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Qualitative Theory of Dynamical Systems

The Hamilton–Jacobi Analysis of Powers of Singular Lagrangians: A Connection Between the Modified Schrödinger and the Navier–Stokes Equations

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Abstract Non-standard Lagrangians have gained recently an increasing interest in the theory of nonlinear differential equations, classical and quantum nonlinear dynamical systems. In this work, we discuss a number of dynamical systems characterized by powers of singular Lagrangians identified to non-standard Lagrangians based on the resulting Hamilton–Jacobi equation. A number of dynamical problems were addressed and a number of statements which support the non-standard Lagrangians formalism were postulated. After connecting the action to a certain complex wave function and in particular for the case of linear potentials, a link is established between the resulting modified Schrödinger equation which describes specific classes of quantum mechanical systems and the Navier–Stokes equation which describes the motion of viscous fluid matters.

Keywords Non-standard Lagrangians · Singular Lagrangians · Hamilton–Jacobi equation · Classical and quantum dynamics · Modified Schrödinger equation · Navier–Stokes equation

Mathematics Subject Classification 70S05 · 70H20

1 Introduction

Newtonian classical mechanics is the most fundamental part of physics which is traced back to Sir Isaac Newton. It continues till the present moment to play important roles

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in modern sciences, e.g. in fluid dynamics, planetary dynamics and astrophysics. In its conventional formulation, Newton's theory describes the relationship between a system assumed to be of mass m and forces acting upon it which generally depend on position x, velocity $\dot{x}$ and time t. The general equation of motion is therefore given by $F(\dot{x}, x, t) = m\ddot{x}$. However, in order to solve this set of 2nd-order differential equations, we need to know the exact form of $F(\dot{x}, x, t)$ which is somewhat complicated in particular when constraint forces are present. An additional problem with Newton's formulation is the natural use of Cartesian coordinates since jumping to a different coordinates system is usually awkward. Given these limitations, Joseph-Louis Lagrange and William Rowan Hamilton introduced a geometric formulation of Newton's mechanics where generalized coordinates and constraint forces are incorporated correctly in the mechanical theory. Besides, symmetries which are easily identified in the Lagrangian and Hamiltonian formalisms, have served to construct the Hamilton–Jacobi theory which plays a leading role in classical and quantum physics and steers physicists to develop the basic mathematical tools of classical field theory [22]. The Hamilton–Jacobi equation is mainly helpful in identifying conserved quantities for mechanical systems. It is the only formulation of classical mechanics in which the particle can be represented as a wave and hence going beyond the usual WKBJ approximation. The similarity between the Hamilton–Jacobi equation and the Schrödinger equation is well-known in physics. It is worth stressing that the interplay between the Hamilton–Jacobi-type equations describing the propagation of wave fronts in quantum mechanics and the Hamilton's equations has been largely investigated in quantum physics. Many emergent physical theories based on this analogy are now under studies in particular fluid dynamics. In fact, the interpretation of the Schrödinger wave equation in terms of a probability fluid dates back to 1927 [54] when Madelung convert the Schrödinger equation into a set of two nonlinear equations: the quantum Hamilton–Jacobi equation and the continuity equation by decomposing the complex wavefunction into amplitude and phase [36]. In other words, the Schrödinger wave equation takes a hydrodynamic form. It was observed that the velocity field satisfies the Euler equations for an ideal fluid and the quantum potential is proportional to the pressure field of the frictionless fluid. This hydrodynamic analogy offers new insights with regard to the wave equation [45,68,69] and it has been used to describe the carrier transport in open quantum systems. In parallel of this analogy, Bohm suggested that the quantum potential is a term within the Schrödinger equation which steers the movement of quantum particles and depends on the curvature of the amplitude of the wave function [13]. An alternative to Madelung's and Bohm's interpretations of the quantum potential was explored recently in [36] in which the quantum potential introduced to take into account the dissipative term in the equations of motion of the probability fluid. In that case, the velocity field does not satisfy the Euler's equation for an ideal fluid but the Navier–Stokes equation for a viscous fluid, i.e. the quantum potential is responsible for the occurrence of viscosity. This approach is interesting since classical friction in a fluid yields quantum effects and the fluid described by the classical Navier–Stokes equation becomes a wavefunction satisfying the Schrödinger equation. This framework suggests an alternative appealing description to the Schrödinger equation. It is noteworthy that although the connection between the classical Navier–Stokes equation and the Schrödinger equation, the

resulting systems of partial differential equations obtained differ completely from the quantum Navier–Stokes–Poisson equations derived from the collisional Wigner-BGK model using the moment method and the Chapman–Enskog expansion [48,72–74] used in quantum fluid models. In the present work, we will prove that, based on the notion of the notion of singular non-standard Lagrangians (NSL), a connection between a modified Schrödinger equation and the Navier–Stokes equation may be achieved starting from classical arguments. Our motivations are based on the following observations:

In fact, the majority of real-world physical systems, including gas dynamics, fluid mechanics, plasma physics, biological dynamics, thermodynamics and many more fields, are modeled by nonlinear differential equations. Despite the tremendous success of Lagrangian and Hamiltonian theories, attempts to describe nonlinear dynamics are doomed to failure. Moreover, these fundamental theories fail to describe non-conservative dynamical systems. One can use the inverse scheme which start with the equation of motion and then construct a consistent Lagrangian function. This is known as the "inverse problem" nevertheless it requires a complicated mathematical analysis to deal with it [46]. Recently, this problem was alleviated by means of a special class of Lagrangians known as "non-standard Lagrangians (NSL)" which allow mathematicians and physicists to identify several classes of equations of motion that admit a Lagrangian description [20,56,57]. There exist many motivations to deal with NSL despite their irregular physical forms since neither the ordinary kinetic term nor the classical potential function is introduced in the Lagrangian. Quantum field theorists used NSL in color confinement problem to describe large distances interactions in the region of applicability of classical theory [1]. NSL plays as well a significant role in theory of nonlinear differential equations, e.g. nonlinear 2nd order Riccati equation [14], the Liénard-type nonlinear differential equation [18,19], also in classical and quantum theories [2,15,16,27,29–31,52,63,76]. In fact, NSL arise in different forms, nevertheless in this paper, we will focus on power-law NSL of the form $L^{1+\varepsilon}$ introduced in [29] where ε is a real parameter. This type of Lagrangian is motivating for many reasons. In fact, by considering the action $S = \int L^{1+\varepsilon}(\dot{x}, x, t)dt$, the variational principle leads to a generalized Euler–Lagrange equation which contains higher-order derivative terms and has the general form:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\varepsilon}{L}\frac{\partial L}{\partial \dot{x}}\left(\frac{\partial L}{\partial t} + \dot{x}\frac{\partial L}{\partial x} + \ddot{x}\frac{\partial L}{\partial \dot{x}}\right).$$

This form is interesting since for singular Lagrangians of the form $L = a\dot{x} + bx$ ((a, b) are real or complex parameters) the resulting equation of motion is:

$$a^2\varepsilon\ddot{x} + ab(\varepsilon - 1)\dot{x} - b^2x = 0.$$

Obviously, for $\varepsilon = 1$ and $b^2 = -a^2$, this equation is reduced to $\ddot{x} + x = 0$ which is the equation of motion of a harmonic oscillator derived at the moment from a complex singular Lagrangian. However, for $\varepsilon \neq 1$, the differential equation describes dissipative oscillators. Therefore, this technique works with singular or degenerate

Lagrangians yet it fails if the standard formalism is used since the Hamilton equations of motion cannot be obtained by the common procedure [17]. This problem was in fact solved using Dirac's method which consists of constructing canonical equations at the expense of introducing undetermined variables [21,28]. Alternative solutions were addressed in literature [4,22] yet the topic still required more plausible solutions. In a recent work [31], it was observed that the power-law NSL $L^{1+\varepsilon}$ modifies the Hamilton–Jacobi equation and consequently the Schrödinger equation. A number of interesting physical properties not obtained in the traditional formalism were derived accordingly. The Hamilton–Jacobi formulation for singular systems was addressed in literature through different arguments and based on standard Lagrangians [7,59]. It is worth mentioning that for $b^2 = -a^2$ the Lagrangian $L = a\dot{x} + bx$ is complexified. However, complex Lagrangians are not forbidden in literature since they occur not only in quantum but also in classical systems [9–11,32,33]. It was observed recently that within complexified classical mechanics, the Ostrogradsky theorem which predicts boundless kinetic terms may be evaded and lead to real energy spectrum [60].

