Analysis of the Predator–Prey Interactions: A Stochastic Model Incorporating Disease Invasion

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Abstract
Environmental noise and infectious diseases are important factors affecting the development of the population. This paper develops a mathematical system to investigate the impacts of environmental noise and infectious diseases on predator–prey interactions. The globally unique positive solution is confirmed by using conventional methods. The stochastic uniform boundedness of the solution is obtained under certain conditions. Sufficient conditions for the persistence and extinction are given to measure the level of population size. Asymptotic dynamics of the solutions are carried out by two criteria parameters. The long-term dynamics of the solutions are demonstrated by numerical simulations. The results show that small-intensity environmental perturbations can cause population size to fluctuate around a certain level, while high-intensity environmental perturbations may lead to population extinction.

Keywords Predator–prey model · Stochastic noise · Persistence and extinction · Asymptotic dynamics · Infectious disease

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1 Introduction

Interaction between predator and prey is one of the cornerstones in bio and ecosystems. Understanding the mechanism behind is of great significance for maintaining species diversity, and therefore it remains high attention in many fields of science [26–28,31,

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35]. Mathematically, the basic frameworks of this kind of interaction can be described as

$$\begin{cases} \frac{dS}{dt} = \mu(S) - f(S, Y), \\ \frac{dY}{dt} = \delta f(S, Y) - M_Y(Y). \end{cases} \tag{1}$$

The biological meanings of parameters and functions appeared are listed in Table 1. System (1) and its variants have extensively studied with attempts to understand the predator–prey interactions [19,20,22,23]. However, the system (1) does not include the effect of infectious diseases on predator–prey interactions explicitly. Recently, Hethcote et al. [14] reported that the prevalence of infectious diseases is one of the main factors that is troubling the development of the population community. The spread of diseases in populations can lead to an increase in mortality, and predators can catch more infected prey. Thus, it has practical significance to gain deep insights into understanding the transmission dynamics of diseases in predator–prey interactions, but there are limited literature in this regards [5,13,16,17,25].

May [24] pointed out that birth rate, mortality rate and some other parameters related to population interactions could be affected by environmental noise. The biosphere environment is often changeable, and stochastic noise is also the cause leading to the extinction of individuals. By running a stochastic system several times, we can obtain the distribution of the predicted number of individuals, while a deterministic system will give a single predicted value. Traditionally, there are two common ways to introduce stochastic factors into deterministic population models. One is to assume that the predator–prey interactions are subject to some small random fluctuations. Physically, these small random fluctuations can be described by white noise [4,6,9,11]. Another is to assume that the predator–prey interactions are subject to sudden catastrophic shocks including earthquake, flood, and drought. These catastrophic shocks can be described mathematically by the Lévy process [2,21]. Several recent studies on predator–prey interactions have focused on the effects of environmental noise [1,7,10]. They played an irreplaceable role in studying the effects of diseases on predator–prey interactions. However, most of these models have considered either white noise or Lévy noise alone. Based on the work of [32], here we shall propose a new model, which allows us to examine the effects of white noise, Lévy noise as well as the diseases on the predator–prey interactions.

Table 1 The biological meanings of parameters and functions in (1)

Symbols	Biological meaning
$S(t)$	The abundance of the prey
$Y(t)$	The abundance of the predator
$\mu(S)$	The growth rate of the prey in the absence of predation
$f(S, Y)$	The attack rate of the predator on the prey
$\delta f(S, Y)$	The conversion rate from the prey to the predator
$M_Y(Y)$	The mortality rate of the predator

The paper is arranged as follows. We will, in Sect. 2, formulate the system and then prove some preliminary results. It is then followed by exploring the asymptotic dynamics of the model. Numerical simulations will be conducted in Sect. 4, to illustrate the dynamics of the stochastic system. Finally, we conclude our study with a brief discussion in Sect. 5.

2 Model Formulation and Preliminaries

Let $I(t)$ be the abundance of the infected prey, function $g(S, I)$ the infection rate of $I(t), b(I, Y)$ the attack rate of $Y(t)$ on $I(t), mb(I, Y), 0 \leq m \leq 1$ the conversion rate from the infected prey to the predator, and $M_I(I)$ the mortality rate of the infected prey. Supposing that the diseases spread only among the prey yields

$$\begin{cases} \frac{dS}{dt} = \mu(S) - f(S, Y) - g(S, I), \\ \frac{dI}{dt} = g(S, I) - b(I, Y) - M_I(I) \\ \frac{dY}{dt} = \delta f(S, Y) + mb(I, Y) - M_Y(Y), \end{cases} \quad (2)$$

where we take

$$\mu(S) = rS(t)\left(1 - \frac{S(t)}{K}\right),$$

as in [32], $f(S, Y) = 0$ and $b(I, Y) = pIY, \ p > 0$, indicating that healthy prey has an absolute escape advantage over infected prey, and $g(S, I) = \beta SI$. In general, depending on the case in question, $M_I(I)$ and $M_Y(Y)$ may take different forms, such as the linear mortality [37], the quadratic mortality [3,12] and the hyperbolic mortality [36]. In this paper, the mortality rates take forms of

$$M_I(t) = cI(t) + wI^2(t), \ M_Y(t) = dY(t) + hY^2(t), \quad (3)$$

where c denotes the disease-related death rate of the infected prey. w is the density dependence of the infected prey, d is the death rate of $Y(t)$, h is the density dependence of the predator. Please note that when $w = h = 0$, Eq. (3) are linear mortality rates; however, when $c = d = 0$ they quadratic mortality rates. The mortality rate of Eq. (3) has been used in modeling ecosystems of marine bays in [12]. With these specifics, we reach

$$\begin{cases} \frac{dS}{dt} = rS(t)\left(1 - \frac{S(t)}{K}\right) - \beta S(t)I(t), \\ \frac{dI}{dt} = \beta S(t)I(t) - pI(t)Y(t) - cI(t) - wI^2(t), \\ \frac{dY}{dt} = mpI(t)Y(t) - dY(t) - hY^2(t). \end{cases} \quad (4)$$

Using the similar argument as [8], we can prove

Proposition 2.1 *Let* $R_0 = \frac{K\beta}{c}$ *and* $R_1 = \frac{mp}{d} \frac{r(K\beta - c)}{K\beta^2 + wr}$. *Then, for system (4), we know that*

(1) if $R_0 < 1$, the disease-free equilibrium $E_1(K, 0, 0)$ is globally asymptotically stable (GAS);

(2) if $R_0 > 1$ and $R_1 < 1$, there exists a unique boundary equilibrium $E_2(\overline{S}, \overline{I}, 0) = \left(\frac{K(wr+\beta c)}{K\beta^2+wr}, \frac{r(K\beta-c)}{K\beta^2+wr}, 0 \right)$, which is GAS, while E_1 is unstable;

(3) if $R_1 > 1$, there exists a unique positive equilibrium

$$E_3(S^*, I^*, Y^*) = \left(\frac{K(r - \beta I^*)}{r}, \frac{K\beta rh + dpr - crh}{Kh\beta^2 + mp^2r + wrh}, \frac{mpI^* - d}{h} \right),$$

which is GAS, while both E_1 and E_2 are unstable.

