyset$. By Lemma A.3 one has $\omega_V(\mathbf{x}_0) \subset S$. Therefore, Lemma A.5 (i) yields $\omega_V(\mathbf{x}_0) \subset \omega_V^{(w)}(\mathbf{x}_0)$, so, $\omega_V^{(w)}(\mathbf{x}_0) \cap S \neq \emptyset$. Finally, from Theorem 3.1 it follows that $\omega_V^{(w)}(\mathbf{x}_0) \subset S$. Hence, Lemma A.5 (ii) implies $\omega_V^{(w)}(\mathbf{x}_0) \subset \omega_V(\mathbf{x}_0)$, which means $\omega_V(\mathbf{x}_0) = \omega_V^{(w)}(\mathbf{x}_0)$. This completes the proof. □

Remark 3.3 We point out that (ii) part of the theorem is an essential difference between finite and infinite dimensional cases. Since, at the finite dimensional setting, we always have $\omega_V(\mathbf{x}_0) = \omega_V^{(w)}(\mathbf{x}_0) \neq \emptyset$ [6,9].

Now it would be interesting to know the cardinality of $\omega_V^{(w)}(\mathbf{x}_0)$. Next result clarifies this question.

Proposition 3.4 *Let $V \in \mathcal{L}^+ \cup \mathcal{L}^-$. Then $\left|\omega_V^{(w)}(\mathbf{x}_0)\right| = 1$ for any $\mathbf{x}_0 \in S$.*

Proof It is clear that $\omega_V(\mathbf{x}_0) = \{\mathbf{x}_0\}$ for any $\mathbf{x}_0 \in Fix(V)$. So, we prove the assumption of Theorem only for $\mathbf{x}_0 \in S \backslash Fix(V)$.

Let $\mathbf{x}_0 \in S \backslash Fix(V)$. Take a sequence $\{\mathbf{b}_{\downarrow}^{(n)}\}_{n \geq 1} \subset c_0$ defined by

$$b_k^{(n)} = \begin{cases} \frac{1}{k}, & k \leq n; \\ 0, & k > n. \end{cases}, \quad k \in \mathbb{N}$$

Then by definition 2.6 we infer that $\varphi_{\mathbf{b}_{\downarrow}^{(n)}}$ (for every $n \geq 1$) on $\mathbf{B}_1^+$ is a quasi Lyapunov function for $V \in \mathcal{L}^+ \cup \mathcal{L}^-$.

Assume that $\mathbf{x}, \mathbf{y} \in \omega_V^{(w)}(\mathbf{x}_0)$. Then the argument of the proof of Theorem 3.1 yields

$$\varphi_{\mathbf{b}_{\downarrow}^{(n)}}(\mathbf{x}) = \varphi_{\mathbf{b}_{\downarrow}^{(n)}}(\mathbf{y}) \quad \text{for any } n \geq 1,$$

which means $\mathbf{x} = \mathbf{y}$. This completes the proof. □

Remark 3.5 The proved result reveals crucial information about the limiting set (in the weak topology) of the dynamics of stochastic operators. Such kind of result never observed in the theory of quadratic stochastic operators. Most of the results concern the norm convergence of the trajectories (see, for example, [2,4,9]).

Theorem 3.6 *Let $V \in \mathcal{L}^+ \cup \mathcal{L}^-$. Then for any $\mathbf{x}_0 \in S$ the following statements hold:*

(i) V is weak ergodic at point $\mathbf{x}_0$;
(ii) if $V \in \mathcal{L}^+$ then V is ergodic at point $\mathbf{x}_0$;
(iii) if $V \in \mathcal{L}^-$ then is ergodic at point $\mathbf{x}_0$ iff $\omega_V(\mathbf{x}_0) \neq \emptyset$.

Remark 3.7 Thanks to Theorem 3.6 we infer that if $\omega_V(\mathbf{x}_0) \neq \emptyset$ then V is weak ergodic, but not ergodic (w.r.t. ℓ^1-norm) at that point, while it is weak ergodic. This is an essential difference between finite and infinite dimensional settings. For an explicit example we refer to Example 2.5. This is the first example in the theory of nonlinear operators, in the infinite dimensional setting, for which the operator is not ergodic while it is weakly ergodic. We point out that there several results about non-ergodicity of finite dimensional Volterra operators (see, for example, [10,21]).