Motivated by these outcomes, the main objective of the present paper is to discuss several implications of the Hamilton–Jacobi equation which result from the power-law NSL $L^{1+\varepsilon}$ in particular when the Lagrangian is singular. We will prove the importance of the resulting Hamilton–Jacobi equation at classical and quantum levels. The jump from classical to quantum dynamics will be made following Bohm approach [13] which is closed linked to the work of Madelung. Since Bohm's and Madelung's approaches result on a hydrodynamical form of the Schrödinger equation, it will be motivating to derive the hydrodynamical form of the modified Schrödinger equation that results from the power-law NSL $L^{1+\varepsilon}$. The strategy we will follow is therefore completely different from most previous approaches. A comparison between both hydrodynamical forms will be crucial accordingly. The paper is structured as follows: In Sect. 2, we review briefly the basic setups of the power-law NSL $L^{1+\varepsilon}$, then we construct the Hamilton–Jacobi equation and we discuss a number of classical dynamical problems in particular for systems characterized by actions of powers of singular Lagrangians; in Sect. 3, we consider a special class of singular Lagrangian and we discuss its implication in quantum dynamics by connecting the action of the theory to a certain complex wave function; finally in Sect. 4 conclusions are given.

2 Hamilton–Jacobi Formulation: Classical Dynamical Systems

We start by reviewing the basic concepts: given a mechanical system with N degrees of freedom characterized by the power-law NSL $L^{1+\varepsilon}$, the corresponding Hamiltonian functions are defined as the Legendre transformation [42]:

$$H(x, p, t) = \sum_{k=1}^{N} \dot{x}_k \cdot p_k - L^{1+\varepsilon}(x, p), \tag{1}$$

where

$$p = \frac{\partial L^{1+\xi}}{\partial \dot{x}} = (1+\varepsilon)L^{\varepsilon}\frac{\partial L}{\partial \dot{x}}, \tag{2}$$

is the non-standard generalized momenta [29]. It is notable that in this approach:

$$\dot{p} = \frac{\partial L^{1+\xi}}{\partial x} = (1+\varepsilon)L^{\varepsilon}\frac{\partial L}{\partial x}. \tag{3}$$

Remark 2.1 One may logically ask "how can be reasonable that the Hamiltonian $H(x, p, t)$ *has the proper physical dimension?" In fact, it should be pointed out that in our approach, in order to obtain a physical Hamiltonian dimension, one must introduce a certain parameter* A *in the Hamiltonian function with dimension* $[ML^2T^{-2}]^{-\varepsilon}$ *in a way the Hamiltonian recovers the correct energy dimension* $[E] = [ML^2T^{-2}]$. (M, L, T) *refer respectively to the mass, length and time. We effectively recuperate the dimensional problem, yet in our approach we set* $\mathrm{A} = 1$ *for mathematical simplicity.*

The Hamilton's equations of motion in the present formalism are: $\dot{x}_k = \partial H/\partial p_k$ and $\dot{p}_k = -\partial H/\partial x_k$ augmented by the constraint:

$$\frac{\partial H}{\partial t} = -(1+\varepsilon)L^{\varepsilon}\frac{\partial L}{\partial t}. \tag{4}$$

The couple (x_k, p_k) is considered as independent variables. Let us look into transformations which do not change the form of Hamilton's equations:

$$(\dot{x}_k, \dot{p}_k) = \left(\frac{\partial H}{\partial p_k}, -\frac{\partial H}{\partial x_k}\right). \tag{5}$$

To do this, we consider the transformations $(X_k(p, q), P_k(p, q))$ such that:

$$(\dot{X}_k, \dot{P}_k) = \left(\frac{\partial H'}{\partial P_k}, -\frac{\partial H'}{\partial X_k}\right), \tag{6}$$

where

$$H'(X, P, t) = \sum_{k=1}^{N} \dot{X}_k \cdot P_k - L'^{1+\varepsilon}(X, P). \tag{7}$$

We require that both Hamiltonians yield the same equations of motion, i.e. the two Hamiltonians differ only by a total derivative of a function $F = F(x, p, X, P, t)$ with $4n + 1$ variables with respect to time which mathematically means that:

$$\dot{x}_k p_k - H = \dot{X}_k P_k - H' + \dot{F}. \tag{8}$$

If we choose $F = F(x, P, t)$ then one may check that $X_k = \partial F/\partial P_k$, $p_k = \partial F/\partial q_k$ and $H' = H + \partial F/\partial t$ [67]. By denoting $S = S(x, P, t)$ and putting $H' = 0$ we get straightforwardly $H + \partial S/\partial t = 0$. However, if $H' = 0$ we obtain $(\dot{X}_k, \dot{P}_k) = (0, 0)$ which gives $(X_k, P_k) = (\alpha_k, \beta_k)$; (α, β) are integration constants with $\alpha_k = \partial S/\partial \beta_k$ and $p_k = \partial S/\partial q_k$. The Hamilton–Jacobi equation is then:

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0, \tag{9}$$

which contains $2n+1$ derivatives and therefore the solution contains $2n+1$ constants.

Remark 2.2 For the case of singular Lagrangian of the form $L = a\dot{x} + bx$, the action is given by $S = \int L^{1+\varepsilon} dt = \int (a\dot{x} + bx)^{1+\varepsilon} dt$ and for that reason we entitle the term $(a\dot{x} + bx)^{1+\varepsilon}$ as a NSL. Evidently, for $\varepsilon = 1$ and $b = ia$, the NSL is $a^2(\dot{x} + ix)^2$ which is the square of the singular Lagrangian $L = a(\dot{x} + ix)$.

To illustrate, we choose once more the Lagrangian $L = a\dot{x} + bx$ with $b^2 = -a^2$, i.e. $b = ia$. The corresponding Hamiltonian is deduced from $H(x, p, t) = \sum_{k=1}^{N} \dot{x}_k \cdot p_k - L^{1+\varepsilon}$ and takes the form:

$$H(x, p) = \frac{p}{a}\left(\left(\frac{p}{a(1+\varepsilon)}\right)^{\frac{1}{\varepsilon}} - bx\right) - \left(\frac{p}{a(1+\varepsilon)}\right)^{\frac{1+\varepsilon}{\varepsilon}}. \tag{10}$$

The Hamilton–Jacobi equation is $H(x, \partial S/\partial x) + \partial S/\partial t = 0$ and we naturally assume a solution of the form $S = A(t) + B(x)$. Consequently we obtain $H(x, \partial B/\partial x) + \partial A/\partial t = 0$. Since the Lagrangian is time-independent, then the Hamiltonian is constant and is equal to the energy E, i.e. $\partial H/\partial t = -(1+\varepsilon)L^{\varepsilon}\partial L/\partial t$. Therefore $A = -Et$ and we find:

$$E = \frac{1}{a}\frac{dB}{dx}\left(\left(\frac{1}{a(1+\varepsilon)}\frac{dB}{dx}\right)^{\frac{1}{\varepsilon}} - bx\right) - \left(\frac{1}{a(1+\varepsilon)}\frac{dB}{dx}\right)^{\frac{1+\varepsilon}{\varepsilon}}. \tag{11}$$

In particular for $\varepsilon = 1$ we obtain:

$$E = \left(\frac{1}{2a}\frac{dB}{dx}\right)^2 - \frac{b}{a}x\frac{dB}{dx}. \tag{12}$$

After arrangement, we get:

$$\frac{dB}{dx} = 2ia^2x \pm 2a\sqrt{E - a^2x^2}. \tag{13}$$

Performing the integration, we find:

$$B = ia^2x^2 \pm \left(\frac{E}{2}\sin^{-1}\left(\frac{ax}{\sqrt{E}}\right) + \frac{ax}{2}\sqrt{E - a^2x^2}\right), \tag{14}$$

where we have assumed the integration constant equal to zero. The action takes then the form:

$$S = -Et + ia^2x^2 \pm \left(\frac{E}{2}\sin^{-1}\left(\frac{ax}{\sqrt{E}}\right) + \frac{ax}{2}\sqrt{E - a^2x^2}\right). \tag{15}$$

