



# The Ball-Based Origami Theorem and a Glimpse of Holography for Traversing Flows

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## Abstract

This paper describes a mechanism by which a transversally generic flow  $v$  on a smooth connected  $(n + 1)$ -dimensional manifold  $X$  with boundary produces a compact  $n$ -dimensional  $CW$ -complex  $\mathcal{T}(v)$ , which is homotopy equivalent to  $X$  and such that  $X$  embeds in  $\mathcal{T}(v) \times \mathbb{R}$ . The  $CW$ -complex  $\mathcal{T}(v)$  captures some residual information about the smooth structure on  $X$  (such as the stable tangent bundle of  $X$ ). Moreover,  $\mathcal{T}(v)$  is obtained from a simplicial *origami map*  $O : D^n \rightarrow \mathcal{T}(v)$ , whose source space is a ball  $D^n \subset \partial X$ . The fibers of  $O$  have the cardinality  $(n + 1)$  at most. The knowledge of the map  $O$ , together with the restriction to  $D^n$  of a Lyapunov function  $f : X \rightarrow \mathbb{R}$  for  $v$ , make it possible to reconstruct the topological type of the pair  $(X, \mathcal{F}(v))$ , where  $\mathcal{F}(v)$  is the 1-foliation, generated by  $v$ . This fact motivates the use of the word “holography” in the title. In a qualitative formulation of the holography principle, for a massive class of ODE’s on a given compact manifold  $X$ , the solutions of the appropriately staged boundary value problems are topologically rigid.

**Keywords** Manifolds with boundary · Traversing flows · Trajectory spaces · Boundary value problems · Holography

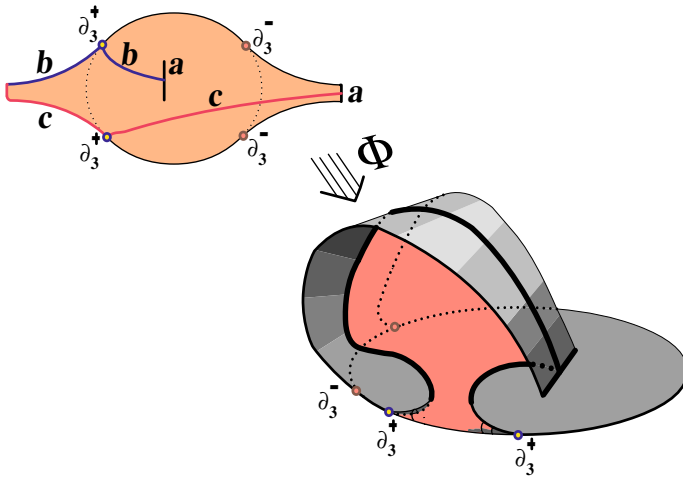
## 1 Introduction

The results of this paper are relatively direct implications of our previous study in [9,10] of boundary generic and transversally generic vector flows on manifolds with boundary. The present results support one general *holography principle* which animates our recent research; it may be vaguely phrased as follows: for an open subspace in the space of all ODE’s on a given compact connected smooth manifold  $X$  with boundary, the solutions of the appropriately staged boundary value problems are *topologically rigid* ([12,13]). Crudely, the open subspace consists of vector flows that admit a Lyapunov function. We call such flows *traversing* (see Definition 2.1 below).

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**Fig. 1** An origami map  $O : D^2 \rightarrow K$  of the 2-disk onto a collapsible 2-complex  $K$ . Note the pairs of arcs in  $D^2$ , marked by the labels  $a, b, c$ ; each pair is identified by  $O$  into a single arc, residing in  $K$ . The map  $O$  is a 3-to-1 at most, and his generic fiber is a singleton

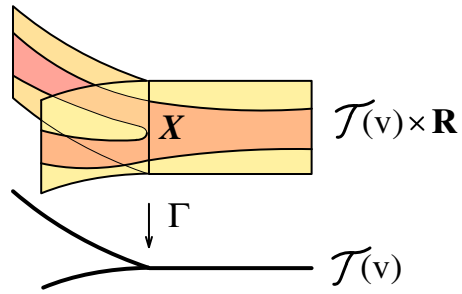
This paper contains two types of claims about the traversing vector flows on compact connected smooth  $(n + 1)$ -dimensional manifolds  $X$  with boundary.

The Origami Theorem 3.1 belongs to the first type. It employs a *traversally generic* (see Definition 2.4) vector field  $v$  to produce a homotopy  $n$ -dimensional model  $\mathcal{T}(v)$  of  $X$ , a model which retains even some surrogate smooth structure of  $X$ . That model  $\mathcal{T}(v)$  is the space of  $v$ -trajectories. So we get an obvious map  $\Gamma : X \rightarrow \mathcal{T}(v)$  which, upon a close examination, turns out to be a homotopy equivalence. Its fibers are closed intervals or singletons. Similar  $n$ -dimensional “shadows” of  $(n + 1)$ -dimensional manifolds, with fibers being graphs<sup>1</sup> of a particular type, have been studied in [7,15]. In Theorem 3.1, we prove that, for an open set of traversing vector fields, the space of trajectories  $\mathcal{T}(v)$  may be obtained as the image of a  $n$ -ball  $D^n$  under a continuous surjection  $O : D^n \rightarrow \mathcal{T}(v)$  with finite fibers of cardinality  $n + 1$  at most. The generic  $O$ -fiber has the cardinality 1. So, speaking informally, we obtain the trajectory space  $\mathcal{T}(v)$  by “folding” the ball  $D^n$ , while identifying some points on its boundary  $\partial D^n$  with other points in  $D^n$ . Thus the word “origami” in the title (see Fig. 1 for an example of such an origami map  $O$  for  $n = 2$ ). Remarkably, not only the homotopy type of  $X$ , but even its stable characteristic classes (like the Pontryagin classes) are encoded in the origami map  $O$ . However, deciphering this information seems to be challenging.

The Origami Theorem 3.1 has another intriguing aspect. It implies that any compact connected  $(n + 1)$ -dimensional smooth manifold  $X$  with boundary admits an *embedding*  $\alpha : X \hookrightarrow \mathcal{T} \times \mathbb{R}$ , where  $\mathcal{T}$  is a compact  $n$ -dimensional  $CW$ -complex, which is homotopy equivalent to  $X$ , and  $\mathbb{R}$  denotes the number line (see Fig. 2). Moreover, the typically *singular* space  $\mathcal{T}$  is the image of the ball  $D^n$  under a continuous map  $O$ ,

<sup>1</sup> i.e., compact 1-dimensional  $CW$ -complexes.

**Fig. 2** The embedding  $\alpha(f, v)$  of  $X$  into the product  $\mathcal{T}(v) \times \mathbb{R}$



whose generic fiber is a singleton. Thus the singularities of the  $X$ -shadow  $\mathcal{T}$ , forced by the topology of  $X$ , admit a simple resolution  $D^n$ .

Given a traversing vector field  $v$  on a compact connected smooth manifold  $X$  with boundary, we denote by  $\mathcal{F}(v)$  the oriented 1-dimensional foliation, generated by  $v$ . Its leaves are the un-parametrized  $v$ -trajectories.

