Qualitative Theory of Dynamical Systems (2020) 19:16
https://doi.org/10.1007/s12346-020-00367-4

Eventually Shadowable Points

Meihua Dong[1] · Woochul Jung[2] · Carlos Morales[3]

Received: 8 February 2019 / Accepted: 2 November 2019 / Published online: 27 January 2020

Abstract
We study the *eventually shadowable points* namely points for which every pseudo orbit passing through then can be eventually shadowed (Good and Meddaugh in Ergod Theory Dyn Syst 38(1):143–154, 2018). We will prove the following results: the set of eventually shadowable points of a surjective continuous map of a compact metric space is invariant (possibly empty or noncompact) and the map has the eventual shadowing property if and only if every point is eventually shadowable. The chain recurrent and nonwandering sets coincide when every chain recurrent point is eventually shadowable. A surjective continuous map of a compact metric space has the eventual shadowing property if and only if the set of eventually shadowable points has a full measure with respect to every ergodic invariant probability measure. If there is an eventually shadowable point for which the associated Li–Yorke set equals the whole space, then the map has the eventual shadowing property. Proximal or transitive maps with eventually shadowable points have the eventual shadowing property. The eventually shadowable and shadowable points coincide for surjective equicontinuous maps on compact metric spaces. In particular, a surjective equicontinuous map of a compact metric space has the eventual shadowing property if and only if it has the shadowing property.

Keywords Eventual shadowing property · Shadowable point · Homeomorphism · Metric space

Mathematics Subject Classification Primary 37C50; Secondary 54H20

1 Introduction

The *shadowing property* means that approximated trajectories may be followed by true ones as close as we want. This is very important in practice since it implies that numeric orbits can be represented by true ones. Because of this it has been widely stud-

✉ Carlos Morales
morales@impa.br

Extended author information available on the last page of the article

ied in the literature [14] and several variations have been proposed. One of these is the pointwise pseudo orbit tracing property due to Li [11]. Wang and Yang [15] studied it and extend some results on chaotic behavior known for systems having the shadowing property to those with the pointwise pseudo orbit tracing property. Fakhari and Ghane defined *ergodic shadowing* and showed that it is equivalent to both shadowing and chain mixing [8]. Dastjerdi and Hosseini [6] introduced further notions like the *thick shadowing property* and noted that every chain transitive system with this property has the shadowing property too. They also defined $\underline{d}$*-shadowing* and proved that this together with thick shadowing implies transitivity. Brian, Meddaugh and Raines [5] proved that shadowing and thick shadowing are equivalent for chain transitive systems, but not in general (e.g. the map $f(x) = x + \frac{1}{4}|\sin(\pi x)|$ in $[0, 2]$). They also asked if shadowing implies thick shadowing for general systems (see Question 3.3 in [5]). This question was answered in positive by Oprocha [13] who further gave a complete characterization of the relationship between thick shadowing and shadowing: a continuous map of a compact metric space $f : X \to X$ has the thick shadowing property if and only if it has the shadowing property and satisfies $CR(f) = \Omega(f) = \overline{Rec(f)}$ where $CR(f)$, $\Omega(f)$ and $\overline{Rec(f)}$ are the chain recurrent set, the nonwandering set and the closure of the recurrent points respectively. He also proved that the thick shadowing property is equivalent to the $(\mathbb{N}, \mathcal{F}_{cf})$-shadowing property (c.f. Theorem 4.5 in [13]). Good and Meddaugh [9] introduced the *eventual shadowing property* and noted that it is precisely the $(\mathbb{N}, \mathcal{F}_{cf})$-shadowing mentioned by Oprocha [13]. Since the eventual shadowing property is just the pointwise pseudo-orbit tracing property (we nevertheless keep the former name in what follows), it follows that the thick and eventual shadowing properties are equivalent. In [12] it was defined *shadowable point* as the pointwise version of the shadowing property. Several properties of shadowable points were obtained elsewhere [10,12]. These results suggest to consider pointwise versions of the aforementioned versions of the shadowing property.

In this paper we will consider the pointwise version of the eventual shadowing property namely the *eventually shadowable points*. These are points for which every pseudo-orbit passing through them can be eventually shadowed. We prove that the set of eventually shadowable points of a surjective continuous map of a compact metric space is invariant (possible empty or noncompact) and the map has the eventual shadowing property if and only if every point is eventually shadowable. The chain recurrent and nonwandering sets coincide when every chain recurrent point is eventually shadowable. A surjective continuous map of a compact metric space has the eventual shadowing property if and only if the set of eventually shadowable points has full measure with respect to every ergodic invariant probability measure. Next if there is an eventually shadowable point for which the associated Li–Yorke set equals to the whole space, then the map has the eventual shadowing property. Finally, we prove that proximal or transitive maps with eventually shadowable points have the eventual shadowing property. Let us state our results in a precise way.