Using $\alpha = \partial S/\partial\beta = \partial S/\partial E$ we find $x = \sqrt{2\bar{E}}\sin(\alpha + t)/a$ where $\bar{E} = 2E$. Consequently we recover the usual solution to the equation of harmonic oscillator starting

from a singular complexified Lagrangian characterized by a complexified Hamiltonian and a complexified energy. In other words, the solution to the corresponding complexified singular Lagrangian exhibit analogous behavior to the harmonic oscillator. This result is expected to have interesting quantum mechanical effects. This is not surprising since early works showed that a number of quantum dynamical systems are characterized by complex Hamiltonians and give rise to complexified energies [11,23,39,40]. Then we can launch the following 1st-statement:

Statement 1 *Given the singular complexified Lagrangian* $L = a(\dot{x} \pm ix)$ *and the classical action* $S = \int L^2 dt$. *The corresponding Hamilton–Jacobi equation results on a 2nd-order differential equation which coincides with the harmonic oscillators and is characterized by a complexified Hamiltonian and a complex energy.*

As a second illustration, we consider the singular Lagrangian $L = A\dot{x} + B\dot{y} + C\sqrt{y}$ where (A, B, C) are real or complex parameters. The conjugate momenta are given by:

$$p_x = (1+\varepsilon)A\left(A\dot{x} + B\dot{y} + C\sqrt{y}\right)^{\varepsilon}, \tag{16}$$

and

$$p_y = (1+\varepsilon)B\left(A\dot{x} + B\dot{y} + C\sqrt{y}\right)^{\varepsilon}, \tag{17}$$

which gives $A = B$ and $p_x = p_y$. The Hamiltonian takes now the following form:

$$H = \left(\frac{1}{A}\left(\frac{p_x}{(1+\varepsilon)A}\right)^{\frac{1}{\varepsilon}} - \frac{C}{A}\sqrt{y}\right)p_x - \left(\frac{p_x}{(1+\varepsilon)A}\right)^{\frac{1+\varepsilon}{\varepsilon}}. \tag{18}$$

The corresponding Hamilton–Jacobi equation is:

$$\left(\frac{1}{A}\left(\frac{1}{(1+\varepsilon)A}\frac{\partial S}{\partial x}\right)^{\frac{1}{\varepsilon}} - \frac{C}{A}\sqrt{y}\right)\frac{\partial S}{\partial x} - \left(\frac{1}{(1+\varepsilon)A}\frac{\partial S}{\partial x}\right)^{\frac{1+\varepsilon}{\varepsilon}} + \frac{\partial S}{\partial t} = 0. \tag{19}$$

Since this is not a completely separable equation we write $S = \sqrt{y}S_x + yS_t$ with $H = E$. Accordingly, for $\varepsilon = 1$ we can write Eq. (19) as:

$$\left(\frac{1}{2A}\frac{dS_x}{dx}\right)^2 - \frac{C}{A}\frac{dS_x}{dx} - E = 0. \tag{20}$$

This gives $S_x = (2AC \pm \sqrt{4A^2C^2 + 4A^2E})x$ after dropping the irrelevant constant of integration. The total action is therefore:

$$S = (2AC \pm \sqrt{4A^2C^2 + 4A^2E})x\sqrt{y} - yEt. \tag{21}$$

Accordingly we get:

$$p_x = \pm 2A\sqrt{C^2 + E}\sqrt{y}, \tag{22}$$

and

$$p_y = (2AC \pm 2A\sqrt{C^2 + E})x - 2\sqrt{y}Et. \tag{23}$$

Since $p_x = p_y$, we find after algebra:

$$\sqrt{y} = \frac{AC \pm A\sqrt{C^2 + E}}{Et \pm A\sqrt{C^2 + E}} x. \tag{24}$$

We can take $E = -C^2$ with $C = i \in \mathbb{C}$ which give rise to a complexified singular Lagrangian with real energy spectra. The action is therefore given by:

$$S = 2iAx\sqrt{y} - yEt. \tag{25}$$

Taking the constants A and E as new momentum variables, we have:

$$q_x = \frac{\partial S}{\partial A} = 2ix\sqrt{y}, \tag{26}$$

and

$$q_y = \frac{\partial S}{\partial E} = -2\sqrt{y}Et. \tag{27}$$

Since from Eq. (24) we have $\sqrt{y} = -Ax/Ct$ we get:

$$x^2 = -\frac{q_x t}{2A}, \tag{28}$$

and

$$\sqrt{y} = \frac{q_y}{2C^2 t}, \tag{29}$$

which give:

$$y = \left(\frac{Aq_y}{q_x x^2}\right)^2, \tag{30}$$

with $A < 0$. The dynamics is therefore hyperbolic and is similar to the scattering problem which occurs in atomic physics [64]. This result leads unavoidably to the conclusion that the particle trajectory deviates from a parabola since the Lagrangian is $L = A(\dot{x} + \dot{y}) + i\sqrt{y}$ and the action of the theory $S \propto \int L^2 dt$ contains coupled terms whereas in the standard approach, the action $S \propto \int (A\dot{x}^2 + B\dot{y}^2 + Cy)dt$. The following 2nd-statement may be addressed:

Statement 2 *Given the singular complexified Lagrangian $L = A\dot{x} + B\dot{y} + i\sqrt{y}$ and the classical action $S = \int L^2 dt$. The corresponding Hamilton–Jacobi equation results on a hyperbolic dynamics similar to a scattering process and is characterized by a real energy.*

One may argue that if the Hamilton–Jacobi equation is not separable then the system can be integrable if the Lagrangian is concurrently of power-law form and complexified. It is notable that in the Hamilton–Jacobi approach, integrable dynamical systems are characterized by a complete solution of the Hamilton–Jacobi equation in the Liouville sense. In the case of compact energy level sets, this requires finding smooth and

invertible action-angle variables by means of the corresponding coordinate transformations. In many cases, this can be done by means of symmetries since they provide constants of motion which in their turn leads to find the action-angle coordinates. However, for bounded orbits, trajectories may be quasi-periodic similar to those occurring in the quantum Toda lattice and accordingly the action-angle coordinates may not exist [43].

As a third illustration we consider $L = \sqrt{\dot{x}}$. The Hamiltonian is then given by:

$$H = p\left(\frac{2p}{(1+\varepsilon)}\right)^{\frac{2}{\varepsilon-1}} - \left(\frac{2p}{(1+\varepsilon)}\right)^{\frac{2(\varepsilon+1)}{\varepsilon-1}}, \tag{31}$$

with $\varepsilon \neq 1$. For $\varepsilon = 3$, Eq. (31) is reduced to:

$$H = \frac{p^2}{2} - \frac{p^4}{16}. \tag{32}$$

The corresponding Hamilton–Jacobi equation is then $H(x, \partial S/\partial x) + \partial S/\partial t = 0$ and since we have a separable solution we assume normally a solution of the form $S = A(t) + B(x)$ which gives $H(x, \partial B/\partial x) + \partial A/\partial t = 0$ with $A = -Et$. We get therefore:

$$E = \frac{1}{2}\left(\frac{dB}{dx}\right)^2 - \frac{1}{16}\left(\frac{dB}{dx}\right)^4, \tag{33}$$

which in its turn gives:

$$\frac{dB}{dx} = \pm 2\sqrt{1 \mp \sqrt{1-E}}. \tag{34}$$

After simple integration, we find:

$$B(x) = \left(\pm 2\sqrt{1 \mp \sqrt{1-E}}\right)x, \tag{35}$$

and the action takes then the form:

$$S = -Et \pm 2\sqrt{1 \mp \sqrt{1-E}}\,x. \tag{36}$$

Using $\alpha = \partial S/\partial\beta = \partial S/\partial E$ we find $x = \pm\sqrt{1-E}(\alpha + t)$ which corresponds for a linear motion characterized by $0 < E < 1$. It is obvious from Eq. (31) that for $\varepsilon = -3$ (inverse Lagrangians), we obtain $H = p - i\sqrt{p}$ which is a complexified Hamiltonian. Using similar steps we find:

$$B(x) = \frac{\left(-2E - 1 \pm \sqrt{1+4E}\right)x}{2}, \tag{37}$$

and therefore:

$$x = \frac{\alpha + t}{-1 \pm \frac{1}{\sqrt{1+4E}}}. \tag{38}$$

The following statement holds accordingly:

Statement 3 *Given the singular complexified Lagrangian* $L = \sqrt{\dot{x}}$ *and the classical action* $S = \int L^4 dt$. *The consequent Hamilton–Jacobi equation results on a linear dynamics characterized by a complexified Hamiltonian and a real bounded energy* $0 < E < 1$.