Proof of Theorem 3.6 (i) From Proposition 3.4 we infer that V is weak ergodic at any point of S.

(ii) Let $V \in \mathcal{L}^+$. Then, Theorem 3.2 (i) and Proposition 3.4 yield $|\omega_V(\mathbf{x}_0)| = 1$ for any $\mathbf{x}_0 \in S$. This implies the ergodicity of V at $\mathbf{x}_0$.

(iii) Let $V \in \mathcal{L}^-$. At first, we suppose that $\omega_V(\mathbf{x}_0) \neq \emptyset$. Then, from Theorem 3.2 and Proposition 3.4 we find $|\omega_V(\mathbf{x}_0)| = 1$. Hence, V is ergodic at $\mathbf{x}_0$.

Now, let us assume $\omega_V(\mathbf{x}_0) = \emptyset$. We are going to establish that V is not ergodic at $\mathbf{x}_0$. Suppose that V is ergodic at $\mathbf{x}_0$. This means that there exists $\mathbf{a} \in S$ such that

$$\frac{1}{n}\sum_{k=0}^{n} V^k(\mathbf{x}_0) \xrightarrow{\|\cdot\|} \mathbf{a}, \quad \text{as } n \to \infty.$$

Then, Lemma A.5 implies

$$\frac{1}{n}\sum_{k=0}^{n} V^k(\mathbf{x}_0) \xrightarrow{\text{p.w.}} \mathbf{a}, \quad \text{as } n \to \infty. \tag{3.3}$$

Hence, from Proposition 3.4 together with (3.3) we obtain

$$V^n(\mathbf{x}_0) \xrightarrow{\text{p.w.}} \mathbf{a}, \quad \text{as } n \to \infty.$$

From the last one and noting $\omega_V(\mathbf{x}_0) = \emptyset$ one gets $\|\mathbf{a}\| < 1$. This contradicts to $\mathbf{a} \in S$. So, we conclude that V is not ergodic at $\mathbf{x}_0$. □

4 Description of Linear Stochastic Operators from $\mathcal{L}^+$ and $\mathcal{L}^-$

Example 2.10 shows that there exists a linear operator T such that $T \in \mathcal{L}^-$. This naturally leads us to the description of the set of all linear stochastic operators belonging to $\mathcal{L}^+$ (resp. $\mathcal{L}^-$). In this section we do such kind of description.

Let $\mathbb{T} = (t_{ij})$ be an infinite dimensional matrix. By T denote a linear operator from ℓ_1 to itself defined by

$$T(\mathbf{x}) = \mathbf{x}^t\mathbb{T} \tag{4.1}$$

here $\mathbf{x}^t$ is the standard transpose of a vector.

The equality (4.1) means

$$(T(\mathbf{x}))_k = \sum_{i=1}^{\infty} t_{ik}x_i, \qquad \forall k \in \mathbb{N}.$$

The following Proposition is useful our next investigations.

Proposition 4.1 *Let* $\mathbb{T} = (t_{ij})_{i,j=1}^{\infty}$. *Then the following statements are equivalent:*

(i) T *is a stochastic;*
(ii) $T(S_r) \subset S_r$ *for some* $r > 0$*;*
(iii) It holds

$$t_{ij} \geq 0, \quad \sum_{k=1}^{\infty} t_{ik} = 1, \qquad \forall i, j \in \mathbb{N}. \tag{4.2}$$

Proof Due to the linearity T the implications (i)⇔(ii) are obvious. So, it is enough to establish $T(S) \subset S$ if and only if it holds (4.2).

Let us assume that $T(S) \subset S$. Then for any $i \geq 1$, $T(\mathbf{e}_i) = (t_{i1}, t_{i2}, t_{i3}, \ldots)$ which, due to $T(\mathbf{e}_i) \in S$, implies $t_{ij} \geq 0$ for every $j \geq 1$ and $\sum_{k=1}^{\infty} t_{ik} = 1$. From the arbitrariness of i we arrive at (4.2).