Denote by X a compact metric space, by $f : X \to X$ a continuous map, by $\mathbb{N}$ the set of positive integers, and by $\mathbb{N}_0$ the set of non-negative integers. We say that $A \subset X$ is *forward invariant* if $f(A) \subset A$ and *invariant* if $f^{-1}(A) = A$.

Given $\delta > 0$ we say that a sequence $\xi = (\xi_n)_{n \in \mathbb{N}_0}$ of X is a δ-pseudo-orbit of f if $d(f(\xi_n), \xi_{n+1}) \leq \delta$ for all $n \in \mathbb{N}_0$. We say that ξ can be ε-shadowed (for some given $\varepsilon > 0$) if there is $x \in X$ such that $d(f^n(x), \xi_n) \leq \varepsilon$ for every $n \in \mathbb{N}_0$. f has the *shadowing property* [4] if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit can be ε-shadowed. f has the *eventual shadowing property* [9] if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit $(\xi_n)_{n\in\mathbb{N}_0}$ can be *eventually ε-shadowed* i.e. there are $x \in X$ and $N \in \mathbb{N}_0$ such that $d(f^n(x), \xi_n) \leq \epsilon$ for every $n \geq N$.

Hereafter we say that a sequence $(\xi_n)_{n\in\mathbb{N}_0}$ in X is through some subset $U \subset X$ if $\xi_0 \in U$. If U reduces to a singleton $\{x\}$ we just say that the sequence is through x. We say that $x \in X$ is a forward shadowable point (or simply a *shadowable point*, [10,12]) if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit through x can be ϵ-shadowed. The set of shadowable points is denoted by $Sh^+(f)$. These definitions motivate the following one.

Definition 1.1 We say that $x \in X$ is *eventually shadowable* if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit through x can be eventually ϵ-shadowed. Denote by $ESh(f)$ the set of eventually shadowable points.

Let us present a related example.

Example Clearly $Sh^+(f) \subset ESh(f)$ but it may happen that $Sh^+(f) \neq ESh(f)$. For instance, $Sh^+(f) = ESh(f)$ in the case when f is a surjective equicontinuos map (by Theorem 1.6 below) and $Sh^+(f) \neq ESh(f)$ if f is the map in Example 5 p. 146 of [9].

We say that $x \in X$ is *nonwandering* if for every $\epsilon > 0$ there are $z \in X$ and $k \geq 1$ such that such that $d(z, x) < \epsilon$ and $d(f^k(z), x) < \epsilon$. Denote by $\Omega(f)$ the set of nonwandering points. We say that $A \subset X$ is an invariant set if $f(A) = A$.

Given $\delta > 0$, a *δ-chain* is a finite sequence $x_0, x_1, \ldots, x_k \in X$ with $d(f(x_i), x_{i+1}) < \delta$ for every $0 \leq i \leq k-1$. If $x_0 = x$ and $x_k = y$ for some $x, y \in X$, we say that the δ-chain is from x to y and $x \in X$ is *chain recurrent* if for every $\delta > 0$ there is a δ-chain from x to itself. Denote by $CR(f)$ the chain recurrent set i.e. the set of chain recurrent points. It is clear that $\Omega(f) \subset CR(f)$ and the converse may be false.

Define the *omega-limit set* of $x \in X$ by

$$\omega(x) = \left\{ y \in X : y = \lim_{i\to\infty} f^{n_i}(x) \text{ for some subsequence } n_i \to \infty \right\}.$$

We say that $A \subset X$ is *transitive* or a *limit cycle* if there is $x \in A$ (resp. $x \in X$) such that $A = \omega(x)$. If X is a transitive set (or a limit cycle) we say that f is a transitive map.

Example Recall that a subset A is *internally chain transitive* if for every $x, y \in A$ and $\delta >$ there is a δ-chain in A from x to y. Every internal chain transitive set with eventually shadowable points can be accumulated by limit cycles with respect to the Hausdorff metric of compact subsets of X. This can be seen as a pointwise version of Theorem 4 in [9] (with similar proof).

We say that $(x, y) \in X \times X$ is a Li–Yorke pair (c.f. [2]) whenever $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ and $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$. Denote $LY(x) = \{y \in X : (x, y)$ is a Li–Yorke pair of $f\}$ the *associated Li–Yorke set* of x. f is called *Li–Yorke chaotic* if there exists an uncountable subset S of X such that any two distinct points $x, y \in S$ form a Li–Yorke pair of f (in such a case S is called a Li–Yorke chaotic set). f is called *completely scrambled* if X is a Li–Yorke chaotic set. For the most recent development about completely scrambled systems including examples see [3] and references therein.

Let $x, y \in X$, x and y are called proximal if $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$. Denote by $P(x) = \{y \in X : x, y\ are\ proximal\}$ the proximal cell of x [1]. f is called proximal if $P(x) = X$ for each $x \in X$. We say that f is *equicontinuous* if for every $\epsilon > 0$ there is $\delta > 0$ such that if $x, y \in X$ and $d(x, y) < \delta$, then $d(f^n(x), f^n(y)) < \epsilon$ for every $n \in \mathbb{N}_0$.