Theorem 4.1 validates the holographic principle, crudely stated above. We are interested in the  $v$ -generated residual data on the boundary  $\partial X$  that will make it possible to reconstruct  $(X, \mathcal{F}(v))$ , up to (1) a homeomorphism, or up to (2) a diffeomorphism of  $X$ . In Theorem 4.1, we present a simple solution of the first problem for any traversing vector field  $v$  on  $X$ . The second problem is much more subtle; it was partially solved in [12] for many *boundary generic* (see Definition 2.2) and traversing vector fields. However, the solution of (2) there is quite involved. So it seems appropriate to present here a simple argument for solving (1).

In Corollary 4.3, we combine the results from the Holography Theorem 4.1 with the Origami Theorem 3.1 to conclude that the origami map  $O : D^n \rightarrow \mathcal{T}(v)$ , together with the restriction  $f_+^\partial : D^n \rightarrow \mathbb{R}^2$  of the Lyapunov function  $f : X \rightarrow \mathbb{R}$  for  $v$ , allows for a reconstruction of the topological type of the pair  $(X, \mathcal{F}(v))$  — another instance of holography at work.

## 2 Trivia About Traversing Flows on Manifolds with Boundary

For the reader convenience, we start with presenting few basic definitions and facts related to the *boundary generic*, *traversing*, and *traversally generic* vector fields on manifolds with boundary.

**Definition 2.1** Let  $X$  be a compact connected smooth  $(n + 1)$ -dimensional manifold with boundary. A vector field  $v$  on  $X$  is called *traversing* if each  $v$ -trajectory is either a closed interval with its both ends residing in  $\partial X$ , or a singleton, also residing in  $\partial X$ .

We denote by  $\mathcal{V}_{\text{trav}}(X)$  the space of all traversing fields on  $X$ . □

The proof of the following key lemma can be found in [9], Lemma 4.1.

**Lemma 2.1** A vector field  $v$  on  $X$  is traversing if and only if it admits a smooth Lyapunov function  $f : X \rightarrow \mathbb{R}$ , such that  $df(v) > 0$  in  $X$ . □

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<sup>2</sup> In fact, we may assume that  $D^n \subset \partial X$ .

In contrast with the vector flows that have fixed points, for a generic traversing vector flow  $v$  on a compact connected manifold  $X$ , the space of its trajectories  $\mathcal{T}(v)$  is not pathological: in fact, it is a compact  $CW$ -complex of dimension  $\dim(X) - 1$ . Such trajectory space  $\mathcal{T}(v)$  is homology equivalent to  $X$  (Theorem 5.1, [11]). Moreover, it also shares with  $X$  all stable characteristic classes of a surrogate “tangent bundle”  $\tau(\mathcal{T}(v))$ .

We consider an important subclass of traversing vector fields which we call *traversally generic* (see Definition 2.4). The traversally generic vector fields form an *open and dense subset* of the space of all traversing vector fields on a given  $X$  ([10]). Crudely, any trajectory of a traversally generic vector field admits a special system of coordinates  $(u, x_0, \dots, x_m)$ , with  $u$  being the flow-aligned coordinate and  $m < \dim(X)$ , so that the boundary  $\partial X$  is given by a polynomial in  $u$  equation of degree  $\leq 2 \cdot \dim(X)$  (see formula (2.4) below). In order to introduce the notion of a traversally generic vector field accurately, we need to describe below some relevant combinatorial structures. This description will take some space.

For a traversally generic vector field  $v$ , the trajectory space  $\mathcal{T}(v)$  is naturally stratified by closed subspaces, labeled by the elements  $\omega$  of an *universal* poset  $\Omega_{\ell}^{\bullet}(n)$ , which depends only on  $\dim(X) = n + 1$ . The elements  $\omega \in \Omega_{\ell}^{\bullet}(n)$  are represented by sequences of natural numbers  $(\omega_1, \dots, \omega_{\ell})$  such that  $\sum_{i=1}^{\ell} (\omega_i - 1) \leq n$  and  $\sum_{i=1}^{\ell} \omega_i \leq 2n + 2$  (see [11], Sect. 2, for the definition of the partial order in  $\Omega_{\ell}^{\bullet}(n)$  and its properties).

Such sequences  $\omega$  correspond to combinatorial patterns that describe the way in which  $v$ -trajectories  $\gamma \subset X$  intersect the boundary  $\partial X$ . For a boundary generic vector field  $v$  (see Definition 2.2), each intersection point  $a \in \gamma \cap \partial X$  acquires a well-defined *multiplicity*  $m(a) \leq n + 1$ , a natural number that reflects *the order of tangency* of  $\gamma$  to  $\partial X$  at the point  $a$  (see Definition 2.3 for the expanded definition of  $m(a)$ ). So the finite  $v$ -ordered set  $\gamma \cap \partial X$ , together with the multiplicities of its points, can be viewed as a *divisor*  $D_{\gamma}$  on the trajectory  $\gamma$ . Then  $\omega$ , associated with  $\gamma$ , is just the ordered sequence of multiplicities  $\{m(a)\}_{a \in \gamma \cap \partial X}$ .

The support of the divisor  $D_{\gamma}$  is: **(1)** either a singleton  $a$ , in which case  $m(a) \equiv 0 \pmod{2}$ , or **(2)** the minimum and maximum points of  $\sup D_{\gamma}$  have *odd* multiplicities, and the rest of the points have *even* multiplicities.

For a boundary generic traversing vector field  $v$  and its trajectory  $\gamma$ , let

$$m(\gamma) =_{\text{def}} \sum_{a \in \gamma \cap \partial_1 X} m(a) \quad \text{and} \quad m'(\gamma) =_{\text{def}} \sum_{a \in \gamma \cap \partial_1 X} (m(a) - 1). \quad (2.1)$$

Similarly, for  $\omega =_{\text{def}} (\omega_1, \omega_2, \dots, \omega_i, \dots)$  we introduce the *norm* and the *reduced norm* of  $\omega$  by the formulas:

$$|\omega| =_{\text{def}} \sum_i \omega_i \quad \text{and} \quad |\omega|' =_{\text{def}} \sum_i (\omega_i - 1). \quad (2.2)$$

Throughout this text, we assume that  $X$  is embedded in a larger smooth manifold  $\hat{X}$ , and the vector field  $v$  is extended to a non-vanishing vector field  $\hat{v}$  in  $\hat{X}$ . We treat the pair  $(\hat{X}, \hat{v})$  as a *germ*, containing  $(X, v)$ .

