



Normalization of Hamiltonian and Nonlinear Stability of Triangular Equilibrium Points in the Photogravitational Restricted Three Body Problem with P–R Drag in Non-resonance Case

Ram Kishor¹ · M. Xavier James Raj² · Bhola Ishwar³

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Abstract

Normal forms of Hamiltonian are very important to analyze the nonlinear stability of a dynamical system in the vicinity of invariant objects. This paper presents the normalization of Hamiltonian and the analysis of nonlinear stability of triangular equilibrium points in non-resonance case, in the photogravitational restricted three body problem under the influence of radiation pressures and P–R drags of the radiating primaries. The Hamiltonian of the system is normalized up to fourth order through Lie transform method and then to apply the Arnold–Moser theorem, Birkhoff normal form of the Hamiltonian is computed followed by nonlinear stability of the equilibrium points is examined. Similar to the case of classical problem, we have found that in the presence of assumed perturbations, there always exists one value of mass parameter within the stability range at which the discriminant D_4 vanish, consequently, Arnold–Moser theorem fails, which infer that triangular equilibrium points are unstable in nonlinear sense within the stability range. Present analysis is limited up to linear effect of the perturbations, which will be helpful to study the more generalized problem.

✉ Ram Kishor
kishor.ram888@gmail.com

M. Xavier James Raj
xavierjamesraj@gmail.com

Bhola Ishwar
ishwar_bhola@hotmail.com

¹ Department of Mathematics, Central University of Rajasthan, NH-8, Bandarsindari, Kishangarh, Ajmer, Rajasthan 305801, India

² Applied Mathematics Division, Vikram Sarabhai Space Centre, Thiruvananthapuram 695022, India

³ Department of Mathematics, B. R. A. Bihar University, Muzaffarpur 842001, India

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1 Introduction

Since the time of Poincaré, invariant objects are very much important to understand the behavior of a dynamical system, especially, phase space. Moreover, there are many possible approaches to find the invariant objects, whereas the normal forms (truncated) are very useful because these can give integrable approximations to the dynamics under appropriate hypothesis [15]. Because of the approximation of true dynamics by the normal forms, invariant objects of the initial system get approximated also, accordingly [17,36]. The approximate first integrals are those quantities, which are almost preserved through the system's flow. This shows that the surface levels by the flow are almost invariant. Some informations about the dynamics can be obtained through this property. To minimize the overflow and complexity in the computations, an appropriate approach is to use of power series or Fourier sines, or a combination of both to represent the object. Because in many cases they needed only a few numbers of terms to maintain the good accuracy. Some other approach can also be found in Gómez et al. [10], Jorba and Masdemont [16], in which trigonometric series is used. The normal forms of the Hamiltonian system up to some finite order is necessary to study the nonlinear stability of the equilibrium points using Arnold–Moser theorem in non-resonance case. They also help to know the behavior of dynamics in the neighborhood of the invariant objects. Many researchers have described the different method to find the normal forms of the Hamiltonian of the dynamical system [2,6,8,15,19,29,39]. In the normal forms, the central idea is to find suitable transforms of the phase coordinates, which can convert the Hamiltonian system in its simplest form up to a finite order of accuracy. Normalization of Hamiltonian is obtained to change the Hamiltonian into its simplest form using the method of Lie transforms [6,15].

Because of radiating primary in the present problem under the analysis, force due to radiation pressure came into existence [32,35], which acts in opposite direction to the gravitational attraction force of the primary. Concept of Poynting–Roberston drag is came into the picture when, Poynting [30] investigated the effect of radiation pressure on the moving particle in interplanetary space and Robertson [34] modified the Poynting's theory through the principle of relativity. In the analysis of Roberston, he considered only first order terms in the expression related to the ratio of velocity of the particle to that of the light. The radiation force is expressed as

$$\vec{F} = F_p \left(\frac{\vec{R}}{R} - \frac{\vec{V} \cdot \vec{R} \vec{R}}{cR^2} - \frac{\vec{V}}{c} \right), \quad (1)$$

where F_p is the radiation pressure force due to radiating primary; \vec{R} is the position vector of the particle relative to the radiating primary; \vec{V} is the velocity of the particle; and c is the speed of the light. First term of the Eq. (1) denotes the radiation pressure, second term represents the Doppler shift due to the motion of the particle, whereas

third term corresponds to the absorption and subsequent re-emission part of induced radiation. The combined form of the last two terms of the Eq. (1) known as Poynting–Robertson (P–R) drag. Chernikov [4] analyzed the photogravitational restricted three body problem (RTBP) with P–R drag under the frame of Sun-planet-particle system and found that non-collinear (triangular) equilibrium points are unstable. Effect of P–R drag including radiation pressure is described by Schuerman [35]. A similar analysis is presented by Murray [28] and Ragos and Zafiroopoulos [31] to observed the effect of P–R drag in the context of existence and stability of the equilibrium points. Kushvah et al. [20] examined the nonlinear stability in the generalized photogravitational RTBP with P–R drag of first primary and oblateness of secondary and found that triangular equilibrium points are unstable, whereas Mishra and Ishwar [27] investigated about the stability of non-collinear equilibrium points in the photogravitational elliptic RTBP with P–R drag. Kushvah et al. [21] and Kishor and Kushvah [18] have analyzed the effect of radiation pressure force on the existence and linear stability of the equilibrium points in the generalized photogravitational Chermnykh-like problem with a disc. They found that the effect of perturbation factors are significant. In literature, many researchers have analyzed the photogravitational RTBP in nonlinear sense by considering one or two perturbations at a time [1,13,22,24,38], but very few of them have considered the problem under the combined influence of few perturbations [19,20]. Ishwar and Sharma [14] have discussed about the nonlinear stability of out of plane equilibrium points in the RTBP with oblate primary and found that L_6 point is stable in nonlinear sense. Raj and Ishwar [33] have obtained diagonalized form of the Hamiltonian with P–R drag. Kishor and Kushvah [19] have studied nonlinear stability of triangular equilibrium points in the Chermnykh-like problem, in the presence of radiation pressure, oblateness and a disc. They found that these perturbations affect the numerical results significantly.

Due to above reasons in addition to wide applications of the RTBP in mission design, we are motivated to study the problem under the influence of the radiation pressures and P–R drags of both primary and secondary. In the present study, we are interested to compute the fourth order normalized Hamiltonian and utilizing them to analyze the nonlinear stability of triangular equilibrium points using Arnold–Moser theorem in non-resonance case. Because of both primary and secondary radiating, the problem under analysis includes the four perturbing parameters in the form of mass reduction factors q_1 , q_2 due to the radiation pressures of the primaries and P–R drags W_1 , W_2 of both the primaries, respectively. The paper is organized as follows: In Sect. 2, we have formulated the problem and found the equations of motion. Section 3 presents the second order normalized Hamiltonian of the problem under analysis. Nonlinear stability analysis is discussed in Sect. 4. Section 5 is devoted to Birkhoff normal form and application of Arnold–Moser theorem in non-resonance case. Results are concluded in Sect. 6. For algebraic and numerical computations, Mathematica® [40] software package is used. The results of this study may be used to describe more generalized problem under the influence of other perturbations such as albedo, solar wind drag, Stokes drag etc. [12,37].