As a final illustration in this section, we discuss dissipative dynamics using the Hamilton–Jacobi equation and the singular Lagrangian $L = a\dot{x} + bx$. In fact, in the Hamiltonian formulation of dissipative systems, several methodologies have been used in literature, e.g. the Rayleigh dissipation method where the friction force is proportional to the velocity (see [63] and references therein); Bateman method which aims to include supplementary coordinates in the Lagrangian [6]; Bauer argument which states that "the equations of motion of a dissipative linear dynamical system with constant coefficients are not given by a variational principle" [8]; methods based on fractional calculus of variations [34] among others (see [3] and references therein). The Hamilton–Jacobi equation was also used to explore dissipative systems [5,26,44, 47,55,61]. As stated in the introductory text of this work, if we consider the Lagrangian $L = a\dot{x} + bx$ then the resulting equation of motion in our approach is:

$$a^2\varepsilon\ddot{x} + ab(\varepsilon - 1)\dot{x} - b^2 x = 0, \tag{39}$$

which describes dissipation for $\varepsilon \neq 1$. In particular for $b = i$, $a = (i \pm \sqrt{3})/2$ and $\varepsilon = (1 \pm i\sqrt{3})/4$, the differential equation is reduced to $\ddot{x} + \dot{x} + x = 0$ and describes unforced underdamping harmonic oscillators. For $\varepsilon = 2$, the differential equation takes the form $2a^2\ddot{x} + ab\dot{x} - b^2 x = 0$ which has the characteristic equation $2a^2 s^2 + abs - b^2 = 0$ with characteristic roots $(-b \pm 3b)/2$. Undoubtedly, one can choose different values of (a, b, ε) so that the differential Eq. (39) describes overdamping oscillations, e.g. $(a, b, \varepsilon) \approx (4.8i, i, -0.04)$ which results into $\ddot{x} + 5\dot{x} + x = 0$.

In particular for $a = i$ and $b = 1$ the system describes an exotic oscillator which is critically damped and characterized by a complex damping term. Such types of oscillators are discussed in literature and are connected to quantization (see [24,39] and references therein). The corresponding momentum is given by:

$$p = (1 + \varepsilon)a\,(a\dot{x} + bx)^{\varepsilon}\,, \tag{40}$$

and the canonical Hamiltonian has the form of Eq. (10). For $\varepsilon = 2$, the Hamiltonian is reduced to:

$$H(x, p) = \frac{2}{3}\sqrt{\frac{1}{3}}\left(\frac{1}{a}\right)^{\frac{3}{2}} p^{\frac{3}{2}} - \frac{b}{a}px. \tag{41}$$

The Hamilton–Jacobi equation is $H(x, \partial S/\partial x) + \partial S/\partial t = 0$ and a solution of the form $S = A(t) + B(x)$ exists accordingly. Since the Lagrangian is time-dependent, then $A = -Et$ and we obtain:

$$E = \frac{2}{3}\sqrt{\frac{1}{3}}\left(\frac{1}{a}\right)^{\frac{3}{2}}\left(\frac{dB}{dx}\right)^{\frac{3}{2}} - \frac{b}{a}x\frac{dB}{dx}, \tag{42}$$

which gives:

$$\frac{dB}{dx} = \frac{x^2}{3I^2}\left(1+3\sqrt[3]{\frac{27}{2}\frac{a^2I^4E^2}{b^2x^6}-\frac{9aI^2E}{bx^4}+\frac{3\sqrt{3}}{2}\left(\frac{27a^4I^8E^4}{b^4x^{12}}-\frac{4a^3I^6E^3}{b^3x^9}\right)}+1 - \frac{\sqrt[3]{2}\left(-\frac{6aI^2E}{bx^3}-1\right)}{\sqrt[3]{\frac{27a^2I^4E^2}{b^2x^6}-\frac{18aI^2E}{bx^4}+3\sqrt{3}\left(\frac{27a^4I^8E^4}{b^4x^{12}}-\frac{4a^3I^6E^3}{b^3x^9}\right)}+2}\right), \tag{43}$$

where $I = -(2a/3b)(1/a)^{3/2}\sqrt{1/3}$. For $x >> 1$, we can approximate Eq. (43) by:

$$\frac{dB}{dx} \approx \frac{9b^2ax^2}{4}\left(4+\sqrt[3]{2}+\frac{8\sqrt[3]{2}E}{9b^3}\frac{1}{x^3}\right), \tag{44}$$

and after integration we find:

$$B(x) \approx 3\left(4+\sqrt[3]{2}\right)b^2ax^3+\frac{2\sqrt[3]{2}E}{ab}\ln x, \tag{45}$$

where the integration constant is set equal to zero. The action takes now the form:

$$S = -Et+3\left(4+\sqrt[3]{2}\right)b^2ax^3+\frac{2\sqrt[3]{2}E}{ab}\ln x. \tag{46}$$

Using $\alpha = \partial S/\partial\beta = \partial S/\partial E$ we find:

$$x \propto\propto e^{\frac{ab(\alpha+t)}{2\sqrt[3]{2}}}. \tag{47}$$

For $a = i$ and $b = 1$ we obtain:

$$x \propto e^{\frac{i(\alpha+t)}{2\sqrt[3]{2}}}. \tag{48}$$

We state thus the following:

Statement 4 *Given the singular complexified Lagrangian $L = i\dot{x} + x$ and the classical action $S = \int L^3 dt$. The consequential Hamilton–Jacobi equation results on an exponential dynamics characterized by a complexified Hamiltonian, a complexified energy and a critically damped complex exotic oscillator.*

3 Hamilton–Jacobi Formulation: Quantum Dynamics

In this section, we will discuss a mechanical scenario by considering the singular Lagrangian $L = a\dot{x} - V(x)$ where $V(x)$ is a function of the position identified to a

potential. Our aim is to connect the resulting Hamilton–Jacobi equation to quantum dynamics. The corresponding Hamiltonian is given by:

$$H = \frac{p}{a}\left(\left(\frac{p}{a(1+\varepsilon)}\right)^{\frac{1}{\varepsilon}} + V(x)\right) - (a\dot{x} - V(x))^{1+\varepsilon}. \tag{49}$$

In fact, the implications of the classical Hamilton–Jacobi equations in quantum dynamics is done for the first time by Schrödinger by connecting the action to a complex wave function ψ by the relation $\psi = Ae^{iS/\hbar}$ where $\hbar$ is the Planck's constant which is set equal to one for convenience and A is the wave amplitude [65,66]. Since $p = \partial S/\partial x$, then we get easily $p\psi = -i\partial\psi/\partial x \equiv -i\nabla_x\psi$, i.e. the momentum plays the role of a complex differential operator. Besides $\psi\partial S/\partial t = -i\partial\psi/\partial t$. The Hamiltonian takes therefore the following form:

$$H = -\frac{i}{a}\left(\left(\frac{-i}{a(1+\varepsilon)}\nabla_x\right)^{\frac{1}{\varepsilon}} + V(x)\right)\nabla_x - \left(-\frac{i}{a(1+\varepsilon)}\nabla_x\right)^{\frac{1+\varepsilon}{\varepsilon}}, \tag{50}$$

and the modified Schrödinger equation is now nonlinear and takes the form:

$$i\frac{\partial\psi}{\partial t} = -\frac{i}{a}\left(\left(\frac{-i}{a(1+\varepsilon)}\nabla_x\right)^{\frac{1}{\varepsilon}} + V(x)\right)\nabla_x\psi - \left(-\frac{i}{a(1+\varepsilon)}\nabla_x\right)^{\frac{1+\varepsilon}{\varepsilon}}\psi. \tag{51}$$

Remark 3.1 *For $\varepsilon \neq 1$,* Eq. (51) *is no-longer interpreted as a modified Schrödinger equation since the Laplacian operator will be absent. To clarify, for $\varepsilon = 1/2$* Eq. (51) *is reduced to the* 3rd*-order partial differential equation*:

$$\frac{\partial\psi}{\partial t} = -\frac{20i}{27a^3}\Delta_x\nabla_x\psi - \frac{i}{a}V(x)\nabla_x\psi,$$

whereas for $\varepsilon = 2$, we obtain:

$$i\frac{\partial\psi}{\partial t} = \frac{2\sqrt{3}}{9}\left(\frac{-i}{a}\right)(\nabla_x)^{\frac{3}{2}}\psi - \frac{i}{a}V(x)\nabla_x\psi,$$

which is characterized by a fractional gradient operator. Equation (51) *gives accordingly a family of partial differential equations although we start from a simple singular Lagrangian $L = a\dot{x} - V(x)$.*