Now, we suppose that (4.2) holds. Then for every $\mathbf{x} \in S$, we get $(T(\mathbf{x}))_k \geq 0$, $\forall k \geq 1$ and

$$\sum_{k=1}^{\infty} (T(\mathbf{x}))_k = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} t_{ik} x_i = \sum_{i=1}^{\infty} x_i \sum_{k=1}^{\infty} t_{ik} = \sum_{i=1}^{\infty} x_i = 1.$$

This means that $T(S) \subset S$. This completes the proof. □

Theorem 4.2 *Let* $\mathbb{T} = (t_{ij})_{i,j=1}^{\infty}$ *be a stochastic operator and* T *be the corresponding linear operator* (4.1). *Then* $T \in \mathcal{L}^+$ *(resp.* $T \in \mathcal{L}^-$*) iff* $\mathbb{T}$ *is lower (resp. upper) triangular matrix.*

Proof Since the constructions of the proofs for the cases $T \in \mathcal{L}^+$ and $T \in \mathcal{L}^-$ are similar, therefore, we restrict ourselves to the case $T \in \mathcal{L}^+$.

Sufficiency. Let $(t_{ij})_{i,j=1}^{\infty}$ be a lower triangular stochastic matrix. Then, for every $\mathbf{b}_{\downarrow} \in c_0$ we have

$$\sum_{k=1}^{\infty} t_{ik} b_k \geq b_i, \qquad \forall i \in \mathbb{N}. \tag{4.3}$$

Indeed, due to $b_1 \geq b_2 \geq b_3 \geq \cdots$ and keeping in mind T is lower triangular, one gets

$$\sum_{k=1}^{\infty} t_{ik} b_k = \sum_{k=1}^{i} t_{ik} b_k \geq b_i \sum_{k=1}^{i} t_{ik}, \qquad \forall i \in \mathbb{N}.$$

The last one together with $\sum_{k=1}^{i} t_{ik} = 1$ implies (4.3).

Let $\varphi_{\mathbf{b}_\downarrow}$ be a linear functional given by (2.8). Then for every $\mathbf{x} \in S$ we obtain

$$\varphi_{\mathbf{b}_\downarrow}(T(\mathbf{x})) = \sum_{k=1}^{\infty} b_k \sum_{i=1}^{\infty} t_{ik} x_i = \sum_{i=1}^{\infty} x_i \sum_{k=1}^{i} t_{ik} b_k. \tag{4.4}$$

Hence, from (4.3) and (4.4) it follows that $\varphi_{\mathbf{b}_\downarrow}(T(\mathbf{x})) \geq \varphi_{\mathbf{b}_\downarrow}(\mathbf{x})$, which implies that $T \in \mathcal{L}^+$.

Necessity. Let $T \in \mathcal{L}^+$. Then for all $\mathbf{b}_\downarrow \in c_0$, and $\forall i \in \mathbb{N}$ and for $\varphi_{\mathbf{b}_\downarrow}$ given by (A.2) we have

$$0 \leq \varphi_{\mathbf{b}_\downarrow}(T(\mathbf{e}_i)) - \varphi_{\mathbf{b}_\downarrow}(\mathbf{e}_i) = \sum_{k=1}^{\infty} t_{ik} b_k - b_i.$$

This yields that

$$\sum_{k=1}^{\infty} t_{ik} b_k \geq b_i, \qquad \forall i \in \mathbb{N}. \tag{4.5}$$

Let us suppose that $t_{lm} \neq 0$ for some $m > l \geq 1$. Then

$$\sum_{k=1}^{l} t_{lk} < \sum_{k=1}^{m} t_{lk} \leq \sum_{k=1}^{\infty} t_{lk} = 1. \tag{4.6}$$

Let us take $\tilde{\mathbf{b}}_\downarrow = (\underbrace{1, 1, \ldots, 1}_{l}, 0, 0, \ldots)$. Then by (4.6) one finds

$$\sum_{k=1}^{\infty} t_{lk} \tilde{b}_k = \sum_{k=1}^{l} t_{lk} < 1,$$

which implies $\sum_{k=1}^{\infty} t_{lk} \tilde{b}_k < \tilde{b}_l$. This contradicts to (4.5). So, $(t_{ij})_{i,j=1}^{\infty}$ is lower triangular. □

Remark 4.3 From this result and (4.1), we infer that a stochastic operator T belongs to $\mathcal{L}^+$ iff for every $\mathbf{b}_\downarrow \in c_0$ one has

$$\varphi_{\mathbf{b}_\downarrow}(\mathbf{x}^t \mathbb{T}) \geq \varphi_{\mathbf{b}_\downarrow}(\mathbf{x}) \tag{4.7}$$

for all $\mathbf{x} \in S$.