2 Mathematical Formulation

We consider the photogravitational restricted three body problem with P–R drag, which consists of motion of an infinitesimal mass under the influence of gravitational field and radiation effect of two massive and radiating bodies of masses m_1 and m_2 , ($m_1 > m_2$), respectively, called primaries. Forces, which govern the motion of infinitesimal mass are gravitational attractions, radiation pressures and P–R drags of both the primaries, respectively. It is assumed that gravitational effect of infinitesimal mass on the system is negligible. Units are normalized such as units of mass and distance are taken as the sum of the masses of both the primaries and separation distance between them, respectively, whereas unit of time is the time period of the rotating frame. We suppose that the coordinate of the primaries are $(-\mu, 0)$, $(1 - \mu, 0)$, respectively and that of infinitesimal mass is (x, y) , then the equations of motion [33] are

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x}, \quad (2)$$

$$\ddot{y} - 2\dot{x} = \frac{\partial U}{\partial y}, \quad (3)$$

where

$$\begin{aligned} \frac{\partial U}{\partial x} = & x - \frac{q_1(1-\mu)(x+\mu)}{r_1^3} - \frac{q_2(1-\mu)(x+\mu-1)}{r_2^3} \\ & - \frac{W_1 S_1}{r_1^2} - \frac{W_2 S_2}{r_2^2}, \end{aligned} \quad (4)$$

$$\frac{\partial U}{\partial y} = y - \frac{q_1(1-\mu)y}{r_1^3} - \frac{q_2(1-\mu)y}{r_2^3} - \frac{W_1 S_3}{r_1^2} - \frac{W_2 S_4}{r_2^2}, \quad (5)$$

and further

$$\begin{aligned} S_1 &= \frac{(x+\mu)\{(x+\mu)\dot{x} + y\dot{y}\}}{r_1^2} + \dot{x} - y, \\ S_2 &= \frac{(x+\mu-1)\{(x+\mu-1)\dot{x} + y\dot{y}\}}{r_2^2} + \dot{x} - y, \\ S_3 &= \frac{y\{(x+\mu)\dot{x} + y\dot{y}\}}{r_1^2} + \dot{y} + x + \mu, \\ S_4 &= \frac{y\{(x+\mu-1)\dot{x} + y\dot{y}\}}{r_2^2} + \dot{y} + x + \mu - 1 \end{aligned}$$

with $q_i = 1 - F_{pi}/F_{gi}$, $i = 1, 2$ as mass reduction factors of both the primaries, respectively; F_{pi} , F_{gi} , $i = 1, 2$ are the forces of radiation pressure and gravitational attraction of the respective primaries; $W_1 = [(1 - q_1)(1 - \mu)]/c_d$ and $W_2 = [(1 - q_2)\mu]/c_d$ as P–R drags of both the primaries, respectively; c_d is the speed of light in non-dimensional form; r_1 , r_2 are distances of infinitesimal mass from the first and second primary, which are given as

$$r_1^2 = (x + \mu)^2 + y^2, \quad r_2^2 = (x + \mu - 1)^2 + y^2. \tag{6}$$

The co-ordinates $(x_0, \pm y_0)$ of triangular equilibrium points $L_{4,5}$ are obtained on similar basis as in Raj and Ishwar [33]. To overcome the complexity in the analysis, co-ordinates x_0 and y_0 are linearized with respected to $W_1, W_2, \epsilon_1, \epsilon_2$, keeping in mind that the perturbing parameters lie in $(0, 1)$ so, take $q_1 = 1 - \epsilon_1, q_2 = 1 - \epsilon_2$, where ϵ_1, ϵ_2 are very small. The linearized co-ordinates x_0 and y_0 are

$$x_0 = \frac{1}{2} - \mu - \frac{4W_1(2 - \mu)}{3\sqrt{3}} - \frac{4W_2(1 + 2\mu)}{3\sqrt{3}} - \frac{\epsilon_1}{3} + \frac{\epsilon_2}{3}, \tag{7}$$

$$y_0 = \pm \left[\frac{\sqrt{3}}{2} + \frac{4W_1(2 - 3\mu)}{9} + \frac{4W_2(1 - 3\mu)}{9} - \frac{\epsilon_1}{3\sqrt{3}} - \frac{\epsilon_2}{3\sqrt{3}} \right]. \tag{8}$$

The plus sign corresponds to L_4 , whereas minus sign corresponds to L_5 . The Hamiltonian function of the problem is written as

$$H = p_x \dot{x} + p_y \dot{y} - \frac{\dot{x}^2 + \dot{y}^2}{2} - \frac{x^2 + y^2}{2} - x\dot{y} + \dot{x}y - \frac{(1 - \mu)q_1}{r_1} - \frac{\mu q_2}{r_2} - W_1 S_5 - W_2 S_6, \tag{9}$$

where

$$S_5 = \frac{(x + \mu)\dot{x} + y\dot{y}}{2r_1^2} - \arctan\left(\frac{y}{x + \mu}\right),$$

$$S_6 = \frac{(x + \mu - 1)\dot{x} + y\dot{y}}{2r_1^2} - \arctan\left(\frac{y}{x + \mu - 1}\right).$$

The conjugate momenta p_x, p_y corresponding to generalized co-ordinate x, y respectively, are given as

$$p_x = \dot{x} - y + \frac{W_1(x + \mu)}{2r_1^2} + \frac{W_2(x + \mu - 1)}{2r_2^2}, \tag{10}$$

$$p_y = \dot{y} + x + \frac{W_1 y}{2r_1^2} + \frac{W_2 y}{2r_2^2}. \tag{11}$$

3 Second Order Normal Form of the Hamiltonian

In the present analysis only the stability of L_4 is analyzed, because the dynamics of L_5 is similar to that of L_4 . Only first order terms in the perturbing parameters W_1, W_2, q_1, q_2 are considered for simplifying the complex calculations involved in the problem through out the analysis. The second order normal form of the Hamiltonian of the problem under analysis is obtained in Raj and Ishwar [33] and for self sufficiency

of this paper, we have taken some necessary expressions in appropriate form there to use under this section. Shifting the origin to the triangular equilibrium point L_4 using simple transformations as

$$\begin{aligned}x^* &= x - x_0, \quad y^* = y - y_0, \\p_x^* &= p_x + y_0, \quad p_y^* = p_y - x_0.\end{aligned}$$