Let  $\partial_j X =_{\text{def}} \partial_j X(v)$  denote the locus of points  $a \in \partial_1 X$  such that the multiplicity of the  $v$ -trajectory  $\gamma_a$  through  $a$  at  $a$  is greater than or equal to  $j$ . (By definition,  $\partial_1 X(v) = \partial X$ .) This locus  $\partial_j X$  has a description in terms of an auxiliary function  $z : \hat{X} \rightarrow \mathbb{R}$  that satisfies the following three properties:

- 0 is a regular value of  $z$ ,
- $z^{-1}(0) = \partial_1 X$ , and
- $z^{-1}((-\infty, 0]) = X$ . (2.3)

In terms of  $z$ , the locus  $\partial_j X =_{\text{def}} \partial_j X(v)$  is defined by the equations:

$$\{z = 0, \mathcal{L}_v z = 0, \dots, \mathcal{L}_v^{(j-1)} z = 0\},$$

where  $\mathcal{L}_v^{(k)}$  stands for the  $k$ -th iteration of the Lie derivative operator  $\mathcal{L}_v$  in the direction of  $v$  (see [10]).

The pure stratum  $\partial_j X^\circ \subset \partial_j X$  is defined by the additional constraint  $\mathcal{L}_v^{(j)} z \neq 0$ . The locus  $\partial_j X$  is the union of two loci: (1)  $\partial_j^+ X$ , defined by the constraint  $\mathcal{L}_v^{(j)} z \geq 0$ , and (2)  $\partial_j^- X$ , defined by the constraint  $\mathcal{L}_v^{(j)} z \leq 0$ . The two loci,  $\partial_j^+ X$  and  $\partial_j^- X$ , share a common boundary  $\partial_{j+1} X$ .

**Definition 2.2** We say that a vector field  $v$  on a smooth  $(n + 1)$ -dimensional manifold  $X$  with boundary is *boundary generic* if, for for each  $j \in [1, n + 1]$ , the differential  $j$ -form  $dz \wedge \mathcal{L}_v(dz) \wedge \dots \wedge \mathcal{L}_v^{(j-1)}(dz)$  does not vanish along the locus  $\partial_j X(v)$ .

We denote by  $\mathcal{B}^\dagger(X)$  the space of boundary generic vector fields on  $X$ . □

This definition implies that, for boundary generic vector fields  $v$ , all the loci  $\{\partial_j X(v)\}$  are closed smooth manifolds, and all the loci  $\{\partial_j^\pm X(v)\}$  are compact smooth manifolds. This property of the loci may serve well as a working definition of boundary generic vector fields.

The boundary generic vector fields form an open and dense subset  $\mathcal{B}^\dagger(X)$  in the space of all vector fields on  $X$  that do not vanish on the boundary  $\partial X$  [9].

**Definition 2.3** For a boundary generic vector field  $v$  on  $X$ , the *multiplicity*  $m(a)$ , where  $a \in \partial X$ , is the index  $j$  such that  $a \in \partial_j X^\circ(v)$ . □

As we have mentioned, the characteristic property of *traversally generic* vector fields is that their trajectories admit special flow-adjusted coordinate systems, in which the boundary is given by quite special polynomial equations (see formula (2.4)), and the trajectories are parallel to one of the preferred coordinate axis  $u$  (see [10], Lemma 3.4). According to this lemma, for a traversally generic  $v$  on a  $(n + 1)$ -dimensional  $X$ , the vicinity  $U \subset \hat{X}$  of each  $v$ -trajectory  $\gamma$  of the combinatorial type  $\omega$  has a special coordinate system

$$(u, \vec{x}, \vec{y}) : U \rightarrow \mathbb{R} \times \mathbb{R}^{|\omega|'} \times \mathbb{R}^{n-|\omega|'}$$

By Lemma 3.4 and formula (3.17) from [10], in these coordinates, the boundary  $\partial_1 X$  is given by the polynomial equation:

$$\wp(u, \vec{x}) =_{\text{def}} \prod_i \left[ (u - i)^{\omega_i} + \sum_{\ell=0}^{\omega_i-2} x_{i,\ell} (u - i)^\ell \right] = 0 \tag{2.4}$$

of an even degree  $|\omega|$  in  $u$ . Here  $i \in \mathbb{Z}$  runs over the distinct roots of  $\wp(u, \sim)$ , and  $\vec{x} := \{x_{i,\ell}\}_{i,\ell}$ . At the same time,  $X$  is given by the polynomial inequality  $\{\wp(u, \vec{x}) \leq 0\}$ . Each  $v$ -trajectory in  $U$  is produced by freezing all the coordinates  $\vec{x}, \vec{y}$ , while letting  $u$  to be free.

**Definition 2.4** A traversing vector field  $v$  on a compact connected  $(n + 1)$ -manifold  $X$  with boundary is called *traversally generic* if each  $v$ -trajectory admits local coordinates  $(u, \vec{x}, \vec{y})$  in which  $\partial X$  is given by the equation (2.4) and all the nearby trajectories by the equations  $\{\wp(u, \vec{x}) \leq 0, \vec{x} = \text{const}, \vec{y} = \text{const}'\}$ .

We denote by  $\mathcal{V}^\ddagger(X)$  the space of traversally generic vector fields on  $X$ . ◊

It turns out that any traversally generic vector field is traversing and boundary generic ([10]); in other words,

$$\mathcal{V}^\ddagger(X) \subset \mathcal{B}^\dagger(X) \cap \mathcal{V}_{\text{trav}}(X).$$

In fact,  $\mathcal{V}^\ddagger(X)$  is an *open and dense* (in the  $C^\infty$ -topology) subspace  $\mathcal{V}_{\text{trav}}(X)$  of traversing vector fields (see [10], Theorem 3.5).

We denote by  $X(v, \omega)$  the union of  $v$ -trajectories whose divisors are of a given combinatorial type  $\omega \in \Omega_{[n]}^\bullet$ . Its closure  $\bigcup_{\omega' \leq_\bullet \omega} X(v, \omega')$  is denoted by  $X(v, \omega_{\geq_\bullet})$ .

Let  $\mathcal{T}(v, \omega) \subset \mathcal{T}(v)$  denote the space of trajectories of the combinatorial type  $\omega \in \Omega_{[n]}^\bullet$ . Each pure stratum  $\mathcal{T}(v, \omega)$  is an open smooth manifold and, as such, has a “conventional” tangent bundle (see [12], Lemma 2.1).

### 3 How Traversally Generic Flows Generate the Origami Homotopy Models of Manifolds with Boundary

Let  $X$  be an  $(n + 1)$ -dimensional compact connected smooth manifold with boundary, carrying a traversally generic vector field  $v$  (by [10], such a vector field is always available). Abusing notations, we use the same symbol “ $\gamma$ ” for the  $v$ -trajectory in  $X$  and for the point in the trajectory space  $\mathcal{T}(v)$  it represents.

We introduce a new filtration  $\{\mathcal{T}_{\{max \geq k\}}^+(v)\}_{k \in [1, n+1]}$  of the trajectory space  $\mathcal{T}(v)$  by closed subspaces (actually, by cellular subcomplexes). This stratification is cruder than the stratification  $\{\mathcal{T}(v, \omega_{\geq_\bullet})\}_{\Omega_{[n]}^\bullet}$ . By definition, a trajectory  $\gamma \in \mathcal{T}_{\{max \geq k\}}^+(v)$

if  $\gamma \cap \partial_1^+ X$  contains at least one point  $x$  of multiplicity greater than or equal to  $k$ . Moreover, we insist that this point  $x \in \partial_k^+ X$  (and not in  $\partial_k^- X^\circ$ ). In other words,

$\mathcal{T}_{\{\max \geq k\}}^+(v)$  is exactly the image of  $\partial_k^+ X(v)$  in the trajectory space under the obvious map  $\Gamma : X \rightarrow \mathcal{T}(v)$ . In particular,  $\mathcal{T}_{\{\max \geq 1\}}^+(v) = \mathcal{T}(v)$ .