We point out that as an application of Theorem 4.2 one can construct lots of examples of nonlinear stochastic operators belonging to the classes $\mathcal{L}^+$ and $\mathcal{L}^-$. First, by $\mathcal{L}^+_{lin}$ (resp. $\mathcal{L}^-_{lin}$), we denote the set of all lower (resp. upper) triangular stochastic matrices.

Let $\mathbb{T} : \mathbf{x} \in S \mapsto \mathbb{T}(\mathbf{x}) \in \mathcal{L}^+_{lin}$ be an affine mapping. Now, let us define a mapping $V : S \to \ell^1$ by

$$V(\mathbf{x}) = \mathbf{x}^t \mathbb{T}(\mathbf{x}), \quad \forall \mathbf{x} \in S. \tag{4.8}$$

Due to the stochasticity of $\mathbb{T}(\mathbf{x})$ the mapping V is stochastic as well.

Theorem 4.4 *Let $\mathbb{T} : S \to \mathcal{L}^+_{lin}$ (resp. $\mathbb{T} : S \to \mathcal{L}^-_{lin}$) be an affine mapping. Then the mapping V given by (4.8) belongs to $\mathcal{L}^+$ (resp. $\mathcal{L}^-$).*

Proof Take any $\mathbf{b}_\downarrow \in c_0$, then due to (4.7), we find

$$\begin{aligned} \varphi_{\mathbf{b}_\downarrow}(V(\mathbf{x})) &= \varphi_{\mathbf{b}_\downarrow}(\mathbf{x}^t \mathbb{T}(\mathbf{x})) \\ &\geq \varphi_{\mathbf{b}_\downarrow}(\mathbf{x}) \end{aligned}$$

which means $V \in \mathcal{L}^+$. □

From this result, we may construct a lot examples of polynomial stochastic operators belonging to the classes $\mathcal{L}^+$ and $\mathcal{L}^-$. We notice that in [17] it was investigated (in a finite dimensional setting) b-bistochastic quadratic stochastic operators which have more general form than (4.8) while they belong to $\mathcal{L}^-$. Therefore, we may formulate the following problem:

Problem 4.5 *Describe all polynomial stochastic operators belonging to the class $\mathcal{L}^+$ (resp. $\mathcal{L}^-$).*

Now, as a consequence of Theorems 3.6 and 4.2, one gets the following fact.

Theorem 4.6 *Let $\mathbb{T} = (t_{ij})_{i,j=1}^\infty$ be a stochastic matrix and T be the corresponding operator. Then for any $\mathbf{x}_0 \in S$ the following statements hold:*

(i) if $\mathbb{T}$ is lower triangular then T is ergodic at point $\mathbf{x}_0$;
(ii) if $\mathbb{T}$ is upper triangular then T is weak ergodic at point $\mathbf{x}_0$. Moreover, T is ergodic at point $\mathbf{x}_0$ iff $\omega_T(\mathbf{x}_0) \neq \emptyset$.

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Appendix A. Pointwise Convergence on ℓ^1

In this section is devoted to some properties of point-wise convergence in ℓ^1.

It is known that $S = convh(Extr S)$, where $Extr(S)$ is the extremal points of S and $convh(A)$ is the convex hall of a set A. Any extremal point of S has the following form:

$$\mathbf{e}_k = (\underbrace{0, \ldots, 0, 1}_{k}, 0, 0, \ldots), \quad \forall k \in \mathbb{N}.$$

Here and henceforth we denote

$$riS_r = \{\mathbf{x} \in S_r : x_k > 0,\ k \in \mathbb{N}\},\quad \partial S_r = S_r \setminus riS_r.$$

Let $\{\mathbf{x}^{(n)}\}_{n\geq 1}$ be a sequence in ℓ_1. In what follows we write $\mathbf{x}^{(n)} \xrightarrow{\|\cdot\|} \mathbf{a}$ instead of $\left\|\mathbf{x}^{(n)} - \mathbf{a}\right\| \to 0$.