To incorporate small stochastic noise into the deterministic system (4), for any initial value X_0 and and $0 \le \Delta t \ll 1$ we assume that the solution $X_t = (S_t, I_t, Y_t)'$ is a Markov process with the conditional mean

$$\mathbb{E}[X_{t+\Delta t} - X_t | X = X_0] \approx \begin{bmatrix} rS(1 - \frac{S}{K}) - \beta SI \\ \beta SI - pIY - cI - wI^2 \\ mpIY - dY - hY^2 \end{bmatrix} \Delta t$$

and the conditional variance

$$\text{Var}[X_{t+\Delta t} - X_t | X = X_0] \approx \begin{bmatrix} \sigma_1^2 S^2 \\ \sigma_2^2 I^2 \\ \sigma_3^2 Y^2 \end{bmatrix} \Delta t.$$

With these considerations, we obtain the stochastic version of the system (4) as follows:

$$\begin{cases} dS = \left(rS(t)\left(1 - \frac{S(t)}{K}\right) - \beta S(t)I(t) \right) dt + \sigma_1 S(t)dB_1(t), \\ dI = \left(\beta S(t)I(t) - cI(t) - pI(t)Y(t) - wI^2(t) \right) dt + \sigma_2 I(t)dB_2(t), \\ dY = \left(mpI(t)Y(t) - dY(t) - hY^2(t) \right) dt + \sigma_3 Y(t)dB_3(t), \end{cases} \quad (5)$$

where $B_i(t)$ $(i = 1, 2, 3)$ present the standard Brownian motions with intensities σ_i^2. Similarly, we can obtain the stochastic Lévy predator–prey system as follows accounted for the sudden catastrophic shocks

$$\begin{cases} dS = \left(rS(t)\left(1 - \frac{S(t)}{K}\right) - \beta S(t)I(t) \right) dt + \sigma_1 S(t)dB_1(t) \\ \qquad + \int_{\mathbb{Z}} S(t^-)\gamma_1(u)\widetilde{\Gamma}(dt, du), \\ dI = (\beta S(t)I(t) - cI(t) - pI(t)Y(t) - wI^2(t))dt + \sigma_2 I(t)dB_2(t) \\ \qquad + \int_{\mathbb{Z}} I(t^-)\gamma_2(u)\widetilde{\Gamma}(dt, du), \\ dY = (mpI(t)Y(t) - dY(t) - hY^2(t))dt + \sigma_3 Y(t)dB_3(t) \\ \qquad + \int_{\mathbb{Z}} Y(t^-)\gamma_3(u)\widetilde{\Gamma}(dt, du), \end{cases} \quad (6)$$

where $\widetilde{\Gamma}(dt, du) = \Gamma(dt, du) - \lambda(du)dt$, Γ is a Poisson counting measure on a measureable subset $\mathbb{Z}$ of $(0, \infty)$, and λ is the characteristic measure of the Poisson counting measure Γ with $\lambda(\mathbb{Z}) < \infty$.

Next, we will investigate model (6). For the sake of following discussion, we define

$$b_i = \frac{1}{2}\sigma_i + \int_{\mathbb{Z}} \left[\gamma_i - \ln(1 + \gamma_i)\right],$$

$$\langle X(t)\rangle = \frac{1}{t} \int_0^t X(s)ds, \quad \langle X(t)\rangle_* = \liminf_{t\to+\infty} \langle X(t)\rangle, \quad \langle X(t)\rangle^* = \limsup_{t\to+\infty} \langle X(t)\rangle,$$

and introduce

Definition 2.1 System (6) is called stochastically ultimately bounded if for any $\varepsilon \in (0, 1)$ there exists a $\chi(= \chi(\omega)) > 0$ such that

$$\limsup_{t\to\infty} \mathbb{P}\{|(S(t), I(t), Y(t))| > \chi\} < \varepsilon.$$

3 Main Results

Define

$$V(S, I, Y) = S - 1 - \ln S + I - 1 - \ln I + \frac{1}{m}(Y - 1 - \ln Y).$$

Applying the same argument as in [33], we have the following result, which suggests the solution of (6) is always biological meaningful.

Theorem 3.1 *Given* $(S(0), I(0), Y(0)) \in \mathbb{R}^3_+$. *System* (6) *has a unique positive solution on* $t \geq 0$ *with probability one.*

Theorem 3.2 *The solution of* (6) *determined by Theorem* 3.1 *is stochastically ultimately bounded, i.e.,*

$$\limsup_{t\to\infty} \mathbb{P}\{|(S(t), I(t), Y(t))| > \chi\} \leq \varepsilon$$

provided $\frac{r}{K} > \frac{2\beta}{3}$, $w > \frac{\beta + 2mp}{3}$ *and* $h > \frac{mp}{3}$.

Proof Define $V(S, I, Y) = S^{\frac{1}{2}} + I^{\frac{1}{2}} + Y^{\frac{1}{2}}$. By Itô's formula, we obtain

$$
\begin{aligned}
dV = LVdt &+ \frac{1}{2}\sigma_1 S^{\frac{1}{2}}dB_1(t) + \frac{1}{2}\sigma_2 I^{\frac{1}{2}}dB_2(t) + \frac{1}{2}\sigma_3 Y^{\frac{1}{2}}dB_3(t) \\
&+ S^{\frac{1}{2}} \int_{\mathbb{Z}} [(1 + \gamma_1(u))^{\frac{1}{2}} - 1]\widetilde{\Gamma}(dt, du) + I^{\frac{1}{2}} \int_{\mathbb{Z}} [(1 + \gamma_2(u))^{\frac{1}{2}} - 1]\widetilde{\Gamma}(dt, du) \\
&+ Y^{\frac{1}{2}} \int_{\mathbb{Z}} [(1 + \gamma_3(u))^{\frac{1}{2}} - 1]\widetilde{\Gamma}(dt, du),
\end{aligned}
$$

where

$$LV = \frac{1}{2}S^{\frac{1}{2}}\left(r(1 - \frac{S}{K}) - \beta I\right) + \frac{1}{2}I^{\frac{1}{2}}(\beta S - c - pY - wI) + \frac{1}{2}Y^{\frac{1}{2}}(mpI - d - hY)$$

$$- \frac{1}{8}\sigma_1^2 S^{\frac{1}{2}} + S^{\frac{1}{2}}\int_{\mathbb{Z}}[(1 + \gamma_1(u))^{\frac{1}{2}} - 1 - \frac{1}{2}\gamma_1(u)]\lambda(du)$$

$$- \frac{1}{8}\sigma_2^2 I^{\frac{1}{2}} + I^{\frac{1}{2}}\int_{\mathbb{Z}}[(1 + \gamma_2(u))^{\frac{1}{2}} - 1 - \frac{1}{2}\gamma_2(u)]\lambda(du)$$

$$- \frac{1}{8}\sigma_3^2 Y^{\frac{1}{2}} + Y^{\frac{1}{2}}\int_{\mathbb{Z}}[(1 + \gamma_3(u))^{\frac{1}{2}} - 1 - \frac{1}{2}\gamma_3(u)]\lambda(du)$$