For $\varepsilon = 1$ and $a^2 = 1/2$, we can write Eq. (51) as:

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\Delta_x\psi - i\sqrt{2}V(x)\nabla_x\psi. \tag{52}$$

This is a modified Schrödinger equation which differs from the standard equation because of the presence of the last term which couples the potential to the momentum

in place of the usual term $V(x)\psi$. Here we have set the mass of the particle is equal to one for straightforwardness. Considering the conjugate wave function $\psi^* = Ae^{-i\theta}$, θ being the real phase of the wave function [12,13], we can write Eq. (51) as:

$$i\frac{\partial\psi^*}{\partial t} = \frac{1}{2}\Delta_x\psi^* - i\sqrt{2}V(x)\nabla_x\psi^*. \tag{53}$$

By multiplying Eq. (52) by ψ^* and Eq. (53) by ψ and adding the resulting equations, we obtain:

$$i\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right) = -\frac{1}{2}\left(\psi^*\Delta_x\psi - \psi\Delta_x\psi^*\right) - i\sqrt{2}V(x)\left(\psi^*\nabla_x\psi + \psi\nabla_x\psi^*\right). \tag{54}$$

This equation may be written as:

$$\frac{\partial A^2}{\partial t} + \frac{1}{2i}\left(\psi^*\Delta_x\psi - \psi\Delta_x\psi^*\right) + \sqrt{2}V(x)\nabla_x A^2 = 0. \tag{55}$$

A^2 is interpreted now as a probability density ρ (the particle density). Introducing the density flux J by the relation:

$$divJ = \frac{\psi^*\Delta_x\psi - \psi\Delta_x\psi^*}{2i}, \tag{56}$$

we can write Eq. (55) as:

$$\frac{\partial\rho}{\partial t} + divJ + \sqrt{2}V(x)\nabla_x\rho = 0. \tag{57}$$

Equation (57) is the modified continuity equation since it differs of the continuity equation of standard quantum mechanics, i.e. in the absence of the potential this equation is reduced to the standard continuity equation. In fact, since $\psi = Ae^{i\theta} = Ae^{iS}$ and $p = \nabla_x S$, then we can split Eq. (57) after algebra into real and imaginary parts as follows:

$$\frac{\partial A}{\partial t} = -\frac{1}{2}\left(\Delta_x S + 2\nabla_x A\cdot\nabla_x S\right) - \sqrt{2}V(x)\nabla_x A, \tag{58}$$

$$-\frac{\partial S}{\partial t} = -\frac{1}{2A}\left(\Delta_x A - A\left(\nabla_x S\right)^2\right) + \sqrt{2}V(x)\nabla_x S. \tag{59}$$

If we neglect the 1st-term in the parenthesis of Eq. (58), then this equation will be reduced to the modified Hamilton–Jacobi equation:

$$-A\frac{\partial S}{\partial t} = \frac{A}{2}\left(\nabla_x S\right)^2 + \sqrt{2}V(x)A\nabla_x S. \tag{60}$$

Multiplying Eq. (60) by $2A$ gives:

$$\frac{\partial A^2}{\partial t} + \nabla\left(A^2 \nabla S\right) + \sqrt{2} V(x) \nabla_x A^2. \tag{61}$$

Equation (61) is similar to Eq. (57) since $A^2 = \psi\psi^*$ and $j = A^2 \nabla_x S$. Introducing the velocity field of the quantum probability by the relation $\mathbf{v} = \nabla_x S = \nabla_x \theta$, then the gradient of Eq. (59) for irrotational flow gives:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla_x)\, \mathbf{v} - \sqrt{2} \mathbf{v} \nabla_x V(x) = 0. \tag{62}$$

However, an irrotational flow is described by the following Navier–Stokes equation [36,70]:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla_x)\, \mathbf{v} + \frac{1}{\rho} \nabla_x p - \frac{\zeta + \frac{4\eta}{3}}{\rho} \Delta_x \mathbf{v} = 0. \tag{63}$$

Here ζ and η are respectively the shear viscosity and the bulk viscosity of the fluid and p is the density of the fluid. Equations (62) and (63) are similar if the following relation holds:

$$\nabla_x p + \sqrt{2} \mathbf{j} \nabla_x V(x) = \left(\zeta + \frac{4\eta}{3}\right) \Delta_x \mathbf{v}, \tag{64}$$

where $\mathbf{j} = \rho \mathbf{v}$ is the mass flux which is a classical version of $j = A^2 \nabla_x S$ [71]. For the case of a linear potential, e.g. bouncing ball due to gravity effect and gas in a gravitational field with $V(x) = \sqrt{2} x/2$, Eq. (64) is reduced to:

$$\nabla_x p + \mathbf{j} = \left(\zeta + \frac{4\eta}{3}\right) \Delta_x \mathbf{v}, \tag{65}$$

and eventually this equation states that for the correspondence between both equations to hold, the mass flux added to the gradient of the pressure must be equal to the viscosity term in the Navier–Stokes equation. Equation (65) may be written as:

$$(\Delta_x - \tilde{\rho})\, \mathbf{v} = \nabla_x \tilde{p}, \tag{66}$$

where

$$\tilde{\rho} = \frac{\rho}{\zeta + \frac{4\eta}{3}}, \tag{67}$$

and

$$\tilde{p} = \frac{p}{\zeta + \frac{4\eta}{3}}. \tag{68}$$

If, for instance, the potential is constant, then Eq. (66) is reduced to:

$$\Delta_x \mathbf{v} = \nabla_x \tilde{p}, \tag{69}$$

which states that for the correspondence to hold, the gradient of the pressure must compensate the viscosity term in the Navier–Stokes equation. For a highly viscous

fluid which appear as solid, we obtain the Laplace equation $\Delta_x \mathbf{v} = 0$. All the previous equations are interesting since they offer new constraints of viscosity solutions. Altogether, one may set up the following statement:

Statement 5 *Given the singular Lagrangian* $L = \sqrt{2}(\dot{x} - x)/2$ *and the classical action* $S = \int L^2 dt$. *Assuming that there exist a complex scalar field* $\psi = e^{iS/\hbar}$ *and that the velocity field of the quantum probability is* $\mathbf{v} = \nabla_x S = \nabla_x \theta$, *then for the case of a linear potential, the Hamilton–Jacobi equation leads to a modified Schrödinger equation which coincides with the Navier–Stokes equation for irrotational and viscous flow. This connection holds only if the mass flux added to the gradient of the pressure must match the viscosity term in the Navier–Stokes equation.*

Obviously, the quantum potential is absent in our framework and hence our treatment is classical since its occurrence accounts for most of the differences between classical and quantum physics. The theory describes here describes therefore classical fluids and not quantum fluids.

In Table 1, we summarize for lucidity reasons the main differences between our appraoch and the basic formal aspects mainly described in [36,70].

Hence, we have shown that if the modified Schrödinger equation is valid to describe a physical state of a fluid particle, then it coincides with the Navier–Stokes equation and besides the modified continuity equation implies the existence of the velocity potential. It is important to point out a distinction between our approach and the

Table 1 Comparison between the power-law NSL approach and basic formal aspects

	Power-law NSL $L^{1+\varepsilon}$ approach	Basic formal aspects
Hamiltonian for $\varepsilon = 1$ and $a^2 = 1/2$	$H = -\frac{1}{2}\Delta_x - i\sqrt{2}V(x)\nabla_x$	$H = -\frac{1}{2}\Delta_x + V(x)$
Action of the theory	$S = \int L^{1+\varepsilon} dt = \int (a\dot{x} - V(x))^{1+\varepsilon} dt$	$S = \int L_S dt$ L_S : Schrödinger Lagrangian
Complex wave function	$\psi = Ae^{i\theta} = Ae^{iS}$	$\psi = Ae^{i\theta} = Ae^{iS}$
Continuity equation	$\frac{\partial \rho}{\partial t} + div J + \sqrt{2}V(x)\nabla_x \rho = 0$	$\frac{\partial \rho}{\partial t} + div J = 0$
Velocity field	$\mathbf{v} = \nabla_x S = \nabla_x \theta$	$\mathbf{v} = \nabla_x S = \nabla_x \theta$
Dynamical equation of $\mathbf{v}$	$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla_x)\mathbf{v} - \sqrt{2}\mathbf{v}\nabla_x V(x) = 0$	$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla_x)\mathbf{v} = -\nabla_x U - \nabla_x V$ $U = -\frac{1}{2}\frac{\Delta_x \lvert\psi\rvert}{\lvert\psi\rvert}$ is the quantum potential
Condition for the hydrodynamic correspondence for the case of a linear potential $V(x) = \sqrt{2}x/2$	$\nabla_x p + \mathbf{j} = \left(\zeta + \frac{4\eta}{3}\right)\Delta_x \mathbf{v}$ the mass flux added to the gradient of the pressure must match the viscosity term in the Navier–Stokes equation	the pressure gradient in the Navier–Stokes equation $\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla_x)\mathbf{v} = -\frac{1}{\rho}\nabla_x P - \nabla_x V$ is associated with the gradient term in $\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla_x)\mathbf{v} = -\nabla_x U - \nabla_x V$ suggesting a relationship between the density and pressure