With these definitions we can present our results.

Theorem 1.2 *If $f : X \to X$ is a continuous map of a compact metric space, then:*

1. *$ESh(f)$ is a forward invariant set (empty or non-empty possibly noncompact). If in addition f is surjective, then $ESh(f)$ is invariant;*
2. *f has the eventual shadowing property if and only if $ESh(f) = X$;*
3. *if $CR(f) \subset ESh(f)$, then $CR(f) = \Omega(f)$. If in addition f is surjective, then f has the eventual shadowing property.*

Theorem 1.3 *A surjective continuous map of a compact metric space has the eventual shadowing property if and only if the set of eventually shadowable points has full measure with respect to every ergodic invariant probability measure.*

Theorem 1.4 *A proximal or transitive surjective map with eventually shadowable points of a compact metric space has the eventual shadowing property.*

Theorem 1.5 *If there is an eventually shadowable point for which the associated Li–Yorke set equals to the whole space, then the map has the eventual shadowing property. In particular, a completely scramble continuous map of a compact uncountable metric space has either the eventual shadowing property or no eventually shadowable points.*

Theorem 1.6 *If $f : X \to X$ is a surjective equicontinuous map of a compact metric space, then $ESh(f) = Sh^+(f)$.*

Corollary 1.7 *A surjective equicontinuous map of a compact metric space has the eventual shadowing property if and only if it has the shadowing property.*

We finish with an example related to this corollary.

Example In Example 6 p. 147 in [9] it was proved that the *irrational* circle rotations does not have the eventual shadowing property. But indeed *there is no circle rotation with the eventual shadowing property*. This follows from Corollary 1.7 since all such maps are surjective equicontinuous without the shadowing property [12].

Remark 1.8 Indeed, Corollary 1.7 is a consequence of Theorem 4.5 in [13] and the known fact that all surjective equicontinuous map are chain recurrent. On the other hand, Item (3) of Theorem 1.2 implies the identity $CR(f) = \Omega(f)$ for eventual shadowing maps $f : X \to X$ proved in the aforementioned Theorem 4.5 of [13].

This paper is organized as follows. In Sect. 2 we give some preliminary lemmas. In Sect. 3 we prove the above theorems.

The authors would like to thank the anonymous referee for her/his invaluable remarks to improve the proofs in this work. They also thank the Chungnam National University in Daejeon, Republic of Korea, for its kindly hospitality during the preparation of this work.

2 Preliminary Lemmas

Denote by $B[x, \delta]$ the closed δ-ball of a metric space X centered at x.

Lemma 2.1 *Let $f : X \to X$ be a continuous map of a compact metric space. Then, $x \in ESh(f)$ if and only if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit $(\xi_n)_{n\in\mathbb{N}_0}$ through $B[x, \delta]$ can be eventually ϵ-shadowed.*

Proof We only have to prove the necessity. Suppose by contradiction that $x \in ESh(f)$ but there are $\epsilon > 0$ and a sequence of $\frac{1}{k}$-pseudo-orbits $\xi^k = (\xi^k_n)_{n\in\mathbb{N}_0}$ with $d(x, \xi^k_0) \leq \frac{1}{k}$ such that ξ^k cannot be eventually ϵ-shadowed for every $k \in \mathbb{N}$. For this ϵ we let δ be given by the fact that x is eventually shadowable. Since X is compact and f continuous, f is uniformly continuous and so there is a k sufficiently large so that

$$\max\left\{d(f(x), f(\xi^k_0)), \frac{1}{k}\right\} \leq \frac{\delta}{2}.$$

Define the sequence $\hat{\xi} = (\hat{\xi_n})_{n\in\mathbb{N}_0}$ by

$$\hat{\xi_n} = \begin{cases} \xi^k_n & \text{if } n \neq 0 \\ x & \text{if } n = 0. \end{cases}$$

Since $d(f(\hat{\xi_n}), \hat{\xi}_{n+1}) = d(f(\xi^k_n), \xi^k_{n+1})$ for $n \neq 0$ and, for $n = 0$,

$$d(f(\hat{\xi}_0), \hat{\xi}_1) = d(f(x)), \xi^k_1) \leq d(f(x), f(\xi^k_0)) + d(f(\xi^k_0), \xi^k_1) \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

So, $\hat{\xi}$ is a δ-pseudo-orbit through x. Then, $\hat{\xi}$ can be eventually ϵ-shadowed i.e. there are $y \in X$ and $N \in \mathbb{N}$ such that $d(f^n(y), \hat{\xi}_n) \leq \epsilon$ for every $n \geq N$. It follows that $d(f^n(y), \xi^k_n) = d(f^n(y), \hat{\xi}_n) \leq \epsilon$ for $n \geq N$. Hence ξ^k can be eventually ϵ-shadowed which is absurd. This completes the proof. □