$$\leq - \frac{r}{2K}S^{\frac{3}{2}} + S^{\frac{1}{2}}\left(\frac{1}{2}r - \frac{1}{8}\sigma_1^2 + \int_{\mathbb{Z}}[(1 + \gamma_1(u))^{\frac{1}{2}} - 1 - \frac{1}{2}\gamma_1(u)]\lambda(du)\right)$$

$$- \frac{1}{2}wI^{\frac{3}{2}} + I^{\frac{1}{2}}\left(\frac{1}{2}c - \frac{1}{8}\sigma_2^2 + \int_{\mathbb{Z}}[(1 + \gamma_2(u))^{\frac{1}{2}} - 1 - \frac{1}{2}\gamma_2(u)]\lambda(du)\right)$$

$$- \frac{1}{2}hY^{\frac{3}{2}} + Y^{\frac{1}{2}}\left(-\frac{1}{2}d - \frac{1}{8}\sigma_3^2 + \int_{\mathbb{Z}}[(1 + \gamma_3(u))^{\frac{1}{2}} - 1 - \frac{1}{2}\gamma_3(u)]\lambda(du)\right)$$

$$+ \frac{1}{2}\beta I^{\frac{1}{2}}S + \frac{1}{2}mpY^{\frac{1}{2}}I.$$

By the Hölder inequation $ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \frac{1}{p} + \frac{1}{q} = 1(p, q > 1)$, we have

$$I^{\frac{1}{2}}S \leq \frac{1}{3}I^{\frac{3}{2}} + \frac{2}{3}S^{\frac{3}{2}}, \; Y^{\frac{1}{2}}I \leq \frac{1}{3}Y^{\frac{3}{2}} + \frac{2}{3}I^{\frac{3}{2}}.$$

Thus

$$LV \leq -\frac{1}{2}\left(\frac{r}{K} - \frac{2\beta}{3}\right)S^{\frac{3}{2}} - \frac{1}{2}\left(w - \frac{\beta + 2mp}{3}\right)I^{\frac{3}{2}} - \frac{1}{2}\left(h - \frac{mp}{3}\right)Y^{\frac{3}{2}}$$

$$+ S^{\frac{1}{2}}\left(\frac{1}{2}r + 1 - \frac{1}{8}\sigma_1^2 + \int_{\mathbb{Z}}[(1 + \gamma_1(u))^{\frac{1}{2}} - 1 - \frac{1}{2}\gamma_1(u)]\lambda(du)\right)$$

$$+ I^{\frac{1}{2}}\left(\frac{1}{2}c + 1 - \frac{1}{8}\sigma_2^2 + \int_{\mathbb{Z}}[(1 + \gamma_2(u))^{\frac{1}{2}} - 1 - \frac{1}{2}\gamma_2(u)]\lambda(du)\right)$$

$$+ Y^{\frac{1}{2}}\left(-\frac{1}{2}d + 1 - \frac{1}{8}\sigma_3^2 + \int_{\mathbb{Z}}[(1 + \gamma_3(u))^{\frac{1}{2}} - 1 - \frac{1}{2}\gamma_3(u)]\lambda(du)\right)$$

$$- V(S, I, Y)$$

$$\leq H - V(S, I, Y), \; H > 0 \text{ is a constant.}$$

Applying Itô's formula to $e^t V(S, I, Y)$ yields

$$d(e^t V(S, I, Y))$$
$$= e^t[V(S, I, Y)dt + dV(S, I, Y)]$$
$$\leq e^t Hdt + e^t\left[\frac{1}{2}\left(\sigma_1 S^{\frac{1}{2}}dB_1(t) + \sigma_2 I^{\frac{1}{2}}dB_2(t) + \sigma_3 Y^{\frac{1}{2}}dB_3(t)\right)\right.$$

$$+ S^{\frac{1}{2}} \int_{\mathbb{Z}} [(1 + \gamma_1(u))^{\frac{1}{2}} - 1]\tilde{\Gamma}(dt, du) + I^{\frac{1}{2}} \int_{\mathbb{Z}} [(1 + \gamma_2(u))^{\frac{1}{2}} - 1]\tilde{\Gamma}(dt, du)$$

$$+ Y^{\frac{1}{2}} \int_{\mathbb{Z}} [(1 + \gamma_3(u))^{\frac{1}{2}} - 1]\tilde{\Gamma}(dt, du) \Big].$$

Therefore we have

$$e^t \mathbb{E} V(S, I, Y) \leq V(S(0), I(0), Y(0)) + H e^t$$

and

$$\limsup_{t \to +\infty} \mathbb{E} V(S, I, Y) \leq H.$$

By elementary inequality

$$n^{(1-p/2)\wedge 0}|x|^p \leq \sum_{i=1}^{n} x_i^p \leq n^{(1-p/2)\vee 0}|x|^p, \quad \mathbb{R}_+^n := \{x \in \mathbb{R}^n : x_i > 0, 1 \leq i \leq n\},$$

we have

$$\limsup_{t \to +\infty} \mathbb{E}|(S(t), I(t), Y(t))|^{\frac{1}{2}} \leq H.$$

For any $\varepsilon > 0$, set $\chi = \frac{H^2}{\varepsilon^2}$, the Chebyshev inequality implies

$$\mathbb{P}\{|(S(t), I(t), Y(t))| > \chi\} \leq \frac{\mathbb{E}|(S(t), I(t), Y(t))|^{\frac{1}{2}}}{\sqrt{\chi}},$$

namely,

$$\limsup_{t \to \infty} \mathbb{P}\{|(S(t), I(t), Y(t))| > \chi\} \leq \varepsilon.$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

To study the persistence and extinction of Eq. (6), we define:

$$R_{11} = \frac{mpK\beta(r - b_1)}{rmp(c + b_2) + wr(d + b_3)}$$

and

$$R_{12} = \frac{mpK\beta h \left[wr(r - b_1) - K\beta^2(r - b_1) + \beta r(c + b_2) \right] - w^2 r^2 h(d + b_3)}{mp[wr^2 h(c + b_2) + pr (mpK\beta(r - b_1) - mpr(c + b_2) - wr(d + b_3))]}.$$

Theorem 3.3 *For the solution of* (6) *determined by Theorem* 3.1, *we have*

(i) *If* $r < \frac{\sigma_1^2}{2} + \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))] \lambda(du)$, *then we have*

$$\lim_{t \to +\infty} S(t) = 0, \ \lim_{t \to +\infty} I(t) = 0, \ \lim_{t \to +\infty} Y(t) = 0 \ a.s.(almost \ surely),$$

i.e., the predator and prey populations are die out.

(ii) *If* $R_{11} > 1$ *and* $R_{12} > 1$, *then we have*

$$S(t)\rangle_* > 0, \ \langle I(t) \rangle_* > 0, \ \langle Y(t) \rangle_* > 0 \ a.s.,$$

i.e., the predator and prey populations are persistence in the mean.