Remark A.1 Note that for any $r > 0$ the sets S_r and B_r are not compact w.r.t. ℓ^1-norm. In the finite dimensional setting, analogues of these sets are compact, and hence, the investigation of the dynamics of nonlinear mappings over these kind of sets use well-known methods and techniques of dynamical systems. In our case, the non compactness (w.r.t. ℓ^1-norm) of the set $\mathbf{B}_r^+$ complicates our further investigation on dynamics of Volterra operators. Therefore, we need such a weak topology on ℓ^1 so that the set $\mathbf{B}_r^+$ would be compact with respect to that topology.

One of weak topologies on ℓ^1 is the Tychonov topology which generates the pointwise convergence. We say that a sequence $\{\mathbf{x}^{(n)}\}_{n\geq 1} \subset \ell^1$ converges *pointwise* to $\mathbf{x} = (x_1, x_2, \ldots) \in \ell^1$ if

$$\lim_{n\to\infty} x_k^{(n)} = x_k \quad \text{for every } k \geq 1.$$

and write $\mathbf{x}^{(n)} \xrightarrow{\text{p.w.}} \mathbf{x}$.

Remark A.2 We notice that the set ℓ^1 is not closed w.r.t. pointwise topology, and its completion is s which is the space of all sequences. It is known that this topology is metrizable by the following metric:

$$\rho(\mathbf{a}, \mathbf{b}) = \sum_{k=1}^{\infty} 2^{-k} \frac{|a_k - b_k|}{1 + |a_k - b_k|}, \quad \mathbf{a}, \mathbf{b} \in s. \tag{A.1}$$

Hence, for a given sequence $\{\mathbf{x}^{(n)}\}_{n\geq 1} \subset s$ the following statements are equivalent:

(i) $\mathbf{x}^{(n)} \xrightarrow{\text{p.w.}} \mathbf{x}$;

(ii) $\mathbf{x}^{(n)} \xrightarrow{\rho} \mathbf{x}$.

In the sequel, we will show that the unit ball of ℓ^1 is compact w.r.t. pointwise convergence, while whole ℓ^1 is not closed in s.

We recall that ℓ^∞ is defined to be the space of all bounded sequences endowed with the norm

$$\|\mathbf{x}\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\}.$$

The following lemma plays a crucial role in our further investigations.

Lemma A.3 *Let* $\{\mathbf{x}^{(n)}\}_{n\geq 1} \subset S_r$*, for some* $r > 0$*. If* $\mathbf{x}^{(n)} \xrightarrow{\|\cdot\|} \mathbf{a}$*, then* $\mathbf{a} \in S_r$.

Proof It is easy to check that $\|\mathbf{x}-\mathbf{y}\| \geq |r-\rho|, \forall \mathbf{x} \in S_r, \ \forall \mathbf{y} \in S_\rho$. This fact together with $\mathbf{x}^{(n)} \xrightarrow{\|\cdot\|} \mathbf{a}$ yields that $\mathbf{a} \in S_r$. □

Proposition A.4 *The set $\mathbf{B}_1^+$ is sequentially compact w.r.t. the pointwise convergence.*

It is clear that $\mathbf{x}^{(n)} \xrightarrow{\|\cdot\|} \mathbf{a}$ implies $\mathbf{x}^{(n)} \xrightarrow{\text{p.w.}} \mathbf{a}$. A natural question arises: is there any equivalence criteria for these two types of convergence on some set? Next result gives a positive answer to this question.