The key step is based on Corollary 3.3 from [9] and Theorems 3.4 and 3.5 from [10]. They claim that, for a given Lyapunov function  $f : X \rightarrow \mathbb{R}$ , there is a considerable freedom to alter the stratification  $\{\partial_k^+ X(v)\}_k$  by varying a given vector field  $v$  within the space  $\mathcal{V}^\ddagger(X)$  (of transversally generic vector fields) so that  $f$  remains a Lyapunov function for the variation. Specifically, these results imply that there is a nonempty open subset  $\mathcal{D}(X)$  of the space  $\mathcal{V}^\ddagger(X)$  such that, for each field  $v \in \mathcal{D}(X)$ , all the strata  $\{\partial_j^+ X\}_j$  are diffeomorphic to closed balls, except for  $\partial_n^+ X$  (which is a finite union of 1-balls) and for the finite set  $\partial_{n+1}^+ X$ .

So let us consider a model filtration

$$Z^0 \subset Z^1 \subset D^2 \subset D^3 \subset \dots \subset D^{n-1} \subset D^n \tag{3.1}$$

of a closed ball  $D^n$ , such that:

- (1) each ball  $D^j \subset \partial D^{j+1}$ ,
- (2)  $Z^1$  is a disjoint union of finitely many arcs in  $\partial D^2$ ,
- (3)  $Z^0 \subset \partial Z^1$  is a finite set.

It turns out that, for  $v \in \mathcal{D}(X)$ , the trajectory space  $\mathcal{T}(v)$  can be produced by an origami-like folding of the ball  $D^n$  (see Fig. 1 for an example of an origami map on a 2-ball).

The theorem below should be compared with Theorem 1 from [5]. That theorem claims that a closed 3-manifold  $X$  has a *spine* (see [6] for the brief description of spines) which is the image of an immersed 2-sphere in general position in  $X$ . Theorem 3.1 should be compared also with somewhat similar Theorem 5.2 in [8], the latter dealing with the flow-generated spines, not trajectory spaces.

**Theorem 3.1** (Trajectory spaces as the ball-based origami) *For  $n \geq 2$ , any compact connected smooth  $(n + 1)$ -manifold  $X$  with boundary admits a transversally generic vector field  $v$  such that:*

- its trajectory space  $\mathcal{T}(v)$  is the image of a closed ball  $D^n \subset \partial_1 X$  under a continuous cellular map  $\Gamma : D^n \rightarrow \mathcal{T}(v)$ , which is  $(n + 1)$ -to-1 at most. A generic fiber of  $\Gamma$  is a singleton.
- The  $\Gamma$ -image in  $\mathcal{T}(v)$  of each ball  $D^k$  from the filtration (3.1), is the space  $\mathcal{T}_{\{\max \geq n+1-k\}}^+(v)$ , and the restriction  $\Gamma|_{D^k}$  is a  $\lceil \frac{n}{n-k} \rceil$ -to-1 map at most. For  $n > 2$ , the maps

$$\Gamma : Z^1 \rightarrow \mathcal{T}_{\{\max \geq n\}}^+(v) \text{ and } \Gamma : Z^0 \rightarrow \mathcal{T}_{\{\max \geq n+1\}}^+(v)$$

are both bijective.

- The restrictions of  $\Gamma$  to  $\partial D^{k+1} \setminus D^k$  are 1-to-1 maps for all  $1 < k < n$ , and so are the restrictions of  $\Gamma$  to  $\partial D^2 \setminus Z^1$  and to  $\partial Z^1 \setminus Z^0$ .
- The vector fields  $v$ , for which the above properties hold, form an open nonempty set  $\mathcal{D}(X)$  in the space  $\mathcal{V}^\ddagger(X)$ , and thus an open set in the space  $\mathcal{V}_{\text{trav}}(X)$  of all traversing fields.

**Proof** If  $\partial_1 X := \partial X$  has several connected components, we pick one of them, say,  $\partial_1^* X$ . The union of the remaining boundary components is denoted by  $\partial_1^{**} X$ . We can construct a Morse function  $f : X \rightarrow \mathbb{R}$  so that it is locally constant on  $\partial_1^{**} X$ , and these constants are the local maxima of  $f$  in a collar  $U$  of  $\partial_1^{**} X$  in  $X$ . Then by finger moves (as in the proof of Lemma 3.2, [9]) we eliminate all critical points of  $f$  without changing  $f$  in  $U$ .

We pick a Riemmanian metric on  $X$  and let  $v$  be the gradient field of  $f$ . Evidently,  $\partial_1^{**} X \subset \partial_1^- X(v)$  and  $\partial_1^+ X(v) \subset \partial_1^* X$ .

By an argument as in [9], Corollary 3.3, in the vicinity of  $\partial_1^* X$ , we can deform the field  $v$  to a new  $f$ -gradient-like field so that all the manifolds  $\partial_1^+ X, \partial_2^+ X, \dots, \partial_{n-1}^+ X$ , residing in the component  $\partial_1^* X$ , will be diffeomorphic to balls, and  $Z^1 \stackrel{\text{def}}{=} \partial_n^+ X$  will consist of a number of arcs. The argument in Corollary 3.3 from [9] constructs such a  $v$  to be boundary generic in the sense of Definition 2.2. Moreover, by Theorem 3.5 from [10], we can further perturb  $v$  inside  $X$ , without changing it on  $\partial_1 X$ , so that the new perturbation will be a transversally generic field. Abusing notations, we continue to denote the new field by  $v$ .

Thus the locus  $\partial_{n+1-k}^+ X(v)$  is diffeomorphic to the disk  $D^k$ , provided  $k \geq 2$ . We notice that, for  $k \geq 2$ , for each point  $x \in D^k = \partial_{n+1-k}^+ X(v)$ , the  $v$ -trajectory  $\gamma_x$  has at least one tangency point of multiplicity  $n + 1 - k$ , residing in  $\partial_1^+ X$ : namely,  $x$  itself. Thus,  $\Gamma$  maps  $D_k$  onto the locus  $\mathcal{T}_{\{\max \geq n+1-k\}}^+(v)$ . Similarly,  $\Gamma : Z^1 \rightarrow \mathcal{T}_{\{\max \geq n\}}^+(v)$ ,  $\Gamma : Z^0 \rightarrow \mathcal{T}_{\{\max \geq n+1\}}^+(v)$  are surjective maps.