Proof Case (i). For the first equation of system (6), we apply the Itô's formula, resulting

$$d \ln S(t) = \left(r - \frac{r}{K} S(t) - \beta I(t) - \frac{1}{2}\sigma_1^2 - \int_{\mathbb{Z}} \left[\gamma_1(u) - \ln(1 + \gamma_1(u)) \right] \lambda(du) \right) dt$$
$$+ \sigma_1 dB_1(t) + \int_{\mathbb{Z}} \ln(1 + \gamma_1(u)) \widetilde{\Gamma}(dt, du),$$

integrating both sides of which gives

$$\frac{1}{t} \ln \frac{S(t)}{S(0)} = r - \frac{1}{2}\sigma_1^2 - \int_{\mathbb{Z}} \left[\gamma_1(u) - \ln(1 + \gamma_1(u)) \right] \lambda(du) - \frac{r}{K} \langle S(t) \rangle - \beta \langle I(t) \rangle$$
$$+ \frac{1}{t}\sigma_1 B_1(t) + \frac{1}{t} \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_1(u)) \widetilde{\Gamma}(dt, du)$$
$$\leq r - \frac{1}{2}\sigma_1^2 - \int_{\mathbb{Z}} \left[\gamma_1(u) - \ln(1 + \gamma_1(u)) \right] \lambda(du) - \frac{r}{K} \langle S(t) \rangle + \frac{1}{t}\sigma_1 B_1(t)$$
$$+ \frac{1}{t} \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_1(u)) \widetilde{\Gamma}(dt, du). \tag{7}$$

By using [21], we obtain $\lim_{t \to +\infty} S(t) = 0 \ a.s.$ Similarly, we have

$$\frac{1}{t} \ln \frac{I(t)}{I(0)} = -c - \frac{1}{2}\sigma_2^2 - \int_{\mathbb{Z}} \left[\gamma_2(u) - \ln(1 + \gamma_2(u)) \right] \lambda(du) + \beta \langle S(t) \rangle - p \langle Y(t) \rangle$$
$$- w \langle I(t) \rangle + \frac{1}{t}\sigma_2 B_2(t) + \frac{1}{t} \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_2(u)) \widetilde{\Gamma}(dt, du)$$
$$\leq -c - \frac{1}{2}\sigma_2^2 - \int_{\mathbb{Z}} \left[\gamma_2(u) - \ln(1 + \gamma_2(u)) \right] \lambda(du) + \beta \langle S(t) \rangle - w \langle I(t) \rangle$$
$$+ \frac{1}{t}\sigma_2 B_2(t) + \frac{1}{t} \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_2(u)) \widetilde{\Gamma}(dt, du). \tag{8}$$

Since $\lim_{t \to +\infty} S(t) = 0$, for sufficiently large T, there is a constant $\varepsilon > 0$ such that

$$-c - \frac{1}{2}\sigma_2^2 - \int_{\mathbb{Z}} \left[\gamma_2(u) - \ln(1 + \gamma_2(u)) \right] \lambda(du) + \varepsilon < 0 \ \text{for} \ t > T$$

Thus,

$$\lim_{t \to +\infty} I(t) = 0 \quad \text{almost surely.}$$

Note that

$$\frac{1}{t} \ln \frac{Y(t)}{Y(0)} = -d - \frac{1}{2}\sigma_3^2 - \int_{\mathbb{Z}} \left[\gamma_3(u) - \ln(1 + \gamma_3(u)) \right] \lambda(du) + mp\langle I(t) \rangle - h\langle Y(t) \rangle$$

$$+ \frac{1}{t}\sigma_3 B_3(t) + \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_3(u)) \widetilde{\Gamma}(dt, du).$$

$$(9)$$

We conclude by $\lim_{t \to +\infty} I(t) = 0$ that

$$\lim_{t \to +\infty} Y(t) = 0 \quad a.s.$$

Case (ii). From (7) and [21], we have

$$\langle S(t) \rangle^* \leq \frac{K(r - b_1)}{r},$$

$$(10)$$

substituting which into (8) gives

$$\frac{1}{t} \ln \frac{I(t)}{I(0)} \leq -c - b_2 + \beta \langle S(t) \rangle^* - w\langle I(t) \rangle + \frac{1}{t}\sigma_2 B_2(t)$$

$$+ \frac{1}{t} \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_2(u)) \widetilde{\Gamma}(dt, du)$$

$$\leq -c - b_2 + \frac{K\beta(r - b_1)}{r} - w\langle I(t) \rangle + \frac{1}{t}\sigma_2 B_2(t)$$

$$+ \frac{1}{t} \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_2(u)) \widetilde{\Gamma}(dt, du).$$

Thus,

$$\langle I(t) \rangle^* \leq \frac{K\beta(r - b_1) - r(c + b_2)}{wr}.$$

$$(11)$$

Then (11) and (7) yield

$$\frac{1}{t} \ln \frac{S(t)}{S(0)} \geq r - b_1 - \frac{r}{K}\langle S(t) \rangle - \beta\langle I(t) \rangle^* + \frac{1}{t}\sigma_1 B_1(t)$$

$$+ \frac{1}{t} \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_1(u)) \widetilde{\Gamma}(dt, du)$$

$$\geq r - b_1 - \frac{K\beta^2(r - b_1) - \beta r(c + b_2)}{wr} - \frac{r}{K}\langle S(t) \rangle + \frac{1}{t}\sigma_1 B_1(t)$$

$$+ \frac{1}{t} \int_0^t \int_{\mathbb{Z}} \ln(1 + \gamma_1(u)) \widetilde{\Gamma}(dt, du).$$

Using a result from [21],

$$\langle S(t)\rangle_* \geq \frac{Kwr(r - b_1) - K^2\beta^2(r - b_1) + K\beta r(c + b_2)}{wr^2} > 0. \qquad (12)$$

Similarly, by (11) and (9) one can obtain

$$\frac{1}{t}\ln\frac{Y(t)}{Y(0)} \leq -d - b_3 + mp\langle I(t)\rangle^* - h\langle Y(t)\rangle + \frac{1}{t}\sigma_3 B_3(t)$$

$$+ \frac{1}{t}\int_0^t\int_{\mathbb{Z}}\ln(1 + \gamma_3(u))\widetilde{\Gamma}(dt, du)$$

$$\leq -d - b_3 + \frac{mpK\beta(r - b_1) - rmp(c + b_2)}{wr} - h\langle Y(t)\rangle + \frac{1}{t}\sigma_3 B_3(t)$$

$$+ \frac{1}{t}\int_0^t\int_{\mathbb{Z}}\ln(1 + \gamma_3(u))\widetilde{\Gamma}(dt, du).$$

Thus,

$$\langle Y(t)\rangle^* \leq \frac{mpK\beta(r - b_1) - rmp(c + b_2) - wr(d + b_3)}{wrh}. \qquad (13)$$