one described in [50]. The later started from the standard Schrödinger equation, then introduced the complex wave function $\psi = Ae^{i\theta} = Ae^{iS}$ and the velocity field of the quantum probability $\mathbf{v} = \nabla_x S = \nabla_x \theta$ which permitted to prove a connection between the continuity equation and the Schrödinger equation; this correlation facilitated to establish in the case of an incompressible fluid with a constant viscosity a relation between the Navier–Stokes equation and the reaction-diffusion equation for the wave function of the de Broglie's wave associated to the moving fluid substance. Whereas in our arguments, we have started from the singular Lagrangian $L = a\dot{x} - V(x)$, then we have constructed the corresponding Hamiltonian which permitted us to derive a modified Schrödinger equation after connecting the action to a wave function ψ by the relation $\psi = e^{iS}$. In particular for $\varepsilon = 1$, $a^2 = 1/2$ and a linear potential, a correlation between the modified Schrödinger and the Navier–Stokes equations exist only if the mass flux added to the gradient of the pressure balance the viscosity term in the Navier–Stokes equation.

The following observation holds: in fact, by introducing the new complex potential $U(x) = -i\sqrt{2}V(x)\nabla_x\psi/\psi$, Eq. (52) will be reduced to the linear Schrödinger equation:

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\Delta_x\psi - U(x)\psi. \tag{70}$$

This complex potential differs from those introduced in Bohm's and Madelung's approaches to quantum mechanics since it is coupled to the classical potential. In that case, the standard Hamiltonian operator of the Schrödinger equation is characterized by the presence of a quantum complex potential $U(x)$ and not the classical potential $V(x)$. This is in contrast to Bohm's approach where both the classical and the quantum potential are present in the Hamiltonian. Using Bohm's suggestion and letting again $\psi = Ae^{iS}$, the linear complex Schrödinger equation is splitted to the following two real equations:

$$\frac{\partial S}{\partial t} + \frac{(\nabla_x S)^2}{2} - \frac{1}{2}\frac{(\nabla_x A)^2}{A} + W(x) = 0, \tag{71}$$

$$\frac{\partial A^2}{\partial t} + \nabla\cdot\left(A^2\nabla S\right) = 0, \tag{72}$$

where

$$W(x) = -i\sqrt{2}V(x)\left(\frac{\nabla_x A}{A} + \nabla_x S\right). \tag{73}$$

Obviously, the Bohm's quantum potential is modified and is given by:

$$Q(x) = -\frac{1}{2}\frac{(\nabla_x A)^2}{A} - i\sqrt{2}V(x)\left(\frac{\nabla_x A}{A} + \nabla_x S\right). \tag{74}$$

Obviously, it is a complexified quantum potential. In short, we can say that in the Bohmian interpretation, the particle is under the influence of A, S and the classical potential which is coupled simultaneously to A and S. In the absence of the classical potential, $Q(x)$ is reduced to the standard Bohm's quantum potential. The following statement then holds accordingly:

Statement 6 *Starting from the Schrödinger equation* $i\partial\psi/\partial t = -\Delta_x\psi/2 - U(x)\psi$ *with* $U(x) = -i\sqrt{2}V(x)\nabla_x\psi/\psi$ *and following the Bohm's approach, we get a complexified quantum potential which is coupled to* A, S *and the classical potential.*

Introducing once more the velocity field of the quantum probability $\mathbf{v} = \nabla_x S$, then the gradient of Eq. (71) for irrotational flow gives:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla_x)\,\mathbf{v} + \nabla_x Q = 0. \tag{75}$$

Comparing this equation with Eq. (63) gives:

$$\frac{1}{\rho}\nabla_x p - \frac{\zeta + \frac{4\eta}{3}}{\rho}\Delta_x \mathbf{v} = \nabla_x Q. \tag{76}$$

and eventually this equation states that for the correspondence between Eqs. (63) and (75) to hold, the gradient of the pressure must match the viscosity term in the fluid's equations of motion added to the gradient of the Bohm's quantum potential. These results are summarized in Table 2 and compared with the results obtained in [36]:

Our approach has two different hydrodynamical correspondences: the 1st one is free from quantum potential (summarized in Table 1) and the 2nd one is characterized by a modified complexified quantum potential (summarized in Table 2). Although the quantum potential has multiples implications in different fields of sciences, e.g. solid state physics and theoretical chemistry, its nature is still not understood [51]. In the basic formal aspects the quantum potential arises from the action of Laplacian operator on the wave function in the standard Schrödinger equation and it is therefore it is interpreted as a kinetic-like energy term [62].

One more point is that Eq. (64) is absent in the arguments of [50]. The similarity between both approaches lies on the form of the velocity of the fluid particle which is given by $\mathbf{v} = \nabla_x S = \nabla_x\theta = -i\nabla_x\psi/\psi$ known as the Cole-Hopf transformation and which is used to linearize the Burgers equation. Replacing the relation $\mathbf{v} = \nabla_x\theta$ into Eq. (63) and introducing a new scalar function $\bar{\psi}$ such that $\theta = -2((3\zeta + 4\eta)/3\rho)\ln\bar{\psi}$, then the Navier–Stokes equation is reduced to the reaction-diffusion equation:

$$\frac{\partial\bar{\psi}}{\partial t} - \nu\Delta_x\bar{\psi} = \frac{1}{2\left(\zeta + \frac{4\eta}{3}\right)\rho}\bar{\psi}\Delta p, \tag{77}$$

also known as the Einstein–Kolmogorov equation [50]. Here $\Delta p = p - p_0$, i.e. the difference between the actual pressure and the initial pressure. It is notable that $\bar{\psi}$ and ψ represent the same state of the particle since $\theta = -2((3\zeta + 4\eta)/3\rho)\ln\bar{\psi} = -i\ln\psi$. One can check that Eq. (77) may also be written as:

$$\frac{\partial\psi}{\partial t} - \nu\Delta_x\psi = \frac{1}{2\left(\zeta + \frac{4\eta}{3}\right)\rho}\psi\Delta p, \tag{78}$$

Table 2 Comparison between the power-law NSL approach with a new complex quantum potential and Bohm's approach

	Power-law NSL $L^{1+\varepsilon}$ approach with a new complex quantum potential	Bohm's approach and Schrödinger versus Navier–Stokes
Schrödinger equation	$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\Delta_x\psi - U(x)\psi$ $U(x) = -i\sqrt{2}V(x)\frac{\nabla_x\psi}{\psi}$	$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\Delta_x\psi - V(x)\psi$
Quantum potential	$Q(x) = -\frac{1}{2}\frac{(\nabla_x A)^2}{A} - i\sqrt{2}V(x)\left(\frac{\nabla_x A}{A} + \nabla_x S\right)$	$Q(x) = -\frac{1}{2}\frac{(\nabla_x A)^2}{A}$
Complex wave function	$\psi = Ae^{i\theta} = Ae^{iS}$	$\psi = Ae^{i\theta} = Ae^{iS}$
Continuity equation	$\frac{\partial\rho}{\partial t} + divJ = 0$	$\frac{\partial\rho}{\partial t} + divJ = 0$
Velocity field	$\mathbf{v} = \nabla_x S = \nabla_x\theta$	$\mathbf{v} = \nabla_x S = \nabla_x\theta$
Dynamical equation of $\mathbf{v}$	$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla_x)\mathbf{v} + \nabla_x Q = 0$	$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla_x)\mathbf{v} = -\nabla_x U - \nabla_x V$ $U = -\frac{1}{2}\frac{\Delta_x\lvert\psi\rvert}{\lvert\psi\rvert}$ is the quantum potential
Condition for the hydrodynamic correspondence	$\frac{1}{\rho}\nabla_x p - \frac{\zeta+\frac{4\eta}{3}}{\rho}\Delta_x\mathbf{v} = \nabla_x Q$ the gradient of the pressure must match the viscosity term in the fluid's equations of motion added to the gradient of the Bohm's quantum potential	the pressure gradient in the Navier–Stokes equation $\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla_x)\mathbf{v} = -\frac{1}{\rho}\nabla_x P - \nabla_x V$ is associated with the gradient term in $\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla_x)\mathbf{v} = -\nabla_x U - \nabla_x V$ suggesting a relationship between the density and pressure