Lemma 2.2 *A continuous map of a compact metric space $f : X \to X$ has the eventual shadowing property if and only if $ESh(f) = X$.*

Proof We only have to prove the sufficiency. Then, suppose that $ESh(f) = X$ and choose $\epsilon > 0$. By Lemma 2.1 for every $x \in X$ there is $\delta_x > 0$ such that every δ_x-pseudo-orbit through the ball $B[x, \delta_x]$ can be eventually ϵ-shadowed. Since X is compact, we can cover X with finitely many of such balls namely

$$X = \bigcup_{i=1}^{l} B[x_i, \delta_{x_i}].$$

Take $\delta = \min\{\delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_l}\}$ and let $(\xi_n)_{n\in\mathbb{N}_0}$ be a δ-pseudo-orbit. Clearly, $\xi_0 \in B[x_i, \delta_{x_i}]$ for some $1 \le i \le l$. So, $\{\xi_n\}_{n\in\mathbb{N}}$ is a δ-pseudo-orbit through $B[x_i, \delta_{x_i}]$. This implies that $(\xi_n)_{n\in\mathbb{N}_0}$ is a δ_{x_i}-pseudo-orbit through $B[x_i, \delta_{x_i}]$. Then, $(\xi_n)_{n\in\mathbb{N}_0}$ can be eventually ϵ-shadowed proving the result. □

Lemma 2.3 *If f is a continuous map of a compact metric space X, then $\omega(x) \subset ESh(f)$ for every $x \in ESh(f)$.*

Proof Fix $x \in ESh(f)$, $y \in \omega(x)$ and $\epsilon > 0$. For this ϵ we let $\delta > 0$ and N be given by the fact that x is eventually shadowable. Since $y \in \omega(x)$, there exists $k \in \mathbb{N}$ such that $f^k(x) \in B(y, \delta)$. Let $(y_i)_{n\in\mathbb{N}_0}$ be a δ-pseudo orbit with $y_0 = y$. Define the sequence $(z_i)_{n\in\mathbb{N}_0}$ by $z_i = f^i(x)$ when $i = 0, 1, \ldots, k-1$ and $z_i = y_{i-k}$ when $i \ge k$. Since $d(f(z_{k-1}), z_k) = d(f^k(x), y) < \delta$, we have that $(z_i)_{n\in\mathbb{N}_0}$ is a δ-pseudo orbit with $z_0 = x$. Then, there is $z \in X$ and N such that $d(f^i(z), z_i) \le \epsilon$ for $i \ge N$. Suppose $i \ge \max(0, N-k)$. Then, $i + k \ge \max(k, N)$ so $i + k \ge k$ thus $z_{i+k} = y_i$. It follows that

$$d(f^i(f^k(z)), y_i) = d(f^{i+k}(z), y_i) = d(f^{i+k}(z), z_{i+k}) \overset{(i+k\ge N)}{\le} \epsilon,$$

so $(y_i)_{n\in\mathbb{N}_0}$ can be eventually ε-shadowed. Hence $y \in ESh(f)$ proving the result. □

The orbit of $x \in X$ under $f : X \to X$ is defined by $O_f(x) = \{f^n(x) : n \in \mathbb{N}\}$.

Lemma 2.4 *Suppose that $f : X \to X$ is a surjective continuous map. Given an $x \in X$ if $\overline{O_f(x)} \cap ESh(f) \ne \emptyset$, then $x \in ESh(f)$.*

Proof Fix $\epsilon > 0$. By Lemma 2.1 for $y \in \overline{O_f(x)} \cap ESh(f)$ there exists $\delta > 0$ such that for every δ-pseudo-orbit through $B[y, \delta]$ can be eventually ϵ-shadowed. Since $y \in \overline{O_f(x)}$ there is $M \in \mathbb{N}$ such that $d(f^M(x), y) \le \frac{\delta}{2}$. For this M, fix $0 < \delta' < \delta$ such that for every δ'-pseudo-orbit $(\xi_n)_{n\in\mathbb{N}_0}$ through x satisfying $d(f^M(x), \xi_M) < \frac{\delta}{2}$. Then, we take δ'-pseudo-orbit through x and define $\hat{\xi} = (\hat{\xi}_n)_{n\in\mathbb{N}_0}$ by $\hat{\xi_n} = \xi_{n+M}$ for all $n \in \mathbb{N}$. Then, $\hat{\xi}$ is a δ-pseudo-orbit with $\hat{\xi_0} = \xi_M$ and so through $B[y, \delta]$.