Substituting these variables in Hamiltonian (9), we get new Hamiltonian H^* . Now, expanding the new Hamiltonian using Taylor's series about the origin, which is now, the triangular equilibrium point, H^* can be written as

$$H^* = H_0^* + H_1^* + H_2^* + H_3^* + \cdots + H_n^* + \cdots, \quad (12)$$

where

$$H_n^* = \sum H_{ijkl} x^{*i} y^{*j} p_x^{*k} p_y^{*l}, \quad (13)$$

such that $i + j + k + l = n$. Since, the origin is the triangular equilibrium point, H_1^* must vanish, whereas H_0^* is constant, and hence, it can be dropped out as it is irrelevant to the dynamics. The quadratic Hamiltonian H_2^* , which is to be normalized first and then to be used for higher order normalization, is given as

$$H_2^* = \frac{p_x^{*2} + p_y^{*2}}{2} + y^* p_x^* - x^* p_y^* + E x^{*2} + G x^* y^* + F y^{*2}, \quad (14)$$

where

$$E = \frac{1}{8} + \frac{4W_1}{\sqrt{3}} + \frac{2W_1}{\sqrt{3}} + \frac{\epsilon_1}{4} - \frac{\epsilon_2}{2}, \quad (15)$$

$$F = -\frac{5}{8} - \frac{4W_1}{\sqrt{3}} - \frac{2W_1}{\sqrt{3}} - \frac{\epsilon_1}{4} + \frac{\epsilon_2}{2}, \quad (16)$$

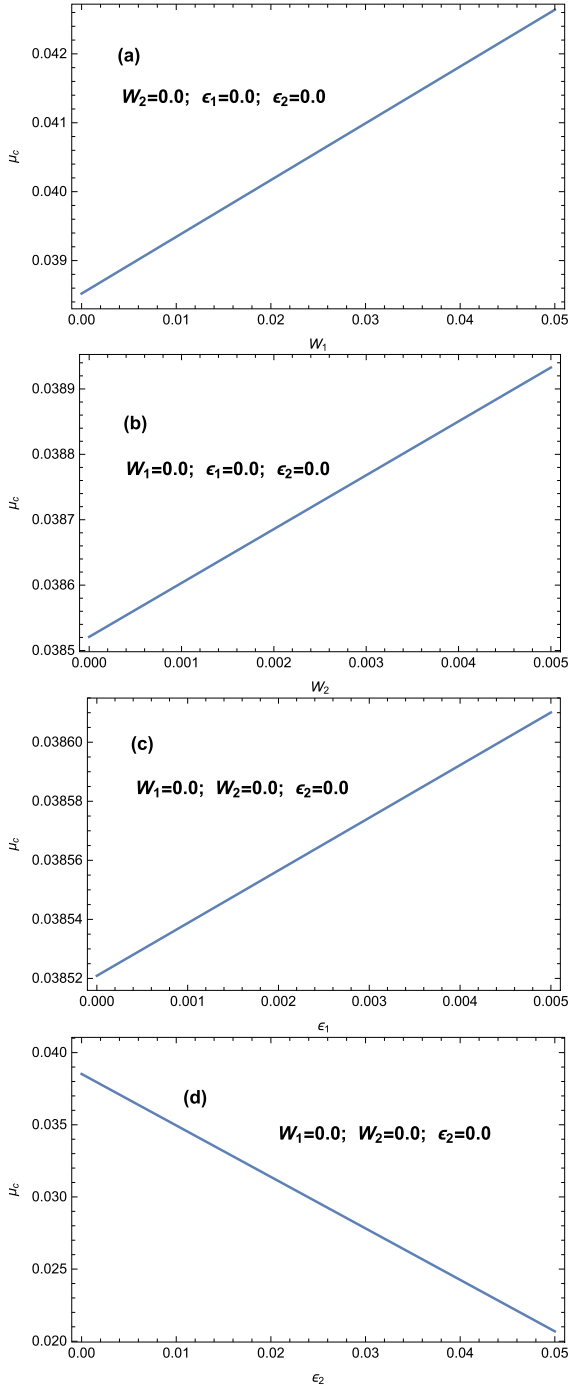
$$G = -\gamma \left(1 - \frac{32W_1}{9\sqrt{3}} - \frac{16W_1}{9\sqrt{3}} - \frac{2\epsilon_1}{9} + \frac{4\epsilon_2}{9} \right), \quad (17)$$

$$\text{with } \gamma = \frac{3\sqrt{3}}{4}(1 - 2\mu). \quad (18)$$

In the present study, the problem is dealt with four perturbation parameters in the form of P-R drag and radiation pressure of both the primaries. Hence, the coefficient H_{ijkl} for $i, j, k, l = 0, 1, 2, 3, 4$ such that $i + j + k + l = 4$ in (13) can be bifurcated into five parts such as H_{ijk11} , H_{ijk12} , H_{ijk13} , H_{ijk14} , and H_{ijk15} , which corresponds to the terms in classical case, terms with P-R drag of first primary W_1 , P-R drag of second primary W_2 , radiation pressure of first primary $\epsilon_1 = 1 - q_1$ and radiation pressure of second primary $\epsilon_2 = 1 - q_2$, respectively. Thus,

$$H_{ijkl} = H_{ijk11} + H_{ijk12} + H_{ijk13} + H_{ijk14} + H_{ijk15}. \quad (19)$$

Fig. 1 Variation of critical mass ratio μ_c with respect to **a** W_1 , **b** W_2 , **c** ϵ_1 and **d** ϵ_2



It is noted that if there is no perturbations in the system, i.e. $W_1 = W_2 = \epsilon_1 = \epsilon_2 = 0$, then $H_{ijkl} = H_{ijkl1}$, which is nothing but the coefficient of the Hamiltonian in classical case.

Hamiltonian equations of motion of the infinitesimal mass in matrix form is written as

$$\begin{bmatrix} \dot{x}^* \\ \dot{y}^* \\ \dot{p}_x^* \\ \dot{p}_y^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -2E & -G & 0 & 1 \\ -G & -2F & -1 & 0 \end{bmatrix} \begin{bmatrix} x^* \\ y^* \\ p_x^* \\ p_y^* \end{bmatrix}. \quad (20)$$

The characteristic equation of the system (20) is

$$\lambda^4 + 2(E + F + 1)\lambda^2 + (4EF - G^2 - 2E - 2F + 1) = 0. \quad (21)$$