Equations (12), (13) and (8) yield

$$\frac{1}{t}\ln\frac{I(t)}{I(0)} \geq -c - b_2 + \beta\langle S(t)\rangle_* - p\langle Y(t)\rangle^* - w\langle I(t)\rangle + \frac{1}{t}\sigma_2 B_2(t)$$

$$+ \frac{1}{t}\int_0^t\int_{\mathbb{Z}}\ln(1 + \gamma_2(u))\widetilde{\Gamma}(dt, du)$$

$$\geq -c - b_2 + -w\langle I(t)\rangle + \frac{K\beta wr(r - b_1) - K^2\beta^3(r - b_1) + K\beta^2 r(c + b_2)}{wr^2}$$

$$- \frac{mp^2 K\beta(r - b_1) - rmp^2(c + b_2) - wrp(d + b_3)}{wrh} + \frac{1}{t}\sigma_2 B_2(t)$$

$$+ \frac{1}{t}\int_0^t\int_{\mathbb{Z}}\ln(1 + \gamma_2(u))\widetilde{\Gamma}(dt, du).$$

Note that

$$\frac{K\beta h\left[wr(r - b_1) - K\beta^2(r - b_1) + \beta r(c + b_2)\right]}{wr^2 h(c + b_2) + pr\left(mpK\beta(r - b_1) - mpr(c + b_2) - wr(d + b_3)\right)} > 1.$$

We have

$$\langle I(t)\rangle_* \geq \frac{1}{w^2 r^2 h}\Big[K\beta h\left(wr(r - b_1) - K\beta^2(r - b_1) + \beta r(c + b_2)\right) - wr^2 h(c + b_2)$$

$$- pr\left(mpK\beta(r - b_1) - mpr(c + b_2) - wr(d + b_3)\right)\Big] > 0. \qquad (14)$$

Substituting Eq. (14) into Eq. (9), we have

$$
\frac{1}{t} \ln \frac{Y(t)}{Y(0)} \geq -d - b_3 + mp\langle I(t)\rangle_* - h\langle Y(t)\rangle + \frac{1}{t}\sigma_3 B_3(t)
$$
$$
+ \frac{1}{t}\int_0^t \int_{\mathbb{Z}} \ln(1+\gamma_3(u))\widetilde{\Gamma}(dt,du)
$$
$$
\geq -d - b_3 + \frac{mp}{w^2 r^2 h}\big[K\beta h(wr(r-b_1) - K\beta^2(r-b_1) + \beta r(c+b_2))
$$
$$
- wr^2 h(c+b_2) - pr(mpK\beta(r-b_1) - mpr(c+b_2) - wr(d+b_3))\big]
$$
$$
- h\langle Y(t)\rangle + \frac{1}{t}\sigma_3 B_3(t) + \frac{1}{t}\int_0^t \int_{\mathbb{Z}} \ln(1+\gamma_3(u))\widetilde{\Gamma}(dt,du).
$$

Since $R_{12} > 1$, we obtain

$$
\langle Y(t)\rangle_* \geq \frac{1}{w^2 r^2 h^2}\Big[mp\big[K\beta h\big(wr(r-b_1) - K\beta^2(r-b_1) + \beta r(c+b_2)\big)
$$
$$
- wr^2 h(c+b_2) - pr\,(mpK\beta(r-b_1) - mpr(c+b_2) - wr(d+b_3))\big]
$$
$$
- w^2 r^2 h(d+b_3)\Big] > 0.
$$

The proof is complete. □

Notes. Theorem 3.3 shows us the dynamics of birth and death of the prey-predator system. By Theorem 3.3, if $r < \frac{\sigma_1^2}{2} + \int_{\mathbb{Z}}[\gamma_1(u) - \ln(1+\gamma_1(u))]\lambda(du)$, then the predator and prey are die out; while if $R_{11} > 1$ and $R_{12} > 1$, then the predator and prey are persistence in mean. Note that the right end of inequality $r < \frac{\sigma_1^2}{2} + \int_{\mathbb{Z}}[\gamma_1(u) - \ln(1+\gamma_1(u))]\lambda(du)$ is a function of the intensity of noise and is positively correlated with the intensity of noise. This shows that as long as the intensity of the stochastic noise is large enough, we have $r < \frac{\sigma_1^2}{2} + \int_{\mathbb{Z}}[\gamma_1(u) - \ln(1+\gamma_1(u))]\lambda(du)$, which means that large environmental disturbances can lead to the extinction of the population.

Theorem 3.4 *For the solution of* (6), *we have*

(i) *if* $R_0 < 1$, *then*

$$
F_1(S, I, V) := \limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \Big[(S(s) - K)^2 + I^2(s) + Y^2(s)\Big]ds \leq \frac{\delta_1}{\Theta};
$$

(ii) *if* $R_0 > 1$ *and* $R_1 < 1$, *then*

$$
F_2(S, I, V) := \limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \Big[(S(s) - \overline{S})^2 + (I(s) - \overline{I})^2 + Y^2(s)\Big]ds \leq \frac{\delta_2}{\Theta};
$$

(iii) if $R_1 > 1$, then

$$F_3(S, I, V) := \limsup_{t \to \infty} \frac{1}{t} \mathbb{E} \int_0^t \left[(S(s) - S^*)^2 + (I(s) - I^*)^2 + (Y(s) - Y^*)^2 \right] ds$$

$$\leq \frac{\delta_3}{\Theta},$$

where $\Theta = \min \left\{ \frac{r}{K}, w, \frac{h}{m} \right\}$ and

$$\delta_1 = \frac{1}{2} \sigma_1^2 K + K \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))] \lambda(du),$$

$$\delta_2 = \frac{\sigma_1^2 \overline{S}}{2} + \frac{\sigma_2^2 \overline{I}}{2} + \overline{S} \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))] \lambda(du)$$

$$+ \overline{I} \int_{\mathbb{Z}} [\gamma_2(u) - \ln(1 + \gamma_2(u))] \lambda(du),$$

$$\delta_3 = \frac{1}{2} \sigma_1^2 S^* + \frac{1}{2} \sigma_2^2 I^* + \frac{1}{2m} \sigma_3^2 Y^* + S^* \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))] \lambda(du)$$

$$+ I^* \int_{\mathbb{Z}} [\gamma_2(u) - \ln(1 + \gamma_2(u))] \lambda(du)$$

$$+ \frac{1}{m} Y^* \int_{\mathbb{Z}} [\gamma_3(u) - \ln(1 + \gamma_3(u))] \lambda(du).$$

Proof Case (iii). Since (S^*, I^*, Y^*) is the positive equilibrium point of the system (4), we have

$$r \left(1 - \frac{S^*}{K} \right) = \beta I^*, \quad \beta S^* - c - pY^* - wI^* = 0, \quad mpI^* - d - hY^* = 0.$$

Define the Lyapunov function:

$$V(S, I, Y) = S - S^* - S^* \ln \frac{S}{S^*} + I - I^* - I^* \ln \frac{I}{I^*} + \frac{1}{m} \left(Y - Y^* - Y^* \ln \frac{Y}{Y^*} \right).$$