which is "the reaction-diffusion equation for the wave function corresponding to the de Broglie's wave associated to the moving fluid particle with velocity $\mathbf{v}$ [50]". Although there are a number of similarities between our approach and the one discussed in [50], the arguments differ totally as stated previously. Substituting $\mathbf{v} = \nabla_x\theta$ into Eq. (69) which describes the motion of the particle in constant potential field, we obtain using:

$$\Delta_x\mathbf{v} = \nabla_x(\nabla_x\cdot\mathbf{v}) - \nabla_x\times\nabla_x\times\mathbf{v} = \nabla_x\Delta_x\theta, \tag{79}$$

since $\nabla_x\times\nabla_x\equiv 0$:

$$\nabla_x(\Delta_x\theta) = \nabla_x\tilde{p}, \tag{80}$$

or

$$\Delta_x\theta = \Delta\tilde{p}. \tag{81}$$

Substituting $\theta = -i\ln\psi$ into Eq. (81) gives:

$$(\Delta_x - i\Delta\tilde{p})\psi - \frac{(\nabla_x\psi)^2}{\psi} = 0, \tag{82}$$

which is a nonlinear 2nd-order differential equation where the solution is given by:

$$\psi(x) = c_2 e^{c_1 x + \frac{i\Delta\tilde{p}x^2}{2}}. \tag{83}$$

$c_i, i = 1, 2, ...$ are constants of integration. Assuming initial conditions $\psi(0) = 1$ and $\psi'(0) = 0$, the solution is given by $\psi(x) = e^{i\Delta\tilde{p}x^2/2}$ and it differs completely from the standard solution of the 2nd-order linear differential equation $(\Delta_x + k^2)\psi = 0$ which is $\psi_{\pm}(x) \propto e^{\pm ikx}$ where $k = \sqrt{2E}$, E being the total energy of the particle (in units $\hbar = c = 1$ and assuming the mass of the particle equal to unity). In Fig. 1, we plot the numerical solutions of $\psi(x) = e^{i\Delta\tilde{p}x^2/2}$ and $\psi(x) = e^{ikx}$ for $\Delta\tilde{p} = k = 1$:

We observe that the solution $\psi(x) = e^{ix^2/2}$ corresponds to fast oscillations (super-oscillatory final state) in contrast to the normal oscillatory mode of $\psi(x) = e^{ikx}$.

Remark 3.2 *If we introduce a new complex wave function ϕ such that $\psi = e^{i\alpha\phi}$, $\alpha \in \mathbb{R}$, then* Eq. (82) *is reduced to the Poisson equation $\alpha\Delta_x\phi = \Delta\tilde{p}$ and hence the $\Delta\tilde{p}/\alpha$ is reduced to a source term. For $\alpha = 1$ then $\psi \equiv \theta$.*

In order to show the difference between the modified Schrödinger equation obtained in our framework and the standard one used for classical waves and fluids, we introduce the new complex wave function $\phi = V(x)\psi$ which permits us to write Eq. (52) after some algebra as:

$$i\frac{\partial\phi}{\partial t} = -\frac{1}{2}\Delta_x\phi + \left(\frac{1}{2V}\frac{d^2V}{dx^2} - \frac{1}{V^2}\left(\frac{dV}{dx}\right)^2 + i\sqrt{2}\frac{dV}{dx}\right)\phi + \left(\frac{1}{V}\frac{dV}{dx} - i\sqrt{2}V\right)\nabla_x\phi. \tag{84}$$

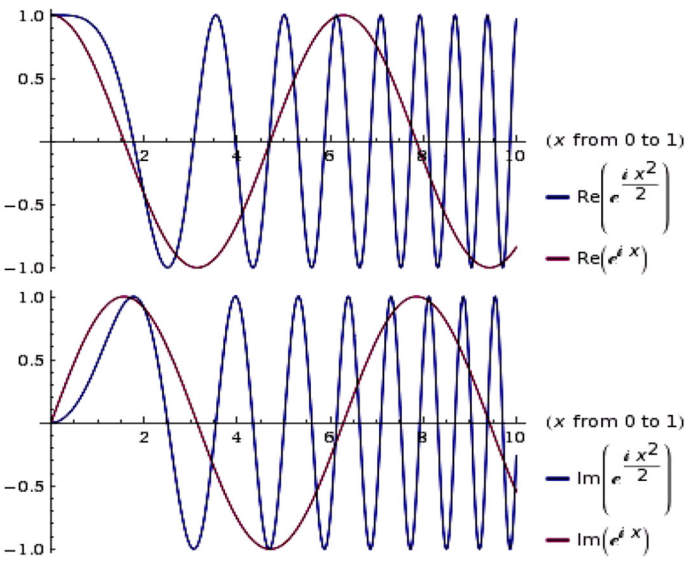

Fig. 1 Variations of real and imaginary parts of $\psi(x) = e^{ix^2/2}$ and $\psi(x) = e^{ix}$

If we choose the imaginary inverse potential $V(x) = ix^{-1}/\sqrt{2}$, then Eq. (84) is reduced to:

$$i\frac{\partial\phi}{\partial t} = -\frac{1}{2}\Delta_x\phi + \frac{1}{2x^2}\phi, \tag{85}$$

which amazingly is similar the Schrödinger equation in the presence of an inverse square real potential. In fact, Schrödinger equation with imaginary potentials was explored in literature to study the propagation of wave in gain media [53,75]. Inverse potentials are of interest since they represent an intermediate threshold between those potentials giving rise to ordinary stationary states and singular potentials particle falling to the center with a non-lower-bounded energy [25,41]. By the method of variables separation, we set $\phi = f(t)g(x)$ and we obtain easily:

$$i\frac{1}{f}\frac{\partial f}{\partial t} = -\frac{1}{2g}\Delta_x g + \frac{1}{2x^2} = E, \tag{86}$$

where E is a constant (identified to the energy). The LHS part of Eq. (86) gives $f(t) = e^{-iEt}$ whereas the RHS gives the following 2nd-order differential equation:

$$\Delta_x g + \left(2E - \frac{1}{x^2}\right) g = 0, \tag{87}$$

and the solution is given by:

$$g(x) = \sqrt{x}\left(c_3 J_{\frac{\sqrt{5}}{2}}\left(\sqrt{2E}x\right) + c_4 Y_{\frac{\sqrt{5}}{2}}\left(\sqrt{2E}x\right)\right). \tag{88}$$

$J_n(z)$ and $Y_n(z)$ are respectively the Bessel functions of 1st and 2nd kinds. The general solution is then given by:

$$\phi = e^{-iEt}\sqrt{x}\left(c_3 J_{\frac{\sqrt{5}}{2}}\left(\sqrt{2E}x\right) + c_4 Y_{\frac{\sqrt{5}}{2}}\left(\sqrt{2E}x\right)\right), \tag{89}$$

and its corresponding real and imaginary parts are plotted in Figs. 2 and 3 respectively after setting $E = 1$ and solving Eq. (87) with initial conditions $g(1) = 0$ and $g'(1) = 1$ for numerical illustration:

However, if we start from the standard Schrödinger equation and introduce the inverse complex potential $V(x) = ix^{-1}/\sqrt{2}$, it is easy to check that Eq. (85) will be reduced to:

$$i\frac{\partial\phi}{\partial t} = -\frac{1}{2}\Delta_x\phi + \frac{i}{\sqrt{2}x}\phi, \tag{90}$$

and the general solution is obtained from the method of variables separation and takes at the end the form:

$$\phi = xe^{-iEt}e^{-\sqrt{-2E}x}\left(c_5 {}_1F_1\left(1 + \frac{i}{4\sqrt{-E}}; 2; 2\sqrt{-2E}x\right)\right.$$
$$\left. + c_6 U\left(1 + \frac{i}{4\sqrt{-E}}; 2, 2\sqrt{-2E}x\right)\right). \tag{91}$$