We notice that, due to the convexity of the flow in their neighborhoods (or rather, due to the convexity of the boundary with respect to the flow), the points of  $\partial_j^- X$  are “protected” in the following sense: no  $v$ -trajectory can reach  $\partial_2^- X(v)$ , unless the trajectory is a singleton which belongs to  $\partial_2^- X(v)$  in the first place, no  $v$ -trajectory can reach  $\partial_3^- X(v)$ , unless the trajectory is a singleton in  $\partial_3^- X(v)$ , and so on ... In particular, no  $v$ -trajectory through a point of  $\partial_1^+ X(v)$  can reach  $\partial_2^- X(v)$ , unless the trajectory is a singleton which belongs to  $\partial_2^- X(v)$  in the first place, no  $v$ -trajectory through a point of  $\partial_2^+ X(v)$  can reach  $\partial_3^- X(v)$ , unless the trajectory is a singleton  $\partial_3^- X(v)$ , and so on. The claim also follows from Theorem 2.2, [10].

Therefore all the maps  $\{\Gamma : \partial_j^- X(v) \setminus \partial_{j+1} X(v) \rightarrow \mathcal{T}(v)\}_j$  are 1-to-1. So the claim in the third bullet has been validated.

For a transversally generic  $v$ , by Corollary 5.1 from [11], the map  $\Gamma : \partial_1 X \rightarrow \mathcal{T}(v)$  is  $(n + 2)$ -to-1 at most. Since each trajectory, distinct from a singleton, must exit through  $\partial_1^- X$  at a point of an odd multiplicity, the same argument shows that  $\Gamma : \partial_1^+ X \rightarrow \mathcal{T}(v)$  is  $(n + 1)$ -to-1 at most. Moreover, a generic fiber of  $\Gamma : \partial_1^+ X \rightarrow \mathcal{T}(v)$  is a singleton. For  $v \in \mathcal{V}^\ddagger(X)$ , the tangent spaces  $\{T(\partial_j^+ X^\circ)\}_j$  to the strata  $\{\partial_j^+ X^\circ\}_j$  along each trajectory  $\gamma$  must form, with the help of the flow, a stable configuration in the germ of a  $n$ -section  $S$ , transversal to  $\gamma$  (see [10], Definition 3.2, which is a more conceptual version of Definition 2.4). Since  $\dim(T_x(\partial_j^+ X)) = n + 1 - j$  for every point  $x \in \gamma \cap \partial_j^+ X^\circ$  and since the flow-generated images of the spaces  $\{T_x(\partial_j^+ X)\}_{x \in \gamma \cap \partial_j^+ X^\circ}$  must be in general position in a  $n$ -dimensional space  $S$ , the cardinality of the set  $\gamma \cap \partial_j^+ X^\circ$  cannot exceed  $\lceil \frac{n}{j-1} \rceil = \lceil \frac{n}{n-k} \rceil$ , provided  $k < n$ . The statement in second bullet has been established.



By the second bullet of Theorem 3.4, [10], the smooth topological type of the stratification  $\{\partial_j X(v)\}_j$  is stable under perturbations of  $v$  within the space  $\mathcal{B}^\dagger(X)$  of boundary generic fields. The same argument shows that  $\{\partial_j^+ X(v)\}_j$  is stable as well. Thus, for all fields  $v'$ , sufficiently close to  $v$ , the stratification  $\{\partial_j^+ X(v')\}_j$  will remain as in (3.1). By Theorem 3.5 from [10], all vector fields, sufficiently close to a transversally generic vector field, will remain transversally generic. Therefore this fact gives the desired control of the cardinality for the fibers of the maps  $\Gamma : \partial_j^+ X(v') \rightarrow \mathcal{T}(v')$  and of the smooth topology of the stratification  $\{\partial_j^+ X(v')\}_j$  within an open neighborhood of  $v$  in  $\mathcal{V}^\ddagger(X)$ .  $\square$

**Remark 3.1** Recall that the trajectory space  $\mathcal{T}(v)$  in the Origami Theorem 3.1 is not only weakly homotopy equivalent to the manifold  $X$  (see [11], Theorem 5.1), but also carries a  $n$ -bundle  $\tau$ , whose pull-back under  $\Gamma$  is stably isomorphic to the tangent bundle  $TX$  ([12], Lemma 2.1). As a result,  $\tau$  and  $TX$  share all stable characteristic classes. So all this information about  $X$  is hidden in a subtle way in *the geometry of the origami map*  $\Gamma : D^n \rightarrow \mathcal{T}(v)$ .  $\diamond$

**Definition 3.1** We say that a function  $h : \mathcal{T}(v) \rightarrow \mathbb{R}$  is *smooth*, if its pull-back  $\Gamma^*h : X \rightarrow \mathbb{R}$ , under the obvious map  $\Gamma : X \rightarrow \mathcal{T}(v)$ , is smooth.  $\diamond$

The Origami Theorem 3.1 oddly resembles the Noether Normalization Lemma in the Commutative Algebra [16], however, with the direction of the ramified morphism being reversed. Recall that, in its algebro-geometrical formulation, the Normalization Lemma states that any affine variety is a branched covering over an affine space. In contrast, in our setting, many trajectory spaces  $\mathcal{T}(v)$ —rather intricate objects—have a simple and *universal* ramified cover—the ball.

To explain this analogy, for a transversally generic field  $v$ , consider the Lie derivation  $\mathcal{L}_v$  of the algebra  $C^\infty(X)$  of smooth functions on  $X$ . Its kernel  $C^\infty(\mathcal{T}(v))$ <sup>3</sup> is a part of the exact sequence of vector spaces:

$$0 \rightarrow C^\infty(\mathcal{T}(v)) \rightarrow C^\infty(X) \xrightarrow{\mathcal{L}_v} C^\infty(X) \rightarrow \dots$$

By definition,  $C^\infty(\mathcal{T}(v))$ , the algebra of smooth functions on the space of trajectories, can be identified with the algebra of all smooth functions on  $X$  that are constant along each  $v$ -trajectory.

When a transversally generic  $v$  is such that  $\partial_1^+ X(v)$  is diffeomorphic to  $D^n$ , then employing Theorem 3.1 and with the help of the finitely ramified surjective map

$$\Gamma^\partial : D^n = \partial_1^+ X(v) \subset X \xrightarrow{\Gamma} \mathcal{T}(v),$$

we get the induced monomorphism  $(\Gamma^\partial)^* : C^\infty(\mathcal{T}(v)) \rightarrow C^\infty(D^n)$  of algebras, where the target algebra  $C^\infty(D^n)$  of smooth functions on the  $n$ -ball is *universal* for a given dimension  $n$ .

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<sup>3</sup>  $C^\infty(\mathcal{T}(v))$  is just a subalgebra of  $C^\infty(X)$ , not an ideal.