Applying Itô's formula to the system (6), we have

$$dV = LV dt + \sigma_1 (S - S^*) dB_1(t) + \int_{\mathbb{Z}} [\gamma_1(u)S - S^* \ln(1 + \gamma_1(u))] \widetilde{\Gamma}(dt, du)$$

$$+ \sigma_2 (I - I^*) dB_2(t) + \int_{\mathbb{Z}} [\gamma_2(u)I - I^* \ln(1 + \gamma_2(u))] \widetilde{\Gamma}(dt, du)$$

$$+ \frac{1}{m} \sigma_3 (Y - Y^*) dB_3(t) + \frac{1}{m} \int_{\mathbb{Z}} [\gamma_3(u)Y - Y^* \ln(1 + \gamma_3(u))] \widetilde{\Gamma}(dt, du),$$

$$(15)$$

where

$$
\begin{aligned}
LV &= (S - S^*)\left(r(1 - \frac{S}{K}) - \beta I\right) + \frac{\sigma_1^2 S^*}{2} + S^* \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))]\lambda(du) \\
&\quad + (I - I^*)(\beta S - c - pY - wI) + \frac{\sigma_2^2 I^*}{2} + I^* \int_{\mathbb{Z}} [\gamma_2(u) - \ln(1 + \gamma_2(u))]\lambda(du) \\
&\quad + (Y - Y^*)(pI - \frac{d}{m} - \frac{h}{m}Y) + \frac{\sigma_3^2 Y^*}{2m} + \frac{Y^*}{m} \int_{\mathbb{Z}} [\gamma_3(u) - \ln(1 + \gamma_3(u))]\lambda(du) \\
&= r(S - S^*) - \frac{r}{K}(S - S^*)S - \beta(S - S^*)I + \beta(I - I^*)S - c(I - I^*) \\
&\quad - p(I - I^*)Y - w(I - I^*)I + p(Y - Y^*)I - \frac{d}{m}(Y - Y^*) - \frac{h}{m}(Y - Y^*)Y \\
&\quad + \frac{1}{2}\left(\sigma_1^2 S^* + \sigma_2^2 I^* + \frac{1}{m}\sigma_3^2 Y^*\right) + S^* \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))]\lambda(du) \\
&\quad + I^* \int_{\mathbb{Z}} [\gamma_2(u) - \ln(1 + \gamma_2(u))]\lambda(du) + \frac{1}{m}Y^* \int_{\mathbb{Z}} [\gamma_3(u) - \ln(1 + \gamma_3(u))]\lambda(du) \\
&= -\frac{r}{K}(S - S^*)^2 - w(I - I^*)^2 - \frac{h}{m}(Y - Y^*)^2 + \frac{1}{2}\left(\sigma_1^2 S^* + \sigma_2^2 I^* + \frac{1}{m}\sigma_3^2 Y^*\right) \\
&\quad + S^* \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))]\lambda(du) + I^* \int_{\mathbb{Z}} [\gamma_2(u) - \ln(1 + \gamma_2(u))]\lambda(du) \\
&\quad + \frac{1}{m}Y^* \int_{\mathbb{Z}} \left[\gamma_3(u) - \ln(1 + \gamma_3(u))\right]\lambda(du).
\end{aligned}
$$

Integrating on both sides of Eq. (15) and then taking expectation result in

$$
\begin{aligned}
\mathbb{E}\int_0^t &\left\{\frac{r}{K}(S(\theta) - S^*)^2 + w(I(\theta) - I^*)^2 + \frac{h}{m}(Y(\theta) - Y^*)^2\right\}d\theta \\
&\leq \mathbb{E}V(0) + \left[\frac{1}{2}\left(\sigma_1^2 S^* + \sigma_2^2 I^* + \frac{1}{m}\sigma_3^2 Y^*\right) + S^* \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))]\lambda(du) \right. \\
&\quad \left. + I^* \int_{\mathbb{Z}} [\gamma_2(u) - \ln(1 + \gamma_2(u))]\lambda(du) + \frac{1}{m}Y^* \int_{\mathbb{Z}} [\gamma_3(u) - \ln(1 + \gamma_3(u))]\lambda(du)\right]t.
\end{aligned}
$$
(16)

Dividing Eq. (16) by t and letting $t \to \infty$, we have

$$
\limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left[(S(s) - S^*)^2 + (I(s) - I^*)^2 + (Y(s) - Y^*)^2\right]ds \leq \frac{\delta_3}{\Theta}.
$$

The proof of case (iii) is complete. By defining the Lyapunov functions $V(S, I, Y) = S - K - K\ln\frac{S}{K} + I + \frac{1}{m}Y$ and $V(S, I, Y) = S - \overline{S} - \overline{S}\ln\frac{S}{\overline{S}} + I - \overline{I} - \overline{I}\ln\frac{I}{\overline{I}} + \frac{1}{m}Y$, respectively, the proofs of cases (i) and (ii) are similar to case (iii), so we omit them here. □

Notes. Theorem 3.4 shows that under different conditions, the solution of the stochastic predator–prey system can fluctuate (around the equilibrium of the ODEs system (4)) in different states, and the amplitude of the fluctuation is positively correlated with the
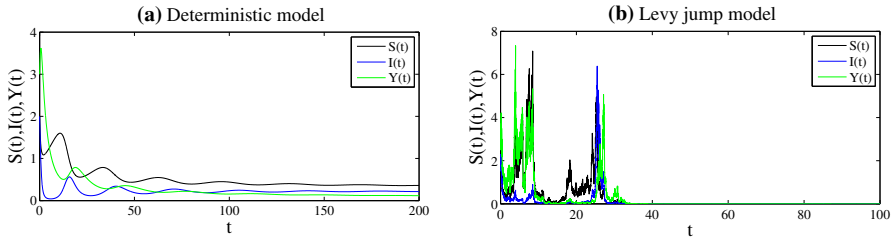
Fig. 1 The trajectories of (6) and (4) under the conditions of Theorem 3.3 (i). **a** The persistence of the ODE system (4); **b** the predator and prey populations predicted by (6) go extinct with probability 1

intensity of stochastic noise. In particular, when the intensity of the noise is zero, i.e., $\sigma_i = \gamma_i(u) = 0$ $(i = 1, 2, 3)$, we have

$$LV \leq -\frac{r}{K}(S - S^*)^2 - w(I - I^*)^2 - \frac{h}{m}(Y - Y^*)^2 \leq 0,$$

which indicates that the interior solution $E_3(S^*, I^*, Y^*)$ of the ODEs model (4) is GAS provided that $R_1 > 1$. Similarly, we can obtain that the boundary equilibrium $E_1(K, 0, 0)$ is GAS provided that $R_0 = \frac{K\beta}{c} < 1$, and the disease-free equilibrium $E_2(\overline{S}, \overline{I}, 0)$ is GAS provided that $R_0 > 1$ and $R_1 < 1$. Therefore, we generalised the global stability of the ODEs system (4).

4 Numerical Simulation

In this section, all simulations are carried out with ©Matlab2013b, the initial value is $(2, 2, 3)$ and the numerical method is based on [15].