By assumption, we can choose $z' \in X$ and $N' \in \mathbb{N}$ with $d(f^{n'}(z'), \hat{\xi}_{n'}) \le \epsilon$ for all $n' \ge N'$. Since f is surjective, there exists $z \in X$ such that $f^M(z) = z'$. Now, we take $N = N' + M$ and fix $n \ge N$. This implies that for $n - M \ge N'$ and $d(f^{n'}(z'), \hat{\xi}_{n'}) = d(f^{n'}(z'), \xi_{n'+M}) \le \epsilon$ if we choose for $n' = n - M$. Also

$$f^n(z) = f^{n-M}(f^M(z)) = f^{n-M}(z') = f^{n'}(z')$$

Hence, we can choose $N = N' + M$ such that

$$d(f^n(z), \xi_n) = d(f^{n'}(z'), \xi_{n'+M}) \le \epsilon, \qquad \forall n \ge N.$$

This ends the proof. □

Remark 2.5 Lemma 2.4 is false for forward shadowable points (instead of eventually shadowable points). This was pointed out in [10].

Lemma 2.6 *If $f : X \to X$ is a continuous map of a compact metric space, then $ESh(f)$ is a forward invariant set of f. If in addition f is surjective, then $ESh(f)$ is an invariant set.*

Proof Take $x \in ESh(f)$ and $\epsilon > 0$. For this ϵ we let $\delta > 0$ be given by the fact that x is eventually shadowable. Let $\xi = (\xi_n)_{n\in\mathbb{N}_0}$ be a δ-pseudo orbit through $f(x)$. Define $\hat{\xi} = (\hat{\xi}_n)_{n\in\mathbb{N}_0}$ by

$$\hat{\xi}_n = \begin{cases} x & \text{if } n = 0 \\ \xi_{n-1} & \text{if } n \ge 1. \end{cases}$$

Since $d(f(\hat{\xi}_0), \hat{\xi}_1) = d(f(x), f(x)) = 0 < \delta$ and, in addition, $d(f(\hat{\xi}_n), \hat{\xi}_{n+1}) = d(f(\xi_{n-1}), \xi_n) < \delta$ for $n \ge 1$, we have that $\hat{\xi}$ is a δ-pseudo orbit through $\hat{\xi}_0 = x$. Then, there are $y \in X$ and $N \ge 0$ such that $d(f^n(y), \hat{\xi}_n) < \epsilon$ for every $n \ge N$. If $N = 0$, then $d(f^n(f(y)), \xi_n) = d(f^{n+1}(y), \hat{\xi}_{n+1}) < \epsilon$ for every $n \ge 0$ and if $N \ge 1$, $d(f^n(f(y)), \xi_n) = d(f^{n+1}(y), \hat{\xi}_{n+1}) < \epsilon$ for $n \ge N-1$. Therefore, $f(x) \in ESh(f)$ proving $f(ESh(f)) \subset ESh(f)$. In particular,

$$ESh(f) \subset f^{-1}(ESh(f)). \tag{2.1}$$

Now assume that f is surjective. If $x \in ESh(f)$ since f is surjective, $x = f(x')$ for some $x' \in X$. Then, $x \in \overline{O_f(x')} \cap ESh(f)$ so $\overline{O_f(x')} \cap ESh(f) \neq \emptyset$ thus $x' \in ESh(f)$ by Lemma 2.4. Therefore, $ESh(f) \subset f(ESh(f))$ and so $f^{-1}(ESh(f)) \subset ESh(f)$. This combined with (2.1) shows that $ESh(f)$ is invariant. □

A point $x \in X$ is *minimal* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\{r \in \mathbb{N} : d(f^r(x), x) < \epsilon\} \cap \{k, k+1, \ldots, k+N-1\} \neq \emptyset, \qquad \forall k \in \mathbb{N}.$$

Denote by $M(f)$ the set of all minimal points of f.

Lemma 2.7 *A continuous map of a compact metric space $f : X \to X$ has the eventual shadowing property if and only if $M(f) \subset ESh(f)$.*

Proof We just need to prove the sufficiency. Suppose $M(f) \subset ESh(f)$. As is known (see Theorem 3, p.67 in [1]) for every point $y \in X$ there is $x \in M(f)$ such that $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$. So, there exists a sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers with $\lim_{k\to\infty} d(f^{n_k}(x), f^{n_k}(y)) = 0$. Since X is compact, without loss of generality, we assume that $\lim_{k\to\infty} f^{n_k}(x) = z$(otherwise, take a subsequence). By Lemma 2.3 we have $z \in ESh(f)$ and $\lim_{k\to\infty} f^{n_k}(y) = z$ which implies $z \in \omega(y)$. Thus $\overline{O_f(y)} \cap ESh(f) \neq \emptyset$, we obtain by Lemma 2.4 that $y \in ESh(f)$. Thus $ESh(f) = X$, which together with Lemma 2.2 implies that f has the eventual shadowing property. This ends the proof. □

Lemma 2.8 *If $f : X \to X$ is a continuous map of a compact metric space, then $CR(f) \cap ESh(f) \subset \Omega(f)$.*