Solving the simplified discriminant of the characteristic equation (21) as

$$(E + F + 1)^2 - (4EF - G^2 - 2E - 2F + 1) = 0, \quad (22)$$

we have the value of critical mass ratio $0 < \mu_c \leq (1/2)$ as

$$\mu_c = 0.0385209 + 0.0823761W_1 + 0.0823761W_2 + 0.0178349\epsilon_1 - 0.356699\epsilon_2, \quad (23)$$

which is similar to that of Kushvah et al. [20] and Kishor and Kushvah [18] and agree with the classical value $\mu_c = 0.0385209$. Figure 1a–d shows the variations of critical mass ratio μ_c with respect to perturbing parameters W_1 , W_2 , ϵ_1 and ϵ_2 , respectively. We observed that the effects of the perturbations in question are significant. As, system will be stable when four roots of the characteristic equation (21) are pure imaginary, which is possible when the mass parameter μ satisfy the condition $0 < \mu < \mu_c$. Since, we are analyzing the nonlinear stability within the range of linear stability $0 < \mu < \mu_c$, it is obvious to assume that roots of the characteristic equation (21) are pure imaginary. Suppose, the roots of the characteristic equation (21) are $\pm i\omega_1$ and $\pm i\omega_2$, where ω_1 , ω_2 can be obtained by solving the equation

$$\omega^4 - 2(E + F + 1)\omega^2 + (4EF - G^2 - 2E - 2F + 1) = 0. \quad (24)$$

Motion corresponds to frequencies ω_1 , $\omega_2 \in \mathbb{R}$ are known as long and short periodic motion of infinitesimal mass at L_4 with periods of $2\pi/\omega_1$ and $2\pi/\omega_2$, respectively. Frequencies ω_1 , ω_2 corresponding to the long and short periodic motion are related to each other by the relations

$$\omega_1^2 + \omega_2^2 = 2E + 2F + 2, \quad (25)$$

$$\omega_1^2\omega_2^2 = 4EF - G^2 - 2E - 2F + 1. \quad (26)$$

Substituting the values of E , F and G from Eqs. (15–17), we get

$$\omega_1^2 + \omega_2^2 = 1, \tag{27}$$

$$\omega_1^2 \omega_2^2 = \frac{27}{16} \gamma^2 - 4\sqrt{3}W_1 - 2\sqrt{3}W_2 - \frac{2\epsilon_1}{3} + \frac{3\epsilon_2}{4}, \tag{28}$$

where the values of ω_1 and ω_2 are

$$\omega_1 = \sqrt{-1 + \sqrt{1 - 4\delta}}, \quad \omega_2 = \sqrt{-1 - \sqrt{1 - 4\delta}}, \tag{29}$$

with

$$\delta = \frac{27}{16} \gamma^2 - 4\sqrt{3}W_1 - 2\sqrt{3}W_2 - \frac{3\epsilon_1}{4} + \frac{3\epsilon_2}{2}. \tag{30}$$

The real normalized Hamiltonian of the Hamiltonian (14) up to second order is given as [33]

$$H_2 = \omega_1 \frac{\mathbf{x}^2 + \mathbf{p}_x^2}{2} + \omega_2 \frac{\mathbf{y}^2 + \mathbf{p}_y^2}{2}, \tag{31}$$

which is complexified by using the co-ordinate transformations

$$\mathbf{x} = \frac{X + iP_X}{\sqrt{2}}, \tag{32}$$

$$\mathbf{y} = \frac{-Y + iP_Y}{\sqrt{2}}, \tag{33}$$

$$\mathbf{p}_x = \frac{iX + P_X}{\sqrt{2}}, \tag{34}$$

$$\mathbf{p}_y = \frac{iY - P_Y}{\sqrt{2}} \tag{35}$$

and changed as

$$H_2 = i\omega_1 X P_X - i\omega_2 Y P_Y, \tag{36}$$

Finally, symplectic matrix \mathbf{C} of the symplectic transformations, which are used to obtain the complex normal form of Hamiltonian is given as [33]

$$\mathbf{C} = [s_{ij}], \quad 1 \leq i, j \leq 4 \tag{37}$$

with

$$s_{11} = 0 = s_{12}, \quad s_{13} = \frac{1 - 2F + \omega_1^2}{\sqrt{d(\omega_1)}},$$

$$\begin{aligned}
s_{14} &= \frac{1 - 2F + \omega_1^2}{\sqrt{d(\omega_2)}}, \quad s_{21} = \frac{2\omega_1}{\sqrt{d(\omega_1)}}, \\
s_{22} &= \frac{2\omega_1}{\sqrt{d(\omega_2)}}, \quad s_{23} = \frac{G}{\sqrt{d(\omega_1)}}, \\
s_{24} &= \frac{G}{\sqrt{d(\omega_2)}}, \quad s_{31} = \frac{\omega_1^3 - (2F + 1)\omega_1}{\sqrt{d(\omega_1)}}, \\
s_{32} &= \frac{\omega_2^3 - (2F + 1)\omega_2}{\sqrt{d(\omega_2)}}, \quad s_{33} = \frac{-G}{\sqrt{d(\omega_1)}}, \\
s_{34} &= \frac{-G}{\sqrt{d(\omega_2)}}, \quad s_{41} = \frac{G\omega_1}{\sqrt{d(\omega_1)}}, \quad s_{42} = \frac{G\omega_2}{\sqrt{d(\omega_2)}}, \\
s_{43} &= \frac{1 - 2F - \omega_1^2}{\sqrt{d(\omega_1)}}, \quad s_{44} = \frac{1 - 2F - \omega_2^2}{\sqrt{d(\omega_2)}},
\end{aligned}$$

where $d(\omega_i)$ for $i = 1, 2$ is obtained from the following equation

$$d(\omega) = \omega \left[\omega^4 - (2E + 6F)\omega^2 + (4EF + 4F^2 - 2E + 2F - 2) \right]. \quad (38)$$

4 Nonlinear Stability in Non-resonance Case

Nonlinear stability of the equilibrium points can be described in two cases, one as resonance case and other as non-resonance case. For resonance case, the nonlinear stability is studied through the theorems of Markeev and Sokolskii [23] as in Goździewski [11] and for non-resonance case, it is analyzed through the Arnold–Moser theorem. In the present analysis the nonlinear stability of the perturbed triangular equilibrium point in non-resonance case will be studied through Arnold–Moser theorem [25,26], which is described as follows:

Consider the Hamiltonian expressed in action variables I_1, I_2 and angles variables ϕ_1, ϕ_2 as,

$$K = K_2 + K_4 + \dots + K_{2m} + K_{2m+1}, \quad (39)$$

in which: (i) K_{2m} is homogeneous polynomial of degree m in action variables I_1, I_2 and K_{2m+1} is higher degree polynomial than m (ii) $K_2 = \omega_1 I_1 - \omega_2 I_2$ with $\omega_{1,2}$ as positive constants (iii) $K_4 = -(AI_1^2 + BI_1 I_2 + CI_2^2)$, where A, B, C are constants to be determined. Since, K_2, K_4, \dots, K_{2m} are functions of I_1 and I_2 , the Hamiltonian (39) follows the Birkhoff normal form [2] up to the terms m . This can be obtained with some non-resonance condition on the frequencies ω_1, ω_2 . To state the Arnold–Moser theorem, we assume that K is in the required form.