Lemma A.5 *Let $\{\mathbf{x}^{(n)}\}_{n\geq 1}$ be a sequence on S_r. Then the following statements are equivalent:*

(1) $\mathbf{x}^{(n)} \xrightarrow{\|\cdot\|} \mathbf{a}$ *and* $\mathbf{a} \in S_r$*;*

(2) $\mathbf{x}^{(n)} \xrightarrow{\text{p.w.}} \mathbf{a}$ *and* $\mathbf{a} \in S_r$*.*

Recall that a functional $\varphi : \ell^1 \to \mathbb{R}$ is called *pointwise continuous* if for any $\mathbf{a} \in \ell^1$ and any sequence $\{\mathbf{x}^{(n)}\}_{n\geq 1} \subset \ell^1$ with $\mathbf{x}^{(n)} \xrightarrow{\text{p.w.}} \mathbf{a}$ one has $\varphi(\mathbf{x}^{(n)}) \to \varphi(\mathbf{a})$.

Now we provide a criteria for linear functionals to be pointwise continuous.

Given $\mathbf{b} \in \ell^\infty$, let us define

$$\varphi_{\mathbf{b}}(\mathbf{x}) = \sum_{k=1}^{\infty} b_k x_k, \quad \mathbf{x} \in \ell^1. \tag{A.2}$$

Lemma A.6 *Let $\mathbf{b} \in \ell^\infty$, then the linear functional $\varphi_{\mathbf{b}}$ is pointwise continuous on $\mathbf{B}_1^+$ iff $\mathbf{b} \in c_0$.*

Proof Assume that $\varphi_{\mathbf{b}}$ is a pointwise continuous. Consider the sequence $\{\mathbf{e}_n\}_{n\geq 1}$ for which one has $\mathbf{e}_n \xrightarrow{\text{p.w.}} \mathbf{0}$, where $\mathbf{0} = (0, 0, \ldots)$. From $\varphi_{\mathbf{b}}(\mathbf{e}_n) = b_n$, $\varphi_{\mathbf{b}}(\mathbf{0}) = 0$ and the pointwise continuity of $\varphi_{\mathbf{b}}$ implies $b_n \to 0$ as $n \to \infty$.

Now let us suppose that $b_k \to 0$ as $k \to \infty$, and take any sequence $\{\mathbf{x}^{(n)}\}_{n\geq 1} \subset \mathbf{B}_r^+$ such that $\mathbf{x}^{(n)} \xrightarrow{\text{p.w.}} \mathbf{x}$. We will show that $\varphi_{\mathbf{b}}(\mathbf{x}^{(n)}) \to \varphi_{\mathbf{b}}(\mathbf{x})$. If $\|\mathbf{b}\|_\infty = 0$ then nothing to proof. So, we consider $\|\mathbf{b}\|_\infty \neq 0$.

Take an arbitrary positive number ε. Then there exists an integer $m \geq 1$ such that $|b_k| < \frac{\varepsilon}{4r}$ for all $k > m$. The pointwise convergence $\mathbf{x}^{(n)} \xrightarrow{\text{p.w.}} \mathbf{x}$ implies the existence of an integer n_0 such that

$$|x_k^{(n)} - x_k| < \frac{\varepsilon}{2\,\|\mathbf{b}\|_\infty}, \quad k \in \{1, \ldots, m\}, \quad \forall n > n_0.$$

Consequently, we have

$$\left|\varphi_{\mathbf{b}}(\mathbf{x}^{(n)}) - \varphi_{\mathbf{b}}(\mathbf{x})\right| \leq \left|\sum_{k\leq m} b_k(x_k^{(n)} - x_k)\right| + \left|\sum_{k>m} b_k(x_k^{(n)} - x_k)\right|$$

$$
\begin{aligned}
&\leq \sum_{k\leq m}\left|b_k(x_k^{(n)}-x_k)\right|+\sum_{k>m}\left|b_k(x_k^{(n)}-x_k)\right| \\
&\leq \|\mathbf{b}\|_\infty \sum_{k\leq m}\left|x_k^{(n)}-x_k\right|+\frac{\varepsilon}{4r}\sum_{k>m}\left|x_k^{(n)}-x_k\right| \\
&< \|\mathbf{b}\|_\infty \cdot \frac{\varepsilon}{2\,\|\mathbf{b}\|_\infty}+\frac{\varepsilon}{4r}\cdot 2r \\
&= \varepsilon, \quad \text{for all } \ n > n_0.
\end{aligned}
$$

This yields the desired assertion. □

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