Proof Take $x \in CR(f) \cap ESh(f)$ and $\epsilon > 0$. For this ϵ let δ be given by the fact that x is eventually shadowable. As $x \in CR(f)$, there is a δ-chain $x_0, \ldots, x_k$ from x to itself. Define $\xi = (\xi_n)_{n\in\mathbb{N}_0}$ by $\xi_{pk+i} = x_i$ for $p \in \mathbb{N}_0$ and $0 \le i \le k-1$. Then, ξ is a δ-pseudo orbit through x and so there are $y \in X$ and $N \in \mathbb{N}_0$ such that $d(f^n(y), \xi_n) < \epsilon$ for $n \ge N$. Write $N = p_N k + i_N$ and define $z = f^{N+k-i_N}(y)$. It follows that

$$\xi_{N+k-i_n} = \xi_{p_N k+i_N+k-i_N} = \xi_{(p_N+1)k} = x$$

and similarly $\xi_{N+2k-i_n} = x$. Then,

$$d(z, x) = d(f^{N+k-i_N}(y), \xi_{N+k-i_N}) < \epsilon$$

and

$$d(f^k(z), x) = d(f^{N+2k-i_N}(y), \xi_{N+2k-i_N}) < \epsilon.$$

As ϵ is arbitrary, $x \in \Omega(f)$. □

Lemma 2.9 *If $f : X \to X$ is a continuous map of a compact metric space, then $ESh(h \circ f \circ h^{-1}) = h(ESh(f))$ for every homeomorphism of metric spaces $h : X \to Y$.*

Proof Let $y \in h(ESh(f))$ and $\epsilon > 0$. We have $y = h(x)$ for some $x \in ESh(f)$. Take $\epsilon' > 0$ such that if $a, b \in X$ and $d(a, b) < \epsilon'$ then $d(h(a), h(b)) < \epsilon$. For this ϵ' we take $\delta' > 0$ from the fact that x is eventually shadowable with respect to f. For this δ' we let $\delta > 0$ be such that if $c, d \in Y$ and $d(c, d) < \delta$ then $d(h^{-1}(c), h^{-1}(d)) < \delta'$.

Now let $\xi = (\xi_n)_{n\in\mathbb{N}_0}$ be a δ-pseudo orbit of $h \circ f \circ h^{-1}$ through y. Then, $d(h(f(h^{-1}(\xi_n), \xi_{n+1}) < \delta$ so $d(f(h^{-1}(\xi_n), h^{-1}(\xi_{n+1}) < \delta'$ thus $(h^{-1}(\xi_n))_{n\in\mathbb{N}_0}$ is a δ'-pseudo orbit of f through $h^{-1}(\xi_0) = h^{-1}(y) = x$. It follows that there are $z \in X$ and $N \in \mathbb{N}_0$ such that $d((f^n(z), h^{-1}(\xi_n)) < \epsilon'$ for $n \ge N$. Then, $d(h \circ f \circ h^{-1})^n(h(z)), \xi_n) < \epsilon$ for $n \ge N$ proving $y \in ESh(h \circ f \circ h^{-1})$. We conclude that $h(ESh(f)) \subset ESh(h \circ f \circ h^{-1})$. So,

$$h(ESh(f)) = h(ESh(h^{-1} \circ (h \circ f \circ h^{-1}) \circ h)) \subset ESh(h \circ f \circ h^{-1})$$

proving the result. □

Given a metric space X we denote by X^{deg} the set of points $x \in X$ for which the connected component of X containing x is $\{x\}$. Recall also that a surjective map $i : X \to X$ is an isometry if $d(i(x), i(y)) = d(x, y)$ for every $x, y \in X$.

Lemma 2.10 *If $f : X \to X$ is an isometry of a compact metric space, then $ESh(f) \subset X^{deg}$.*

Proof Otherwise, choose $x \in ESh(f) \setminus X^{deg}$. Since $x \notin X^{deg}$, the connected component F of X containing x has positive diameter $diam(F)$. Fix any $k \in \mathbb{N}$ and let $0 < \delta < \frac{diam(F)}{4}$ be given by the fact that x is eventual shadowable with respect to f for $\epsilon = \frac{diam(F)}{4}$.

Since F is compact and connected, there is a sequence $q_0, \ldots, q_r \in F$ such that $q_0 = q_r = x$, $d(q_i, q_{i+1}) < \frac{\delta}{2}$ for $0 \leq i \leq r-1$ and $F \subset \bigcup_{i=0}^{r} B[q_i, \delta]$.

Given $k \in \mathbb{N}_0$ and $0 \leq i \leq r-1$, we define $q_{kr+i} = q_i$. This way we obtain a sequence $q_1, q_2, \ldots, q_i, \cdots \in F$ such that $d(q_i, q_{i+1}) < \frac{\delta}{2}$ for $i \geq 0$ and

$$F \subset \bigcup_{i=N}^{\infty} B[q_i, \delta], \qquad \forall N \in \mathbb{N}_0. \tag{2.2}$$