Arnold–Moser Theorem: *The origin is stable for the system whose Hamiltonian is (39) provided for some $v, 2 \leq v \leq m, D_{2v} = K_{2v}(\omega_2, \omega_1) \neq 0$.*

Since, for Arnold–Moser theorem, Birkhoff normal form of the Hamiltonian is necessary and for Birkhoff normal form, assumption of non-resonance on fre-

quencies is required. The non-resonance condition of frequencies as in Deprit and Deprit-Bartholome [9], Kishor and Kushvah [19] is that if ω_1, ω_2 are frequencies of infinitesimal mass in linear dynamics and $\sigma \in \mathbb{Z}$ such that $\sigma \geq 2$, then

$$\sigma_1\omega_1 + \sigma_2\omega_2 \neq 0 \tag{40}$$

for all $\sigma_1, \sigma_2 \in \mathbb{Z}$ satisfying $|\sigma_1| + |\sigma_2| \leq 2\sigma$. This is also, called as condition of irrationality, which insures that there exists a symplectic normalizing transformation which transform the Hamiltonian (12) in the form of Hamiltonian (39). Coefficients of the normalized Hamiltonian are independent on the integer σ as well as to the transformation obtained. In specific

$$\det \begin{pmatrix} \frac{\partial^2 K}{\partial I_1^2} & \frac{\partial^2 K}{\partial I_1 \partial I_2} & \frac{\partial K}{\partial I_1} \\ \frac{\partial^2 K}{\partial I_2 \partial I_1} & \frac{\partial^2 K}{\partial I_2^2} & \frac{\partial K}{\partial I_2} \\ \frac{\partial K}{\partial I_1} & \frac{\partial K}{\partial I_2} & 0 \end{pmatrix}_{I_1, I_2=0} \tag{41}$$

is invariant of the Hamiltonian (39) with respect to the symplectic transformation considered. The nonlinear stability of perturbed triangular equilibrium points is analyzed through the Arnold–Moser theorem under these conditions. In classical case frequencies ω_1, ω_2 satisfy the condition $0 < \omega_2 < (1/\sqrt{2}) < \omega_1 < 1$. Therefore, if $\sigma = 2$, then irrationality condition (40) fails for following pairs of integers $\sigma_1 = 1, \sigma_2 = -2, \sigma_1 = -1, \sigma_2 = 2, \sigma_1 = 1, \sigma_2 = -3$ and $\sigma_1 = -1, \sigma_2 = 3$. First, two pairs of integers with condition (40) yield $(\omega_1/\omega_2) = (1/2)$ and last two pairs of integers give $(\omega_1/\omega_2) = (1/3)$, which are also known as second and third order resonance of the frequencies respectively. If $(\omega_1/\omega_2) = (1/2)$ or $\omega_1 = 2\omega_2$, then from Eqs. (27–28), we get

$$\frac{4}{25} = \frac{27}{16}\gamma^2 - 4\sqrt{3}W_1 - 2\sqrt{3}W_2 - \frac{2\epsilon_1}{3} + \frac{3\epsilon_2}{4}. \tag{42}$$

Simplifying Eq. (42), we have a quadratic equation in μ as

$$\frac{27}{16}\mu^2 - \frac{27}{16}\mu + \left(\sqrt{3}W_1 + \frac{\sqrt{3}W_2}{2} + \frac{3\epsilon_1}{16} - \frac{3\epsilon_2}{8} + \frac{1}{25} \right) = 0. \tag{43}$$

The solution $\mu = \mu_{c1}$ of Eq. (43) within the stability range $0 < \mu < \mu_c$ is

$$\mu_{c1} = 0.0242939 + 1.078820W_1 + 0.539409W_2 + 0.116785\epsilon_1 - 0.233571\epsilon_2. \tag{44}$$

This means, Arnold–Moser theorem fails at $\mu_{c1} \in (0, \mu_c)$. If $(\omega_1/\omega_2) = (1/3)$ or $\omega_1 = 3\omega_2$, then proceeding on similar basis, we find that Arnold–Moser theorem fails at $\mu = \mu_{c2}$, where

$$\mu_{c2} = 0.013516 + 1.054920W_1 + 0.527459W_2 + 0.114198\epsilon_1 - 0.228396\epsilon_2. \tag{45}$$

Table 1 μ_{c1} and μ_{c2} at different values of perturbing parameters

W_1	W_1	ϵ_1	ϵ_2	μ_{c1}	μ_{c2}
0.000	0.0000	0.000	0.00	0.024294	0.013516
0.005	0.0000	0.000	0.00	0.029688	0.018791
0.010	0.0000	0.000	0.00	0.035082	0.024065
0.015	0.0000	0.000	0.00	0.040476	0.029340
0.000	0.0040	0.000	0.00	0.026452	0.015626
0.000	0.0045	0.000	0.00	0.026721	0.015890
0.000	0.0050	0.000	0.00	0.026991	0.016153
0.000	0.0000	0.001	0.00	0.024411	0.013630
0.000	0.0000	0.002	0.00	0.024528	0.013744
0.000	0.0000	0.003	0.00	0.024644	0.013859
0.000	0.0000	0.000	0.01	0.021958	0.011232
0.000	0.0000	0.000	0.02	0.019623	0.008948
0.000	0.0000	0.000	0.03	0.017287	0.006664
0.005	0.0040	0.001	0.01	0.029627	0.018731

Equations (44–45) are similar to that of the results in Deprit and Deprit-Bartholome [9], Kishor and Kushvah [18] and agree with classical result in the absence of perturbing parameters. To see the effects of perturbing parameters on μ_{c1} and μ_{c2} , its numerical values are computed and presented in Table 1. From Table 1, it is clear that the values of μ_{c1} and μ_{c2} are very much affected from radiation pressures and P–R drags of the primaries.

5 Fourth Order Normalized Hamiltonian

Since, Birkhoff's normal form up to fourth order of the Hamiltonian is necessary to apply the Arnold–Moser theorem, which is computed from second order normalized Hamiltonian (13) using Lie transform method described in Coppola and Rand [5,7], Jorba [15], Celletti [3], Kishor and Kushvah [19]. As, in the paper of Coppola and Rand [7] as well as in the book of Celletti [3], higher order normalized Hamiltonian is

$$K = K_2 + K_3 + K_4 + \cdots + K_n + \cdots, \quad (46)$$

where

$$K_n = \sum K_{ijkl} X^i Y^j P_X^k P_Y^l \quad (47)$$

such that $i + j + k + l = n$. Quadratic part of K is $K_2 = H_2$, whereas K_n through the n th step of Lie transform is given as

$$K_n = \frac{1}{n} \{H_2, G_n\} + (\text{known terms}), \quad (48)$$

where Lie bracket of normalized quadratic Hamiltonian H_2 and generating function G_n is defined as

$$\{H_2, G_n\} = \frac{\partial H_2}{\partial X} \frac{\partial G_n}{\partial P_X} - \frac{\partial H_2}{\partial P_X} \frac{\partial G_n}{\partial X} + \frac{\partial H_2}{\partial Y} \frac{\partial G_n}{\partial P_Y} - \frac{\partial H_2}{\partial P_Y} \frac{\partial G_n}{\partial Y}. \tag{49}$$

Using H_2 from Eq. (13), it reduces to

$$\{H_2, G_n\} = i\omega_1 \left(P_X \frac{\partial G_n}{\partial P_X} - X \frac{\partial G_n}{\partial X} \right) + i\omega_2 \left(P_Y \frac{\partial G_n}{\partial P_Y} - Y \frac{\partial G_n}{\partial Y} \right). \tag{50}$$