We show the extinction and persistence of the system (6). The simulations of the ODEs system (4) were also studied as a comparison. To proceed, we set $\sigma_i = 0.5, \gamma_i(u) = 0.1, i = 1, 2, 3, r = 0.12, K = 4, \beta = 0.5, m = 1, p = 0.5, d = 0.1, h = 0.1, c = 0.1, w = 0.1$. By straightforward calculation we obtain that:

$$R_1 = \frac{mp}{d} \frac{r(K\beta - c)}{K\beta^2 + wr} = 1.126 > 1$$

and

$$r = 0.12 < \frac{\sigma_1^2}{2} + \int_{\mathbb{Z}} [\gamma_1(u) - \ln(1 + \gamma_1(u))]\lambda(du) = 0.142.$$

By Theorem 3.3 (i), one can see that equilibrium $E_3(0.3415, 0.2195, 0.0976)$ of system (4) is GAS, and the populations in the stochastic system (6) are die out. Figure 1a shows that the ODE system (4) admits a positive equilibrium, while Fig. 1b shows that stochastic solution of the predator and prey populations go to zero with probability one.
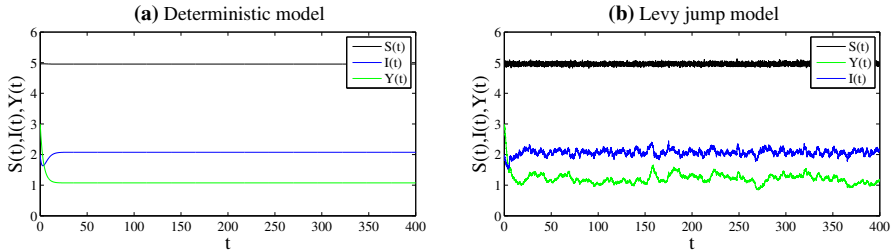
Fig. 2 The trajectories of the stochastic system (6) and the ODEs system (4) under the conditions of Theorem 3.3 (ii). **a** The persistence of the ODE system (4); **b** the persistence in mean of the stochastic system (6)

Next, we set $\sigma_i = 0.02$, $\gamma_i(u) = 0.01$, $i = 1, 2, 3$, $r = 22$, $K = 5$, $\beta = 0.1$, $m = 1$, $p = 0.1$, $d = 0.1$, $h = 0.1$, $c = 0.01$, $w = 0.182$. It follows that:

$$R_1 = 2.6591 > 1, \quad R_{11} = 1.0637 > 1, \quad R_{12} = 1.7401 > .1$$

By Theorem 3.3 (ii), we know that the unique positive equilibrium $E_3(4.9528, 2.0755, 1.0755)$ of (4) is GAS, and the populations in (6) are persistence in mean. Figure 2a shows that the ODE system (4) admits a positive equilibrium, while Fig. 2b shows that stochastic solution of the predator and prey populations fluctuates around the deterministic equilibrium point E_3.

We now numerically illustrate the asymptotic dynamics of (6) with $\sigma_i = 0.3$, $i = 1, 2, 3$, $r = 1$, $w = 0.2$. The other parameters are given by:

(1) $K = 1$, $\beta = 0.2$, $m = 0.2$, $p = 0.3$, $d = 0.1$, $h = 0.2$, $c = 0.3$, $w = 0.2$;
(2) $K = 3$, $\beta = 0.4$, $m = 0.2$, $p = 0.5$, $d = 0.6$, $h = 0.1$, $c = 0.3$, $w = 0.2$;
(3) $K = 4$, $\beta = 0.3$, $m = 0.8$, $p = 0.6$, $d = 0.3$, $h = 0.2$, $c = 0.1$, $w = 0.2$.

In case (1), by direct calculation we obtain that $R_0 = 0.6667 < 1$, it follows that the ODEs model (4) admits a disease-free equilibrium $E_1(K, 0, 0) = (1, 0, 0)$. To see the effects of noise on the stochastic system (6), we choose $\gamma_i = 0, 0.2$ and 0.3 respectively, $i = 1, 2, 3$. By straightforward calculations we obtain that $F_1(S, I, V) \leq \frac{\delta_1}{\Theta} = 0.225, 0.3134$ and 0.4132, respectively. Figure 3a shows that the ODEs system (4) admits a disease-free equilibrium $E_1(K, 0, 0) = (1, 0, 0)$. Figure 3b, d describe that the level of the susceptible prey in the stochastic model vibrates around the solution of the ODEs model, while the infected prey and predator go to zero with probability one.

In case (2), by direct calculation we obtain that $R_0 = 4 > 1$, $R_1 = 0.2206 < 1$, it follows that the ODEs model (4) admits a boundary equilibrium $E_2(\overline{S}, \overline{I}, 0) = (1.4118, 1.3235, 0)$. To see the effects of noise on the stochastic system (6), we choose $\gamma_i = 0, 0.1$ and 0.2 respectively, $i = 1, 2, 3$. By straightforward calculations we obtain that $F_2(S, I, V) \leq \frac{\delta_2}{\Theta} = 0.6154, 0.6796$ and 0.8572, respectively. Figure 4a shows that the ODEs system (4) admits a boundary equilibrium $E_2(\overline{S}, \overline{I}, 0) = (1.4118, 1.3235, 0)$. Figure 4b–d show that stochastic solutions of the susceptible prey and infected prey fluctuate around the solution of the ODEs model, while the predator go to zero with probability one.
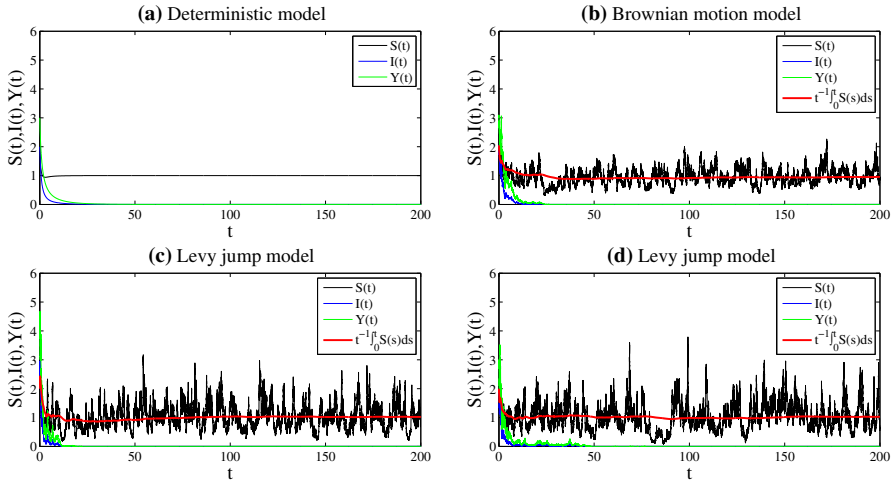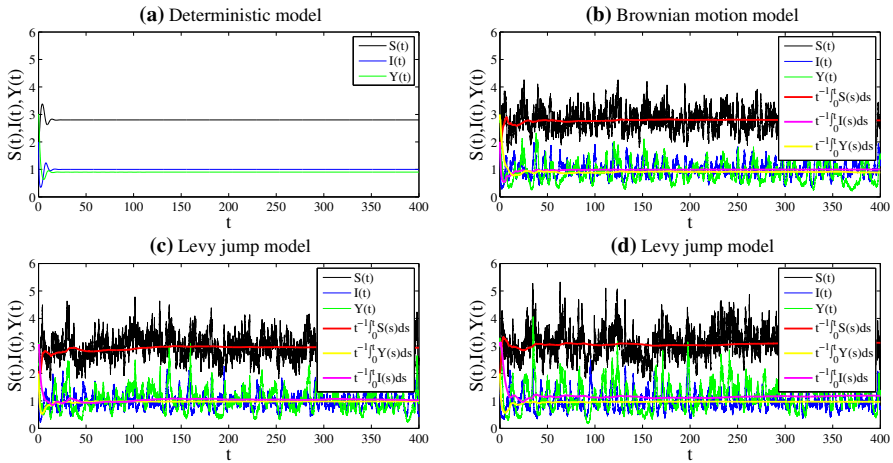
Fig. 3 The trajectories of the stochastic system (6) and the ODEs system (4) under the conditions of Theorem 3.4 (i). **a** The deterministic model; **b** the Brownian motion model; **c** the Lévy jump model with $\gamma_i = 0.2$; **d** the Lévy jump model with $\gamma_i = 0.3$, $i = 1, 2, 3$
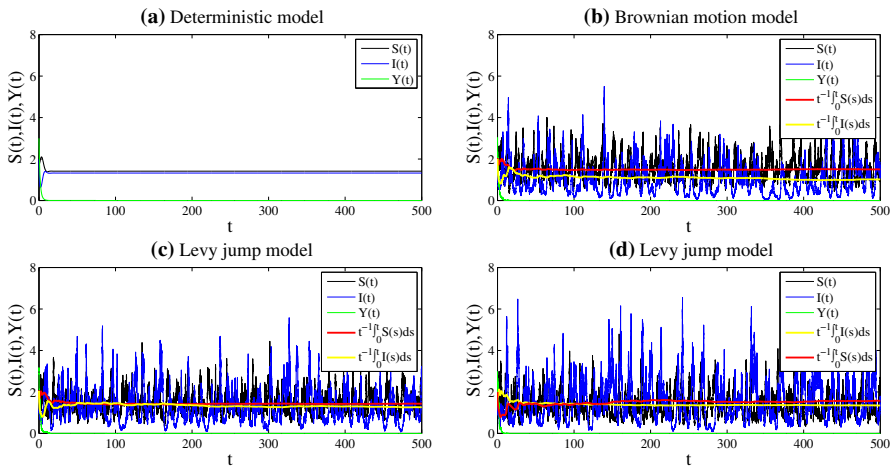
Fig. 4 The trajectories of the stochastic system (6) and the ODEs system (4) under the conditions of Theorem 3.4 (ii). **a** The deterministic model; **b** the Brownian motion model; and the Lévy jump model, **c** for $\gamma_i = 0.1$ and **d** for $\gamma_i = 0.2$, $i = 1, 2, 3$