Define the sequence $\xi = (\xi_i)_{i \in \mathbb{N}_0}$ by $\xi_i = f^i(q_i)$ for $i \geq 0$. Since f is an isometry, $d(f(\xi_i), \xi_{i+1}) = d(f^{i+1}(q_i), f^{i+1}(q_i)) = d(q_i, q_{i+1}) < \delta$ for every $i \geq 0$. Hence ξ is a δ-pseudo orbit through $\xi_0 = q_0 = x$ so there are $y \in X$ and $N \geq 0$ such that $d(f^i(y), \xi_i) < \frac{diam(F)}{4}$ for $i \geq N$. Since f is an isometry, $d(y, f^{-i}(\xi_i)) < \frac{diam(F)}{4}$ and so $d(y, q_i) < \frac{diam(F)}{4}$ for all $i \geq N$. Take $z, w \in F$. By (2.2) there are $i, j \geq N$ such that $d(z, q_i) < \delta$ and $d(w, q_j) \leq \delta$. Then,

$$\begin{aligned} d(z, w) &\leq d(z, q_i) + d(y, q_i) + d(y, q_j) + d(w, q_j) \\ &< \frac{diam(F)}{2} + \frac{diam(F)}{2} \\ &= diam(F). \end{aligned}$$

Since F is compact, this a contradiction which proves the lemma. □

3 Proof of the Theorems

Proof of Theorem 1.2 Items (1) and (2) follow from Lemmas 2.6 and 2.2 respectively. To prove Item (3) suppose that $CR(f) \subset ESh(f)$. Since $\Omega(f) \subset CR(f)$ we have $\Omega(f) \subset ESh(f)$. It follows from Lemma 2.8 that

$$\Omega(f) = \Omega(f) \cap ESh(f) \subset CR(f) \cap ESh(f) \subset \Omega(f)$$

hence $CR(f) \cap ESh(f) = \Omega(f)$ and then $CR(f) = CR(f) \cap ESh(f) = \Omega(f)$. Finally, suppose that f is surjective. Since $\Omega(f) \subset ESh(f)$ and $\overline{O_f(x)} \cap \Omega(f) \neq \emptyset$ for every $x \in X$, one has $\overline{O_f(x)} \cap ESh(f) \neq \emptyset$ for every $x \in X$. Since f is surjective, Lemma 2.4 implies $ESh(f) = X$ so f has the eventual shadowing property by Item (2). □

Proof of Theorem 1.3 If f has the eventual shadowing property, $ESh(f) = X$ and so $ESh(f)$ has full measure with respect to every Borel probability measure and, in particular, with respect to the ergodic invariant ones. Conversely, assume that $ESh(f)$ has full measure with respect to every ergodic invariant measure. Take $x \in M(f)$. By definition $\overline{O_f(x)}$ is minimal so there is an ergodic invariant measure μ supported on $\overline{O_f(x)}$. Since μ is ergodic and invariant, we have $\mu(ESh(f)) = 1$ by hypothesis. If $\overline{O_f(x)} \cap ESh(f) = \emptyset$, then $\mu(\overline{O_f(x)} \cup ESh(f)) = 2$ which is absurd. This implies $\overline{O_f(x)} \cap ESh(f) \neq \emptyset$ hence $x \in ESh(f)$ by Lemma 2.4. Therefore, $M(f) \subset ESh(f)$ and so f has the eventual shadowing property by Lemma 2.7. □

Proof of Theorem 1.4 Let $f : X \to X$ be a continuous map with $ESh(f) \neq \emptyset$. First suppose that f is proximal. If $x \in X$, then $P(x) = X$. Since $ESh(f) \neq \emptyset$, there is an eventually shadowable point $y \in P(x)$. Then, there is an integer sequence $n_i \to \infty$ such that $\lim_{i\to\infty} d(f^{n_i}(x), f^{n_i}(y)) \to 0$ as $i \to \infty$. Since X is compact, we can assume that $f^{n_i}(x) \to z$ for some $z \in X$. Then, $f^{n_i}(y) \to z$ and so $z \in \omega(y) \cap \omega(x)$. Since $y \in ESh(f)$, $z \in ESh(f)$ by Lemma 2.3. It follows that $\omega(x) \cap ESh(f)$ and so $x \in ESh(f)$ by Lemma 2.4. This proves $ESh(f) = X$ so f has the eventual shadowing property by Lemma 2.2.

Now suppose that f is transitive. Then, there is a point x such that $\overline{O_f(x)} = \omega(x) = X$. Since $ESh(f) \neq \emptyset$, $\omega(x) \cap ESh(f) \neq \emptyset$ so $\overline{O_f(x)} \cap ESh(f)$ thus $x \in ESh(f)$ by Lemma 2.4. It follows that $\omega(x) \subset ESh(f)$ by Lemma 2.3 and then $ESh(f) = X$. Consequently f has the eventual shadowing property by Lemma 2.2. □

Proof of Theorem 1.5 Let $x \in ESh(f)$ with $LY(x) = X$. This implies $P(x) = X$ and so by the previous proof we have $P(x) \subset ESh(f)$, so $ESh(f) = X$. It follows from Lemma 2.2 that f has the shadowing property. This ends the proof. □