The choice of generating function G_n is such that the above partial differential operator on G_n remove large possible number of terms from the expression of K_n . As, each terms of the K_n is of the form $\alpha X^i Y^j P_X^k P_Y^l$, where α is constant, we can assume terms in G_n of the form $\beta X^i Y^j P_X^k P_Y^l$, where constant β is to be determined. Therefore, we obtain that

$$\frac{\{H_2, G_n\}}{n} = \frac{i\beta}{n} [(k - i)\omega_1 - (l - j)] X^i Y^j P_X^k P_Y^l, \tag{51}$$

and hence,

$$\beta = \frac{i\alpha}{[(k - i)\omega_1 - (l - j)]}, \quad i + j + k + l = n. \tag{52}$$

This shows that even in the non-resonance case, the term of the form $X^i Y^j P_X^i P_Y^j$ in K_n can not be deleted because of vanishing denominator in (52) at $i = k, j = l$, whereas in the resonance case some additional non-removable terms occur while solving the generating function G_n . Hence, in non-resonance case, the Hamiltonian of the present problem can be written in the form of (46), in which

$$K_2 = i\omega_1 X P_X - i\omega_2 Y P_Y, \tag{53}$$

$$K_3 = 0, \tag{54}$$

$$K_4 = \frac{AX^2 P_X^2 + BX P_X Y P_Y + CY^2 P_Y^2}{2}, \tag{55}$$

where $A = 2K_{2020}$, $B = 2K_{1111}$, and $C = 2K_{0202}$. Using action variables $I_1 = iX P_X$ and $I_2 = iY P_Y$ in Eqs. (53–55), we get

$$K_2 = \omega_1 I_1 - \omega_2 I_2, \tag{56}$$

$$K_3 = 0, \tag{57}$$

$$K_4 = - \left(AI_1^2 + BI_1 I_2 + CI_2^2 \right). \tag{58}$$

Thus, normalized Hamiltonian up to fourth order is

$$K(I_1, I_2) = K_2 + K_3 + K_4 \\ = \omega_1 I_1 - \omega_2 I_2 - K_{2020} I_1^2 + K_{1111} I_1 I_2 + K_{0202} I_2^2, \quad (59)$$

which agree with that of Deprit and Deprit-Bartholome [9], Kushvah et al. [20], Kishor and Kushvah [19].

From Eq. (59), it is clear that fourth order normalized Hamiltonian is the function of only action variables I_1, I_2 , which shows that these are in Birkhoff normal form. The coefficients K_{ijkl} used in the Eq. (55 or 58) can be written into 5 parts such as $K_{ijkl1}, K_{ijkl2}, K_{ijkl3}, K_{ijkl4}$ and K_{ijkl5} for $i, j, k, l = 0, 1, 2, 3, 4$ such that $i + j + k + l = 4$. These coefficients corresponds to the term of classical part, terms with P–R drags W_1 and W_2 of first and second primary, radiation pressures $\epsilon_1 = 1 - q_1$ and $\epsilon_2 = 1 - q_2$ of first and second primary, respectively. In the absence of perturbing parameters i.e. for $W_1 = W_2 = \epsilon_1 = \epsilon_2 = 0, K_{ijkl} = K_{ijkl1}$. Therefore, K_{2020}, K_{1111} and K_{0202} become

$$K_{2020} = K_{20201} + K_{20202} + K_{20203} + K_{20204} + K_{20205}, \quad (60)$$

$$K_{1111} = K_{11111} + K_{11112} + K_{11113} + K_{11114} + K_{11115}, \quad (61)$$

$$K_{0202} = K_{02021} + K_{02022} + K_{02023} + K_{02024} + K_{02025}. \quad (62)$$

The algebraic expressions of above 15 coefficients on right hand sides of Eqs. (60–62) are too complicated and huge to be placed here hence, we avoid to present in the paper. These are utilized to compute the determinant $D_4 = K_4(\omega_2, \omega_1)$ for applying the Arnold–Moser theorem. For the simplicity, D_4 is expressed as

$$D_4 = \left(\frac{A_1}{B_1}\right) + \left(\frac{A_2}{B_2}\right) W_1 + \left(\frac{A_3}{B_3}\right) W_2 + \left(\frac{A_4}{B_4}\right) \epsilon_1 + \left(\frac{A_5}{B_5}\right) \epsilon_2, \quad (63)$$

where $A_i, B_i, i = 1, 2, 3, 4, 5$ are numerator and denominator of the coefficients, which correspond to classical part, P–R drags W_1 and W_2 of the primaries, radiation pressure $\epsilon_1 = 1 - q_1$ and $\epsilon_2 = 1 - q_2$ of the primaries, respectively. On simplification, we found that

$$A_1 = -35 + 541\omega_1^2\omega_2^2 - 644\omega_1^4\omega_2^4, \quad (64)$$

$$A_2 = 26244(2262 - 653b) - 27(5292162 - 4787719b)\omega_1^2\omega_2^2 \\ - 2(402982614 - 10430203b)\omega_1^4\omega_2^4 + 32(12457908 - 1490819b)\omega_1^6\omega_2^6 \\ + 1024(67581 + 1634b)\omega_1^8\omega_2^8, \quad (65)$$

$$A_3 = -78732(416 - 241\sqrt{3}b) - 27(3181248 + 4414649\sqrt{3}b)\omega_1^2\omega_2^2 \\ - 6(72610776 + 10390609\sqrt{3}b)\omega_1^4\omega_2^4 + 32(6808752 - 212191\sqrt{3}b)\omega_1^6\omega_2^6 \\ + 1024(36828 + 997\sqrt{3}b)\omega_1^8\omega_2^8, \quad (66)$$

$$A_4 = 8748(195\sqrt{3} - 584b) - 27(465795\sqrt{3} - 1556744b)\omega_1^2\omega_2^2$$

$$+2 \left(10722915\sqrt{3} - 11609036b \right) \omega_1^4 \omega_2^4 + 32 \left(200970\sqrt{3} - 103079b \right) \omega_1^6 \omega_2^6 - 512 \left(2565\sqrt{3} - 3217b \right) \omega_1^8 \omega_2^8, \tag{67}$$

$$A_5 = -8748 \left(507\sqrt{3} - 688b \right) + 27 \left(315819\sqrt{3} - 1533728b \right) \omega_1^2 \omega_2^2 + 2 \left(3085838\sqrt{3} + 759212b \right) \omega_1^4 \omega_2^4 - 32 \left(941526\sqrt{3} - 283835b \right) \omega_1^6 \omega_2^6 - 512 \left(10251\sqrt{3} + 877b \right) \omega_1^8 \omega_2^8, \tag{68}$$

$$B_1 = 8 \left(1 - 4\omega_1^2 \omega_2^2 \right) \left(4 - 25\omega_1^2 \omega_2^2 \right), \tag{69}$$

$$B_2 = 864ab, \tag{70}$$

$$B_3 = B_2, \tag{71}$$

$$B_4 = 1152ab, \tag{72}$$

$$B_5 = 576ab, \tag{73}$$

$$a = \left[\omega_1^2 \omega_2^2 \left(1 - 4\omega_1^2 \omega_2^2 \right) \left(4 - 25\omega_1^2 \omega_2^2 \right) \left(117 + 16\omega_1^2 \omega_2^2 \right) \right], \tag{74}$$