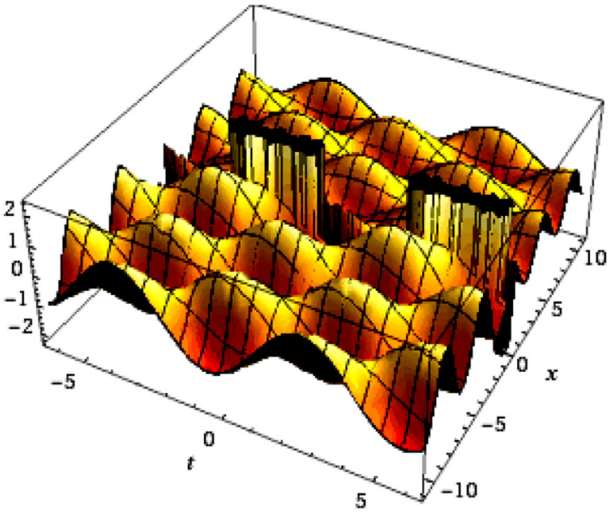

Fig. 2 Plot of the real part solution of Eq. (89)

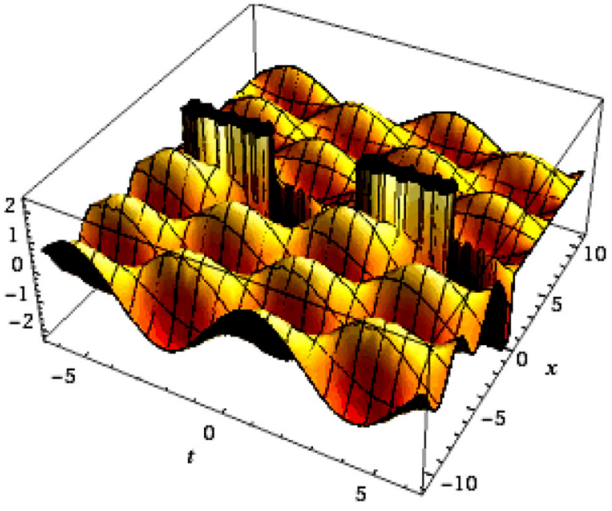

Fig. 3 Plot of the imaginary part solution of Eq. (89)

${}_1F_1(a;b;x)$ is the Kummer confluent hypergeometric function and $U(a,b,x)$ is the confluent hypergeometric function of the 2nd kind. The real and imaginary parts of Eq. (91) are plotted in Figs. 4 and 5 respectively after setting $E = 1$ and solving spatial differential equations for initial conditions $g(1) = 0$ and $g'(1) = 1$:

Obviously, the dynamics differ since Figs. 4 and 5 describe the solutions of the standard Schrödinger equation in the presence of an inverse complex potential whereas Figs. 2 and 3 describe the solutions of the modified Schrödinger equation in the presence of an inverse square real potential.

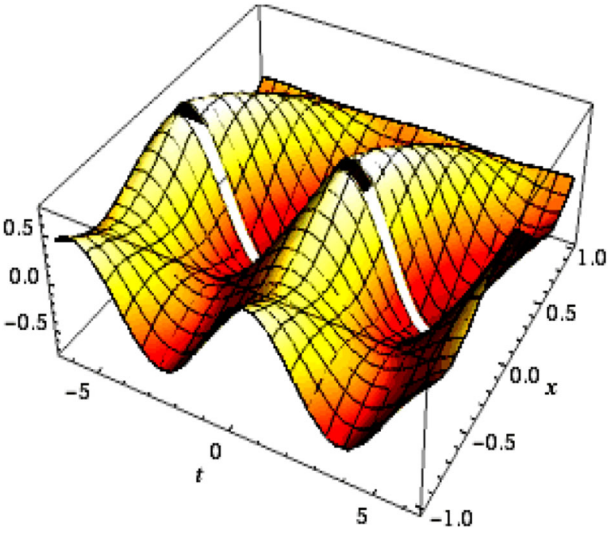

Fig. 4 Plot of the real part solution of Eq. (91)

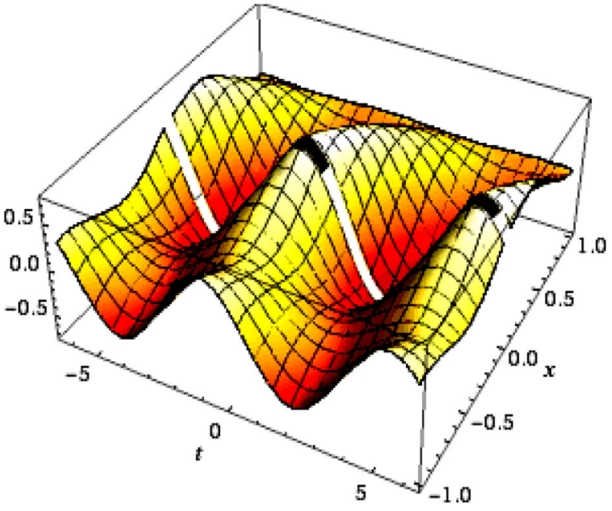

Fig. 5 Plot of the imaginary part solution of Eq. (91)

Note that in Eq. (85), the Schrödinger operator $-\Delta_x + x^{-2}$ is scaling invariant and represents a mathematical mean for certain important spectral inequalities such as Strichartz estimates [35,49]. Inverse square potentials appear also in association with 2D Schrödinger operators with Aharonov–Bohm-type magnetic field [37,38]. The following statement then holds:

Statement 7 *Starting from a singular Lagrangian and introducing the new complex wave function* $\phi = V(x)\psi$ *where* $\psi = e^{iS}$, $S = \int L^2 dt$ *and* $V(x) = ix^{-1}/\sqrt{2}$,

we obtain a modified Schrödinger equation characterized by a scaling invariant Schrödinger operator $-\Delta_x + x^{-2}$.

At the end we note that Eq. (84) may be written as: $E\phi = H\phi$ where $H = p^2/2 + i\sigma p + U(x)$. Here $U(x)$ represents the terms inside the 1st parenthesis in the RHS of Eq. (84) and σ represents the terms inside the 2nd parenthesis of the same equation. We observe that in contrast to the relativistic case where the Hamiltonian involves quadratic and quartic terms on the momentum [58], the Hamiltonian involves in our framework quadratic and linear terms on p.

4 Conclusions

In this paper, we have discussed the power of singular Lagrangians which are identified to non-standard Lagrangians. We have established seven different statements which prove that the Hamilton–Jacobi formulation of the power of singular Lagrangians offers new features not realized in the standard approach although singular Lagrangians for a real system are not reported in classical mechanics. The Hamilton–Jacobi equation was derived from the common procedure and various differential equations which exhibit various properties were obtained for different forms of singular Lagrangians. At the classical level, it was observed that for special forms of complexified singular Lagrangians, the corresponding dynamical system holds a number of interesting properties which are pointed out in Statements 1 to 4. These statements offer a number of motivating features which support the physics of NSL in general and singular Lagrangians in particular. Nevertheless, the main outcomes of the present work concern: 1st: the connection between the modified Schrödinger and the Navier–Stokes equations for irrotational and viscous flow starting from a singular Lagrangian characterized by a linear potential (pointed out in Statement 5); 2nd: the emergence of a modified complexified quantum potential which is coupled to the wave amplitude, the gradient of the action or the velocity field and the classical potential (pointed out in Statement 6), 3rd: the emergence of a modified Schrödinger equation characterized by the scaling invariant Schrödinger operator $-\Delta_x + x^{-2}$ which is used in spectral theories (pointed out in Statement 7). Our approach is characterized by two different views: the 1st one is free from quantum potential (Bohm's theory without quantum potential) and still offers a correspondence between the Schrödinger equation and the Navier–Stokes equation whereas the 2nd one is characterized by a modified complex quantum potential. Each view holds its corresponding conditions for the hydrodynamic correspondence. Our results led to another approach for solving classical problems characterized by singular complexified Lagrangians. In literature, there exist a number of classical and quantum models in which the equations of motion are derived from singular Lagrangians. However, our mathematical framework is totally different since the action, the corresponding Euler–Lagrange equation, the Hamiltonian and the canonical equations of motions hold different mathematical structures. In this paper, we deepened into our formalism and demonstrated that singular Lagrangians offer new features at both classical and quantum levels that deserve to be taken seriously. It will be of interest to expand the present work to additional dynamical problems, using

different types of potentials and demonstrate further the importance of singular and NSL in classical and quantum dynamical systems.

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