Any point-trajectory  $\gamma \in \mathcal{T}(v)$  gives rise to the maximal ideal  $\mathfrak{m}_\gamma \triangleleft C^\infty(\mathcal{T}(v))$ , comprising the smooth functions on  $\mathcal{T}(v)$  that vanish at  $\gamma$ . On the other hand, if  $\mathfrak{m} \triangleleft C^\infty(\mathcal{T}(v))$  is a maximal ideal and a function  $h \in \mathfrak{m}$  does not vanish on the compact  $\mathcal{T}(v)$ , then the function  $1 = (\frac{1}{h}) \cdot h \in \mathfrak{m}$ , so that  $\mathfrak{m} = C^\infty(\mathcal{T}(v))$ . Thus every maximal ideal  $\mathfrak{m} \triangleleft C^\infty(\mathcal{T}(v))$ , distinct from the algebra itself, is of the form  $\mathfrak{m}_\gamma$ .

We already used the fact that the map  $\Gamma^\partial$  is finitely ramified with fibers of cardinality  $(n + 1)$  at most ([11], Corollary 5.1). Therefore, for any maximal ideal  $\mathfrak{m} \triangleleft C^\infty(\mathcal{T}(v))$ , its  $\Gamma^\partial$ -induced image  $(\Gamma^\partial)^*(\mathfrak{m})$  is the intersection  $\bigcap_i n_i$  of  $(n + 1)$  maximal ideals  $n_i \triangleleft C^\infty(D^n)$  at most. Also, the image  $(\Gamma^\partial)^*(\mathfrak{m})$  of a generic maximal ideal  $\mathfrak{m} \triangleleft C^\infty(\mathcal{T}(v))$  is a maximal ideal  $\mathfrak{n} \triangleleft C^\infty(D^n)$ , since the set  $\mathcal{T}(v, (11))$  of trajectories of the combinatorial type (11) is open and dense in  $\mathcal{T}(v)$ .

One can think of smooth vector fields on  $X$  as derivatives of the algebra  $C^\infty(X)$ . We denote the space of such operators by the symbol  $\mathcal{D}(X)$ . Let  $\mathcal{C}^+(X) \subset C^\infty(X)$  denote the open cone, formed by all strictly positive functions. The gradient-like (traversing) vector fields  $v$  correspond to derivatives  $\mathcal{L}_v \in \mathcal{D}(X)$  such that  $\mathcal{L}_v(f) \in \mathcal{C}^+(X)$  for some  $f \in C^\infty(X)$ .

By Theorem 3.5 from [10], the transversally generic fields form a nonempty open set  $\mathcal{V}^\ddagger(X)$  in the space of all vector fields and an open and dense set in the space of all traversing vector fields. Therefore, the previous considerations lead to the following reformulation of Theorem 3.1.

**Corollary 3.1** (The origami resolutions  $C^\infty(\mathbf{D}^n)$  for the kernels of special derivatives of the algebra  $C^\infty(\mathbf{X})$ ) *Let  $C^\infty(D^n)$  denote the algebra of smooth functions on the  $n$ -ball.*

*For any  $(n + 1)$ -dimensional smooth connected and compact manifold  $X$  with boundary, there exists an open nonempty subset  $\text{Der}^\circ(X) \subset \text{Der}(X)$  of the space of derivatives  $\mathcal{L} : C^\infty(X) \rightarrow C^\infty(X)$  that possess the following properties:*

- *for each  $\mathcal{L} \in \text{Der}^\circ(X)$ , there exists a function  $f \in C^\infty(X)$  such that  $\mathcal{L}(f) \in \mathcal{C}^+(X)$ , the positive cone,*
- *for each  $\mathcal{L} \in \text{Der}^\circ(X)$ , there exists a monomorphism of algebras<sup>4</sup>*

$$(\Gamma^\partial)^* : \ker(\mathcal{L}) \rightarrow C^\infty(D^n)$$

*such that, for any maximal ideal  $\mathfrak{m} \triangleleft \ker(\mathcal{L})$ , the image  $(\Gamma^\partial)^*(\mathfrak{m})$  is an intersection  $\bigcap_i n_i$  of  $n + 1$  maximal ideals  $n_i \triangleleft C^\infty(D^n)$  at most. For any maximal ideal  $\mathfrak{m}$  from an open in Zariski topology set in the spectrum  $\text{Spec}(\ker(\mathcal{L}))$ , the image  $(\Gamma^\partial)^*(\mathfrak{m})$  is a maximal ideal. □*

We would like very much to learn the answer to the following question:

**Question 3.1** *Let  $X$  be a compact connected smooth manifold with boundary. Let  $v$  be a traversing and boundary generic vector field on  $X$ .*

*Describe the image, under the restriction map induced by the inclusion  $\partial X \subset X$ , of the algebra  $\ker(\mathcal{L}_v)$  in the algebra  $C^\infty(\partial X)$ . □*

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<sup>4</sup> that is induced by the origami map  $\Gamma : D^n \rightarrow \mathcal{T}(v)$

### 4 A Glimpse of Holography

We devote this section to the fundamental phenomenon of the *holography* of traversing flows. Recall that we are concerned with the ability to reconstruct the manifold  $X$  and the traversing flow  $v$  on it (or rather, the 1-dimensional oriented foliation  $\mathcal{F}(v)$ , generated by  $v$ ) in terms of some data, generated by the flow on the boundary  $\partial X$ . This kind of problem is in the focus of an active research in Differential Geometry, where it is known under the name of *geodesic inverse scattering problem* (see [1–4], [17–21]). In this context, the idea is to apply the holography principle to the geodesic flow on the tangent spherical bundle of a given Riemannian manifold with boundary. On many special occasions, this application allows for a reconstruction of the manifold and the metric from the scattering data [13].

The main result of this section, Theorem 4.1, describes some boundary data, sufficient for a reconstruction of the pair  $(X, \mathcal{F}(v))$ , up to a homeomorphism. The reader interested in further developments of these ideas may glance at the paper [12,13], and the forthcoming book [14].

First, we introduce one basic construction (see Fig. 2) which will be very useful throughout our investigations below.

**Lemma 4.1** *Consider a traversing vector field  $v$  on a compact smooth connected manifold  $X$  with boundary and a Lyapunov function  $f : X \rightarrow \mathbb{R}$ ,  $df(v) > 0$ .*

*Any such pair  $(v, f)$  generates an embedding  $\alpha(v, f) : X \subset \mathcal{T}(v) \times \mathbb{R}$ , where  $\mathcal{T}(v)$  denotes the trajectory space.*

*For any smooth<sup>5</sup> map  $\beta : \mathcal{T}(v) \rightarrow \mathbb{R}^N$ , the composite map*

$$A(v, f) : X \xrightarrow{\alpha} \mathcal{T}(v) \times \mathbb{R} \xrightarrow{\beta \times id} \mathbb{R}^N \times \mathbb{R}$$

*is smooth.*

*Any two embeddings,  $\alpha(f_1, v)$  and  $\alpha(f_2, v)$ , are isotopic through homeomorphisms, provided that  $df_1(v) > 0$ ,  $df_2(v) > 0$ .*

**Proof** Since  $f$  is strictly increasing along the  $v$ -trajectories, any point  $x \in X$  is determined by the  $v$ -trajectory  $\gamma_x$  through  $x$  and the value  $f(x)$ . Therefore,  $x$  is determined by the point  $\gamma_x \times f(x) \in \mathcal{T}(v) \times \mathbb{R}$ . By the definition of the quotient topology in  $\mathcal{T}(v)$ , the correspondence  $\alpha(f, v) : x \rightarrow \gamma_x \times f(x)$  is a continuous map.