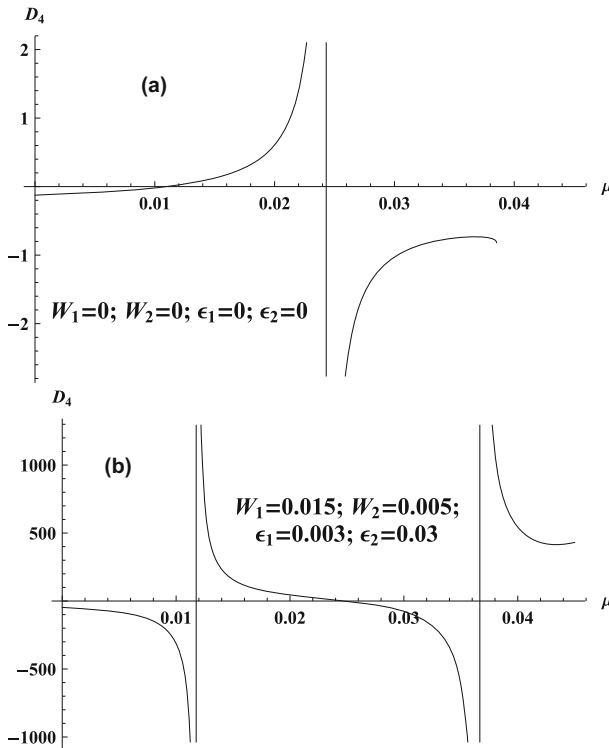


Fig. 2 Zero (μ_0) of the determinant D_4 within the stability range $0 < \mu < \mu_c$ at: **a** $W_1 = W_2 = \epsilon_1 = \epsilon_2 = 0$ (classical case); **b** $W_1 = 0.015, W_2 = 0.005, \epsilon_1 = 0.003, \epsilon_2 = 0.03$ (perturbed case)

$$b = \sqrt{(27 - 16\omega_1^2\omega_2^2)}. \tag{75}$$

In the absence of perturbing parameters,

$$D_4 = \frac{-35 + 541\omega_1^2\omega_2^2 - 644\omega_1^4\omega_2^4}{8(1 - 4\omega_1^2\omega_2^2)(4 - 25\omega_1^2\omega_2^2)}, \tag{76}$$

which agree with the classical result [9,19,20,25]. In order to analyze the nonlinear stability of triangular equilibrium points in non-resonance case using Arnold–Moser theorem, we plot the determinant D_4 with respect to the mass parameter μ to insure the value of $D_4 = K_4(\omega_2, \omega_1)$. From Figs. 2, 3, 4, 5 and 6, it is clear that within the linear stability range $0 < \mu < \mu_c$, there exists one value of mass parameter $\mu = \mu_0$, called the zero of D_4 , at which D_4 vanish in each case. Thus, Arnold–Moser theorem fails, which insure that in non-resonance case, triangular equilibrium points of the problem under analysis are unstable in nonlinear sense within the linear stability range $0 < \mu < \mu_c$. To see the effect of perturbing parameters, we have computed values of the zero (μ_0) of D_4 and critical mass ratio (μ_c) at different values of perturbing parameters $W_1, W_2, \epsilon_1, \epsilon_2$ and results are placed in Table 2. From Table 2, it is noticed

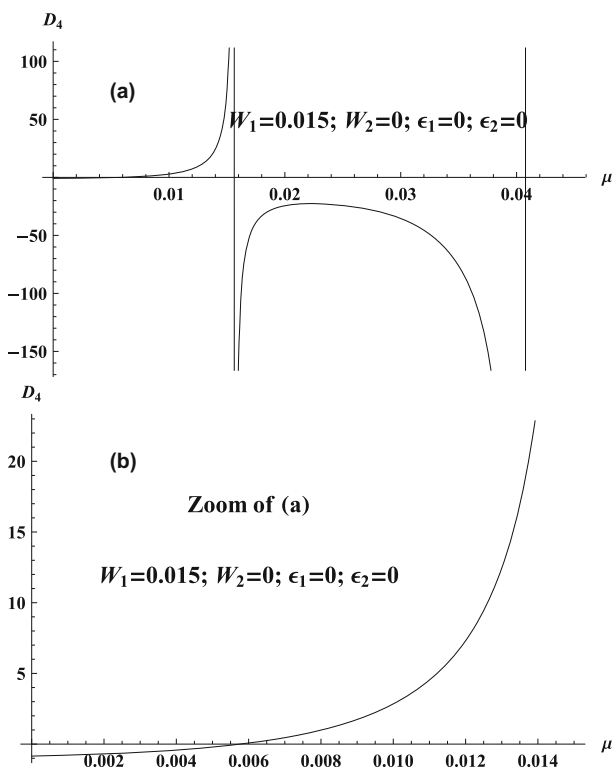


Fig. 3 Zero (μ_0) of the determinant D_4 within the stability range $0 < \mu < \mu_c$ at: **a** $W_1 = 0.015, W_2 = \epsilon_1 = \epsilon_2 = 0$ (only in presence of P–R drag of first primary); **b** zoom of specified region of **a**

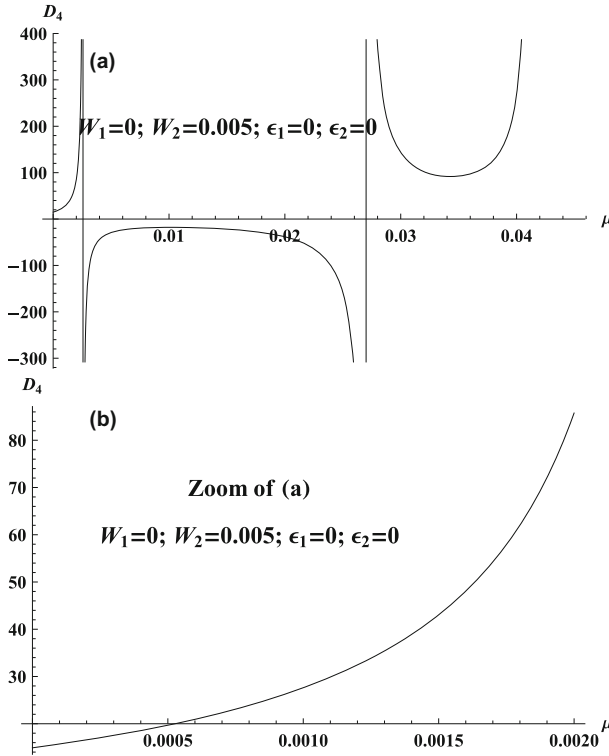


Fig. 4 Zero (μ_0) of the determinant D_4 within the stability range $0 < \mu < \mu_c$ at: **a** $W_2 = 0.005$, $W_1 = \epsilon_1 = \epsilon_2 = 0$ (only in presence of P-R drag of second primary); **b** zoom of specified region of **a**