Proof of Theorem 1.6 Let $f : X \to X$ be a surjective equicontinuous map of a compact metric space. It is known that $Sh^+(f) = X^{deg}$ (e.g. [12]). Define $Y = X$ with the metric $\rho(x, y) = \sup_{n\in\mathbb{N}_0} d(f^n(x), f^n(y))$, for $x, y \in X$. Clearly $d \leq \rho$ and since f is equicontinuous, it follows that for every $\epsilon > 0$ there is $\delta > 0$ such that $d(x, y) < \delta$ implies $\rho(x, y) < \epsilon$. Therefore, the identity $h : X \to Y$ is a homeomorphism. In particular, Y is a compact metric space. Now define $g : Y \to Y$ by $g(x) = f(x)$. Since h is the identity, we have $g = h \circ f \circ h^{-1}$. Since

$$\rho(g(x), g(y)) = \sup_{n\geq 1} d(f^n(x), f^n(y)) \leq \sup_{n\in\mathbb{N}_0} d(f^n(x), f^n(y)) = \rho(x, y)$$

for every $x, y \in Y$, Y is compact and $g : Y \to Y$ is surjective, we have that $g : Y \to Y$ is an isometry (e.g. [7]). It follows that $ESh(g) \subset Y^{deg}$ by Lemma 2.10. But $g = h \circ f \circ h^{-1}$ so $ESh(g) = h(ESh(f)) = ESh(f)$ by Lemma 2.9 thus $ESh(f) \subset Y^{deg}$. As h is an homeomorphism, $Y^{deg} = X^{deg}$ therefore $ESh(f) \subset X^{deg}$. As $X^{deg} = Sh^+(f)$ (see [12]), we get $ESh(f) \subset Sh^+(f)$ proving the result. □

Acknowledgements Funding to CAM was provided by National Research Foundation of Korea (Grant No. BP No 2018H1D3A2001632) and Conselho Nacional de Desenvolvimento Científico e Tecnológico (Grant No. 303389/2015-0).

References

1. Auslander, J.: Minimal flows and their extensions. North-Holland Mathematics Studies, vol. 153. North-Holland, Amsterdam (1988)
2. Blanchard, F., Glasner, E., Kolyada, S., Maass, A.: On Li–Yorke pairs. J. Reine Angew. Math. **547**, 51–68 (2002)
3. Boroński, J.P., Kupka, J., Oprocha, P.: A mixing completely scrambled system exists. Ergod. Theory Dyn. Syst. **39**(1), 62–73 (2019)
4. Bowen, R.: ω-limit sets for axiom A diffeomorphisms. J. Differ. Equ. **18**(2), 333–339 (1975)
5. Brian, W.R., Meddaugh, J., Raines, B.E.: Chain transitivity and variations of the shadowing property. Ergod. Theory Dyn. Syst. **35**(7), 2044–2052 (2015)
6. Dastjerdi, D.A., Hosseini, M.: Sub-shadowings. Nonlinear Anal. **72**(9–10), 3759–3766 (2010)
7. Edrei, A.: On mappings which do not increase small distances. Proc. Lond. Math. Soc. **2**(3), 272–278 (1952)
8. Fakhari, A., Ghane, F.H.: On shadowing: ordinary and ergodic. J. Math. Anal. Appl. **364**(1), 151–155 (2010)
9. Good, C., Meddaugh, J.: Orbital shadowing, internal chain transitivity and ω-limit sets. Ergod. Theory Dyn. Syst. **38**(1), 143–154 (2018)
10. Kawaguchi, N.: Quantitative shadowable points. Dyn. Syst. **32**, 504–518 (2017)
11. Li, M.-J.: Pointwise pseudo-orbit tracing property and its application. J. Math. Res. Expos. **25**(1), 23–30 (2005)
12. Morales, C.A.: Shadowable points. Dyn. Syst. **31**, 347–356 (2016)
13. Oprocha, P.: Shadowing, thick sets and the Ramsey property. Ergod. Theory Dyn. Syst. **36**(5), 1582–1595 (2016)
14. Pilyugin, S.-Yu.: Shadowing in Dynamical Systems. Lecture Notes in Mathematics, 1706. Springer, Berlin (1999)
15. Wang, Q., Yang, R.S.: Pointwise pseudo-orbit tracing property and chaos. J. Math. Res. Expos. **28**(2), 413–420 (2008)

Affiliations

Meihua Dong[1] · Woochul Jung[2] · Carlos Morales[3]

Meihua Dong
meihua0511@163.com

Woochul Jung
wcjung@cnu.ac.kr

1 Department of Mathematics, College of Science, Yanbian University, No. 977, Gongyuan Road, Yanji City 133002, Jilin Province, China

2 Department of Mathematics, Chungnam National University, Daejeon 34134, Republic of Korea

3 Instituto de Matemática, Universidade Federal do Rio de Janeiro, P.O. Box 68530, Rio de Janeiro 21945-970, Brazil