In fact,  $\alpha(f, v)$  is a smooth map in the spirit of Definition 3.1: more accurately, for any map  $\beta : \mathcal{T}(v) \rightarrow \mathbb{R}^N$ , given by  $N$  smooth functions on  $\mathcal{T}(v)$ , the composite map  $A(v, f) : X \rightarrow \mathbb{R}^N \times \mathbb{R}$  is smooth. The verification of this fact is on the level of definitions.

For a fixed  $v$ , the condition  $df(v) > 0$  defines an open convex cone  $\mathcal{C}^+(v)$  in the space  $C^\infty(X)$ . Thus,  $f_1$  and  $f_2$  can be linked by a path in  $\mathcal{C}^+(v)$ , which results in  $\alpha(f_1, v)$  and  $\alpha(f_2, v)$  being homotopic through homeomorphisms. □

**Remark 3.1** By examining Fig. 2, we observe an interesting phenomenon: the embedding  $\alpha : X \subset \mathcal{T}(v) \times \mathbb{R}$  does not extend to an embedding of a larger manifold  $\hat{X} \supset X$ ,

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<sup>5</sup> see Definition 3.1.

where  $\hat{X} \setminus X \approx \partial_1 X \times [0, \epsilon)$ . In other words,  $\alpha(\partial_1 X)$  has no outward “normal field” in the ambient  $\mathcal{T}(v) \times \mathbb{R}$ . In that sense,  $\alpha(\partial_1 X)$  is *rigid* in  $\mathcal{T}(v) \times \mathbb{R}$ !  $\square$

**Corollary 4.1** *For  $n \geq 2$ , any compact connected smooth  $(n + 1)$ -manifold  $X$  with boundary admits an embedding  $\alpha : X \rightarrow \mathcal{T} \times \mathbb{R}$ , where  $\mathcal{T}$  is a CW-complex which is the image of the  $n$ -ball  $D^n$  under a continuous map, whose fibers are of the cardinality  $n + 1$  at most and whose generic fiber is of the cardinality 1. Moreover,  $\alpha$  is a homotopy equivalence.*

**Proof** We combine the Origami Theorem 3.1 with Lemma 4.1 to validate the first claim of the lemma.

Since  $\Gamma = p \circ \alpha$ , where  $p : \mathcal{T} \times \mathbb{R} \rightarrow \mathcal{T}$  is the obvious projection, and  $\Gamma$  is a homotopy equivalence by Theorem 5.1 from [11], so is the map  $\alpha$ .  $\square$

**Corollary 4.2** *Let  $v$  be a traversing vector field on a compact smooth connected manifold  $X$  with boundary and  $f : X \rightarrow \mathbb{R}$  its Lyapunov function. Let  $X^\circ$  denote the interior of  $X$ .*

*Then the embedding*

$$\alpha(f, v) : \partial_1 X \longrightarrow (\mathcal{T}(v) \times [0, 1]) \setminus \alpha(f, v)(X^\circ)$$

*is a homology equivalence. As a result, the space*

$$(\mathcal{T}(v) \times [0, 1]) \setminus \alpha(f, v)(X^\circ)$$

*is a Poincaré complex of the formal dimension  $\dim(X) - 1$ .*

**Proof** Put  $\alpha =_{\text{def}} \alpha(f, v)$ . Let us compare the homology long exact sequences of the two pairs:

$$X \supset \partial_1 X \quad \text{and} \quad \mathcal{T}(v) \times [0, 1] \supset (\mathcal{T}(v) \times [0, 1]) \setminus \alpha(X^\circ).$$

They are connected by the vertical homomorphisms that are induced by  $\alpha$ . Using the excision property,

$$\alpha_* : H_*(X, \partial_1 X) \rightarrow H_*(\mathcal{T}(v) \times [0, 1], (\mathcal{T}(v) \times [0, 1]) \setminus \alpha(X^\circ))$$

are isomorphisms. On the other hand, since by Theorem 5.1, [11],  $\Gamma : X \rightarrow \mathcal{T}(v)$  is a homology equivalence,  $\alpha_* : H_*(X) \rightarrow H_*(\mathcal{T}(v) \times [0, 1])$  are isomorphisms. Therefore by the Five Lemma,

$$\alpha_* : H_*(\partial_1 X) \rightarrow H_*((\mathcal{T}(v) \times [0, 1]) \setminus \alpha(X^\circ))$$

must be isomorphisms as well. Since  $\partial_1 X$  is a closed  $n$ -manifold, it is a Poincaré complex of formal dimension  $n$ , and thus so is the space  $(\mathcal{T}(v) \times [0, 1]) \setminus \alpha(f, v)(X^\circ)$ .  $\square$

**Definition 4.1** Let  $v$  be a traversing vector field on  $X$ . Given two points  $x, y \in \partial_1 X$ , we write  $y \succ_v x$  if both points belong to the same  $v$ -trajectory  $\gamma \subset X$  and, moving from  $x$  in the  $v$ -direction along  $\gamma$ , we can reach  $y$ .

The relation  $y \succ_v x$  introduces a *partial order*  $\succ_v$  in the set  $\partial_1 X$ . ◇

Adding an extra ingredient to the partial order  $\succ_v$  (equivalently, to the origami construction), allows for a reconstruction of the topological type of the pair  $(X, \mathcal{F}(v))$  from the flow-generated information, *residing on the boundary*  $\partial X$ . The new ingredient is the restriction of the Lyapunov function  $f : X \rightarrow \mathbb{R}$  to the boundary.

**Theorem 4.1** (Topological Holography of Traversing Flows) *Let  $v$  be a traversing vector field  $v$  on a compact connected smooth manifold  $X$  with boundary, and let  $f : X \rightarrow \mathbb{R}$  be its Lyapunov function.*

*Then the partial order  $\succ_v$  on  $\partial_1 X$ , together with the restriction  $f^\partial : \partial X \rightarrow \mathbb{R}$  of a Lyapunov function  $f : X \rightarrow \mathbb{R}$ , allows for a reconstruction of the topological type of the pair  $(X, \mathcal{F}(v))$ .*

**Proof** The validation of the theorem is based on Lemma 4.1. First, we observe that the partial order  $\succ_v$  allows for a reconstruction of the trajectory space  $\mathcal{T}(v)$  and of the quotient map  $\Gamma^\partial : \partial_1 X \rightarrow \mathcal{T}(v)$ . Indeed, we declare two points  $x, y \in \partial_1 X$  *equivalent* if  $y \succ_v x$  or  $x \succ_v y$ . This equivalence relation  $\sim_v$  produces the quotient map  $\Gamma^\partial : \partial_1 X \rightarrow (\partial_1 X) \setminus \sim_v$ . Its target may be identified with the space  $\mathcal{T}(v)$  since, for a traversing  $v$ , every trajectory  $\gamma$  is determined by its intersection  $\gamma \cap \partial_1 X$ .