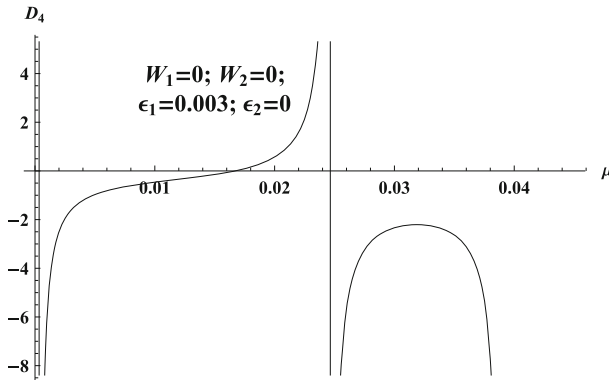


Fig. 5 Zero (μ_0) of the determinant D_4 within the stability range $0 < \mu < \mu_c$ at: $\epsilon_1 = 1 - q_1 = 0.003$, $W_1 = W_2 = \epsilon_2 = 0$ (only in presence of radiation pressure of first primary)

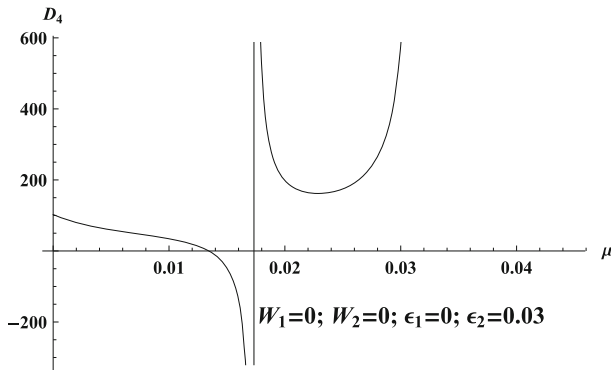


Fig. 6 Zero (μ_0) of the determinant D_4 within the stability range $0 < \mu < \mu_c$ at $\epsilon_2 = 1 - q_2 = 0.03$, $W_1 = W_2 = \epsilon_1 = 0$ (only in presence of radiation pressure of second primary)

that on increase in the values of W_1 , W_2 , ϵ_1 , value of critical mass μ_c increases but the value of μ_0 is nonzero in each case. On the other hand, on increase in the value of ϵ_2 , μ_c decreases with nonzero μ_0 . Thus, from the Figs. 2, 3, 4, 5 and 6 as well as from the Table 2, it is clear that radiation pressure and P–R drag of both the primaries affect the linear stability range of the problem significantly. The nonzero value of the zero (μ_0) of the determinant D_4 in the Arnold–Moser theorem under non-resonance case, insure the instability of triangular equilibrium points, within the range of stability $0 < \mu < \mu_c$.

6 Conclusions

We have considered the photogravitational restricted three body problem in the presence of radiation pressure force and P–R drag of both the massive bodies, which are radiating in nature. Analysis of nonlinear stability of the triangular equilibrium points is performed in non-resonance case using Arnold–Moser theorem under the influence of four perturbing parameters in the form of P–R drags W_1 , W_2 and mass reduction factors q_1 , q_2 , of both the primaries. First, we have normalized the Hamiltonian of the problem up to order four using Lie transform method and then Birkhoff normal form of the Hamiltonian constructed, which is necessary to apply the Arnold–Moser theorem in non-resonance case. The determinant D_4 of the Arnold–Moser theorem is computed analytically under the consideration of only linear order terms of perturbing parameters, which agree with that of Deprit and Deprit-Bartholome [9], Meyer and Schmidt [25], Kushvah et al. [20], Kishor and Kushvah [19] in the absence of perturbing parameters. To apply the Arnold–Moser theorem in non-resonance case, we have plotted the determinant D_4 with respect to the mass parameter μ within the stability range $0 < \mu < \mu_c$. It is observed that in presence as well as in absence of perturbing parameters, there exist a nonzero value of $\mu = \mu_0$ at which D_4 vanish (Figs. 2, 3, 4, 5, 6), which insure that triangular equilibrium points are unstable in nonlinear sense. The effect of perturbing parameters are also analyzed and it is found that on increasing the values of W_1 , W_2 , ϵ_1 , critical mass ratio μ_c increases,

Table 2 Zero (μ_0) of D_4 and critical mass ratio μ_c at different values of perturbing parameters

W_1	W_1	ϵ_1	ϵ_2	μ_0	μ_c
0.000	0.0000	0.000	0.00	0.010950	0.0385209
0.005	0.0000	0.000	0.00	0.000305	0.0393447
0.010	0.0000	0.000	0.00	0.000876	0.0401684
0.015	0.0000	0.000	0.00	0.005844	0.0409922
0.020	0.0000	0.000	0.00	0.010970	0.0418159
0.025	0.0000	0.000	0.00	0.016230	0.0426397
0.000	0.0040	0.000	0.00	0.001122	0.0388504
0.000	0.0045	0.000	0.00	0.000440	0.0388916
0.000	0.0050	0.000	0.00	0.000529	0.0389328
0.000	0.0055	0.000	0.00	0.000625	0.0389740
0.000	0.0060	0.000	0.00	0.000727	0.0390152
0.000	0.0000	0.001	0.00	0.014260	0.0385387
0.000	0.0000	0.002	0.00	0.015750	0.0385566
0.000	0.0000	0.003	0.00	0.016580	0.0385744
0.000	0.0000	0.004	0.00	0.017320	0.0385922
0.000	0.0000	0.005	0.00	0.017860	0.0386101
0.000	0.0000	0.000	0.01	0.018040	0.0381642
0.000	0.0000	0.000	0.02	0.015740	0.0378075
0.000	0.0000	0.000	0.03	0.013380	0.0374508
0.000	0.0000	0.000	0.04	0.011130	0.0370941
0.000	0.0000	0.000	0.05	0.008876	0.0367374
0.015	0.0050	0.003	0.031	0.02483	0.0438750

with the existence of nonzero μ_0 in each case, whereas on increasing the value of ϵ_2 , μ_c decreases with the existence of nonzero μ_0 (Fig. 1; Table 2). A similar trend is also seen in case of μ_{c1} and μ_{c2} (Table 1). Thus, we conclude that due to radiation pressure and P–R drag of both the primaries, the linear stability range of the problem get changed, significantly. Also, due to existence of nonzero value of the zero (μ_0) of the determinant D_4 in the Arnold–Moser theorem under non-resonance case, within the range of stability $0 < \mu < \mu_c$, triangular equilibrium points are unstable in nonlinear sense. Present analysis is limited up to first order terms of the perturbing parameter, which may be extended to higher order inclusion of the terms. The results obtained can help to analyze the more generalized problem under the influence of other perturbations such as albedo, solar wind drag, Stokes drag etc.

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