In case (3), by direct calculation we obtain that $R_1 = 3.1429 > 1$, it follows that the ODEs model (4) admits a positive equilibrium $E_3(S^*, I^*, Y^*) = (2.8, 1, 0.9)$. To see the effects of noise on the stochastic system (6), we choose $\gamma_i = 0, 0.1$ and 0.2 respectively, $i = 1, 2, 3$. By straightforward calculations we obtain that $F_3(S, I, V) \leq \frac{\delta_3}{\Theta} = 0.7852$, 0.8671 and 1.0937, respectively. Figure 5a shows that the ODEs system (4) admits a positive equilibrium $E_3(S^*, I^*, Y^*) = (2.8, 1, 0.9)$. Figure 5b–d show

Fig. 5 The trajectories of the stochastic system (6) and the ODEs system (4) under the conditions of Theorem 3.4 (iii). **a** The deterministic model; **b** the Brownian motion model; **c** the Lévy jump model with $\gamma_i = 0.1$; **d** the Lévy jump model with $\gamma_i = 0.2$, $i = 1, 2, 3$

that stochastic solutions of the predator and prey fluctuate around the deterministic equilibrium point.

Note that in Fig. 1a, b, all parameters are the same except for the intensity of stochastic noise. This indicates that high-intensity environmental disturbances can lead to the extinction of the population. In addition, Figs. 2, 3, 4 and 5 show that when the stochastic noise is sufficiently small, the environmental disturbance does not cause the extinction of the population, but causes the density of the population to oscillate within a certain small range. In summary, our results that: (i) High-intensity environmental disturbances may lead to population extinction; (ii) Small-intensity environmental disturbances may not cause population extinction, but will cause population size to fluctuate within a certain interval. And the amplitude of the fluctuation is positively correlated with the intensity of the environmental disturbance.

5 Discussion

This paper formulates a stochastic system to study the interactions between predator and prey populations. The model is incorporating the disease invasion and sudden catastrophic shocks. The globally unique positive solution is confirmed by using conventional methods. The stochastic uniform boundedness of the solution is obtained under certain conditions. Sufficient conditions for the persistence and extinction are given to measure the level of population size. Asymptotic dynamics of the solutions are carried out by two criteria parameters. The long-term dynamics of the solutions are demonstrated by numerical simulations. By comparing the stochastic system with the corresponding ODE system, we find that: (i) when the intensity of environmental disturbance is large enough, environmental disturbance may lead to the extinction of the population; (ii) when the intensity of environmental disturbances is small enough,

environmental disturbances do not cause extinction of the population but cause the population to fluctuate around a certain level, and the amplitude of the fluctuations is proportional to the intensity of environmental disturbances. Our findings indicate that: (i) when the intensity of environmental disturbances is small, the impact of environmental disturbances on the population can be ignored, and the deterministic model can be used to estimate the dynamics of the population. However, when the intensity of environmental disturbance is large, the impact of environmental disturbance on the dynamics of the population cannot be ignored. Otherwise, the estimation result may be inaccurate.

Many researchers have made great efforts in studying the dynamics of population and infectious diseases [29,30]. Notably, Lipsitch et al. [18] studied the transmission mechanism of a SIRS infectious disease, but the results they have obtained were somewhat different from the actual situation. One persuasive reason is that they ignored the effects of noise (such as earthquake, flood, drought, typhoon, tsunami). May [24] indicated that the predation rate, environmental accommodation, and other factors could be affected by environmental noise. The biosphere environment in which the population located is often highly stochastic, and stochastic noise is also the cause leading to the extinction of individuals. Hence the effects of noise cannot be ignored. Accordingly, we introduced the Lévy jump into the proposed model and studied the dynamics of the model where noise plays a crucial rule. Compared to the stochastic system driven by Brownian motion in Ref. [34], our system is driven simultaneously by Brownian motion and Poisson motion. Therefore, the model in Ref. [34] can be used to study the effects of small environmental disturbances (such as wind and rain) on population dynamics, and our model can be used to study the effects of small environmental disturbances on population dynamics, as well as to study large environmental disturbances (such as sudden volcanoes and floods) on population dynamics. Therefore, our theoretical analysis is much more complicated than the theoretical analysis in Ref. [34].

The system developed in this paper unifies much of the previous work. It encompasses the influential work of Xiao et al. [32] in understanding the asymptotic stability of the model with the disease in prey and the more recent work on the random disturbance in a predator–prey model [34]. The results may help for the further study of such systems with singular diffusion, indicating that the variables or parameters are subject to the same environmental noise.

Compliance with Ethical Standards

Conflict of interest We declare that we have no conflict of interest.

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