As in Lemma 4.1, using  $f^\partial$ , we construct an embedding

$$\alpha^\partial = \alpha(v, f^\partial) | : \partial X \subset \mathcal{T}(v) \times \mathbb{R}.$$

Then  $\alpha^\partial(\partial X)$  divides  $\mathcal{T}(v) \times \mathbb{R}$  into two domains, one of which is compact. That compact domain  $\mathcal{X}$  is  $\alpha(v, f)(X)$ . Since  $\alpha(v, f) : X \rightarrow \mathcal{X}$  is a homeomorphism, we managed to reconstruct the topological type of  $X$  from the boundary data  $(\succ_v, f^\partial)$  (in the end, from  $(\Gamma^\partial, f^\partial)$ ).

Evidently,  $\mathcal{X}$  is equipped with a 1-dimensional foliation  $\mathcal{G}$ , generated by the product structure in the ambient  $\mathcal{T}(v) \times \mathbb{R}$ .

By its construction, the homeomorphism  $\alpha(v, f)$  maps each leaf of  $\mathcal{F}(v)$  to a leaf of  $\mathcal{G}$ , while preserving their orientations. Thanks to  $\alpha(v, f)$ , the pair  $(\mathcal{X}, \mathcal{G})$ , which we have recovered from the boundary data  $(\succ_v, f^\partial)$ , has the same topological type as the original pair  $(X, \mathcal{F}(v))$ .

Note that, for a given pair  $(\succ_v, f^\partial)$ , the homeomorphism  $\alpha(v, f)$  is far from being unique. Even, for a fixed pair  $(X, v)$ , we may vary the Lyapunov function  $f$ , while keeping  $f^\partial$  fixed. However, the space  $\text{Lyap}(v, f^\partial)$  of such Lyapunov functions  $f$  is convex, and thus contractible. Therefore, for any two  $f_1, f_2 \in \text{Lyap}(v, f^\partial)$ , the embeddings  $\alpha(v, f_1)$  and  $\alpha(v, f_2)$  are homotopic through homeomorphisms that map  $\mathcal{F}(v)$  to  $\mathcal{G}$ . □

**Example 4.1** Let  $Y$  be a closed surface of genus  $g$ . Let  $X$  be obtained from  $Y$  by deleting an open 2-ball. Consider a transversally generic vector field  $v$  on the surface  $X$  together with its Lyapunov function  $f : X \rightarrow \mathbb{R}$  (such a pair  $(v, f)$  does exist).

Then Theorem 4.1 claims that the order relation  $\succ_v$  on the circle  $\partial X$ , together with the function  $f^\partial : \partial X \rightarrow \mathbb{R}$ , is sufficient for a reconstruction of the pair  $(X, \mathcal{F}(v))$ , up to a homeomorphism which is the identity on  $\partial X$ . In particular, the genus  $g$  may be reconstructed from the boundary data  $(\succ_v, f^\partial)$ .  $\diamond$

**Example 4.2** Let  $Y$  be a closed surface, equipped with a smooth vector field  $w$ . Consider the product  $\hat{X} =_{\text{def}} Y \times \mathbb{R}$ . Let  $\hat{v}$  be a vector field on  $\hat{X}$ , whose  $Y$ -component is the given  $w$  and whose  $\mathbb{R}$ -component is 1. Evidently,  $\hat{v}$  admits a Lyapunov function  $T : \hat{X} \rightarrow \mathbb{R}$ , defined by the obvious projection.

Let  $X \subset \hat{X}$  be a compact connected domain with a smooth boundary. Thanks to  $T$ , the  $\hat{v}$ -flow is traversing on the 3-fold  $X$  and defines the partial order  $\succ_{\hat{v}}$  on the surface  $\partial X$ . We denote by  $v$  the restriction of  $\hat{v}$  to  $X$ .

Then Theorem 4.1 claims that the order relation  $\succ_v$  on the surface  $\partial X$ , together with the function  $f^\partial : \partial X \rightarrow \mathbb{R}$ , allows for a reconstruction of the pair  $(X, \mathcal{F}(v))$ , up to a homeomorphism which is the identity on  $\partial X$ .

If we treat  $T$  as *time* and  $Y$  as *space*, it is natural to interpret  $\succ_v$  as the *causality relation* between the events-points on the boundary  $\partial X$ .  $\diamond$

**Remark 4.1** The question whether the data  $(\succ_v, f^\partial)$  are sufficient for a reconstruction of the *differentiable* or even *smooth* topological type of the pair  $(X, \mathcal{F}(v))$  seems to be much more delicate. We suspect that the positive answer to it will depend on our ability to answer Question 3.1.

Under certain assumptions about  $v$  (such as some restrictions on the combinatorial types of  $v$ -trajectories), the answer is positive [12].  $\square$

**Corollary 4.3** *Let a traversally generic vector field  $v$  on  $X$  be such that  $\partial_1^+ X(v) \approx D^n$ .<sup>6</sup> Then the origami map  $\Gamma^\partial : D^n \rightarrow \mathcal{T}(v)$ , together with the restriction  $f_+^\partial : D^n \rightarrow \mathbb{R}$  of the Lyapunov function  $f : X \rightarrow \mathbb{R}$ , allow for a reconstruction of the topological type of the pair  $(X, \mathcal{F}(v))$ .*

**Proof** To validate the claim of the theorem, we combine Theorems 3.1 and 4.1.

Given a traversing  $v$  and a function  $f_+^\partial : \partial_1^+ X(v) \rightarrow \mathbb{R}$ , let  $\text{Lyap}(v, f_+^\partial)$  be the space of Lyapunov functions  $f : X \rightarrow \mathbb{R}$  such that  $f|_{\partial_1^+ X(v)} = f_+^\partial$ . Again,  $\text{Lyap}(v, f_+^\partial)$  is a convex contractible space.

We assume that the function  $f_+^\partial$  is known and is generated by *some* (unknown)  $f \in \text{Lyap}(v)$ . By the properties of the Lyapunov function  $f$ , we may assume that  $f_+^\partial$  extends to a function  $f^\partial : \partial_1 X \rightarrow \mathbb{R}$  so that, for any  $x, y \in \partial_1 X$ ,  $y \succ_v x$ , the inequality  $f(x) < f(y)$  is valid.

By Theorem 3.1, the image  $\Gamma^\partial(D^n)$  of the Origami map  $\Gamma^\partial$  is the trajectory space  $\mathcal{T}(v)$ , and the fibers of  $\Gamma^\partial$  may be identified with the  $(\sim_v)$ -equivalence classes of points in  $D^n$ . Moreover, since  $v$  is traversally generic, by Theorem 5.1 from [11],  $\mathcal{T}(v)$  is a compact  $CW$ -complex.

Now, as in Lemma 4.1, using  $f^\partial$ , we construct an embedding  $\alpha(v, f^\partial) : \partial X \subset \mathcal{T}(v) \times \mathbb{R}$ . The rest of the argument is similar to the argument we used to prove Theorem 4.1.  $\square$

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<sup>6</sup> By Theorem 4.1, such vector field exists.

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